

OPTION VALUATION UNDER A REGIME-SWITCHING MODEL USING THE
FAST FOURIER TRANSFORM

by

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ABSTRACT

ELHAM SOHRABI. Option Valuation Under A Regime-Switching Model Using the Fast Fourier Transform. (Under the direction of Dr. Adriana Ocejo Monge)

The global financial crisis had severe implications on the real economy. For the US alone, Luttrell et al. [40] estimate output losses in the range of 6 to 14 trillion USD. It is hence not surprising that policy makers are keen to develop models which can issue warning signals ideally sufficiently early to implement policies that increase the resilience of financial institutions and ultimately mitigate at least some of the risks and costs associated with financial crises. Hence regime-switching models have been used extensively to identify business cycle turning points. Specifically, regime-switching models have the capability to incorporate the changes of the model dynamics brought by the changing macroeconomics conditions. Regime-switching models typically use the states of a modulating Markov chain to represent the states of an economy, depicted by some macroeconomics indicators. By adopting this methodology, regime-switching models can incorporate the impacts of structural changes in macroeconomics conditions on asset price dynamics and the stochastic evolution of investment opportunity sets, for example. Consequently, it is practical to consider the valuation of financial derivatives under regime-switching models.

In this thesis, we consider valuation of different types of options where the underlying asset price or commodity spot price is governed by a regime-switching model. We adopt an observable, continuous-time, finite-state Markov chain. We mostly focus on obtaining analytical formula of the so-called characteristic function for logarithm of commodity spot price, futures price and stock price.

Chapter 1 is organized as follows. Section 1 describes options and its pricing model. Section 2 provides a literature review for regime-switching model, stochastic interest rate models and fast Fourier transform (FFT). Section 3 presents a brief

introduction of the basic definitions and mathematical tools to be used in this thesis such as Markov chain setup model and FFT. We describe how to obtain analytical pricing formula using inverse Fourier transform and discretize the pricing formula via FFT.

The outline of Chapter 2 is as follows. In Section 1, we first briefly introduce the motivation behind developing a fast Fourier transform approach for option pricing when the underlying asset process is governed by a regime-switching model. Section 2 describes the risk-neutral world and the asset price dynamics where under risk-neutral probability measure follows a regime-switching geometric Brownian motion. Section 3 presents the derivation for obtaining an analytical pricing formula for the two-state case and general case via the inverse Fourier transform. Then Section 4 calculates the inverse Fourier transform via FFT, providing an easier and faster way to calculate options prices. Section 5 introduces other numerical methods to compare with FFT results. As usual, we try to implement Monte Carlo simulation since it frequently serves as a benchmark for testing other numerical methods. Furthermore, a novel semi-Monte Carlo simulation algorithm is presented by Liu et. al. at [39] that can also be used as benchmark values in numerical experiments. To price our path dependent European call options, we require the stock price trajectory $\{S_t\}_{t \in [0, T]}$. Finally, we reported numerical results in Section 6 & 7 and provide further remarks and conclusions about the chapter in section 8. All of our proofs and Python programming are placed in the Appendix. It is our hope that this information when combined with some familiarity with the language, or at least an error checking IDE, that any reader will be able to replicate our results with little trouble.

In Chapter 3, we first state our motivation in section 1. Section 2 presents the Markovian regime-switching Ornstein-Uhlenbeck model. In this section, we discuss a Markovian regime-switching extension to the Ornstein-Uhlenbeck model for evaluating European-style commodity options and futures options. The main feature of our

model is that model parameters, the mean-reverting level and the volatility of the commodity spot price, are governed by an observable continuous-time, finite-state, Markov chain. In Section 3, we first consider the valuation of commodity options and then the valuation of commodity futures options using inverse Fourier transform. The final section provides concluding remarks. All proofs in this chapter are standard and involve the use of standard mathematical techniques.

We also extend our work in chapter 4 to investigate the pricing of European-style commodity options and futures options with a Markovian regime-switching Vasicek stochastic interest rate model. The parameters of this model, including the mean-reversion level, the volatility of the stochastic interest rate, and the volatility of the commodity spot price are modulated by an observable, continuous-time, finite-state Markov chain. We start with introducing a risk-neutral probability measure. To take the zero-coupon bond value as the numéraire, a measure change technique is applied to change the risk-neutral probability measure into a T -forward measure. We then obtained a closed-form expression for the characteristic function of the logarithmic commodity price and futures price. Eventually, chapter 5 shows future directions and some potential future works.

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CHAPTER 1: INTRODUCTION

1.1 Overview

A “derivative” is a contract between two or more parties whose value is based on an agreed-upon underlying financial asset (like a security) or set of assets (like an index). Common underlying instruments include bonds, commodities, currencies, interest rates, market indexes and stocks. Derivatives have been about to be as the virtually important monetary instruments for centuries. The valuation of derivatives has been a long-lasting issue. There are many different types of derivatives. The most common derivative types are futures contracts, forward contracts, options and swaps. Amongst the different kinds of derivatives, options play a carrying a lot of weight role in the financial market. Options contracts have been known for decades. The Chicago Board Options Exchange was established in 1973, which set up a regime using standardized forms and terms and trade through a guaranteed clearing house. Trading activity and academic interest has increased since then (Brealey and Myers [4]).

Today, many options are created in a standardized form and traded through clearing houses on regulated options exchanges, while other over-the-counter options are written as bilateral, customized contracts between a single buyer and seller, one or both of which may be a dealer or market-maker. Hull [31] defines an option as follows: “An option is a financial derivative that represents a contract sold by one party (the option writer) to another party (the option holder). The contract offers the buyer the right, but not the obligation, to buy (call) or sell (put) a security or other financial

asset at an agreed-upon price (the strike price) during a certain period of time or on a specific date (exercise date)".

There are many different types of options that can be traded and these can be categorized in a number of ways. In a very broad sense, there are two main types: calls and puts. Calls give the buyer the right to buy the underlying asset, while puts give the buyer the right to sell the underlying asset. Along with this clear distinction, options are also usually classified based on whether they are American style or European style. This has nothing to do with geographical location, but rather when the contracts can be exercised.

Options can be further categorized based on the method in which they are traded, their expiration cycle, and the underlying security they relate to. There are also other specific types and a number of exotic options that exist. Options and financial products with embedded-option features have become so important that we can hardly find an investment portfolio without these products. Consequently, the valuation of such well known financial derivatives deserves in a superior way attention.

After the introduction of well-known Black and Scholes [3]'s work, option valuation has played a vital role in the development of modern finance. The valuation of options has been a theoretically and practically important topic in the area of finance. In mathematical finance, the Black-Scholes equation is a partial differential equation (PDE) governing the price evolution of a European call or European put under the Black-Scholes model. The well-known Black-Scholes-Merton have attracted a lot of attention for quite a while due to the easy implementation of the closed-form option pricing formula. However, numerous empirical studies have revealed that the Black-Scholes-Merton model doesn't satisfy the ability to describe some vital features of the underlying assets, like no dividends are paid out during the life of the option, there are no transaction costs in buying the option, the risk-free rate and volatility of the underlying are known and constant, the returns on the underlying

are normally distributed. In order to overcome all the mentioned shortcomings and improve the efficiency of the Black-Scholes-Merton model, both academic researchers and industry practitioners have dedicated efforts to extend the Black-Scholes-Merton model in various possible directions, including jump-diffusion models (Merton [42]), stochastic volatility models (Hull and White [32]; Wiggins [50]; Heston [30]; etc.), regime-switching models, etc.

1.2 Literature review

In this dissertation, we investigate options valuation under Regime-Switching (RS) model with stochastic interest rate using the fast Fourier transform (FFT). Therefore, it's beneficial to know about the background of RS model, Stochastic interest rate and FFT in the following subsections. This section presents a brief literature review of RS model, Stochastic interest rate and FFT to be used in the subsequent chapters.

1.2.1 Regime-Switching model

The global financial crisis had severe implications on the real economy. For the US alone, Luttrell et al. [40] estimate output losses in the range of 6 to 14 trillion USD. It is hence not surprising that policy makers are keen to develop models which can issue warning signals ideally sufficiently early to implement policies that increase the resilience of financial institutions and ultimately mitigate at least some of the risks and costs associated with financial crises. Hence RS models have been used extensively to identify business cycle turning points. Specifically, RS models have the capability to incorporate the changes of the model dynamics brought by the changing macroeconomics conditions. Consequently, RS models have attracted considerable interests and have been applied to various financial areas, such as option pricing, bond pricing, stock returns, etc. The history of RS models can be traced back to the works of Quandt [44] and Goldfeld and Quandt [27]. The RS model by Hamilton [29]

is one of the most popular nonlinear time series models in the literature.

1.2.2 Stochastic interest rate models

An interest rate is the rate at which interest is paid by a borrower for the use of money that they borrow from a lender. Interest rates are fundamental to a capitalist society. Interest rates are normally expressed as a percentage rate over the period of one year. Interest rates are also a tool of monetary policy and are taken into account when dealing with variables like investment, inflation, and unemployment. In traditional actuarial investigations, the interest rate is assumed to be deterministic and hence there is only one source of uncertainty, the mortality uncertainty, to be considered. Concerns about the effects of including a stochastic interest rate in the model have been growing during the last decade. The literature has tended to focus on annuities and the model adopted to describe the interest rate uncertainty, in a continuous framework, has usually involved the use of a Brownian motion. When the market rates are high, volatility is expected to be high or when interest rates are low, volatility will be low. Therefore, different stochastic interest rate models have been proposed and helped to overcome the disadvantage of the constant interest rate assumption under the Black-Scholes-Merton model. Some popular stochastic interest rate models include those proposed by Vasicek [49], Cox et al. [11], Hull and White [33], among others. One common feature of these models is the mean-reverting property of the interest rate. The short-term effectiveness of these models were justified by many empirical studies. Due to the advantages of RS models, it is reasonable to expect that RS stochastic interest rate models may improve the long-term effectiveness of the existing stochastic interest rate models. Examples of RS stochastic interest rate diffusion models can be found in Elliott and Mamon [17], Elliott and Wilson [22] and Elliott and Siu [18]. By adopting the method of stochastic flows, Siu [48] considered the valuation of a bond under a jump-augmented Vasicek model. A partial differential

equation approach was applied in Shen and Siu [46] to obtain an exponential affine formula for a zero-coupon bond.

1.2.3 The fast Fourier transform

Various techniques have been devised to determine the valuation of financial derivatives. Among all, the Fourier transform has been widely applied to the valuation of financial derivatives. The faster calculation speed of the discrete Fourier transform against for example monte carlo simulation may be one of the main reasons why FFT method attracts so much attention from both academics and industry. The first of these Fourier methods is actually the application of the Gil-Palaez inversion formula to finance. This idea originates from Heston [30]. However, singularities in the integrand prevent it to be an accurate method. The second attempt, more recent technique, was first proposed by Carr and Madan [7] by applying the FFT method to price European-style options under the variance gamma (VG) model. Since then, the FFT method has been applied to the valuation of options under different models. For example, Benhamou [1] discussed the valuation of discrete Asian options in non-lognormal density cases. Dempster and Hong [13] presented a two-dimensional FFT and considered the valuation of spread options under a three-factor stochastic volatility model. Černý [7] discussed applications of the FFT in finance. By adopting the FFT technique, Liu et al. [39] investigated the valuation of options under a regime-switching model and Wong and Guan [51] considered the valuation of American options under a Lévy process.

FFT relies on the availability of the so-called characteristic function of the logarithm of the stock price. Given any such characteristic function, one can develop a simple analytic expression for the Fourier transform of the option value. Indeed, for a wide class of stock models characteristic functions have been obtained in closed form even if the risk-neutral densities (or probability mass functions) themselves are not

available explicitly such as all the mentioned studies in the above paragraph. The tremendous speed of the FFT allows option prices for a huge number of strikes to be evaluated very rapidly. Although the FFT approach is significantly faster than other numerical methods, such as finite difference method and Monte Carlo simulation, it still has approximation errors when we adopt a discrete sum to approximate the integral. To control the approximation errors, Carr and Madan [6] discussed the selection of the upper limit of the integral and gave a sufficient condition to guarantee the square integrability property of the dampened pricing formula. Numerical errors in discretizing the pricing formula were discussed in Lee [37]. Liu et al. [39] also showed that the errors are small.

1.3 Preliminaries

This section presents a brief introduction of the basic definitions and tools to be used in the subsequent chapters. The contents in this section are mainly based on Elliott et al. [15] and Carr and Madan [6].

1.3.1 Markov chain

Let's consider a continuous-time economy with a finite time horizon \mathcal{T} , i.e., $\mathcal{T} := [0, T]$, where $T < \infty$. Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, where \mathcal{P} is a risk-neutral probability measure. We assume the state of an economy is modeled by a continuous-time, finite-state, observable Markov Chain $\alpha(\cdot) := \{\alpha(t) | t \in \mathcal{T}\}$ defined on $(\Omega, \mathcal{F}, \mathcal{P})$. The state space of the chain is denoted by $S := \{s_1, s_2, \dots, s_N\}$, representing N different observable states of an economy. Without loss of generality, using the convention in Elliott et al. [15], we identify the state space of the chain with a finite set of standard unit vectors $\varepsilon := \{e_1, e_2, \dots, e_N\} \subset \mathbb{R}^N$, where the j -th component of e_i is the Kronecker delta δ_{ij} , for each $i, j = 1, 2, \dots, N$. Let $Q := [q_{ij}]_{i,j=1,2,\dots,N}$ denote the generator or rate matrix of the chain $\alpha(\cdot)$. Then, Elliott et

al. [15] in Lemma 1.1 Appendix B obtained the following martingale representation for the chain $\alpha(\cdot)$:

$$\alpha(t) = \alpha(0) + \int_0^t Q \alpha(u) du + M(t). \quad (1.1)$$

Here $\{M(t)|t \in \mathcal{T}\}$ is an \mathbb{R}^N -valued martingale with respect to the natural filtration $\mathcal{F}^\alpha := \{\mathcal{F}^\alpha(t)|t \in \mathcal{T}\}$ generated by Markov chain $\{\alpha(t)|t \in \mathcal{T}\}$ under the probability measure \mathcal{P} .

1.3.2 The fast Fourier transform

FFT was introduced in Carr and Madan [6]. For the sake of completeness, we present a brief introduction of the application of FFT. Define by S_T the value of the underlying asset at the maturity time T and K the strike price. Suppose that the price of a T -maturity European-style call option at time 0 is given by

$$C(0, T, K) = \mathbb{E}\left\{e^{-rT}(S_T - K)^+\right\} \quad (1.2)$$

where $\mathbb{E}[\cdot]$ denote the expectation under the risk-neutral probability measure.

Let $X_T = \ln S_T$ and $\kappa = \ln K$ denote the logarithmic of the asset price at time T and the strike value, respectively. Then

$$C(0, T, \kappa) = \mathbb{E}\left\{e^{-rT}(e^{X_T} - e^\kappa)^+\right\}. \quad (1.3)$$

Now let's derive the Fourier transform of $C(0, T, \kappa)$. Assume $f(x)$ is the probability density function of X_T :

$$\hat{C}(0, T, u) = \int_{-\infty}^{\infty} e^{-iu\kappa} C(0, T, \kappa) d\kappa$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} e^{-iu\kappa} \mathbb{E}\{e^{-rT} (e^{X_T} - e^{\kappa})^+\} d\kappa \\
&= e^{-rT} \int_{-\infty}^{\infty} e^{-iu\kappa} \int_{-\infty}^{\infty} (e^x - e^{\kappa})^+ f(x) dx d\kappa \\
&= e^{-rT} \int_{-\infty}^{\infty} e^{-iu\kappa} \int_{\kappa}^{\infty} (e^x - e^{\kappa}) f(x) dx d\kappa \\
&= e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^x e^{-iu\kappa} (e^x - e^{\kappa}) d\kappa f(x) dx \tag{1.4}
\end{aligned}$$

where we changed the order of integration by Fubini's theorem and used the result $\int_x^{\infty} e^{-iu\kappa} (e^x - e^{\kappa})^+ d\kappa = 0$. If we evaluate the inner integral of (1.4), we have

$$\begin{aligned}
\int_{-\infty}^x e^{-iu\kappa} (e^x - e^{\kappa}) d\kappa &= \int_{-\infty}^x e^{-iu\kappa} e^x d\kappa - \int_{-\infty}^x e^{-iu\kappa} e^{\kappa} d\kappa \\
&= e^x \frac{e^{-iu\kappa}}{-iu} \Big|_{-\infty}^x - \frac{e^{(1-iu)\kappa}}{(1-iu)} \Big|_{-\infty}^x. \tag{1.5}
\end{aligned}$$

As we can see, the first term of (1.5) is undetermined due to $\lim_{\kappa \rightarrow -\infty} e^{-iu\kappa} \neq 0$, while in second term of (1.5) $\lim_{\kappa \rightarrow -\infty} e^{(1-iu)\kappa}$ converges to zero.

To get around the “undetermined” problem, Carr and Madan [6] introduced a damping parameter ρ to modify the call option price. The so-called dampened call price is defined as

$$c(0, T, \kappa) = \exp(\rho\kappa) C(0, T, \kappa).$$

The Fourier transform of the dampened call option price is given by

$$\hat{c}(0, T, u) = \int_{-\infty}^{\infty} e^{-iu\kappa} c(0, T, \kappa) d\kappa.$$

Next we are showing that the dampening parameter ρ can force convergence,

thereby permitting a computable Fourier transform.

$$\begin{aligned}
\hat{c}(0, T, u) &= \int_{-\infty}^{\infty} e^{-iu\kappa} c(0, T, \kappa) d\kappa \\
&= \int_{-\infty}^{\infty} e^{-iu\kappa} e^{\rho\kappa} \mathbb{E}\{e^{-rT} (e^{X_T} - e^{\kappa})^+\} d\kappa \\
&= e^{-rT} \int_{-\infty}^{\infty} e^{-iu\kappa} \int_{-\infty}^{\infty} e^{\rho\kappa} (e^x - e^{\kappa})^+ f(x) dx d\kappa \\
&= e^{-rT} \int_{-\infty}^{\infty} e^{\rho\kappa} e^{-iu\kappa} \int_{\kappa}^{\infty} (e^x - e^{\kappa}) f(x) dx d\kappa \\
&= e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^x e^{(\rho-iu)\kappa} (e^x - e^{\kappa}) d\kappa f(x) dx. \tag{1.6}
\end{aligned}$$

Let's compare the inner integral in (1.4) to (1.6). It can be seen that the term $\int_{-\infty}^x e^{-iu\kappa} (e^x - e^{\kappa}) d\kappa$ has been changed to $\int_{-\infty}^x e^{(\rho-iu)\kappa} (e^x - e^{\kappa}) d\kappa$.

Now if we evaluate the inner integral of (1.6), we have

$$\begin{aligned}
\int_{-\infty}^x e^{(\rho-iu)\kappa} (e^x - e^{\kappa}) d\kappa &= \int_{-\infty}^x e^{(\rho-iu)\kappa} e^x d\kappa - \int_{-\infty}^x e^{(\rho-iu)\kappa} e^{\kappa} d\kappa \\
&= e^x \frac{e^{(\rho-iu)\kappa}}{(\rho-iu)} \Big|_{-\infty}^x - \frac{e^{(\rho+1-iu)\kappa}}{(\rho+1-iu)} \Big|_{-\infty}^x \\
&= e^x \frac{e^{(\rho-iu)x}}{(\rho-iu)} - \frac{e^{(\rho+1-iu)x}}{(\rho+1-iu)} \\
&= \frac{e^{(\rho+1-iu)x}}{(\rho-iu)(\rho+1-iu)}. \tag{1.7}
\end{aligned}$$

Given $\rho > 0$, the exponential terms vanish for $\kappa = -\infty$:

$$\lim_{\kappa \rightarrow -\infty} e^{(\rho-iu)x} = \lim_{\kappa \rightarrow -\infty} e^{(\rho+1-iu)x} = 0. \tag{1.8}$$

One can easily prove that $\rho > 0$ is to ensure the square integrability for call options and $\rho < 0$ is to ensure the square integrability for put options.

The pricing formula for the European-style call option can be obtained via the

inverse Fourier transform as follows:

$$C(0, T, \kappa) = \frac{e^{-\rho\kappa}}{2\pi} \int_{-\infty}^{\infty} e^{iu\kappa} \hat{c}(0, T, u) du = \frac{e^{-\rho\kappa}}{\pi} \int_0^{\infty} e^{iu\kappa} \hat{c}(0, T, u) du. \quad (1.9)$$

where the second term comes from the fact that call option is a real number. This implies the Fourier transform $\hat{c}(0, T, u)$ is odd in its imaginary part so that

$$\text{Im}\{\hat{c}(0, T, u)\} = -\text{Im}\{\hat{c}(0, T, -u)\} \quad (1.10)$$

and even in its real part so that

$$\text{Re}\{\hat{c}(0, T, u)\} = \text{Re}\{\hat{c}(0, T, -u)\}. \quad (1.11)$$

Thus this allows to rewrite the pricing integral as the second term in (1.9).

To derive the Fourier transform of the dampened call option price, one standard way is to utilize the relationship between the Fourier transform of the dampened call option price and the characteristic function of the logarithmic asset price. In this thesis, under RS models, the conditional characteristic function of the logarithmic asset price given \mathcal{F}_T^α has to be derived first. To illustrate the method, we present the details in the present context.

Let $f_{\mathcal{F}_T^\alpha}(x)$ denote the conditional density function of X_T given \mathcal{F}_T^α . Then for each $t \in \mathcal{T}$ and $u \in \mathbb{R}$

$$\begin{aligned} \hat{c}(0, T, u) &= \int_{-\infty}^{\infty} e^{-iu\kappa} c(0, T, \kappa) d\kappa \\ &= \int_{-\infty}^{\infty} e^{-iu\kappa} e^{\rho\kappa} \mathbb{E}\{e^{-rT} (e^{X_T} - e^\kappa)^+\} d\kappa \\ &= \mathbb{E}\left\{ \int_{-\infty}^{\infty} e^{-iu\kappa} e^{\rho\kappa} \mathbb{E}\{e^{-rT} (e^{X_T} - e^\kappa)^+ | \mathcal{F}_T^\alpha\} d\kappa \right\} \\ &= \mathbb{E}\left\{ \int_{-\infty}^{\infty} e^{-iu\kappa} e^{\rho\kappa} e^{-rT} \int_{\kappa}^{\infty} (e^x - e^\kappa) f_{\mathcal{F}_T^\alpha}(x) dx d\kappa \right\} \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \left\{ \int_{-\infty}^{\infty} e^{-rT} f_{\mathcal{F}_T^\alpha}(x) \int_{-\infty}^x (e^x e^{(\rho-iu)\kappa} - e^{(1+\rho-iu)\kappa}) d\kappa dx \right\} \\
&= \mathbb{E} \left\{ \int_{-\infty}^{\infty} e^{-rT} f_{\mathcal{F}_T^\alpha}(x) \left(\frac{e^{(1+\rho-iu)x}}{(\rho-iu)} - \frac{e^{(1+\rho-iu)x}}{(1+\rho-iu)} \right) dx \right\} \\
&= \mathbb{E} \left\{ e^{-rT} \left(\frac{\phi_{\mathcal{F}_T^\alpha}(-i(1+\rho)-u)}{(\rho-iu)} - \frac{\phi_{\mathcal{F}_T^\alpha}(-i(1+\rho)-u)}{(1+\rho-iu)} \right) \right\} \\
&= \frac{e^{-rT} \mathbb{E} \left\{ \phi_{\mathcal{F}_T^\alpha}(-i(1+\rho)-u) \right\}}{(\rho-iu)(1+\rho-iu)} \tag{1.12}
\end{aligned}$$

where

$$\phi_{\mathcal{F}_T^\alpha}(\nu) = \mathbb{E}\{e^{i\nu X_T} | \mathcal{F}_T^\alpha\} = \int_{-\infty}^{\infty} e^{i\nu x} f_{\mathcal{F}_T^\alpha}(x) dx \tag{1.13}$$

is the conditional characteristic function of X_T given \mathcal{F}_T^α .

The third equality in (1.12) holds by the well-known property of conditional expectations ($\mathbb{E}\{\mathbb{E}\{X|Y\}\} = \mathbb{E}\{X\}$), and fifth equality holds by Fubini's theorem since the modified call price is bounded.

There are many ways to define the discrete Fourier transform (DFT), varying in the sign of the exponent, normalization, etc. Since we are going to use Python implementation, therefore we are following the same definition for DFT given in Python package numpy. The DFT is defined in Python is

$$A_k = \sum_{m=0}^{n-1} a_m \exp \left\{ -2\pi i \frac{mk}{n} \right\}, \quad k = 0, \dots, n-1. \tag{1.14}$$

The inverse DFT is defined as

$$a_m = \frac{1}{n} \sum_{k=0}^{n-1} A_k \exp \left\{ 2\pi i \frac{mk}{n} \right\} \quad m = 0, \dots, n-1.$$

It differs from the Fourier transform by the sign of the exponential argument and the default normalization by $\frac{1}{n}$.

Given the Fourier transform function $\hat{c}(0, T, u)$, the modified call option price

$c(0, T, \kappa)$ can be obtained by the inverse Fourier transform as described in (1.9)

$$c(0, T, \kappa) = \frac{\exp(-\rho\kappa)}{\pi} \int_0^\infty e^{iu\kappa} \hat{c}(0, T, u) du, \quad -\infty < \kappa < \infty. \quad (1.15)$$

Set $u_j = j\Delta_u$, $j = 0, 1, \dots, N-1$, where Δ_u is the grid size in the variable u . Then (1.15) can be approximated by the following summation:

$$c(0, T, \kappa) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} e^{iu_j\kappa} \hat{c}(0, T, u_j) \Delta_u. \quad (1.16)$$

Next, let Δ_κ be the grid size in κ and choose a grid along the log strike κ as below:

$$\kappa_l = (l - \frac{N}{2})\Delta_\kappa, \quad l = 0, 1, \dots, N-1. \quad (1.17)$$

Then

$$\begin{aligned} c(0, T, \kappa_l) &\approx \frac{1}{\pi} \sum_{j=0}^{N-1} e^{iu_j\kappa_l} \hat{c}(0, T, u_j) \Delta_u, \quad l = 0, 1, \dots, N-1 \\ &= \frac{1}{\pi} \sum_{j=0}^{N-1} e^{ij\Delta_u(l - \frac{N}{2})\Delta_\kappa} \hat{c}(0, T, u_j) \Delta_u, \quad l = 0, 1, \dots, N-1 \\ &= \frac{1}{\pi} \sum_{j=0}^{N-1} e^{ijl\Delta_u\Delta_\kappa} e^{-ij\frac{N}{2}\Delta_u\Delta_\kappa} \hat{c}(0, T, u_j) \Delta_u, \quad l = 0, 1, \dots, N-1. \end{aligned} \quad (1.18)$$

If we set

$$\Delta_u\Delta_\kappa = \frac{2\pi}{N}, \quad (1.19)$$

then we have

$$c(0, T, \kappa_l) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} e^{ijl\frac{2\pi}{N}} e^{-ij\pi} \hat{c}(0, T, u_j) \Delta_u, \quad l = 0, 1, \dots, N-1. \quad (1.20)$$

Using Simpson's rule for numerical integration, define a sequence of weighting factors by

$$w(j) = \begin{cases} \frac{1}{3}, & \text{if } j = 0, \\ \frac{4}{3}, & \text{if } j \text{ is odd,} \\ \frac{2}{3}, & \text{if } j \text{ is even.} \end{cases}$$

Then

$$c(k_l) \approx \frac{\Delta_u}{\pi N} \sum_{j=0}^{N-1} e^{ijl\frac{2\pi}{N}} e^{-ij\pi} \hat{c}(0, T, u_j) w(j) N, \quad l = 0, 1, \dots, N-1. \quad (1.21)$$

Comparing (1.21) with (1.14), it is easily seen that $c(0, T, \kappa)$ can be obtained by taking the Fourier transform of the sequence $\{e^{-ij\pi} \hat{c}(0, T, u_j) w(j) N\}$, for $j = 0, 1, \dots, N-1$.

The fast Fourier transform algorithm developed by Cooley and Tukey [9] and later extended by many others provide a more efficient algorithm for calculating DFT or inverse DFT with sample points that are powers of two. That is, $N = 2^p$, $p \in \{1, 2, \dots\}$. The Cooley-Tukey FFT algorithm can reduce the number of multiplications from N^2 to $N \log N$.

CHAPTER 2: FFT APPROACH FOR PRICING A EUROPEAN CALL OPTION UNDER A REGIME-SWITCHING MODEL

In this chapter, we introduce the FFT approach to option valuation, where the underlying asset price is governed by a regime-switching geometric Brownian motion (GBM).

2.1 Motivation

FFT is a numerical approach for pricing options which utilizes the characteristic function of the underlying instrument's price process. FFT is a significant computational method in scientific computing and it has been widely applied to financial engineering, specifically in options pricing. FFT approach makes use of the characteristic function of the underlying asset price. The use of the FFT method is motivated by the following reasons: the algorithm has speed advantage (especially over Monte Carlo Simulation and PDE). This enables the Fourier transform algorithm to calculate prices accurately for a whole range of strikes. The characteristic function of the log-price is known and has a simple form for many models considered in literature while the density is often not known in the closed form. The models meet this requirement include the stochastic volatility models, the affine jump diffusions, and the exponential Lévy models, among others; see Carr and Madan [6], Carr and Wu [7], and Duffie et al. [113] for detailed discussions of these models. However, FFT approach is only applicable to problems for which the characteristic functions of the underlying price process can be obtained analytically. Because of its prevalence, increasing research efforts have been devoted to the FFT approach in option pricing.

For example, Carr and Madan [6] illustrated the fundamental idea of using FFT for valuing European options based on the Black-Scholes setting and applied it to the variance gamma (VG) model (see Madan et al. [41]). Černý [7] presented a detailed discussion on the implementation of FFT to option pricing.

Along another line, considerable attention has been focused on the RS diffusion models for asset prices recently. In this setting, model parameters (rate of return, volatility, and risk-free interest rate) are assumed to depend on a finite-state, observable Markov chain, whose states represent different “states of the world” or regimes, which can describe various randomly changing economical factors. By incorporating an observable Markov chain into the formulation, the RS framework can capture the effect of those less frequent but significant events that have impact on the individual asset price behavior (especially for long-term dynamics). This is a major advantage compared with other models, see Yao et al. [52], and Zhang [54], among others for discussions on considerations leading to this modelling approach.

In this chapter, the FFT approach is applied for pricing European-style call options, where the underlying asset price is governed by a regime-switching geometric Brownian motion (RSGBM). An FFT method for the RS model is developed. For the two states case, numerical result is provided, however, for the general case where the number of states is more than two ($m > 2$), the fundamental matrix solution is not known explicitly. We use monte-carlo simulation as well as a novel method called semi-MC simulation to compare with our FFT results for the two case. We also use analytical solution for the case where the drift and interest rate don't depend on time and compare the results with what obtained by FFT, MCS and Semi-MCS.

2.2 Regime-switching model and risk-neutral option pricing

We consider a continuous-time economy with a finite time horizon $[0, T]$ where $T < \infty$. Suppose that $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, where \mathcal{P} is a risk-neutral

probability measure. Let $\alpha(\cdot) = \{\alpha(t)|t \in [0, T]\}$ be an observable continuous time Markov chain with a finite state space $\mathcal{M} = \{1, \dots, m\}$, which may represent general market trends. For example, when $m = 2$, $\alpha(t) = 1$ may denote a bullish market and $\alpha(t) = 2$ a bearish market.

Let's assume that under the risk-neutral probability measure \mathcal{P} , the dynamics of the underlying asset value, S_t , is given by

$$dS_t = \mu(\alpha(t)) S_t dt + \sigma(\alpha(t)) S_t dW_t, \quad t \geq 0 \quad (2.1)$$

where $S(0) = S_0 > 0$ is the initial underlying asset price, W_t is a standard Brownian motion independent of $\alpha(t)$, and $\mu(\alpha(t))$ and $\sigma(\alpha(t))$ are the risk-free drift rate and volatility of the underlying asset, respectively. We assume that $\mu(j)$ and $\sigma(j)$ are positive constant, for each $j \in \mathcal{M}$.

Under the risk-neutral probability measure \mathcal{P} , the price of T -maturity European-style call options at time 0 with strike price $K > 0$ is given as follows:

$$C(K) = \mathbb{E} \left\{ \exp \left(- \int_0^T r(\alpha(t)) dt \right) (S_T - K)^+ \right\}. \quad (2.2)$$

where the instantaneous risk-free interest rate, $r(\alpha(t))$ also depends on $\alpha(t)$ with $r(j) > 0$, for each $j \in \mathcal{M}$.

Following the notation in Liu et al. [39], let $\kappa = \ln \left(\frac{K}{S_0} \right)$ and $S_T = S_0 e^{X_T}$. Then (2.2) can be written as

$$C(k) = S_0 \mathbb{E} \left\{ \exp \left(- \int_0^T r(\alpha(t)) dt \right) (e^{X_T} - e^\kappa)^+ \right\}. \quad (2.3)$$

2.3 Fourier transform of the option price

In this section, we would like to apply Fourier transform to the European-style call option price given by equation (2.3). Let's first recall the definition of Fourier trans-

form (FT) and inverse Fourier transform (IFT) for continuous functions. The FT of the function f is traditionally denoted by \hat{f} . There are several common conventions for defining the FT of an integrable function $f : \mathbb{R} \mapsto \mathbb{C}$. Here we will use the following definition:

$$\hat{f}(u) = \int_{-\infty}^{\infty} e^{-iux} f(x) dx,$$

for any real number u .

When the independent variable x represents time, the transform variable u represents frequency. Under suitable conditions, f is determined by \hat{f} via IFT:

$$f(x) = \int_{-\infty}^{\infty} e^{iux} \hat{f}(u) du,$$

for any real number x .

For the rest of this dissertation, we use above definitions for FT and IFT. Assume $f(x)$ is the probability density function of X_T . Now let's derive the FT of $C(k)$.

$$\begin{aligned} \hat{C}(u) &= \int_{-\infty}^{\infty} e^{-iuk} C(k) dk \\ &= \int_{-\infty}^{\infty} e^{-iuk} S_0 \mathbb{E} \left\{ \exp \left(- \int_0^T r(\alpha(t)) dt \right) (e^{X_T} - e^k)^+ \right\} dk \\ &= S_0 \int_{-\infty}^{\infty} e^{-iuk} \int_{-\infty}^{\infty} \exp \left(- \int_0^T r(\alpha(t)) dt \right) (e^x - e^k)^+ f(x) dx dk \\ &= S_0 \int_{-\infty}^{\infty} \exp \left(- \int_0^T r(\alpha(t)) dt \right) e^{-iuk} \int_k^{\infty} (e^x - e^k) f(x) dx dk \\ &= S_0 \int_{-\infty}^{\infty} \exp \left(- \int_0^T r(\alpha(t)) dt \right) f(x) \int_{-\infty}^x e^{-iuk} (e^x - e^k) dk dx \end{aligned} \quad (2.4)$$

where we changed the order of integration by Fubini's theorem and used the result

$\int_x^\infty e^{-i u \kappa} (e^x - e^\kappa)^+ d\kappa = 0$. If we evaluate the inner integral of (2.4), we have

$$\begin{aligned} \int_{-\infty}^x e^{-i u \kappa} (e^x - e^\kappa) d\kappa &= \int_{-\infty}^x e^{-i u \kappa} e^x d\kappa - \int_{-\infty}^x e^{-i u \kappa} e^\kappa d\kappa \\ &= e^x \frac{e^{-i u \kappa}}{-i u} \Big|_{-\infty}^x - \frac{e^{(1-i u) \kappa}}{(1-i u)} \Big|_{-\infty}^x. \end{aligned} \quad (2.5)$$

As we can see, the first term of (2.5) is not integrable since $\lim_{\kappa \rightarrow -\infty} e^{-i u \kappa} \neq 0$, while in second term of (2.5) $\lim_{\kappa \rightarrow -\infty} e^{(1-i u) \kappa}$ converges to zero.

To obtain a squared integrable function with respect to κ , Carr and Madan [6] introduced a dampning parameter ρ to modify the call option price. The so-called dampened call price is defined as

$$c(\kappa) = e^{\rho \kappa} \frac{C(k)}{S_0}, \quad -\infty < \kappa < \infty \quad (2.6)$$

where $\rho > 0$ is a prespecified positive number (dampening factor). We explained in details in Chapter 1, Section 1.3.2 why we need a positive dampening factor for call options and a negative dampening factor for put options.

The FT of the dampened call option price is given by:

$$\hat{c}(u) = \int_{-\infty}^{\infty} e^{-i u \kappa} c(\kappa) d\kappa, \quad u \in (-\infty, \infty). \quad (2.7)$$

Once we find a closed form for FT of the dampened call option price, the pricing formula for the European-style call option can be obtained via the IFT as follows:

$$C(\kappa) = \frac{e^{-\rho \kappa} S_0}{2\pi} \int_{-\infty}^{\infty} e^{i u \kappa} \hat{c}(u) du = \frac{e^{-\alpha \kappa} S_0}{\pi} \int_0^{\infty} e^{i u \kappa} \hat{c}(u) du, \quad (2.8)$$

where the second term comes from the fact that call option is a real number. This

implies the FT $\hat{c}(u)$ is odd in its imaginary part so that

$$\text{Im}\{\hat{c}(u)\} = -\text{Im}\{\hat{c}(-u)\} \quad (2.9)$$

and even in its real part so that

$$\text{Re}\{\hat{c}(u)\} = \text{Re}\{\hat{c}(-u)\}. \quad (2.10)$$

Thus this allows to rewrite the pricing integral as the second term in (2.8).

Therefore, all we need to do now is to find a closed form for FT of the dampened call option price. Let \mathcal{F}_T^α be the σ -algebra generated by the Markov chain $\alpha(t)$, $0 \leq t \leq T$, that is, $\mathcal{F}_T^\alpha = \sigma\{\alpha(t), 0 \leq t \leq T\}$. Note that W_t is still a Brownian motion (BM) with respect to the filtration of Markov chain since we consider BM to be independent of Markov chain. Let $f_{\mathcal{F}_T^\alpha}(x)$ be the conditional density function of X_T given \mathcal{F}_T^α .

Then the FT of the dampened call option price, $c(\kappa)$, is calculated as follows:

$$\begin{aligned} \hat{c}(u) &= \int_{-\infty}^{\infty} e^{-iu\kappa} c(\kappa) d\kappa = \int_{-\infty}^{\infty} e^{-iu\kappa} e^{\rho\kappa} \mathbb{E}\left\{e^{-\int_0^T r(\alpha(t))dt} (e^{X_T} - e^\kappa)^+\right\} d\kappa \\ &= \mathbb{E}\left\{\int_{-\infty}^{\infty} e^{-iu\kappa} e^{\rho\kappa} \mathbb{E}\left\{e^{-\int_0^T r(\alpha(t))dt} (e^{X_T} - e^\kappa)^+ | \mathcal{F}_T^\alpha\right\} d\kappa\right\} \\ &= \mathbb{E}\left\{\int_{-\infty}^{\infty} e^{-iu\kappa} e^{\rho\kappa} e^{-\int_0^T r(\alpha(t))dt} \int_{\kappa}^{\infty} (e^x - e^\kappa) f_{\mathcal{F}_T^\alpha}(x) dx d\kappa\right\} \\ &= \mathbb{E}\left\{\int_{-\infty}^{\infty} e^{-\int_0^T r(\alpha(t))dt} f_{\mathcal{F}_T^\alpha}(x) \int_{-\infty}^x (e^x e^{(\rho-iu)\kappa} - e^{(1+\rho-iu)\kappa}) d\kappa dx\right\} \\ &= \mathbb{E}\left\{\int_{-\infty}^{\infty} e^{-\int_0^T r(\alpha(t))dt} f_{\mathcal{F}_T^\alpha}(x) \left(\frac{e^{(1+\rho-iu)x}}{(\rho-iu)} - \frac{e^{(1+\rho-iu)x}}{(1+\rho-iu)}\right) dx\right\} \\ &= \mathbb{E}\left\{e^{-\int_0^T r(\alpha(t))dt} \left(\frac{\phi_{\mathcal{F}_T^\alpha}(-i(1+\rho)-u)}{(\rho-iu)} - \frac{\phi_{\mathcal{F}_T^\alpha}(-i(1+\rho)-u)}{(1+\rho-iu)}\right)\right\} \\ &= \frac{\mathbb{E}\left\{e^{-\int_0^T r(\alpha(t))dt} \phi_{\mathcal{F}_T^\alpha}(-i(1+\rho)-u)\right\}}{(\rho-iu)(1+\rho-iu)} \end{aligned} \quad (2.11)$$

where

$$\phi_{\mathcal{F}_T^\alpha}(u) = \mathbb{E}\{e^{iuX_T} | \mathcal{F}_T^\alpha\} = \int_{-\infty}^{\infty} e^{iux} f_{\mathcal{F}_T^\alpha}(x) dx \quad (2.12)$$

is the conditional characteristic function of X_T given \mathcal{F}_T^α .

Note that all the calculations done in (2.11) is pretty similar with (1.12). For the rest of this dissertation, we do not repeat it again.

Call values are determined by substituting (2.11) into (2.8) and performing the required integration. To find the explicit form for the required integration, we still need to find the expectation in (2.11). Thus we first try to find $\phi_{\mathcal{F}_T^\alpha}(u)$. To do this end, we need to know the distribution of X_T . Recall $S_T = S_0 e^{X_T}$, therefore

$$S_T = S_0 \exp\left(\int_0^T [\mu(\alpha(s)) - \frac{1}{2}\sigma^2(\alpha(s))] ds + \int_0^T \sigma(\alpha(s)) dW_s\right), \quad t \geq 0.$$

To simplify our notation, let's define

$$L_T = \int_0^T \mu(\alpha(t)) dt, \quad V_T = \int_0^T \sigma^2(\alpha(t)) dt, \quad R_T = \int_0^T r(\alpha(t)) dt. \quad (2.13)$$

Then given \mathcal{F}_T^α , X_T has Gaussian distribution with mean $(L_T - \frac{1}{2}V_T)$ and variance V_T . It follows that

$$\phi_{\mathcal{F}_T^\alpha}(u) = \exp\left(iu(L_T - \frac{1}{2}V_T) - \frac{1}{2}u^2V_T\right). \quad (2.14)$$

Plugging (2.14) in (2.11), we have

$$\begin{aligned} \hat{c}(u) = & \frac{1}{(\rho - iu)(1 + \rho - iu)} \mathbb{E}\left\{ \exp\left((1 + \rho)\left(L_T + \frac{1}{2}\rho V_T\right) - R_T \right. \right. \\ & \left. \left. - \frac{1}{2}u^2V_T - iu\left(L_T + \left(\frac{1}{2} + \rho\right)V_T\right)\right) \right\}. \end{aligned} \quad (2.15)$$

As we can see, still we haven't been able to find a closed form for $\hat{c}(u)$ as it is necessary to calculate the expectation with respect to L_T , V_T and R_T in (2.15). Note

that L_T , V_T and R_T are random variables as their value depends on how much time Markov chain spent in state j for example. Therefore, it's useful to define the sojourn time of the Markov chain $\alpha(t)$ in state j during the interval $[0, T]$.

$$T_j = \int_0^T \mathbf{1}_{\{\alpha(t)=j\}} dt, \quad j \in \mathcal{M}. \quad (2.16)$$

Then $\sum_{j=1}^m T_j = T$. Then the three random variables L_T , V_T and R_T defined in (2.13) can be rewritten as

$$\begin{aligned} L_T &= \sum_{j=1}^{m-1} \left(\mu(j) - \mu(m) \right) T_j + \mu(m)T, \\ V_T &= \sum_{j=1}^{m-1} \left(\sigma^2(j) - \sigma^2(m) \right) T_j + \sigma^2(m)T, \\ R_T &= \sum_{j=1}^{m-1} \left(r(j) - r(m) \right) T_j + r(m)T. \end{aligned} \quad (2.17)$$

Using (2.17) in (2.15), we obtain that

$$\hat{c}(u) = \frac{1}{(\rho - iu)(1 + \rho - iu)} \exp(B(u)T) \mathbb{E} \left\{ \exp \left(i \sum_{j=1}^{m-1} A(u, j) T_j \right) \right\} \quad (2.18)$$

where for $j = 1, \dots, m-1$,

$$\begin{aligned} A(u, j) &= -u \left[(\mu(j) - \mu(m)) + \left(\frac{1}{2} + \rho \right) (\sigma^2(j) - \sigma^2(m)) \right] \\ &\quad + \frac{1}{2} u^2 (\sigma^2(j) - \sigma^2(m)) i + \left[(r(j) - r(m)) - \right. \\ &\quad \left. (1 + \rho)(\mu(j) - \mu(m)) - \frac{1}{2} \rho(1 + \rho)(\sigma^2(j) - \sigma^2(m)) \right] i, \\ B(u) &= -iu \left[\mu(m) + \left(\frac{1}{2} + \rho \right) \sigma^2(m) \right] - \frac{1}{2} u^2 \sigma^2(m) + \\ &\quad (1 + \rho) \mu(m) - r(m) + \frac{1}{2} \rho(1 + \rho) \sigma^2(m). \end{aligned} \quad (2.19)$$

Therefore, the determination of $\hat{c}(u)$ reduces to calculating the characteristic function of the random vector $(T_1, \dots, T_{m-1})'$, where $(T_1, \dots, T_{m-1})'$ denotes the transpose of (T_1, \dots, T_{m-1}) . For the two-state case, we only need to find the characteristic function of random sojourn time T_1 , as by finding T_1 implies that $T_2 = T - T_1$. We adopt the same methodology in Liu et al. [39] for the two states case. However, for general case, when $m > 2$, we need to find the characteristic function of the random vector $(T_1, \dots, T_{m-1})'$. We are going to use a modification of proof of lemma 1 in Buffington and Elliott [5].

Let the generator of the Markov chain $\alpha(\cdot)$ be given by an $m \times m$ matrix $Q = (q_{ij})_{m \times m}$ such that $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^m q_{ij} = 0$ for each $i \in \mathcal{M}$. Let

$$\mathbf{I}(t) = \left(\mathbf{1}_{\{\alpha(t)=1\}}, \mathbf{1}_{\{\alpha(t)=2\}}, \dots, \mathbf{1}_{\{\alpha(t)=m\}} \right)' \in \mathcal{R}^{m \times 1}, \quad (2.20)$$

denote the vector of indicator functions. Then it is shown by Yin and Zhang [11, Lemma 2.4, Chapter 2] that

$$M(t) = \mathbf{I}(t) - \int_0^t Q' \mathbf{I}(s) ds \quad (2.21)$$

is a martingale. This implies

$$d\mathbf{I}(t) = Q' \mathbf{I}(t) dt + dM(t). \quad (2.22)$$

Let's simplify our notation in (2.18) in order to determine the characteristic function of the random vector $(T_1, \dots, T_{m-1})'$.

$$\mathbb{E} \left\{ \exp \left(i \sum_{j=1}^{m-1} \theta_j T_j \right) \right\} = \mathbb{E} \left\{ \exp \left(i \sum_{j=1}^{m-1} \theta_j \int_0^T \mathbf{1}_{\{\alpha(t)=j\}} dt \right) \right\}. \quad (2.23)$$

We would like to generalize our method by calculating the following characteristic

function

$$\mathbb{E}\left\{\exp\left(i\sum_{j=1}^{m-1}\theta_j\int_t^u\mathbf{1}_{\{\alpha(t)=j\}}dt\right)\right\} \quad (2.24)$$

for any $u \in [t, T]$. Define a random vector

$$\mathbf{Z}(t, u) = \left(z_1(t, u), z_2(t, u), \dots, z_m(t, u)\right)' \in \mathcal{R}^{m \times 1} \quad (2.25)$$

for any $u \in [t, T]$, where

$$z_j(t, u) = \exp\left(i\sum_{j=1}^{m-1}\theta_j\int_t^u\mathbf{1}_{\{\alpha(s)=j\}}ds\right)\mathbf{1}_{\{\alpha(u)=j\}}. \quad (2.26)$$

Using the definition for $\mathbf{I}(u)$, one can rewrite (2.25) as the following:

$$\mathbf{Z}(t, u) = \exp\left(i\sum_{j=1}^{m-1}\theta_j\int_t^u\mathbf{1}_{\{\alpha(s)=j\}}ds\right)\mathbf{I}(u).$$

Consequently,

$$\begin{aligned} d\mathbf{Z}(t, u) &= \left(i\sum_{j=1}^{m-1}\theta_j\int_t^u\mathbf{1}_{\{\alpha(s)=j\}}ds\right)\mathbf{Z}(t, u)du \\ &\quad + \exp\left(i\sum_{j=1}^{m-1}\theta_j\int_t^u\mathbf{1}_{\{\alpha(s)=j\}}ds\right)d\mathbf{I}(u). \end{aligned} \quad (2.27)$$

We would like to simplify (2.27) by denoting $\Theta = \mathbf{diag}(\theta_1, \theta_2, \dots, \theta_{m-1}, 0)$ where $\mathbf{diag}(\theta_1, \theta_2, \dots, \theta_{m-1}, 0)$ is the diagonal matrix with diagonal entries $\theta_1, \theta_2, \dots, \theta_{m-1}, 0$.

Note

$$\left(i\sum_{j=1}^{m-1}\theta_j\int_t^u\mathbf{1}_{\{\alpha(s)=j\}}ds\right)\mathbf{Z}(t, u) = i\Theta\mathbf{Z}(t, u). \quad (2.28)$$

Using (2.28) and the martingale property (2.22), we have

$$d\mathbf{Z}(t, u) = i\Theta\mathbf{Z}(t, u)du + Q'\mathbf{Z}(t, u)du + \exp\left(i\sum_{j=1}^{m-1}\theta_j\int_t^u \mathbf{1}_{\{\alpha(s)=j\}}ds\right)dM(u), \quad (2.29)$$

then

$$d\mathbf{Z}(t, u) = (i\Theta + Q')\mathbf{Z}(t, u)du + \exp\left(i\sum_{j=1}^{m-1}\theta_j\int_t^u \mathbf{1}_{\{\alpha(s)=j\}}ds\right)dM(u). \quad (2.30)$$

Consequently,

$$\begin{aligned} \mathbf{Z}(t, u) &= \mathbf{Z}(t, t) + \int_t^u (i\Theta + Q')\mathbf{Z}(t, s)ds \\ &\quad + \int_t^u \exp\left(i\sum_{j=1}^{m-1}\theta_j\int_t^s \mathbf{1}_{\{\alpha(\nu)=j\}}d\nu\right)dM(s). \end{aligned} \quad (2.31)$$

The final integral is a martingale, so taking expected values

$$\mathbb{E}\{\mathbf{Z}(t, u)\} = \mathbf{Z}(t, t) + \int_t^u (i\Theta + Q')\mathbb{E}\{\mathbf{Z}(t, s)\}ds. \quad (2.32)$$

Note that $\mathbf{Z}(t, t) = \mathbf{I}(t)$, where $\mathbf{I}(t)$ is $m \times m$ identity matrix. By differentiation, we have

$$d\mathbb{E}\{\mathbf{Z}(t, u)\} = (i\Theta + Q')\mathbb{E}\{\mathbf{Z}(t, u)\}du, \quad \mathbb{E}\{\mathbf{Z}(t, t)\} = \mathbf{I}(t). \quad (2.33)$$

Hence $\mathbb{E}\{\mathbf{Z}(t, u)\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension m :

$$\frac{d\mathbb{E}\{\mathbf{Z}(t, u)\}}{du} = (i\Theta + Q')\mathbb{E}\{\mathbf{Z}(t, u)\}, \quad \mathbb{E}\{\mathbf{Z}(t, t)\} = \mathbf{I}(t). \quad (2.34)$$

Since $(i\Theta + Q')$ is not time dependent, the solution to (2.34) is given by

$$\mathbb{E}\{\mathbf{Z}(t, u)\} = \mathbf{I}(t)\exp\left((i\Theta + Q')(u - t)\right). \quad (2.35)$$

Thus for $t = 0$ and $u = T$, we have

$$\mathbb{E}\{\mathbf{Z}(0, T)\} = \mathbf{I}(0)\exp\left((i\Theta + Q')T\right). \quad (2.36)$$

Define $\mathbf{1}_m = (1, \dots, 1)' \in \mathbb{R}^{m \times 1}$. Note $\langle \mathbf{I}(T), \mathbf{1}_m' \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the scalar product \mathbb{R}^m . Consequently, the characteristic function can be determined by

$$\begin{aligned} \mathbb{E}\left\{\exp\left(i \sum_{j=1}^{m-1} \theta_j T_j\right)\right\} &= \mathbb{E}\left\{\exp\left(i \sum_{j=1}^{m-1} \theta_j \int_0^T \mathbf{1}_{\{\alpha(t)=j\}} dt\right)\right\} \\ &= \mathbb{E}\left\{\exp\left(i \sum_{j=1}^{m-1} \theta_j \int_0^T \mathbf{1}_{\{\alpha(t)=j\}} dt\right) \langle \mathbf{I}(T), \mathbf{1}_m' \rangle\right\} \\ &= \mathbb{E}\left\{\left\langle \exp\left(i \sum_{j=1}^{m-1} \theta_j \int_0^T \mathbf{1}_{\{\alpha(t)=j\}} dt\right) \mathbf{I}(T), \mathbf{1}_m' \right\rangle\right\} \\ &= \mathbb{E}\{\langle \mathbf{Z}(0, T), \mathbf{1}_m' \rangle\} \\ &= \langle \mathbb{E}\{\mathbf{Z}(0, T)\}, \mathbf{1}_m' \rangle \\ &= \langle \mathbf{I}(0)\exp((i\Theta + Q')T), \mathbf{1}_m' \rangle. \end{aligned} \quad (2.37)$$

Setting $\theta_j = A(u, j)$ in (2.37) and then using the result in (2.18), we obtain the Fourier transform $\hat{c}(u)$, which can then be used in the inverse transform to determine the option price.

We closely follows the innovative way proposed by Liu et. al [39] for deriving a simple form for the two-state case. Let the generator of the Markov chain $\alpha(\cdot)$ be given by

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix} \quad (2.38)$$

where λ_1 is the jump rate from state 1 to state 2 and λ_2 is the jump rate from state 2 to state 1. In this case, we need to find the characteristic function of T_1 , the sojourn time in state 1.

Assume the initial state $\alpha(0) = j_0$. Define

$$\phi_{j_0}(\theta, T) = \mathbb{E}\{e^{i\theta T_1} | \alpha(0) = j_0\}, \quad j_0 = 1, 2. \quad (2.39)$$

Then $\phi_1(\theta, T)$ and $\phi_2(\theta, T)$ satisfy the following system of integral equations (see the Appendix A):

$$\begin{aligned} \phi_1(\theta, T) &= e^{i\theta T} e^{-\lambda_1 T} + \int_0^T e^{i\theta t} \phi_2(\theta, T-t) \lambda_1 e^{-\lambda_1 t} dt, \\ \phi_2(\theta, T) &= e^{-\lambda_2 T} + \int_0^T e^{i\theta t} \phi_1(\theta, T-t) \lambda_2 e^{-\lambda_2 t} dt. \end{aligned} \quad (2.40)$$

In order to find $\phi_1(\theta, T)$ and $\phi_2(\theta, T)$, we are going to use Laplace transforms. Let's denote $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Recall

$$\begin{aligned} \mathcal{L}\{e^{at}\} &= \frac{1}{s-a}, \\ \mathcal{L}\left\{\int_0^t f(t)g(t-u)du\right\} &= F(s) \times G(s). \end{aligned}$$

Taking Laplace transforms, we obtain the following system of algebraic equations:

$$\begin{aligned} \mathcal{L}\{\phi_1(\theta, T)\} &= \frac{1}{s + \lambda_1 - i\theta} + \frac{\lambda_1}{s + \lambda_1 - i\theta} \mathcal{L}\{\phi_2(\theta, T)\}, \\ \mathcal{L}\{\phi_2(\theta, T)\} &= \frac{1}{s + \lambda_2} + \frac{\lambda_2}{s + \lambda_2} \mathcal{L}\{\phi_1(\theta, T)\}. \end{aligned} \quad (2.41)$$

We now solve the pair of equations:

$$\mathcal{L}\{\phi_1(\theta, T)\} = \frac{1 + \lambda_1 \mathcal{L}\{\phi_2(\theta, T)\}}{s + \lambda_1 - i\theta}, \quad (2.42)$$

$$\mathcal{L}\{\phi_2(\theta, T)\} = \frac{1 + \lambda_2 \mathcal{L}\{\phi_1(\theta, T)\}}{s + \lambda_2}. \quad (2.43)$$

Substituting (2.43) in (2.42) to find $\mathcal{L}\{\phi_1(\theta, T)\}$ and (2.42) to (2.43) in order to find $\mathcal{L}\{\phi_2(\theta, T)\}$, we have

$$\begin{aligned} (s + \lambda_1 - i\theta) \mathcal{L}\{\phi_2(\theta, T)\} &= (1 + \lambda_1) \frac{1 + \lambda_2 \mathcal{L}\{\phi_2(\theta, T)\}}{s + \lambda_2} \\ &= \frac{s + \lambda_1 - i\theta + \lambda_2 + \lambda_1 \lambda_2 \mathcal{L}\{\phi_2(\theta, T)\}}{s + \lambda_1 - i\theta}, \end{aligned}$$

$$\begin{aligned} (s + \lambda_2) \mathcal{L}\{\phi_2(\theta, T)\} &= (1 + \lambda_2) \frac{1 + \lambda_1 \mathcal{L}\{\phi_2(\theta, T)\}}{s + \lambda_1 - i\theta} \\ &= \frac{s + \lambda_1 - i\theta + \lambda_2 + \lambda_1 \lambda_2 \mathcal{L}\{\phi_2(\theta, T)\}}{s + \lambda_1 - i\theta}. \end{aligned}$$

Solving the above pair of equations for $\mathcal{L}\{\phi_1(\theta, T)\}$ and $\mathcal{L}\{\phi_2(\theta, T)\}$ implies

$$\begin{aligned} (s^2 + \lambda_1 s + \lambda_2 s - i\theta s - i\theta \lambda_2) \mathcal{L}\{\phi_1(\theta, T)\} - \lambda_2 - \lambda_1 - s &= 0, \\ (s^2 + \lambda_1 s + \lambda_2 s - i\theta s - i\theta \lambda_2) \mathcal{L}\{\phi_2(\theta, T)\} - \lambda_2 - \lambda_1 - s + i\theta &= 0. \end{aligned}$$

Solving the pair of equations finally yields

$$\begin{aligned} \mathcal{L}\{\phi_1(\theta, T)\} &= \frac{s + \lambda_1 + \lambda_2}{s^2 + (\lambda_1 + \lambda_2 - i\theta)s - i\theta \lambda_2}, \\ \mathcal{L}\{\phi_2(\theta, T)\} &= \frac{s + \lambda_1 + \lambda_2 - i\theta}{s^2 + (\lambda_1 + \lambda_2 - i\theta)s - i\theta \lambda_2}. \end{aligned} \quad (2.44)$$

Now let's take inverse Laplace transform to find $\mathcal{L}\{\phi_1(\theta, T)\}$ and $\mathcal{L}\{\phi_2(\theta, T)\}$.

But first, assume s_1 and s_2 are the two roots of the equation

$$s^2 + (\lambda_1 + \lambda_2 - i\theta)s - i\theta \lambda_2 = 0. \quad (2.45)$$

Note that since $\Delta = (\lambda_1 + \lambda_2 - i\theta)^2 + 4i\theta\lambda_2 \neq 0$, there exist two roots for the denominator in (2.44). Hence

$$\begin{aligned}
\phi_1(\theta, T) &= \frac{s + \lambda_1 + \lambda_2}{(s - s_1)(s - s_2)} = \frac{(s + \lambda_1 + \lambda_2)(s_1 - s_2)}{(s - s_1)(s - s_2)(s_1 - s_2)} \\
&= \frac{ss_1 - ss_2 + (\lambda_1 + \lambda_2)s_1 - (\lambda_1 + \lambda_2)s_2 \pm s_1s_2 \pm (\lambda_1 + \lambda_2)s}{(s - s_1)(s - s_2)(s_1 - s_2)} \\
&= \frac{s_1(s - s_2) + (\lambda_1 + \lambda_2)(s - s_2) + (s_1 - s)s_2 + (s_1 - s)(\lambda_1 + \lambda_2)}{(s - s_1)(s - s_2)(s_1 - s_2)} \\
&= \frac{(s_1 + \lambda_1 + \lambda_2)(s - s_2) - (s - s_1)(s_2 + \lambda_1 + \lambda_2)}{(s - s_1)(s - s_2)(s_1 - s_2)} \\
&= \frac{s_1 + \lambda_1 + \lambda_2}{(s_1 - s_2)} \times \frac{1}{s - s_1} - \frac{s_2 + \lambda_1 + \lambda_2}{(s_1 - s_2)} \times \frac{1}{s - s_2} \\
&= \frac{1}{s_1 - s_2} \left((s_1 + \lambda_1 + \lambda_2) \mathcal{L}^{-1} \{ \mathcal{L} \{ e^{s_1 T} \} \} - (s_2 + \lambda_1 + \lambda_2) \mathcal{L}^{-1} \{ \mathcal{L} \{ e^{s_2 T} \} \} \right),
\end{aligned}$$

$$\begin{aligned}
\phi_2(\theta, T) &= \frac{s + \lambda_1 + \lambda_2 - i\theta}{(s - s_1)(s - s_2)} = \frac{(s + \lambda_1 + \lambda_2 - i\theta)(s_1 - s_2)}{(s - s_1)(s - s_2)(s_1 - s_2)} \\
&= \frac{ss_1 - ss_2 + (\lambda_1 + \lambda_2 - i\theta)s_1 - (\lambda_1 + \lambda_2 - i\theta)s_2 \pm s_1s_2 \pm (\lambda_1 + \lambda_2 - i\theta)s}{(s - s_1)(s - s_2)(s_1 - s_2)} \\
&= \frac{s_1(s - s_2) + (\lambda_1 + \lambda_2 - i\theta)(s - s_2) + (s_1 - s)s_2 + (s_1 - s)(\lambda_1 + \lambda_2 - i\theta)}{(s - s_1)(s - s_2)(s_1 - s_2)} \\
&= \frac{(s_1 + \lambda_1 + \lambda_2 - i\theta)(s - s_2) - (s - s_1)(s_2 + \lambda_1 + \lambda_2 - i\theta)}{(s - s_1)(s - s_2)(s_1 - s_2)} \\
&= \frac{s_1 + \lambda_1 + \lambda_2 - i\theta}{(s_1 - s_2)} \times \frac{1}{s - s_1} - \frac{s_2 + \lambda_1 + \lambda_2 - i\theta}{(s_1 - s_2)} \times \frac{1}{s - s_2} \\
&= \frac{1}{s_1 - s_2} \left((s_1 + \lambda_1 + \lambda_2 - i\theta) \mathcal{L}^{-1} \{ \mathcal{L} \{ e^{s_1 T} \} \} \right. \\
&\quad \left. - (s_2 + \lambda_1 + \lambda_2 - i\theta) \mathcal{L}^{-1} \{ \mathcal{L} \{ e^{s_2 T} \} \} \right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\phi_1(\theta, T) &= \frac{1}{s_1 - s_2} ((s_1 + \lambda_1 + \lambda_2) e^{s_1 T} - (s_2 + \lambda_1 + \lambda_2) e^{s_2 T}), \\
\phi_2(\theta, T) &= \frac{1}{s_1 - s_2} ((s_1 + \lambda_1 + \lambda_2 - i\theta) e^{s_1 T} - (s_2 + \lambda_1 + \lambda_2 - i\theta) e^{s_2 T}). \quad (2.46)
\end{aligned}$$

The Fourier transform (2.18) in this case is given by

$$\hat{c}(u) = \frac{\exp(B(u)T)\phi_{j_0}(A(u), T)}{(\rho - iu)(1 + \rho - iu)}, \quad (2.47)$$

where

$$\begin{aligned} A(u) = A(u, 1) = & -u \left[(\mu(1) - \mu(2)) + \left(\frac{1}{2} + \rho\right) (\sigma^2(1) - \sigma^2(2)) \right] \\ & + \frac{1}{2} u^2 (\sigma^2(1) - \sigma^2(2)) i + \left[(r(1) - r(2)) - (1 + \rho) (\mu(1) - \mu(2)) \right. \\ & \left. - \frac{1}{2} \rho (1 + \rho) (\sigma^2(1) - \sigma^2(2)) \right] i, \\ B(u) = & -iu \left[\mu(2) + \left(\frac{1}{2} + \rho\right) \sigma^2(2) \right] - \frac{1}{2} u^2 \sigma^2(2) \\ & + (1 + \rho) \mu(2) - r(2) + \frac{1}{2} \rho (1 + \rho) \sigma^2(2). \end{aligned} \quad (2.48)$$

2.4 FFT algorithm for option pricing

We adopt the approach introduced by Carr and Madan [6]. There are many ways to define the discrete Fourier transform (DFT), varying in the sign of the exponent, normalization, etc. Since we are going to use Python implementation, therefore we are following the same definition for DFT given in Python package numpy. The DFT is defined in Python is

$$A_k = \sum_{m=0}^{n-1} a_m \exp \left\{ -2\pi i \frac{mk}{n} \right\}, \quad k = 0, \dots, n-1. \quad (2.49)$$

The inverse DFT is defined as

$$a_m = \frac{1}{n} \sum_{k=0}^{n-1} A_k \exp \left\{ 2\pi i \frac{mk}{n} \right\} \quad m = 0, \dots, n-1.$$

It differs from the Fourier transform by the sign of the exponential argument and the default normalization by $\frac{1}{n}$.

Given the transform function $\hat{c}(u)$, the modified option price $c(\kappa)$ can be obtained by the inverse Fourier transform as described in (2.8)

$$C(\kappa) = \frac{S_0 \exp(-\rho\kappa)}{2\pi} \int_{-\infty}^{\infty} e^{i\nu\kappa} \hat{c}(u) du = \frac{S_0 \exp(-\rho\kappa)}{\pi} \int_0^{\infty} e^{iu\kappa} \hat{c}(u) du,$$

and the option price is, in view of (2.6), $C(\kappa) = e^{-\rho\kappa} S_0 c(\kappa)$, for $-\infty < \kappa < \infty$.

Set $u_j = j\Delta_u$, $j = 0, 1, \dots, N-1$, where Δ_u is the grid size in the variable u . Then (2.8) can be approximated by the following summation:

$$c(\kappa) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} e^{iu_j\kappa} \hat{c}(u_j) \Delta_u. \quad (2.50)$$

Next, let Δ_κ be the grid size in κ and choose a grid along the modified log strike κ as below:

$$\kappa_l = (l - \frac{N}{2})\Delta_\kappa, \quad l = 0, 1, \dots, N-1. \quad (2.51)$$

Then

$$\begin{aligned} c(\kappa_l) &\approx \frac{1}{\pi} \sum_{j=0}^{N-1} e^{iu_j\kappa_l} \hat{c}(u_j) \Delta_u, \quad l = 0, 1, \dots, N-1 \\ &= \frac{1}{\pi} \sum_{j=0}^{N-1} e^{ij\Delta_u(l - \frac{N}{2})\Delta_\kappa} \hat{c}(u_j) \Delta_u \\ &= \frac{1}{\pi} \sum_{j=0}^{N-1} e^{ij\Delta_u l \Delta_\kappa} e^{-ij\Delta_u \frac{N}{2} \Delta_\kappa} \hat{c}(u_j) \Delta_u. \end{aligned} \quad (2.52)$$

If we set

$$\Delta_u \Delta_\kappa = \frac{2\pi}{N}, \quad (2.53)$$

then we have

$$c(\kappa_l) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} e^{ijl\frac{2\pi}{N}} e^{-ij\pi} \hat{c}(u_j) \Delta_u, \quad l = 0, 1, \dots, N-1. \quad (2.54)$$

Using Simpson's rule for numerical integration, define a sequence of weighting factors by

$$w(j) = \begin{cases} \frac{1}{3}, & \text{if } j = 0, \\ \frac{4}{3}, & \text{if } j \text{ is odd,} \\ \frac{2}{3}, & \text{if } j \text{ is even.} \end{cases}$$

Then

$$c(\kappa_l) \approx \frac{\Delta_u}{\pi N} \sum_{j=0}^{N-1} e^{ijl\frac{2\pi}{N}} e^{-ij\pi} \hat{c}(u_j) w(j) N, \quad l = 0, 1, \dots, N-1. \quad (2.55)$$

Comparing (2.55) with (2.49), it is easy to see that $\{c(\kappa)\}$ can be obtained by taking the Fourier transform of the sequence $\{e^{-ij\pi} \hat{c}(u_j) w(j) N\}$, $j = 0, 1, \dots, N-1$. Details for the code is provided in Appendix B.

2.5 Monte Carlo and Semi-Monte Carlo algorithm for option pricing

Monte Carlo simulations (MCS) are frequently used when closed-form solutions are not available for complex stochastic problems. A Monte Carlo algorithm frequently serves as a benchmark for the “true value” used for testing other numerical methods. The benchmark value is obtained by running a great number of sample paths in simulating the underlying stochastic dynamics. It is very time consuming and therefore not feasible for most practical use in real time.

We employ MCS to price European call options in our two-state Markov-modulated

model. To price non-path dependent options such as European call options, we only require the final stock price at maturity S_T , not the stock price trajectory $\{S_t\}_{t \in [0, T]}$. We therefore implement the following algorithm for obtaining pseudo-random samples from the terminal stock price distribution (under risk-neutral measure). As the first step, we consider simulating the trajectory of the Markov chain $\{\alpha(t)\}_{t \in [0, T]}$, conditional on $\alpha(0) = 1$ (at time $t = 0$ in state 1). Hence we may determine the time until next change of state. Distribution of such a time is exponential. The exponential distribution is often concerned with the amount of time until some specific event occurs (in our case jumping to another state). For example, the amount of time (beginning at $t = 0$) until Markov chain $\alpha(t)$ leaves state 1 has an exponential distribution with rate λ_1 . Let t_1, t_2 & $t_3 \in [0, T]$ denote successive jump times from one state to another. Hence $t_1 \sim \exp(\lambda_1)$ and $P(t = t_1) = \lambda_1 e^{-\lambda_1 t_1}$. We draw a uniform random variable (p) on interval $[0, 1]$ to simulate the probability $P(t = t_1)$ by p . Therefore the exponential random variable t_1 is $-\ln(p/\lambda_1)/\lambda_1$. Let T_1 denote the occupation time of the Markov chain $\alpha(t)$ in state 1 during the time interval $[0, T]$ and that is $T_1 = t_1 + (t_3 - t_2)$. Therefore, we have

$$\begin{aligned} \int_0^T \sigma_s dW_s &= \int_0^{t_1} \sigma_1 dW_s + \int_{t_1}^{t_2} \sigma_2 dW_s + \int_{t_2}^{t_3} \sigma_1 dW_s + \int_{t_3}^T \sigma_2 dW_s \\ &= \sigma_1(W_{t_1} - W_0) + \sigma_2(W_{t_2} - W_{t_1}) + \sigma_1(W_{t_3} - W_{t_2}) + \sigma_2(W_T - W_{t_3}). \end{aligned}$$

Using independent increments property of Brownian motion as well as the fact that the sum of two independent normally distributed random variables is normal, with its mean being the sum of the two means, and its variance being the sum of the two variances, we have

$$\sigma_1(W_{t_1} - W_0) + \sigma_1(W_{t_3} - W_{t_2}) \stackrel{\text{dist}}{\approx} \mathcal{N}\left(0, \sigma_1^2(t_1 + (t_3 - t_2))\right) = \mathcal{N}\left(0, \sigma_1^2 T_1\right).$$

Using above equation implies that

$$\begin{aligned}
(\text{Ln}(S_T)|\mathcal{F}_T) &= \text{Ln}S_0 + \int_0^{T_1} (\mu_1 - \frac{1}{2}\sigma_1^2)ds + \int_{T_1}^T (\mu_2 - \frac{1}{2}\sigma_2^2)ds \\
&\quad + \int_0^{T_1} \sigma_1 dW_s + \int_{T_1}^T \sigma_2 dW_s \\
&= \text{Ln}S_0 + (\mu_1 - \frac{1}{2}\sigma_1^2)T_1 + (\mu_2 - \frac{1}{2}\sigma_2^2)(T - T_1) \\
&\quad + \sigma_1 W(T_1) + \sigma_2(W(T) - W(T_1)) \\
&\stackrel{\text{dist}}{\approx} \mathcal{N}\left(\text{Ln}S_0 + (\mu_1 - \frac{1}{2}\sigma_1^2)T_1 + (\mu_2 - \frac{1}{2}\sigma_2^2)(T - T_1), \sigma_1^2 T_1 + \sigma_2^2(T - T_1)\right).
\end{aligned} \tag{2.56}$$

By using the procedure above and (2.56), we only require one pseudo-random sample from the standard normal distribution, which minimizes the required computational time. To price options using the Monte Carlo algorithm above, let N be the number of replications. For $n = 1, \dots, N$,

1. Obtain the n -th sample path of $\alpha(t)$, for $t \in [0, T]$.
2. Find the occupation time spent at given initial state (In our case, state 1. Note than we only need to find occupation time in one state as we only have two states).
3. Use equation (2.56) to simulate the log of terminal stock price.
4. Calculate $C_n(K)$ for the n th sample path.
5. Calculate the average of $C_n(K)$ by $1/N \sum_{n=1}^N C_n(K)$ to obtain $C(K)$.

A Monte Carlo base algorithm is presented by Liu et al. [39] that can be also used as benchmark values in numerical experiments. It's called semi-Monte Carlo simulation (SMCS). As noted by Buffington and Elliott [4], for a given realization of the Markov chain $\alpha(\cdot) = \{\alpha(t) : 0 \leq t \leq T\}$, the European call option price whose

underlying asset is governed by regime-switching GBM can be calculated by the usual Black-Scholes formula in which the volatility and the interest rate are replaced by the sample path values. SMCS approach only takes random sampling of the Markov chain and then takes advantage of the availability of analytical formula of the conditional price. Recall that from Section 2, \mathcal{F}_T^α denotes the σ -algebra generated by the Markov chain $\alpha(t)$ for $0 \leq t \leq T$. Then the call option price can be calculated by

$$\begin{aligned} C(K) &= \mathbb{E} \left\{ \exp \left(- \int_0^T r(\alpha(t)) dt \right) (S(T) - K)^+ \right\} \\ &= \mathbb{E} \left\{ \mathbb{E} \left\{ e^{-R_T} (S(T) - K)^+ \middle| \mathcal{F}_T^\alpha \right\} \right\}. \end{aligned}$$

The conditional expectation is given by the Black-Scholes formula, that is,

$$\mathbb{E} \left\{ e^{-R_T} (S(T) - K)^+ \middle| \mathcal{F}_T^\alpha \right\} = S_0 e^{-(R_T - L_T)} \mathcal{N} \left(d_1(L_T, V_T) \right) - K e^{-R_T} \mathcal{N} \left(d_2(L_T, V_T) \right)$$

where

$$d_1(L_T, V_T) = \frac{\ln(S_0/K) + L_T + \frac{1}{2}V_T}{\sqrt{V_T}}, \quad d_2(L_T, V_T) = d_1(L_T, V_T) - \sqrt{V_T}$$

and $\mathcal{N}(\cdot)$ is the cumulative standard normal distribution function.

Detail of the code is provided in Appendix C. We consider a two-state ($m = 2$) example. When the underlying Markov chain $\alpha(\cdot)$ has only two states, an analytical formula in terms of an integral with respect to the Bessel function is developed by Guo [28] for the European call option prices. Fuh et al. [26] considered a specific example and compared various methods (binomial tree, Monte Carlo, and an approximation approach presented in their paper) with the analytical results (equation(7)). Here we consider the same example to compare the MCS, Semi-MCS with the analytical results. Details of the code for MCS, analytical results and Semi-MCS are provided

in Appendix C, Appendix D and Appendix E respectively.

2.6 Numerical experiments using MC and analytical method

The parameters used, in this example, are $S_0 = 100$, $K = 90$, $\lambda_1 = \lambda_2 = 1.0$, $\mu_1 = \mu_2 = r_1 = r_2 = 0.1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$. The initial state is $\alpha(0) = 1$ and $N = 100000$ replications are used in the MCS. Table 2.1 lists the numerical results for a range of option expiry times. We then change the initial state from 1 to 2 with the same given parameters and the results are provide in Table 2.2.

It is clear from Table 2.1 and Table 2.2 that the MCS converges to real prices. All the errors show a clear indication of high accuracy. It's worth to mention that Fuh el at. [26] used 50000000 replications in their MCS to obtain the numbers. We only used 100000 replications ($1/50$ of theirs) in the MCS but achieved a much higher degree of accuracy in our Semi-MCS.

Table 2.1: Comparison of Analytical prices, MC and Semi-MC Simulations at state $\alpha(0) = 1$

$T(\text{year})$	Analytical $\alpha(0) = 1$	MC(error) $\alpha(0) = 1$	Semi-MC(error) $\alpha(0) = 1$
0.1	10.999	10.999 (0.000)	10.993 (0.006)
0.2	12.194	12.186 (0.008)	12.165 (0.029)
0.5	15.754	15.671 (0.083)	15.615 (-0.421)
1.0	21.075	20.970 (0.105)	20.723 (-0.648)
2.0	29.943	29.970 (-0.027)	29.289 (0.654)
3.0	37.246	37.422 (-0.176)	36.473 (0.773)

Table 2.2: Comparison of Analytical prices, MC and Semi-MC Simulations at state $\alpha(0) = 2$

$T(\text{year})$	Analytical $\alpha(0) = 2$	MC(error) $\alpha(0) = 2$	Semi-MC(error) $\alpha(0) = 2$
0.1	11.367	11.370 (-0.003)	11.361 (0.006)
0.2	12.914	12.906 (0.008)	12.889 (0.25)
0.5	16.813	16.730 (0.083)	16.720 (0.93)
1.0	22.003	21.749 (0.254)	21.820 (0.183)
2.0	30.383	29.944 (0.439)	30.087 (0.296)
3.0	37.421	37.135 (0.286)	37.062 (0.359)

2.7 Numerical experiments using FFT

In this section, we report numerical results of using FFT for option pricing developed in this chapter. In implementing the FFT, we choose the number of grid points $N = 4096(2^{12})$. That is, we invoke the FFT procedure to calculate 4096 option prices simultaneously. The grid size along the log strike price κ is set to be $\Delta_\kappa = 0.01$. Consequently, $\Delta_u = 0.1534$ by (2.53). We choose the damping factor ρ to be $\rho = 1.0$. All options considered in the examples have maturity $T = 1$ (year). The initial asset price $S_0 = \$100$. We consider a two-state Markov chain model. The parameters are given by $\lambda_1 = 20.0$, $\lambda_2 = 30.0$, $\mu_1 = r_1 = 0.05$, $\mu_2 = r_2 = 0.1$, $\sigma_1 = 0.5$ and $\sigma_2 = 0.3$. Note that, unlike previous example, in this model, the parameters μ , σ and r all vary with different states. Large jump rates λ_1 and λ_2 are chosen so that the system switches frequently during the life of the options. Table 2.3 and Table 2.4 report the results for seven call options with different strike prices (from deep-in-the-money to at-the-money and to deep-out-of-money) obtained using FFT, MC and Semi-MC simulations. Column one in both tables lists the log strike (the strike) for the options. Columns two, three and four in both tables list the FFT, MC and

Semi-MC prices for both $\alpha(0) = 1$ and $\alpha(0) = 2$.

In each case ($\alpha(0) = 1$ and $\alpha(0) = 2$), a single run of FFT algorithm produces 4096 option prices (each one with a different strike price and all other parameters are the same). It took only 0.069 seconds, to run the FFT algorithm. This shows the clear advantage of the FFT.

Table 2.3: Comparison of FFT, MC and Semi-MC simulations at state $\alpha(0) = 1$

$\ln(K/S_0)(K)$	FFT $\alpha(0) = 1$	MC $\alpha(0) = 1$	Semi-MC $\alpha(0) = 1$
-0.3 (74.082)	34.774	34.708	34.773
-0.2 (81.873)	29.696	29.682	29.695
-0.1 (90.484)	24.763	24.766	24.763
0 (100)	20.116	20.095	20.116
0.1 (110.517)	15.881	15.888	15.881
0.2 (122.140)	12.157	12.162	12.155
0.3 (134.986)	9.006	9.019	9.004

Table 2.4: Comparison of FFT, MC and Semi-MC simulations at state $\alpha(0) = 2$

$\ln(K/S_0)(K)$	FFT $\alpha(0) = 2$	MC $\alpha(0) = 2$	Semi-MC $\alpha(0) = 2$
-0.3 (74.082)	34.742	34.661	34.741
-0.2 (81.873)	29.642	29.426	29.642
-0.1 (90.484)	24.688	24.744	24.688
0 (100)	20.022	20.060	20.024
0.1 (110.517)	15.774	15.767	15.772
0.2 (122.140)	12.043	12.086	12.043
0.3 (134.986)	8.893	8.876	8.893

2.8 Concluding remarks

FFT has been used for calculating option prices for a wide range of asset price models. In this chapter, we extended the technique to the class of regime-switching diffusion models and developed the FFT scheme. When the number of states of the driving Markov chain in the model is very large, the calculation of the characteristic function involved in the FFT approach becomes computationally intensive. The speedup of FFT along with acceptable accuracy is a promising direction for future research.

In fact, we have illustrated how the calculation of the call price via the Carr-Madan formula can be done fast and accurately using the fast Fourier Transform. Typically, N is a power of 2 (where N is the number of discretization steps). The number of operations of the FFT algorithm is of the order $\mathcal{O}(N\log(N))$ and this is in contrast to the straightforward evaluation of the sums which give rise to $\mathcal{O}(N^2)$ number of operations.

The methodology cannot only be applied when it's not possible to get an explicit form for characteristic function. However, the fact remains that the FFT is the most fast and efficiently method for options price.

CHAPTER 3: FFT APPROACH FOR VALUATION OF COMMODITY AND FUTURES OPTIONS UNDER A REGIME-SWITCHING MODEL

In this chapter, we use FFT approach to value commodity and futures options price, where the log of the underlying commodity spot price is governed by a regime-switching Ornstein-Uhlenbeck process. This Chapter is organized as follows. In the first section, we state our motivation. Section 2 presents the model dynamics. In Section 3, we first derive the valuation of commodity options via inverse Fourier transform approach. Then the value of futures option price is derived. The final section makes concluding remarks.

3.1 Motivation

In the previous chapter, we described how one can price, very fast and efficiently, European call options where the underlying asset price is governed by a regime-switching geometric Brownian motion using the theory of characteristic functions and FFT. We have developed a solid understanding of the current frameworks for pricing European call options using FFT, and we have provided the mathematical and practical background necessary to apply and implement the technique. FFT method is particularly interesting in case of advanced equity models, like the future contracts, its stochastic volatility extension, and many other models like the Heston model, where no closed-form solutions for call options exist.

An important advantage of the method is that we only need as input the characteristic function of the dynamics of the underlying model. If one wants to switch to another model, only the corresponding characteristic functions needs to be changed

and the actual pricing algorithm remains untouched.

Macroeconomic conditions could have significant impacts on commodity prices. An early work which highlights the link between business cycles and commodity prices was attributed to the paper by Fama and French [23]. From a practical perspective, it is of interest to model and investigate the impacts of structural changes in macroeconomic conditions on commodity prices. The basic idea of regime-switching models is that one set of model parameters is in force at a time depending on the state of the underlying state process at that time. The state process is usually described by a Markov chain.

In this chapter, we consider an observable Markovian regime-switching Ornstein-Uhlenbeck model (MRSOU) for evaluating European-style commodity options and futures options. The main feature of our model is that model parameters, the mean-reverting level and the volatility of the commodity spot price, are governed by an observable continuous-time finite-state Markov chain. We then discuss the valuation of commodity options and commodity futures options using inverse Fourier transform. In the final section, we provide concluding remarks. All proofs in this chapter are standard and involve the use of standard mathematical techniques.

3.2 Model dynamics

We consider a continuous-time economy with a finite time horizon $\mathcal{T} := [0, T^*]$, where $T^* < \infty$. Uncertainties in the economy are described by a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where \mathcal{P} is the real-world probability measure. Let $\alpha(\cdot) = \{\alpha(t) | t \in \mathcal{T}\}$ be an observable continuous-time finite-state Markov chain with state space $\varepsilon = \{e_1, e_2, \dots, e_N\} \subset \mathbb{R}^N$, where e_i is the unit vector in \mathbb{R}^N with one in the i -th position and zero elsewhere. In particular, there could be just two states for $\alpha(\cdot)$, write $\alpha(t) = (1, 0)'$ and $\alpha(t) = (0, 1)'$ for any $t \in \mathcal{T}$, where $(1, 0)'$ denotes the transpose of $(1, 0)$. State 1 and State 2 represent a “good” economy and a “bad” one, respectively.

From Elliott et al.[15] in Appendix B, a martingale representation for the Markov chain $\alpha(\cdot)$ is given by

$$\alpha(t) = \alpha(0) + \int_0^t Q \alpha(u) du + M(t), \quad t \in \mathcal{T} \quad (3.1)$$

where $Q := [q_{ij}]_{N \times N}$ is a constant rate matrix of the Markov chain and $\{M(t)|t \in \mathcal{T}\}$ is a martingale under \mathcal{P} . The element q_{ij} in Q is the constant intensity of the transition of the chain $\alpha(\cdot)$ from State e_j to State e_i .

We consider commodities that are not investment assets, such as agricultural products, oil or metals. Evidence from futures prices (Bessembinder et al. [2]) highlights that the spot prices of these commodities follow mean reverting processes. Hence the main model feature is mean reversion in commodity prices, indeed mean reversion toward a regime-switching mean price level. We now present an observable Markovian Regime-Switching Ornstein-Uhlenbeck model proposed by Schwartz [45] for a commodity futures pricing. In the proposed model by Schwartz [45] θ and σ are constant, which seems restrictive. We use this model to stress the mean reverting character of commodity spot prices. In fact Bessembinder et al. [2] indeed provided evidence from futures prices supporting such mean-reversion in commodity prices. The model in this section extend and complement the one in Schwartz [47]. The spot price process parameters can switch regime. This section presents a commodity futures pricing model that extends one of the models proposed by Schwartz [47].

We define the commodity spot price as $S_t = e^{X_t}$. The process for X_t is assumed to be

$$dX_t = \beta(\theta_t - X_t)dt + \sigma_t dW_t, \quad (3.2)$$

where W_t is a Wiener process under the risk-neutral probability measure. θ_t is the long

term mean level. All future trajectories of X_t will evolve around a mean level θ_t in the long run; and let β be the parameter controlling the speed of mean reversion for the logarithmic commodity price process, where $\beta > 0$. The instantaneous volatility of the model is σ_t . It measures instant by instant the amplitude of randomness entering the system.

We assume that $\{\theta_t|t \in \mathcal{T}\}$ changes over time according to the state process of $\{\alpha(t)|t \in \mathcal{T}\}$ as follows:

$$\theta_t = \langle \theta, \alpha(t) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the scalar product \mathbb{R}^N . Here $\theta = (\theta_1, \theta_2, \dots, \theta_N)' \in \mathbb{R}^N$ with $\theta_i > 0$, for each $i = 1, 2, \dots, N$. In particular, θ_i is the mean-reversion level of the commodity process corresponding to the i -th state of the economic condition.

Define $\{\sigma_t|t \in \mathcal{T}\}$ as the volatility of the commodity price. Again we suppose that this volatility changes over time according to the state process of the economy as follows:

$$\sigma_t = \langle \sigma, \alpha(t) \rangle$$

where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N) \in \mathbb{R}^N$ with $\sigma_i > 0$.

The solution to the stochastic differential equation (3.2) is given by the following equation:

$$X_T = e^{-\beta(T-t)} X_t + \beta \int_t^T \theta_u e^{-\beta(T-u)} du + \int_t^T \sigma_u e^{-\beta(T-u)} dW_u \quad (3.3)$$

3.3 Valuation of commodity futures and options

Our goal in this section is to evaluate the prices of commodity options and futures options at time 0, denoted by $C(0, T)$ and $C_f(0, T, U)$, respectively. The valuation of

the two options at an arbitrary time $t \in [0, T]$, where $T < T^*$ can be conducted in a similar fashion. The mathematical results presented in this section are similar to those in, for example, Fan et al. [25], Section 4 therein. In Fan et al. [25], the FFT approach was used to value European call options in a Markovian regime-switching stochastic interest rate environment. Hence, the expressions of the characteristic functions and the option prices presented here are not exactly the same as those in Fan et al. [25].

Under the risk-neutral probability measure \mathcal{P} , the prices of a T -maturity futures contract and a T -maturity European-style commodity option at time 0 are given as follows:

$$F(0, T) = \mathbb{E}\{S_T\} \quad (3.4)$$

and

$$C(0, T) = \mathbb{E}\left\{e^{-rT} \left(S_T - K\right)^+\right\}, \quad (3.5)$$

where S_T is the terminal commodity price; K is the strike price of the commodity option; \mathbb{E} is the expectation with respect to the risk-neutral probability measure \mathcal{P} . Consider a European-style futures option with a strike price K_f , the terminal payoff function at the maturity time T of the option is $(F(T, U) - K_f)^+$, where $F(T, U)$ represents the futures price with maturity time U at time T . Then the price of the futures option at time 0 is given by

$$C_f(0, T, U) = \mathbb{E}\left\{e^{-rT} \left(F(T, U) - K_f\right)^+\right\}. \quad (3.6)$$

3.3.1 Valuation of commodity options

Following the notation in the previous chapter, write $\kappa = \ln(K)$, the dampened commodity option price is given by

$$c(\kappa) = e^{\rho\kappa} C(O, T) \quad (3.7)$$

where ρ is called the dampening coefficient and assumed to be positive. To obtain a square integrable function, the dampening coefficient ρ is selected and the dampened commodity pricing formula is defined. We derive an explicit formula for the Fourier transform of $c(\kappa)$ next. Let $\mathcal{F}^\alpha := \{\mathcal{F}^\alpha(t) | t \in \mathcal{T}\}$ and $\mathcal{F}^S := \{\mathcal{F}^S(t) | t \in \mathcal{T}\}$ be the natural filtrations generated by $\{\alpha(t) | t \in \mathcal{T}\}$ and $\{S_t | t \in \mathcal{T}\}$ respectively. As usual, we assume that the filtrations given above are right-continuous and \mathcal{P} -complete. We define an enlarged filtration $\mathcal{G} := \{\mathcal{G}_t | t \in \mathcal{T}\}$ by letting $\mathcal{G}_t := \mathcal{F}^\alpha(t) \vee \mathcal{F}^S(t)$. Here $\mathcal{A} \vee \mathcal{B}$ represents the minimal σ -field containing both the σ -fields \mathcal{A} and \mathcal{B} .

Let $f_{\mathcal{F}^\alpha(T)}(x)$ be the conditional density function of X_T given $\mathcal{F}^\alpha(T)$. Then, the dampened commodity Fourier transform is given by

$$\hat{c}(u) = \frac{e^{-rT} \mathbb{E} \left\{ \phi_{\mathcal{F}^\alpha(T)}(-i(1+\rho) - u) \right\}}{(\rho - iu)(1 + \rho - iu)} \quad (3.8)$$

where

$$\phi_{\mathcal{F}^\alpha(T)}(\nu) = \mathbb{E}\{e^{i\nu X_T} | \mathcal{F}^\alpha(T)\} = \int_{-\infty}^{\infty} e^{i\nu x} f_{\mathcal{F}^\alpha(T)}(x) dx \quad (3.9)$$

is the conditional characteristic function of X_T given $\mathcal{F}^\alpha(T)$. See chapter 1 section 1.3.2 for details of calculation in (3.8).

Note that given $\mathcal{F}^\alpha(T)$, X_T has Gaussian distribution with mean

$$\mathbb{E}\{X_T | \mathcal{F}^\alpha(T)\} = e^{-\beta T} X_0 + \beta \int_0^T \theta_u e^{-\beta(T-u)} du \quad (3.10)$$

and variance

$$\text{Var}(X_T|\mathcal{F}^\alpha(T)) = \int_0^T \sigma_u^2 e^{-2\beta(T-u)} du. \quad (3.11)$$

It follows that

$$\begin{aligned} \phi_{\mathcal{F}^\alpha(T)}(\nu) &= \exp\left(i\nu\mathbb{E}\{X_T|\mathcal{F}^\alpha(T)\} - \frac{1}{2}\nu^2\text{Var}(X_T|\mathcal{F}^\alpha(T))\right) \\ &= \exp\left(i\nu(e^{-\beta T}X_0 + \beta \int_0^T \theta_u e^{-\beta(T-u)} du) \right. \\ &\quad \left. - \frac{1}{2}\nu^2 \int_0^T \sigma_u^2 e^{-2\beta(T-u)} du\right). \end{aligned} \quad (3.12)$$

For each $t \in [0, T]$ and $\nu \in \mathbb{R}$, let $G(t, \nu) = (g_1(t, \nu), g_2(t, \nu), \dots, g_N(t, \nu))$, where $g_j(t, \nu)$ for each $j = 1, 2, \dots, N$ is

$$g_j(t, \nu) = i\nu \beta \theta_j e^{-\beta(T-t)} - \frac{1}{2}\nu^2 \sigma_j^2 e^{-2\beta(T-t)}.$$

Therefore,

$$\phi_{\mathcal{F}^\alpha(T)}(\nu) = \exp\left(i\nu e^{-\beta T}X_0 + \int_0^T \langle G(t, \nu), \alpha(t) \rangle dt\right). \quad (3.13)$$

Substituting (3.13) in (3.8) implies

$$\hat{c}(u) = \frac{e^{-rT} \mathbb{E}\left\{\exp\left(i\nu e^{-\beta T}X_0 + \int_0^T \langle G(t, \nu), \alpha(t) \rangle dt\right)\right\}}{(\rho - iu)(1 + \rho - iu)} \quad (3.14)$$

where $\nu = -(u + i(1 + \rho))$. In order to derive an explicit formula for $\hat{c}(u)$, it's necessary to calculate the expectation given in (3.14). To this end, we use a modification to the proof of Lemma A₁ in Buffington and Elliott [5]. Let **diag**($G(t, \nu)$) denote the diagonal matrix with diagonal elements given by the components of $G(t, \nu)$, $\mathbf{1} =$

$(1, 1, \dots, 1)'$ and \mathbf{I} denote the $(n \times n)$ -identity matrix. Let's define

$$W(t, \nu) := \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) \alpha(t), \quad W(0, \nu) = \alpha(0). \quad (3.15)$$

Consequently,

$$dW(t, \nu) = \langle G(t, \nu), \alpha(t) \rangle W(t, \nu) dt + \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) d\alpha(t).$$

Note that under \mathcal{P} ,

$$d\alpha(t) = Q' \alpha(t) dt + dM(t),$$

and that

$$\langle G(t, \nu), \alpha(t) \rangle W(t, \nu) = \mathbf{diag}(G(t, \nu)) W(t, \nu)$$

for all $t \in \mathcal{T}$. Then

$$\begin{aligned} dW(t, \nu) &= \langle G(t, \nu), \alpha(t) \rangle W(t, \nu) dt + \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) Q' \alpha(t) dt \\ &\quad + \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) dM(t) \\ &= \mathbf{diag}(G(t, \nu)) W(t, \nu) dt + Q' W(t, \nu) dt \\ &\quad + \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) dM(t) \\ &= \left[\mathbf{diag}(G(t, \nu)) + Q'\right] W(t, \nu) dt + \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) dM(t). \end{aligned}$$

By taking integration from both sides, we have

$$W(t, \nu) = W(0, \nu) + \int_0^t \left[\mathbf{diag}(G(s, \nu)) + Q'\right] W(s, \nu) ds$$

$$\begin{aligned}
& + \int_0^t \exp \left(\int_0^s \langle G(u, \nu), \alpha(u) \rangle du \right) dM(s) \\
& = \alpha(0) + \int_0^t \left[\mathbf{diag}(G(s, \nu)) + Q' \right] W(s, \nu) ds \\
& + \int_0^t \exp \left(\int_0^s \langle G(u, \nu), \alpha(u) \rangle du \right) dM(s).
\end{aligned}$$

The final integral is a martingale, so taking the expectation under \mathcal{P} gives:

$$\mathbb{E}\{W(t, \nu)\} = \alpha(0) + \int_0^t [\mathbf{diag}(G(s, \nu)) + Q'] \mathbb{E}\{W(s, \nu)\} ds.$$

Hence $\mathbb{E}\{W(t, \nu)\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N with initial condition:

$$\frac{d\mathbb{E}\{W(t, \nu)\}}{dt} = [\mathbf{diag}(G(t, \nu)) + Q'] \mathbb{E}\{W(t, \nu)\}, \quad \mathbb{E}\{W(0, \nu)\} = \alpha(0). \quad (3.16)$$

Suppose $\Phi(t, \nu)$ denotes the fundamental matrix solution of

$$\frac{d\Phi(t, \nu)}{dt} = [\mathbf{diag}(G(t, \nu)) + Q'] \Phi(t, \nu), \quad \Phi(0, \nu) = \mathbf{I}.$$

If $[\mathbf{diag}(G(t, \nu)) + Q'] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Phi(t, \nu)$ is

$$\Phi(t, \nu) = \exp(\Delta t).$$

In general, there exists a unique fundamental matrix solution $\Phi(t, \nu)$ of the linear matrix differential Eq. (3.16) provided that the coefficient matrix is continuous which in our case it is. Now, $\mathbb{E}\{W(t, \nu)\}$ can be represented in terms of the fundamental matrix solution $\Phi(t, \nu)$ as below:

$$\mathbb{E}\{W(t, \nu)\} = \Phi(t, \nu) \alpha(0).$$

Since $\langle \alpha(T), \mathbf{1} \rangle = 1$, we have

$$\begin{aligned}
\mathbb{E} \left\{ \exp \left(\int_0^T \langle G(t, \nu), \alpha(t) \rangle dt \right) \right\} &= \mathbb{E} \left\{ \exp \left(\int_0^T \langle G(t, \nu), \alpha(t) \rangle dt \right) \langle \alpha(T), \mathbf{1} \rangle \right\} \\
&= \mathbb{E} \left\{ \left\langle \exp \left(\int_0^T \langle G(t, \nu), \alpha(t) \rangle dt \right) \alpha(T), \mathbf{1} \right\rangle \right\} \\
&= \left\langle \mathbb{E} \left\{ \exp \left(\int_0^T \langle G(t, \nu), \alpha(t) \rangle dt \right) \alpha(T) \right\}, \mathbf{1} \right\rangle \\
&= \langle \mathbb{E} \{ W(T, \nu) \}, \mathbf{1} \rangle \\
&= \langle \Phi(T, \nu) \alpha(0), \mathbf{1} \rangle.
\end{aligned}$$

Therefore under Markovian regime-switching Ornstein-Uhlenbeck model, the price of the commodity option is given by:

$$\hat{c}(u) = \frac{\exp \left(-rT + i\nu e^{-\beta T} X_0 \right) \langle \Phi(T, \nu) \alpha(0), \mathbf{1} \rangle}{(\rho - iu)(1 + \rho - iu)} \quad (3.17)$$

where $\nu = -(u + i(1 + \rho))$.

3.3.2 Valuation of futures options

In this subsection we consider the valuation of commodity futures options. We wish to evaluate the time-zero value of a standard European call option on the future price $F(T, U)$ with strike price K_f and maturity at time T . That is to evaluate:

$$C_f(0, T, U) = \mathbb{E} \left\{ e^{-rT} \left(F(T, U) - K_f \right)^+ \right\}.$$

As seen before, the dampened commodity futures options price is given by:

$$c_f(\kappa_f) = e^{\rho_f \kappa_f} C_f(0, T, U), \quad (3.18)$$

where $\kappa_f = \ln(K_f)$. Let's define $Y_t := \ln(F(t, U))$ for each $t \in \mathcal{T}$ and let $f_{\mathcal{F}^\alpha(T)}(y)$ be

the conditional density function of Y_T given $\mathcal{F}^\alpha(T)$. Now we derive an explicit formula for the Fourier transform of the dampened commodity futures options.

$$\begin{aligned}
\hat{c}_f(u) &= \int_{-\infty}^{\infty} e^{-iu\kappa_f} c(\kappa_f) d\kappa_f \\
&= \int_{-\infty}^{\infty} e^{-iu\kappa_f} e^{\rho_f \kappa_f} \mathbb{E} \left\{ e^{-rT} (e^{Y_T} - e^{\kappa_f})^+ \right\} d\kappa_f \\
&= \mathbb{E} \left\{ \int_{-\infty}^{\infty} e^{-iu\kappa_f} e^{\rho_f \kappa_f} \mathbb{E} \left\{ e^{-rT} (e^{Y_T} - e^{\kappa_f})^+ \middle| \mathcal{F}^\alpha(T) \right\} d\kappa_f \right\} \\
&= \mathbb{E} \left\{ \int_{-\infty}^{\infty} e^{-iu\kappa_f} e^{\rho_f \kappa_f} e^{-rT} \int_{\kappa_f}^{\infty} (e^y - e^{\kappa_f}) f_{\mathcal{F}^\alpha(T)}(y) dy d\kappa_f \right\} \\
&= e^{-rT} \mathbb{E} \left\{ \int_{-\infty}^{\infty} f_{\mathcal{F}^\alpha(T)}(y) \int_{-\infty}^y (e^y e^{(\rho_f - iu)\kappa_f} - e^{(1 + \rho_f - iu)\kappa_f}) d\kappa_f dy \right\} \\
&= e^{-rT} \mathbb{E} \left\{ \int_{-\infty}^{\infty} f_{\mathcal{F}^\alpha(T)}(y) \left(\frac{e^{(1 + \rho_f - iu)y}}{(\rho_f - iu)} - \frac{e^{(1 + \rho_f - iu)y}}{(1 + \rho_f - iu)} \right) dy \right\} \\
&= e^{-rT} \mathbb{E} \left\{ \left(\frac{\psi_{\mathcal{F}^\alpha(T)}(-i(1 + \rho_f) - u)}{(\rho_f - iu)} - \frac{\psi_{\mathcal{F}^\alpha(T)}(-i(1 + \rho_f) - u)}{(1 + \rho_f - iu)} \right) \right\} \\
&= \frac{e^{-rT} \mathbb{E} \left\{ \psi_{\mathcal{F}^\alpha(T)}(-i(1 + \rho_f) - u) \right\}}{(\rho_f - iu)(1 + \rho_f - iu)} \tag{3.19}
\end{aligned}$$

where

$$\psi_{\mathcal{F}^\alpha(T)}(\nu) = \mathbb{E} \{ e^{i\nu Y_T} | \mathcal{F}^\alpha(T) \} = \int_{-\infty}^{\infty} e^{i\nu y} f_{\mathcal{F}^\alpha(T)}(y) dy \tag{3.20}$$

is the conditional characteristic function of Y_T given $\mathcal{F}^\alpha(T)$. In order to derive an explicit formula for $\hat{c}_f(u)$, we need to derive an analytical formula of the characteristic function of the logarithmic commodity futures price, in other words $\ln(F(T, U))$.

To do so, first we need to derive the time- t price of a T -maturity futures contract.

In other words,

$$F(t, T) = \mathbb{E}\{S_T|\mathcal{G}_t\} = \mathbb{E}\left\{\mathbb{E}\{S_T|\mathcal{F}^\alpha(T) \vee \mathcal{F}^S(t)\}|\mathcal{G}_t\right\}$$

Where we used tower property for the second equality because $\mathcal{G}_t \subset \mathcal{F}^\alpha(T) \vee \mathcal{F}^S(t)$. Since $S_T = e^{X_T}$ and given the initial condition at time t , X_T has a Gaussian distribution with mean

$$\mathbb{E}\{X_T|\mathcal{F}^\alpha(T)\} = e^{-\beta(T-t)}X_t + \beta \int_t^T \theta_u e^{-\beta(T-u)} du, \quad (3.21)$$

and variance

$$\text{Var}(X_T|\mathcal{F}^\alpha(T)) = \int_t^T \sigma_u^2 e^{-2\beta(T-u)} du, \quad (3.22)$$

then, it's easy to see that

$$\begin{aligned} \mathbb{E}\{S_T|\mathcal{F}^\alpha(T)\} &= \exp\left(\mathbb{E}\{X_T|\mathcal{F}^\alpha(T)\} + \frac{1}{2} \text{Var}(X_T|\mathcal{F}^\alpha(T))\right) \\ &= \exp\left(e^{-\beta(T-t)}X_t + \beta \int_t^T \theta_u e^{-\beta(T-u)} du \right. \\ &\quad \left. + \frac{1}{2} \int_t^T \sigma_u^2 e^{-2\beta(T-u)} du\right). \end{aligned} \quad (3.23)$$

Then the time- t price of a T -maturity futures contract is given by

$$\begin{aligned} F(t, T) &= \mathbb{E}\{S_T|\mathcal{G}_t\} = \mathbb{E}\left\{\mathbb{E}\{S_T|\mathcal{F}^\alpha(T) \vee \mathcal{F}^S(t)\}|\mathcal{G}_t\right\} \\ &= \mathbb{E}\left\{\exp\left(e^{-\beta(T-t)}X_t + \beta \int_t^T \theta_u e^{-\beta(T-u)} du \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \int_t^T \sigma_u^2 e^{-2\beta(T-u)} du\right)\middle|\mathcal{G}_t\right\}. \end{aligned}$$

Let's define $N(s) = (n_1(s), n_2(s), \dots, n_N(s))$ for each $s \in [0, T]$, where for each

$j = 1, 2, \dots, N,$

$$n_j(s) = \beta \theta_j e^{-\beta(T-s)} + \frac{1}{2} \sigma_j^2 e^{-2\beta(T-s)}.$$

Therefore

$$F(t, T) = \mathbb{E} \left\{ \exp \left(e^{-\beta(T-t)} X_t + \int_t^T \langle N(s), \alpha(s) \rangle ds \right) \middle| \mathcal{G}_t \right\}. \quad (3.24)$$

In order to derive an explicit formula for $F(t, T)$, It's necessary to calculate the expectation given in (3.24). To do this end, we adopt the same methodology used in previous subsection. Let $\mathbf{diag}(N(t))$ denote the diagonal matrix with diagonal elements given by the components of $N(t)$. Let's define

$$H(t, u) = \exp \left(\int_t^u \langle N(s), \alpha(s) \rangle ds \right) \alpha(u), \quad H(t, t) = \alpha(t).$$

Consequently,

$$dH(t, u) = \langle N(u), \alpha(u) \rangle H(t, u) du + \exp \left(\int_t^u \langle N(s), \alpha(s) \rangle ds \right) d\alpha(u).$$

Note that under \mathcal{P} ,

$$d\alpha(u) = Q' \alpha(u) du + dM(u),$$

and that

$$\langle N(u), \alpha(u) \rangle H(t, u) = \mathbf{diag}(N(u)) H(t, u)$$

for all $t \in \mathcal{T}$. Then

$$\begin{aligned}
dH(t, u) &= \langle N(u), \alpha(u) \rangle H(t, u) du + \exp\left(\int_t^u \langle N(s), \alpha(s) \rangle ds\right) Q' \alpha(u) du \\
&\quad + \exp\left(\int_t^u \langle N(s), \alpha(s) \rangle ds\right) dM(u) \\
&= \mathbf{diag}(N(u)) H(t, u) du + Q' H(t, u) du \\
&\quad + \exp\left(\int_t^u \langle N(s), \alpha(s) \rangle ds\right) dM(u) \\
&= \left[\mathbf{diag}(N(u)) + Q' \right] H(t, u) du + \exp\left(\int_t^u \langle N(s), \alpha(s) \rangle ds\right) dM(u).
\end{aligned}$$

By taking integration from both sides, we have

$$\begin{aligned}
H(t, u) &= H(t, t) + \int_t^u \left[\mathbf{diag}(N(s)) + Q' \right] H(t, s) ds \\
&\quad + \int_t^u \exp\left(\int_t^\tau \langle N(s), \alpha(s) \rangle ds\right) dM(\tau) \\
&= \alpha(t) + \int_t^u \left[\mathbf{diag}(N(s)) + Q' \right] H(t, s) ds \\
&\quad + \int_t^u \exp\left(\int_t^\tau \langle N(s), \alpha(s) \rangle ds\right) dM(\tau).
\end{aligned}$$

The final integral is a martingale, so taking expectation under \mathcal{P} conditioning on \mathcal{G}_t gives:

$$\mathbb{E}\{H(t, u)|\mathcal{G}_t\} = \alpha(t) + \int_t^u \left[\mathbf{diag}(N(s)) + Q' \right] \mathbb{E}\{H(t, s)|\mathcal{G}_t\} ds.$$

Hence $\mathbb{E}\{H(t, u)|\mathcal{G}_t\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N :

$$\frac{d\mathbb{E}\{H(t, u)|\mathcal{G}_t\}}{du} = \left[\mathbf{diag}(N(u)) + Q' \right] \mathbb{E}\{H(t, u)|\mathcal{G}_t\} \quad (3.25)$$

with the initial condition $\mathbb{E}\{H(t, t)|\mathcal{G}_t\} = \alpha(t)$. Suppose $\Psi(t, u)$ denotes the funda-

mental matrix solution of

$$\frac{d\Psi(t, u)}{du} = [\mathbf{diag}(N(u)) + Q'] \Psi(t, u), \quad \Psi(t, t) = \mathbf{I}.$$

If $[\mathbf{diag}(N(u)) + Q'] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Psi(t, u)$ is

$$\Psi(t, u) = \exp(\Delta (u - t)).$$

In general, there exists a unique fundamental matrix solution $\Psi(t, u)$ of the linear matrix differential Eq. (3.25) as the coefficient matrix is continuous. Now, $\mathbb{E}\{H(t, u)|\mathcal{G}_t\}$ can be represented in terms of the fundamental matrix solution $\Psi(t, u)$ as below:

$$\mathbb{E}\{H(t, u)|\mathcal{G}_t\} = \Psi(t, u) \alpha(t).$$

Finally

$$\begin{aligned} F(t, T) &= \mathbb{E}\left\{\exp\left(e^{-\beta(T-t)} X_t + \int_t^T \langle N(u), \alpha(u) \rangle du\right) \middle| \mathcal{G}_t\right\} \\ &= \exp\left(e^{-\beta(T-t)} X_t\right) \mathbb{E}\left\{\exp\left(\int_0^T \langle N(t), \alpha(t) \rangle du\right) \middle| \mathcal{G}_t\right\} \\ &= \exp\left(e^{-\beta(T-t)} X_t\right) \mathbb{E}\left\{\exp\left(\int_t^T \langle N(u), \alpha(u) \rangle du\right) \langle \alpha(T), \mathbf{1} \rangle \middle| \mathcal{G}_t\right\} \\ &= \exp\left(e^{-\beta(T-t)} X_t\right) \mathbb{E}\left\{\left\langle \exp\left(\int_t^T \langle N(u), \alpha(u) \rangle du\right) \alpha(T), \mathbf{1} \right\rangle \middle| \mathcal{G}_t\right\} \\ &= \exp\left(e^{-\beta(T-t)} X_t\right) \left\langle \mathbb{E}\left\{\exp\left(\int_t^T \langle N(u), \alpha(u) \rangle du\right) \alpha(T) \middle| \mathcal{G}_t\right\}, \mathbf{1} \right\rangle \\ &= \exp\left(e^{-\beta(T-t)} X_t\right) \langle \mathbb{E}\{H(t, T)|\mathcal{G}_t\}, \mathbf{1} \rangle \\ &= \exp\left(e^{-\beta(T-t)} X_t\right) \langle \Psi(t, T) \alpha(t), \mathbf{1} \rangle. \end{aligned}$$

Then under the risk-neutral probability measure \mathcal{P} , the time- T price of a U -maturity futures contract is given by:

$$\begin{aligned}
F(T, U) &= \mathbb{E}\{S_U | \mathcal{G}_T\} = \exp\left(e^{-\beta(U-T)} X_T\right) \times \langle \Psi(T, U) \alpha(T), \mathbf{1} \rangle \\
&= \exp\left(e^{-\beta(U-T)} \left[e^{-\beta T} X_0 + \beta \int_0^T \theta_u e^{-\beta(T-u)} du \right. \right. \\
&\quad \left. \left. + \int_0^T \sigma_u e^{-\beta(T-u)} dW_u \right] \right) \times \langle \Psi(T, U) \alpha(T), \mathbf{1} \rangle \\
&= \exp\left(e^{-\beta U} X_0 + e^{-\beta U} \beta \int_0^T \theta_u e^{\beta u} du \right. \\
&\quad \left. + e^{-\beta U} \int_0^T \sigma_u e^{\beta u} dW_u \right) \times \langle \Psi(T, U) \alpha(T), \mathbf{1} \rangle.
\end{aligned}$$

where $T < U < T^*$.

Note that given $\mathcal{F}^\alpha(T)$, $Y_T = \ln(F(T, U))$ has Gaussian distribution with mean

$$\mathbb{E}\{Y_T | \mathcal{F}^\alpha(T)\} = e^{-\beta U} X_0 + \beta \int_0^T \theta_u e^{-\beta(U-u)} du + \ln \langle \Psi(T, U) \alpha(T), \mathbf{1} \rangle \quad (3.26)$$

and variance

$$\text{Var}(Y_T | \mathcal{F}^\alpha(T)) = \int_0^T \sigma_u^2 e^{-2\beta(U-u)} du. \quad (3.27)$$

It follows from (3.26) and (3.27) that

$$\begin{aligned}
\psi_{\mathcal{F}^\alpha(T)}(\nu) &= \exp\left(i \nu \mathbb{E}\{Y_T | \mathcal{F}^\alpha(T)\} - \frac{1}{2} \nu^2 \text{Var}(Y_T | \mathcal{F}^\alpha(T))\right) \\
&= \exp\left(i \nu e^{-\beta U} X_0 + i \nu \beta \int_0^T \theta_u e^{-\beta(U-u)} du\right) \times \langle \Psi(T, U) \alpha(T), \mathbf{1} \rangle^{i \nu} \\
&\quad \times \exp\left(-\frac{1}{2} \nu^2 \int_0^T \sigma_u^2 e^{-2\beta(U-u)} du\right). \quad (3.28)
\end{aligned}$$

Define $Z(\nu) = (Z_1(\nu), Z_2(\nu), \dots, Z_N(\nu))$ and $z(t, \nu) = (z_1(t, \nu), z_2(t, \nu), \dots, z_N(t, \nu))$

for each $\nu \in \mathbb{R}$ and $t \in \tau$, where $Z_j(\nu)$ and $z_j(t, \nu)$ for each $j = 1, 2, \dots, N$ are

$$Z_j(\nu) = Z(e_j, \nu) = \exp\left(i \nu e^{-\beta U} X_0\right) \times \langle \Psi(T, U) e_j, \mathbf{1} \rangle^{i\nu} \quad (3.29)$$

and

$$z_j(t, \nu) = z(t, e_j, \nu) = i \nu \beta e^{-\beta(U-t)} \theta_j - \frac{1}{2} \nu^2 e^{-2\beta(U-t)} \sigma_j^2. \quad (3.30)$$

Therefore,

$$\psi_{\mathcal{F}^\alpha(T)}(\nu) = Z(\alpha(T), \nu) \exp\left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt\right). \quad (3.31)$$

Substituting (3.31) in (3.19) implies

$$\hat{c}_f(u) = \frac{e^{-rT} \mathbb{E}\left\{Z(\alpha(T), \nu) \exp\left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt\right)\right\}}{(\rho_f - iu)(1 + \rho_f - iu)} \quad (3.32)$$

where $\nu = -(u + i(1 + \rho_f))$.

As we have seen before, in order to derive an explicit formula for $\hat{c}_f(u)$, It's necessary to calculate the expectation given in (3.32). Let $\mathbf{diag}(z(t, \nu))$ denote the diagonal matrix with diagonal elements given by the components of $z(t, \nu)$. Let's define

$$\Gamma(t, \nu) = \exp\left(\int_0^t \langle z(s, \nu), \alpha(s) \rangle ds\right) \alpha(t), \quad \Gamma(0, \nu) = \alpha(0). \quad (3.33)$$

Applying Itô differentiation rule to $\Gamma(t, \nu)$ gives us

$$d\Gamma(t, \nu) = \langle z(t, \nu), \alpha(t) \rangle \Gamma(t, \nu) dt + \exp\left(\int_0^t \langle z(s, \nu), \alpha(s) \rangle ds\right) d\alpha(t).$$

Note that under \mathcal{P} ,

$$d\alpha(t) = Q' \alpha(t) dt + dM(t),$$

and that

$$\langle z(t, \nu), \alpha(t) \rangle \Gamma(t, \nu) = \mathbf{diag}(z(t, \nu)) \Gamma(t, \nu)$$

for all $t \in \mathcal{T}$. Then

$$\begin{aligned} d\Gamma(t, \nu) &= \langle z(t, \nu), \alpha(t) \rangle \Gamma(t, \nu) dt \\ &\quad + \exp\left(\int_0^t \langle z(s, \nu), \alpha(s) \rangle ds\right) Q' \alpha(t) dt \\ &\quad + \exp\left(\int_0^t \langle z(s, \nu), \alpha(s) \rangle ds\right) dM(t) \\ &= \mathbf{diag}(z(t, \nu)) \Gamma(t, \nu) dt + Q' \Gamma(t, \nu) dt \\ &\quad + \exp\left(\int_0^t \langle z(s, \nu), \alpha(s) \rangle ds\right) dM(t) \\ &= \left[\mathbf{diag}(z(t, \nu)) + Q' \right] \Gamma(t, \nu) dt \\ &\quad + \exp\left(\int_0^t \langle z(s, \nu), \alpha(s) \rangle ds\right) dM(t). \end{aligned}$$

By taking integration from both sides, we have

$$\begin{aligned} \Gamma(t, \nu) &= \Gamma(0, \nu) + \int_0^t \left[\mathbf{diag}(z(s, \nu)) + Q' \right] \Gamma(s, \nu) ds \\ &\quad + \int_0^t \exp\left(\int_0^s \langle z(u, \nu), \alpha(u) \rangle du\right) dM(s) \\ &= \alpha(0) + \int_0^t \left[\mathbf{diag}(z(s, \nu)) + Q' \right] \Gamma(s, \nu) ds \\ &\quad + \int_0^t \exp\left(\int_0^s \langle G(u, \nu), \alpha(u) \rangle du\right) dM(s). \end{aligned}$$

The final integral is a martingale, so taking expectation under \mathcal{P} gives us:

$$\mathbb{E}\{\Gamma(t, \nu)\} = \alpha(0) + \int_0^t [\mathbf{diag}(z(s, \nu)) + Q'] \mathbb{E}\{\Gamma(s, \nu)\} ds.$$

Hence $\mathbb{E}\{\Gamma(t, \nu)\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N :

$$\frac{d\mathbb{E}\{\Gamma(t, \nu)\}}{dt} = [\mathbf{diag}(z(t, \nu)) + Q'] \mathbb{E}\{\Gamma(t, \nu)\} \quad (3.34)$$

with initial condition $\mathbb{E}\{\Gamma(0, \nu)\} = \alpha(0)$. Suppose $\Upsilon(t, \nu)$ denotes the fundamental matrix solution of

$$\frac{d\Upsilon(t, \nu)}{dt} = [\mathbf{diag}(z(t, \nu)) + Q'] \Upsilon(t, \nu), \quad \Upsilon(0, \nu) = \mathbf{I}.$$

If $[\mathbf{diag}(z(t, \nu)) + Q'] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Phi(t, \nu)$ is

$$\Upsilon(t, \nu) = \exp(\Delta t).$$

In general, there exists a unique fundamental matrix solution $\Upsilon(t, \nu)$ of the linear matrix differential Eq. (3.34) if the coefficient matrix is continuous which in our case it is. Now, $\mathbb{E}\{\Gamma(t, \nu)\}$ can be represented in terms of the fundamental matrix solution $\Upsilon(t, \nu)$ as below:

$$\mathbb{E}\{\Gamma(t, \nu)\} = \Upsilon(t, \nu) \alpha(0).$$

Using $\langle \alpha(T), \mathbf{1} \rangle = 1$ implies:

$$\mathbb{E}\left\{ Z(\alpha(T), \nu) \exp\left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt\right) \right\}$$

$$\begin{aligned}
&= \mathbb{E} \left\{ Z(\alpha(T), \nu) \exp \left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt \right) \langle \alpha(T), \mathbf{1} \rangle \right\} \\
&= \mathbb{E} \left\{ \left\langle \exp \left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt \right) \alpha(T), Z(\nu) \right\rangle \right\} \\
&= \left\langle \mathbb{E} \left\{ \exp \left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt \right) \alpha(T) \right\}, Z(\nu) \right\rangle \\
&= \langle \mathbb{E} \{ \Gamma(T, \nu) \}, Z(\nu) \rangle \\
&= \langle \Upsilon(T, \nu) \alpha(0), Z(\nu) \rangle.
\end{aligned}$$

Therefore under Markovian regime-switching Ornstein-Uhlenbeck model, the price of the futures option is given by:

$$\hat{c}_f(u) = \frac{e^{-rT} \langle \Upsilon(T, \nu) \alpha(0), Z(\nu) \rangle}{(\rho_f - iu)(1 + \rho_f - iu)}, \quad \nu = -(u + i(1 + \rho_f)). \quad (3.35)$$

3.4 Conclusions

In this chapter, we discussed the valuation of European-style call options on commodity spot price and futures price in a Markovian regime-switching Ornstein-Uhlenbeck model. The model parameters were assumed to be modulated by an observable continuous time finite-state Markov chain, whose states represent the states of an economy. The main feature of our study is that the regime-switching effect is emphasized, i.e., the structural changes of macroeconomic conditions could be incorporated in the model. We applied the inverse Fourier transform to evaluate the prices of commodity options and futures options. We may further our work to investigate the valuation of American-style options under Markovian regime-switching Ornstein-Uhlenbeck model, since as we know, most of commodity options traded in NYMEX/CME are American-style options.

CHAPTER 4: FFT APPROACH FOR PRICING COMMODITY AND FUTURES OPTIONS UNDER A REGIME-SWITCHING MODEL WITH STOCHASTIC INTEREST RATE MODEL

In this chapter, we investigate the pricing of European-style commodity and futures options under a Markovian regime-switching Ornstein–Uhlenbeck model with a Markovian regime-switching Vasicek interest rate model. The model parameters, including the mean reversion level, the volatility of the stochastic interest rate, and the volatility of the commodity price process are modulated by an observable continuous-time finite-state Markov chain. We employ the concept of stochastic flows to derive an exponential affine form of the price of a zero-coupon bond. Then, we represent the exponential affine form of the bond price in terms of fundamental matrix solutions of linear matrix differential equations. Furthermore, we give the forward measure when taking the zero-coupon bond as the numéraire. Then we adopt similar methodology used in Chapter 3 to find a closed-form expression for the characteristic function of the logarithmic terminal commodity and futures price.

4.1 Motivation

Option valuation has been an important problem in the theory and practice of financial economics. A major breakthrough in this area was made by Fischer Black, Myron Scholes, and Robert Merton [3]. Despite the practical importance of the Black–Scholes–Merton model, its underlying assumptions, including the constant interest rate and volatility, are not consistent with empirical observations. It is phenomenal that interest rates have become volatile in the past few decades. Many

stochastic interest rate models have been introduced in the literature. Some popular short rate models include those proposed by Vasicek [49], Cox et al. [11], Hull and White [33], amongst others. The main feature of these models is that the short rate process, commonly described as a diffusion process, is mean-reverting. This means that the short rate process will eventually revert to a long-term value. This property is a “stylized” fact of the empirical behavior of interest rates.

Structural changes in economic conditions affect stochastic evolution of interest rates over time. Regime-switching models may be used to describe such impacts. This class of models was popularized by Hamilton [29] in financial econometrics. There has been some interest in pricing bonds and related options in Markovian regime-switching stochastic interest rate models. Elliott and Mamon [17] considered a Vasicek model, with the mean-reverting level being modulated by a continuous-time, finite-state Markov chain, while a regime-switching Hull-White model was considered in Elliott and Wilson [22].

Using the concept of stochastic flows, Elliott and Siu [18] discussed a bond valuation problem under a regime-switching Hull–White short rate model and a regime-switching Cox–Ingersoll–Ross model. Siu [48] proposed a general short rate model incorporating jumps of the interest rate due to some extraordinary market events or economic cycles. More specifically, Siu [48] derived a bond pricing formula under a jump-augmented Vasicek model, a kind of jump-diffusion-type short rate models, using techniques in stochastic flows. Shen and Siu [47] employed a partial differential equation approach to derive exponential-affine formulas for a zero-coupon bond and a longevity bond, respectively, while Shen and Siu [46] considered the valuation of a bond option under a regime-switching Hull–White model. Elliott and Siu [21] considered the valuation of bond options in a Markovian regime-switching Heath–Jarrow–Morton (HJM) model and derived semi-analytical formulas for pricing bond options using the Fourier transform space.

4.2 The model dynamics

As described in the previous chapter, we consider a continuous-time economy with a finite time horizon \mathcal{T} , i.e., $\mathcal{T} := [0, T^*]$, where $T^* < \infty$. Suppose $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space, where \mathcal{P} is a risk-neutral probability measure. Here, we start with a risk-neutral probability as in some literature on stochastic interest rate models. We assume the state of an economy is modeled by an observable continuous-time finite-state Markov chain $\alpha(\cdot) = \{\alpha(t) | t \in \mathcal{T}\}$. The state space of the chain is denoted by $S := \{s_1, s_2, \dots, s_N\}$, representing N different observable states of an economy. Without loss of generality, using the convention in Elliott et al. [15], we identify the state space of the chain with a finite set of standard unit vectors $\varepsilon := \{e_1, e_2, \dots, e_N\} \subset \mathbb{R}^N$, where the j -th component of e_i is the Kronecker delta δ_{ij} , for each $i, j = 1, 2, \dots, N$. Let $Q := [q_{ij}]_{i,j=1,2,\dots,N}$ denote the generator or rate matrix of the chain $\alpha(\cdot)$. Then, Elliott et al. [15] in Lemma 1.1 Appendix B obtained the following martingale dynamics for the chain $\alpha(\cdot)$:

$$\alpha(t) = \alpha(0) + \int_0^t Q \alpha(u) du + M(t) \quad (4.1)$$

for all $t \in \mathcal{T}$. Here $\{M(t) | t \in \mathcal{T}\}$ is an \mathbb{R}^N -valued martingale with respect to the filtration generated by $\alpha(\cdot)$ under the measure \mathcal{P} .

We now present the Markovian regime-switching models for the dynamics of the underlying logarithmic commodity spot price and the stochastic interest rate. Let y' be the transpose of a vector or a matrix y . Denote $\{\mu(t) | t \in \mathcal{T}\}$ and $\{\gamma(t) | t \in \mathcal{T}\}$ as the mean reversion level and the volatility of the short rate process, respectively. Suppose that

$$\mu(t) = \langle \mu, \alpha(t) \rangle$$

and

$$\gamma(t) = \langle \gamma, \alpha(t) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the scalar production \mathbb{R}^N . Here $\mu = (\mu_1, \mu_2, \dots, \mu_N)' \in \mathbb{R}^N$ with $\mu_i > 0$, and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_N)' \in \mathbb{R}^N$ with $\gamma_i > 0$ for each $i = 1, 2, \dots, N$. The mean reversion coefficient η describing the speed of mean reversion for short rate price process is assumed to be a positive constant.

Let $\sigma(t)$ and $\theta(t)$ be the volatility and the mean reversion level of the underlying logarithmic commodity spot price at time t . Again suppose that

$$\sigma(t) = \langle \sigma, \alpha(t) \rangle$$

and

$$\theta(t) = \langle \theta, \alpha(t) \rangle$$

where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)' \in \mathbb{R}^N$ with $\sigma_i > 0$ and $\theta = (\theta_1, \theta_2, \dots, \theta_N)' \in \mathbb{R}^N$ with $\theta_i > 0$ for each $i = 1, 2, \dots, N$. In particular, θ_i is the mean reversion level of the commodity process corresponding to the i -th state of the economic condition for each $i = 1, 2, \dots, N$. Let β be the parameter controlling the speed of mean reversion for the logarithmic commodity price process, where $\beta > 0$.

We define the commodity spot price as $S(t) = e^{X(t)}$. Then, we assume that under the risk-neutral probability measure \mathcal{P} , the dynamics of the underlying logarithmic commodity spot price and the short rate price process are given by

$$dX(t) = \beta(\theta(t) - X(t))dt + \sigma(t)dW_S(t) \quad (4.2)$$

$$dr(t) = \eta(\mu(t) - r(t))dt + \gamma(t)dW_r(t) \quad (4.3)$$

where $W_S := \{W_S(t)|t \in \mathcal{T}\}$ and $W_r := \{W_r(t)|t \in \mathcal{T}\}$ are two standard Brownian motions with respect to their right-continuous, \mathcal{P} -complete, natural filtrations under \mathcal{P} . Furthermore, we suppose that the two Brownian motions W_S and W_r are correlated, and the instantaneous correlation coefficient $\rho(t)$ at time t is given by

$$\rho(t) = \langle W_S, W_r \rangle = \int_0^t \rho(s) ds,$$

where $\rho(t) = \langle \rho, \alpha(t) \rangle$ and $\rho := (\rho_1, \rho_2, \dots, \rho_N) \in \mathbb{R}^N$ with $-1 < \rho_i < 1$.

$\{\langle W_S, W_r \rangle(t)|t \in \mathcal{T}\}$ is the (predictable) quadratic covariation between W_S and W_r . Consequently, the correlation coefficient between the spot price and the short rate depends on the state of an economy.

Let $\mathcal{F}^\alpha = \{\mathcal{F}^\alpha(t)|t \in \mathcal{T}\}$, $\mathcal{F}^S = \{\mathcal{F}^S(t)|t \in \mathcal{T}\}$ and $\mathcal{F}^r = \{\mathcal{F}^r(t)|t \in \mathcal{T}\}$ be the natural filtrations generated by $\{\alpha(t)|t \in \mathcal{T}\}$, $\{S(t)|t \in \mathcal{T}\}$ and $\{r(t)|t \in \mathcal{T}\}$ respectively. As usual, we assume that the filtrations given above are right-continuous and \mathcal{P} -complete. Define two enlarged filtrations $\mathcal{G} = \{\mathcal{G}(t)|t \in \mathcal{T}\}$ and $\mathcal{H} = \{\mathcal{H}(t)|t \in \mathcal{T}\}$ by letting

$$\mathcal{G}(t) := \mathcal{F}^r(t) \vee \mathcal{F}^\alpha(t)$$

and

$$\mathcal{H}(t) := \mathcal{F}^r(t) \vee \mathcal{F}^S(t) \vee \mathcal{F}^\alpha(t).$$

Here $\mathcal{A} \vee \mathcal{B}$ represents the minimal σ -field containing both the σ -fields \mathcal{A} and \mathcal{B} .

4.3 Bond pricing and the forward measure

In this section, we shall employ the concept of stochastic flows to derive an exponential affine form of the price of a zero-coupon bond. Then, we shall represent the

exponential affine form of the bond price in terms of fundamental matrix solutions of linear matrix differential equations. Furthermore, we give the forward measure when taking the zero-coupon bond as the numéraire. To do this end, we adopt the same methodology used in Elliott and Siu [18], Shen and Siu [21 & 22] and Siu [48].

4.3.1 Stochastic flows and bond prices

Let $r_{t,s}(r)$ be a version of stochastic interest rate process $r_{t,s}$, $s \geq t$, with initial condition $r_{t,t}(r) = r \in \mathbb{R}$. Then, from (4.3),

$$r_{t,s}(r) = r + \int_t^s \eta(\mu(u) - r_{t,u}(r))du + \int_t^s \gamma(u)dW_r(u). \quad (4.4)$$

Write

$$D_{t,s} = \frac{\partial r_{r,s}(r)}{\partial r}$$

for the derivative of the map $r \longrightarrow r_{t,u}(r)$. Differentiating (4.4) with respect to r gives

$$D_{t,s} = 1 - \eta \int_t^s D_{t,u}du,$$

with initial condition $D_{t,t} = 1$.

So,

$$D_{t,s} = e^{-\eta(s-t)}.$$

Here, $D_{t,s}$ is a deterministic real-valued process.

The price at time $t \in \mathcal{T}$ of any contingent claim $V \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$ is

$$P(t) = \mathbb{E} \left[\exp \left(- \int_t^T r_{t,u}(r) du \right) V \middle| \mathcal{G}(t) \right]$$

Here, $\mathbb{E}[\cdot]$ represents an expectation with respect to the risk-neutral measure \mathcal{P} . Letting $V(\omega) = 1$, for each $\omega \in \Omega$, the price of a zero-coupon bond at time t with maturity at time T is:

$$P(t, T) = \mathbb{E} \left[\exp \left(- \int_t^T r_{t,u}(r) du \right) \middle| \mathcal{G}(t) \right].$$

Since $(r, \alpha(\cdot))$ is a two-dimensional Markov process with respect to the enlarged filtration $\mathcal{G}(t)$, given that $r(t) = r$ and $\alpha(t) = \alpha$,

$$P(t, T) = \mathbb{E} \left[\exp \left(- \int_t^T r_{t,u}(r) du \right) \middle| \alpha(t) = \alpha, r(t) = r \right] = P(t, T, r, \alpha).$$

Define $B(t, T)$ as the following path integral:

$$B(t, T) = \int_t^T D_{t,u} du = \frac{1}{\eta} (1 - e^{-\eta(T-t)}),$$

so it is a real-valued deterministic process.

Since the exponential is bounded,

$$\begin{aligned} \frac{\partial P(t, T, r, \alpha)}{\partial r} &= \left(- \int_t^T D_{t,u} du \right) \mathbb{E} \left[\exp \left(- \int_t^T r_{t,u}(r) du \right) \middle| \alpha(t) = \alpha, r(t) = r \right] \\ &= -B(t, T) P(t, T, r, \alpha). \end{aligned} \tag{4.5}$$

Integrating (4.5) in r gives

$$P(t, T, r, \alpha) = \tilde{A}(t, T, \alpha) \exp(-B(t, T) r) = \exp \left(A(t, T, \alpha) - B(t, T) r \right),$$

where $A(t, T, \alpha) = \ln[\tilde{A}(t, T, \alpha)]$.

Consider the discounted bond price back to time zero:

$$\tilde{P}(t, T, r, \alpha) = \exp\left(-\int_0^t r_{0,u}(r) du\right) P(t, T, r, \alpha) = \mathbb{E}\left[\exp\left(-\int_0^T r_{t,u}(r) du\right) \middle| \mathcal{G}(t)\right].$$

Here, $\tilde{P}(t, T, r, \alpha)$ is a $(\mathcal{G}, \mathcal{P})$ -martingale.

Write $\tilde{P}_i = \tilde{P}(t, T, r, e_i)$ for $i = 1, 2, \dots, N$ and $\tilde{\mathbf{P}} = (\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_N)' \in \mathbb{R}^N$. Applying Itô's differentiation rule to $\tilde{P}(t, T, r, \alpha)$

$$\begin{aligned} \tilde{P}(t, T, r, \alpha) &= \tilde{P}(0, T, r_0, \alpha_0) + \int_0^t \frac{\partial \tilde{P}}{\partial u} du + \int_0^t \frac{\partial \tilde{P}}{\partial r} \eta(\mu(u) - r(u-)) du \\ &\quad + \int_0^t \frac{\partial \tilde{P}}{\partial r} \gamma(u) dW_r(u) + \int_0^t \langle \tilde{\mathbf{P}}, Q \alpha(u) \rangle du + \int_0^t \langle \tilde{\mathbf{P}}, dM(u) \rangle \\ &= \tilde{P}(0, T, r_0, \alpha_0) + \int_0^t \frac{\partial \tilde{P}}{\partial r} \gamma(u) dW_r(u) + \int_0^t \langle \tilde{\mathbf{P}}, dM(u) \rangle \\ &\quad + \int_0^t \left\{ \frac{\partial \tilde{P}}{\partial u} + \frac{\partial \tilde{P}}{\partial r} \eta(\mu(u) - r(u-)) + \langle \tilde{\mathbf{P}}, Q \alpha(u) \rangle \right\} du \end{aligned}$$

Note that $\tilde{P}(t, T, r, \alpha)$ is a $(\mathcal{G}, \mathcal{P})$ -martingale. So, the bounded variation terms, which are not martingales, in the above stochastic integral representation for $\tilde{P}(t, T, r, \alpha)$ must sum to zero. Therefore,

$$\frac{\partial \tilde{P}}{\partial t} + \frac{\partial \tilde{P}}{\partial r} \eta(\mu(t) - r(t-)) + \frac{1}{2} \frac{\partial^2 \tilde{P}}{\partial r^2} \gamma^2(t) + \langle \tilde{\mathbf{P}}, Q \alpha \rangle = 0$$

Write, for each $i = 1, 2, \dots, N$, $P_i = P(t, T, r, e_i)$ and $\mathbf{P} = (P_1, P_2, \dots, P_N)' \in \mathbb{R}^N$.

Then,

$$\begin{aligned} \exp\left(-\int_0^t r_{0,u}(r) du\right) &\left\{ \frac{\partial P}{\partial t} - r(t-)P + \frac{\partial P}{\partial r} \eta(\mu(t) - r(t-)) \right. \\ &\quad \left. + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \gamma^2(t) + \langle \mathbf{P}, Q \alpha \rangle \right\} = 0 \end{aligned}$$

So, we have the following regime-switching partial differential equation (PDE) for $P(t, T, r, \alpha)$

$$\frac{\partial P}{\partial t} - r(t-)P + \frac{\partial P}{\partial r} \eta(\mu(t) - r(t-)) + \frac{1}{2} \frac{\partial^2 P}{\partial r^2} \gamma^2(t) + \langle \mathbf{P}, Q \alpha \rangle = 0$$

with terminal condition

$$P(T, T, r(T), \alpha(T)) = 1.$$

Equivalently, the vector of bond prices P satisfies the following system of N coupled PDEs

$$\frac{\partial P_i}{\partial t} - r(t-)P_i + \frac{\partial P_i}{\partial r} \eta(\mu_i - r(t-)) + \frac{1}{2} \frac{\partial^2 P_i}{\partial r^2} \gamma_i^2 + \langle \mathbf{P}, Q e_i \rangle = 0$$

with terminal condition

$$P(T, T, r(T), e_i) = 1, \quad i = 1, 2, \dots, N.$$

Note that the bond price has the following Markovian regime-switching exponential affine form

$$P(t, T, r, \alpha) = \exp(A(t, T, \alpha) - B(t, T) r).$$

Recall that $\tilde{A}(t, T, \alpha) = \exp(A(t, T, \alpha))$. Let $\tilde{A}_i = \tilde{A}(t, T, e_i)$ for $i = 1, 2, \dots, N$ and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_N)' \in \mathbb{R}^N$. Note that

$$\begin{aligned} \frac{\partial P}{\partial t} &= P \left(\frac{\partial A}{\partial t} - r \frac{\partial B}{\partial t} \right), \\ \frac{\partial P}{\partial r} &= -B P, \end{aligned}$$

$$\frac{\partial^2 P}{\partial r^2} = B^2 P.$$

Then, $A(t, T, \alpha)$ satisfies the following Markovian regime-switching ordinary differential equation (ODE)

$$\frac{dA}{dt} - \mu(t)(1 - e^{-\eta(T-t)}) + \frac{1}{2\eta^2}\gamma^2(t)(1 - e^{-\eta(T-t)})^2 + e^{-A}\langle \tilde{\mathbf{A}}, Q \alpha \rangle = 0$$

with $A(T, T, \alpha(T)) = 0$.

Write, for each $i = 1, 2, \dots, N$, $A_i = A(t, T, e_i)$ and $\mathbf{A} = (A_1, A_2, \dots, A_N)' \in \mathbb{R}^N$. Then, the vector of coefficients \mathbf{A} satisfies the following system of N coupled ODEs

$$\frac{dA_i}{dt} - \mu_i(1 - e^{-\eta(T-t)}) + \frac{1}{2\eta^2}\gamma_i^2(1 - e^{-\eta(T-t)})^2 + e^{-A_i}\langle \tilde{\mathbf{A}}, Q e_i \rangle = 0 \quad (4.6)$$

with $A(T, T, e_i) = 0$ for each $i = 1, 2, \dots, N$.

Write, for each $i = 1, 2, \dots, N$,

$$F_i(t) = \mu_i(1 - e^{-\eta(T-t)}) + \frac{1}{2\eta^2}\gamma_i^2(1 - e^{-\eta(T-t)})^2.$$

Consider the following diagonal matrix

$$\text{diag}(F(t)) = \text{diag}(F_1(t), F_2(t), \dots, F_N(t)).$$

Substituting $\tilde{A}_i = \exp(A_i)$ for each $i = 1, 2, \dots, N$ into (4.6), $\tilde{\mathbf{A}}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N

$$\frac{d\tilde{\mathbf{A}}(t)}{dt} = \left[\text{diag}(F(t)) - Q' \right] \tilde{\mathbf{A}}(t), \quad \tilde{\mathbf{A}}(T) = \mathbf{1}$$

where $\mathbf{1} = (1, 1, \dots, 1)' \in \mathbb{R}^N$.

Define

$$\Delta(t) = \text{diag}(F(t)) - Q'$$

If we let $\tau = T - t$, then $d\tau = -dt$. So

$$\frac{d\tilde{\mathbf{A}}(\tau)}{d\tau} = -\Delta(\tau) \tilde{\mathbf{A}}(\tau), \quad \tilde{\mathbf{A}}(0) = \mathbf{1}.$$

Suppose $\Phi(t)$ denotes the fundamental matrix solution of

$$\frac{d\Phi(\tau)}{d\tau} = -\Delta(\tau) \Phi(\tau), \quad \Phi(0) = \mathbf{I} \quad (4.7)$$

where \mathbf{I} is the $n \times n$ -identity matrix.

If $\Delta(t) = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Phi(t)$ is $\Phi(t) = \exp(\Delta t)$.

In general, there exists a unique fundamental matrix solution $\Phi(t)$ of the linear matrix differential Eq. (4.7). Now, $\tilde{\mathbf{A}}$ can be represented in terms of the fundamental matrix solution $\Phi(t)$ as below.

$$\tilde{\mathbf{A}}(t) = \Phi(t) \tilde{\mathbf{A}}(0) = \Phi(t) \mathbf{1}.$$

So,

$$A(t, T, \alpha) = \sum_{i=1}^N \ln(\langle \Phi(t) \mathbf{1}, \mathbf{e}_i \rangle) \langle \alpha, \mathbf{e}_i \rangle.$$

Therefore, the bond price is represented as the following Markovian regime-switching

exponential affine form

$$P(t, T, r, \alpha) = \exp \left(\sum_{i=1}^N \ln(\langle \Phi(t) \mathbf{1}, \mathbf{e}_i \rangle) \langle \alpha, \mathbf{e}_i \rangle - B(t, T) r \right).$$

So far, we employed the concept of stochastic flows to derive an exponential affine form of the bond price when the short rate process is governed by a Markovian regime-switching Vasicek model. Our model allowed the market parameters, including the mean-reversion level and the volatility rate to switch over time according to a continuous-time, finite-state Markov chain. We provided a representation to the exponential affine form of the bond price in terms of fundamental matrix solutions of linear matrix differential equations.

4.3.2 Bond pricing and the forward measure

The following lemma was given in Shen and Siu [18, 21 & 22] and gives the dynamics of the underlying commodity spot price, the interest rate, and the Markov chain under a forward measure \mathcal{P}^T to be defined below.

Lemma 4.1. *Let $\Lambda(T)$ denote the Radon–Nikodym derivative defined by*

$$\Lambda(T) = \frac{d\mathcal{P}^T}{d\mathcal{P}} \Big|_{\mathcal{G}(T)} = \frac{\exp(-\int_0^T r(t) dt)}{\mathbb{E}[\exp(-\int_0^T r(t) dt)]} \quad (4.8)$$

Under the following assumptions, we have

1. *The Novikov condition is satisfied.*

$$\mathbb{E} \left[\exp \left(\frac{1}{2} \int_0^T \gamma^2(t) B^2(t, T) dt \right) \right] < \infty.$$

2. $\tilde{A}(t, T, \alpha)$ is a suitable function in the sense that

$$\frac{\tilde{A}(t, T, \alpha(t))}{\tilde{A}(0, T, \alpha(0))} \exp\left(-\int_0^t \frac{\frac{\partial \tilde{A}}{\partial s} + Q \tilde{A}(s, T, \alpha(s))}{\tilde{A}(s, T, \alpha(s))} ds\right)$$

for all $t \in \mathcal{T}$ is a $(\mathcal{G}, \mathcal{P})$ -martingale.

Then, under the forward probability measure \mathcal{P} , the following results hold:

1. The dynamics of the underlying logarithmic of commodity spot price and the short rate are given by

$$dX(t) = \left(\beta(\theta(t) - x(t)) - \rho(t)\gamma(t)\sigma(t)B(t, T) \right) dt + \sigma(t)dW_S^T(t) \quad (4.9)$$

$$dr(t) = \left(\eta(\mu(t) - r(t)) - \gamma^2(t)B(t, T) \right) dt + \gamma(t)dW_r^T(t) \quad (4.10)$$

where

$$W_S^T(t) = W_S(t) + \int_0^t \rho(s)\gamma(s)B(s, T)ds, \quad t \in \mathcal{T},$$

and

$$W_r^T(t) = W_r(t) + \int_0^t \gamma(s)B(s, T)ds, \quad t \in \mathcal{T},$$

are \mathcal{P}^T -standard Brownian motions with instantaneous correlation coefficient

$\rho(t)$ at time t , i.e., $\langle W_S^T, W_r^T \rangle = \int_0^t \rho(s)ds$.

2. The rate matrix of the chain \mathbf{X} is $Q'(t) = [q_{ij}^T(t)]_{i,j=1,2,\dots,N}$

$$q_{ij}^T(t) = \begin{cases} q_{ij} \frac{\tilde{A}(t, T, e_j)}{\tilde{A}(t, T, e_i)} & i \neq j \\ -\sum_{k \neq i} q_{ik} \frac{\tilde{A}(t, T, e_k)}{\tilde{A}(t, T, e_i)} & i = j \end{cases}$$

and the martingale dynamics of the chain is given by

$$\alpha(t) = \alpha(0) + \int_0^t Q'(s) \alpha(s) ds + M^T(t)$$

for all $t \in \mathcal{T}$ where $\{M^T(t) | t \in \mathcal{T}\}$ is an \mathbb{R}^N -valued, $(\mathcal{F}^\alpha, \mathcal{P}^T)$ -martingale.

Proof. The proof is given in Lemma 3.3 of Fan et al. [25] and follows the same arguments in Lemma 3.2 in Shen and Siu [47]. So we do not repeat it again here. \square

4.4 Valuation of commodity futures and options

In this section, we derive the price of a European-style commodity option and futures option under the regime-switching stochastic interest rate model at time 0, denoted by $C(0, T)$ and $C_f(0, T, U)$, respectively. Under the risk-neutral probability measure \mathcal{P} , the prices of a T -maturity futures contract and a T -maturity European-style commodity option at time 0 are given as follows:

$$F(0, T) = \mathbb{E}\{S(T)\} \quad (4.11)$$

and

$$C(0, T) = \mathbb{E}\left\{\left(e^{-\int_0^T r(t)dt}\right)(S(T) - K)^+\right\}, \quad (4.12)$$

where $S(T)$ is the terminal commodity price; K is the strike price of the commodity option; $\mathbb{E}[\cdot]$ is the expectation with respect to the risk-neutral probability measure \mathcal{P} .

Consider a European-style futures option with a strike price K_f , the terminal payoff function at the maturity time T of the option is $(F(T, U) - K_f)^+$, where $F(T, U)$ represents the futures price with maturity time U at time T . Then the price of the futures option at time 0 is given by

$$C_f(0, T, U) = \mathbb{E} \left\{ \left(e^{-\int_0^T r(t) dt} \right) (F(T, U) - K_f)^+ \right\}. \quad (4.13)$$

By change of measures defined in the earlier section, (4.12) and (4.13) become

$$C(0, T) = P(0, T) \mathbb{E}^T \{ (S(T) - K)^+ \}, \quad (4.14)$$

and

$$C_f(0, T, U) = P(0, T) \mathbb{E}^T \{ (F(T, U) - K_f)^+ \} \quad (4.15)$$

where \mathbb{E}^T is an expectation under the forward measure \mathcal{P}^T .

4.4.1 Valuation of commodity options

From now on, all the calculations are pretty similar to chapter 3 section 3.

Following the notation in previous chapter, write $\kappa = \ln(K)$, the dampened commodity option price is given by

$$c(\kappa) = e^{\varrho \kappa} C(O, T) \quad (4.16)$$

where ϱ is called the dampening coefficient and assumed to be positive. To obtain a square integrable function, the dampening coefficient ϱ is selected and the dampened commodity pricing formula is defined. The problem how to choose the value of the coefficient ϱ is explained in details in chapter 2. We derive an explicit formula for the

Fourier transform of $c(\kappa)$ next. Let $f_{\mathcal{F}^\alpha(T)}(x)$ be the conditional density function of $X(T)$ given $\mathcal{F}^\alpha(T)$. Then, the dampened commodity Fourier transform is given by

$$\begin{aligned}
\hat{c}(u) &= \int_{-\infty}^{\infty} e^{-iu\kappa} c(\kappa) d\kappa \\
&= P(0, T) \int_{-\infty}^{\infty} e^{-iu\kappa} e^{\varrho\kappa} \mathbb{E}^T \left\{ (e^{X(T)} - e^\kappa)^+ \right\} d\kappa \\
&= P(0, T) \mathbb{E}^T \left\{ \int_{-\infty}^{\infty} e^{-iu\kappa} e^{\varrho\kappa} \mathbb{E}^T \left\{ (e^{X(T)} - e^\kappa)^+ \middle| \mathcal{F}^\alpha(T) \right\} d\kappa \right\} \\
&= P(0, T) \mathbb{E}^T \left\{ \int_{-\infty}^{\infty} e^{-iu\kappa} e^{\varrho\kappa} \int_{\kappa}^{\infty} (e^x - e^\kappa) f_{\mathcal{F}^\alpha(T)}(x) dx d\kappa \right\} \\
&= P(0, T) \mathbb{E}^T \left\{ \int_{-\infty}^{\infty} f_{\mathcal{F}^\alpha(T)}(x) \int_{-\infty}^x (e^x e^{(\varrho-iu)\kappa} - e^{(1+\varrho-iu)\kappa}) d\kappa dx \right\} \\
&= P(0, T) \mathbb{E}^T \left\{ \int_{-\infty}^{\infty} f_{\mathcal{F}^\alpha(T)}(x) \left(\frac{e^{(1+\varrho-iu)x}}{(\varrho-iu)} - \frac{e^{(1+\varrho-iu)x}}{(1+\varrho-iu)} \right) dx \right\} \\
&= P(0, T) \mathbb{E}^T \left\{ \left(\frac{\phi_{\mathcal{F}^\alpha(T)}(-i(1+\varrho)-u)}{(\varrho-iu)} - \frac{\phi_{\mathcal{F}^\alpha(T)}(-i(1+\varrho)-u)}{(1+\varrho-iu)} \right) \right\} \\
&= \frac{P(0, T) \mathbb{E}^T \left\{ \phi_{\mathcal{F}^\alpha(T)}(-i(1+\varrho)-u) \right\}}{(\varrho-iu)(1+\varrho-iu)} \tag{4.17}
\end{aligned}$$

where

$$\phi_{\mathcal{F}^\alpha(T)}(\nu) = \mathbb{E}^T \{ e^{i\nu X(T)} | \mathcal{F}^\alpha(T) \} = \int_{-\infty}^{\infty} e^{i\nu x} f_{\mathcal{F}^\alpha(T)}(x) dx \tag{4.18}$$

is the conditional characteristic function of $X(T)$ given $\mathcal{F}^\alpha(T)$.

Note that given $\mathcal{F}^\alpha(T)$, $X(T)$ has Gaussian distribution with mean

$$\mathbb{E}^T \{ X(T) | \mathcal{F}^\alpha(T) \} = e^{-\beta T} X(0) + \int_0^T (\beta \theta(u) - \rho(u) \gamma(u) \sigma(u) B(u, T)) e^{-\beta(T-u)} du$$

and variance

$$\text{Var}(X(T)|\mathcal{F}^\alpha(T)) = \int_0^T \sigma^2(u) e^{-2\beta(T-u)} du.$$

It follows that

$$\begin{aligned} \phi_{\mathcal{F}^\alpha(T)}(\nu) &= \exp\left(i\nu \mathbb{E}^T\{X(T)|\mathcal{F}^\alpha(T)\} - \frac{1}{2}\nu^2 \text{Var}(X(T)|\mathcal{F}^\alpha(T))\right) \\ &= \exp\left(i\nu(e^{-\beta T}X(0) + \int_0^T (\beta\theta(u) - \rho(u)\gamma(u)\sigma(u)B(u, T)) e^{-\beta(T-u)} du) \right. \\ &\quad \left. - \frac{1}{2}\nu^2 \int_0^T \sigma^2(u) e^{-2\beta(T-u)} du\right). \end{aligned} \quad (4.19)$$

For each $t \in \mathcal{T}$ and $\nu \in \mathbb{R}$, let $G(t, \nu) = (g_1(t, \nu), g_2(t, \nu), \dots, g_N(t, \nu))$, where $g_j(t, \nu)$ for each $j = 1, 2, \dots, N$ is

$$g_j(t, \nu) = i\nu e^{-\beta(T-t)}(\beta\theta_j - \rho_j\gamma_j\sigma_j B(t, T)) - \frac{1}{2}\nu^2 \sigma_j^2 e^{-2\beta(T-t)}.$$

Therefore,

$$\phi_{\mathcal{F}^X(T)}(\nu) = \exp\left(i\nu e^{-\beta T}X(0) + \int_0^T \langle G(t, \nu), \alpha(t) \rangle dt\right). \quad (4.20)$$

Substituting (4.20) in (4.17) implies

$$\begin{aligned} \hat{c}(u) &= \frac{P(0, T) \mathbb{E}^T\left\{\exp\left(i\nu e^{-\beta T}X(0) + \int_0^T \langle G(t, \nu), \alpha(t) \rangle dt\right)\right\}}{(\varrho - iu)(1 + \varrho - iu)}, \\ \nu &= -(u + i(1 + \varrho)). \end{aligned} \quad (4.21)$$

In order to derive an explicit formula for $\hat{c}(u)$, It's necessary to calculate the expectation given in (4.21). To do this end, we use a modification of proof of lemma

1 in Buffington and Elliott [5]. Let $\mathbf{diag}(G(t, \nu))$ denote the diagonal matrix with diagonal elements given by the components of $G(t, \nu)$, $\mathbf{1} = (1, 1, \dots, 1)'$ and \mathbf{I} denote the $(n \times n)$ -identity matrix. Let's define

$$W(t, \nu) = \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) \alpha(t), \quad W(0, \nu) = \alpha(0). \quad (4.22)$$

Consequently,

$$dW(t, \nu) = \langle G(t, \nu), \alpha(t) \rangle W(t, \nu) dt + \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) d\alpha(t).$$

Note that under \mathcal{P}^T ,

$$d\alpha(t) = Q' \alpha(t) dt + dM^T(t),$$

and that

$$\langle G(t, \nu), \alpha(t) \rangle W(t, \nu) = \mathbf{diag}(G(t, \nu)) W(t, \nu), \quad \forall t \in \mathcal{T}.$$

Then

$$\begin{aligned} dW(t, \nu) &= \langle G(t, \nu), \alpha(t) \rangle W(t, \nu) dt + \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) Q' \alpha(t) dt \\ &\quad + \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) dM^T(t) \\ &= \mathbf{diag}(G(t, \nu)) W(t, \nu) dt + Q' W(t, \nu) dt \\ &\quad + \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) dM^T(t) \\ &= \left[\mathbf{diag}(G(t, \nu)) + Q^T \right] W(t, \nu) dt + \exp\left(\int_0^t \langle G(s, \nu), \alpha(s) \rangle ds\right) dM^T(t). \end{aligned}$$

By taking integration from both sides, we have

$$\begin{aligned}
W(t, \nu) &= W(0, \nu) + \int_0^t \left[\mathbf{diag}(G(s, \nu)) + Q' \right] W(s, \nu) ds \\
&\quad + \int_0^t \exp \left(\int_0^u \langle G(s, \nu), X(s) \rangle ds \right) dM^T(u) \\
&= \alpha(0) + \int_0^t \left[\mathbf{diag}(G(s, \nu)) + Q' \right] W(s, \nu) ds \\
&\quad + \int_0^t \exp \left(\int_0^u \langle G(s, \nu), X(s) \rangle ds \right) dM^T(u).
\end{aligned}$$

The final integral is a martingale, so taking expectation under forward measure \mathcal{P}^T gives:

$$\mathbb{E}^T \{W(t, \nu)\} = \alpha(0) + \int_0^t \left[\mathbf{diag}(G(s, \nu)) + Q' \right] \mathbb{E}^T \{W(s, \nu)\} ds.$$

Hence $\mathbb{E}^T \{W(t, \nu)\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N :

$$\frac{d\mathbb{E}^T \{W(t, \nu)\}}{dt} = \left[\mathbf{diag}(G(t, \nu)) + Q' \right] \mathbb{E}^T \{W(t, \nu)\}, \quad \mathbb{E}^T \{W(0, \nu)\} = \alpha(0). \quad (4.23)$$

Suppose $\Phi(t, \nu)$ denotes the fundamental matrix solution of

$$\frac{d\Phi(t, \nu)}{dt} = \left[\mathbf{diag}(G(t, \nu)) + Q' \right] \Phi(t, \nu), \quad \Phi(0, \nu) = \mathbf{I}.$$

If $\left[\mathbf{diag}(G(t, \nu)) + Q' \right] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Phi(t, \nu)$ is

$$\Phi(t, \nu) = \exp(\Delta t).$$

In general, there exists a unique fundamental matrix solution $\Phi(t, \nu)$ of the linear matrix differential Eq. (4.23). Now, $\mathbb{E}^T \{W(t, \nu)\}$ can be represented in terms of the

fundamental matrix solution $\Phi(t, \nu)$ as below:

$$\mathbb{E}^T \{W(t, \nu)\} = \Phi(t, \nu) \alpha(0).$$

Since $\langle \alpha(T), \mathbf{1} \rangle = 1$, we have

$$\begin{aligned} & \mathbb{E}^T \left\{ \exp \left(\int_0^T \langle G(t, \nu), \alpha(t) \rangle dt \right) \right\} \\ &= \mathbb{E}^T \left\{ \exp \left(\int_0^T \langle G(t, \nu), \alpha(t) \rangle dt \right) \langle \alpha(T), \mathbf{1} \rangle \right\} \\ &= \mathbb{E}^T \left\{ \left\langle \exp \left(\int_0^T \langle G(t, \nu), \alpha(t) \rangle dt \right) \alpha(T), \mathbf{1} \right\rangle \right\} \\ &= \left\langle \mathbb{E}^T \left\{ \exp \left(\int_0^T \langle G(t, \nu), \alpha(t) \rangle dt \right) \alpha(T) \right\}, \mathbf{1} \right\rangle \\ &= \langle \mathbb{E}^T \{W(T, \nu)\}, \mathbf{1} \rangle \\ &= \langle \Phi(T, \nu) \alpha(0), \mathbf{1} \rangle. \end{aligned}$$

Therefore under Markovian regime-switching Ornstein-Uhlenbeck model, the price of the commodity option is given by:

$$\hat{c}(u) = \frac{P(0, T) \exp(i\nu e^{-\beta T} X(0)) \langle \Phi(T, \nu) \alpha(0), \mathbf{1} \rangle}{(\varrho - iu)(1 + \varrho - iu)}, \quad \nu = -(u + i(1 + \varrho)). \quad (4.24)$$

4.4.2 Valuation of futures options

In this subsection we consider the valuation of commodity futures options. We wish to evaluate the time-zero value of a standard European call option on the future price $F(T, U)$ with strike price K_f and maturity at time T . That is to evaluate:

$$C_f(0, T, U) = P(0, T) \mathbb{E}^T \left\{ \left(F(T, U) - K_f \right)^+ \right\}.$$

As seen before, the dampened commodity futures options price is given by:

$$c_f(\kappa_f) = e^{\varrho_f \kappa_f} C_f(0, T, U), \quad (4.25)$$

where $\kappa_f = \ln(K_f)$. Let's define $Y(t) := \ln(F(t, U))$ for all $t \in \mathcal{T}$. The Fourier transform of the dampened commodity futures options is given by

$$\hat{c}_f(u) = \frac{P(0, T) \mathbb{E}^T \left\{ \psi_{\mathcal{F}^\alpha(T)}(-i(1 + \varrho_f) - u) \right\}}{(\varrho_f - iu)(1 + \varrho_f - iu)} \quad (4.26)$$

where

$$\psi_{\mathcal{F}^\alpha(T)}(\nu) = \mathbb{E}^T \{ e^{i\nu Y(T)} | \mathcal{F}^\alpha(T) \} = \int_{-\infty}^{\infty} e^{i\nu y} f_{\mathcal{F}^\alpha(T)}(y) dy \quad (4.27)$$

is the conditional characteristic function of $Y(T)$ given $\mathcal{F}^\alpha(T)$. See chapter 1 section 1.3.2 for details of calculation in (4.26).

In order to derive an explicit formula for $\hat{c}_f(u)$, we need to derive an analytical formula of the characteristic function of the logarithmic commodity futures price, in other words $\ln(F(T, U))$.

To do so, first we need to derive the time- t price of a T -maturity futures contract. In other words,

$$F(t, T) = \mathbb{E}^T \{ S(T) | \mathcal{H}(t) \} = \mathbb{E}^T \{ \mathbb{E}^T \{ S(T) | \mathcal{F}^\alpha(T) \vee \mathcal{F}^S(t) \vee \mathcal{F}^r(t) \} | \mathcal{H}(t) \}$$

where we used tower property for the second equality because $\mathcal{H}(t) \subset \mathcal{F}^\alpha(T) \vee \mathcal{F}^S(t) \vee \mathcal{F}^r(t)$. Since $S(T) = e^{X(T)}$ and under forward measure \mathcal{P}^T , the conditional distribution of $X(T)$ given $\mathcal{F}^\alpha(T)$ is a Gaussian distribution with mean

$$\mathbb{E}^T \{ X(T) | \mathcal{F}^\alpha(T) \} = e^{-\beta(T-t)} X(t) + \int_t^T (\beta \theta(u) - \rho(u) \gamma(u) \sigma(u) B(u, T)) e^{-\beta(T-u)} du$$

and variance

$$\text{Var}(X(T)|\mathcal{F}^\alpha(T)) = \int_t^T \sigma^2(u) e^{-2\beta(T-u)} du,$$

then, it's easy to see that

$$\begin{aligned} & \mathbb{E}^T \{S(T)|\mathcal{F}^\alpha(T) \vee \mathcal{F}^S(t) \vee \mathcal{F}^r(t)\} \\ &= \exp \left(\mathbb{E}^T \{X(T)|\mathcal{F}^\alpha(T)\} + \frac{1}{2} \text{Var}(X(T)|\mathcal{F}^\alpha(T)) \right) \\ &= \exp \left(e^{-\beta(T-t)} X(t) + \int_t^T (\beta \theta(u) - \rho(u) \gamma(u) \sigma(u) B(u, T)) e^{-\beta(T-u)} du \right. \\ & \quad \left. + \frac{1}{2} \int_t^T \sigma^2(u) e^{-2\beta(T-u)} du \right). \end{aligned} \quad (4.28)$$

Then the time- t price of a T -maturity futures contract is given by

$$\begin{aligned} F(t, T) &= \mathbb{E}^T \{S(T)|\mathcal{H}(t)\} = \mathbb{E}^T \{ \mathbb{E}^T \{S(T)|\mathcal{F}^\alpha(T) \vee \mathcal{F}^S(t) \vee \mathcal{F}^r(t)\} | \mathcal{H}(t) \} \\ &= \mathbb{E}^T \left\{ \exp \left(e^{-\beta(T-t)} X(t) + \int_t^T (\beta \theta(u) - \rho(u) \gamma(u) \sigma(u) B(u, T)) e^{-\beta(T-u)} du \right. \right. \\ & \quad \left. \left. + \frac{1}{2} \int_t^T \sigma^2(u) e^{-2\beta(T-u)} du \right) \middle| \mathcal{H}(t) \right\}. \end{aligned}$$

Let's define $N(s) = (n_1(s), n_2(s), \dots, n_N(s))$ for each $s \in [0, T]$, where for each $j = 1, 2, \dots, N$,

$$n_j(s) = (\beta \theta_j - \rho_j \gamma_j \sigma_j B(s, T)) e^{-\beta(T-s)} + \frac{1}{2} \sigma_j^2 e^{-2\beta(T-s)}.$$

Therefore

$$F(t, T) = \mathbb{E}^T \left\{ \exp \left(e^{-\beta(T-t)} X(t) + \int_t^T \langle N(s), \alpha(s) \rangle ds \right) \middle| \mathcal{H}(t) \right\}. \quad (4.29)$$

In order to derive an explicit formula for $F(t, T)$, It's necessary to calculate the expectation given in (4.29). To do this end, we adopt the same methodology used in previous subsection. Let $\mathbf{diag}(N(t))$ denote the diagonal matrix with diagonal elements given by the components of $N(t)$. Let's define

$$H(t, u) = \exp\left(\int_t^u \langle N(s), \alpha(s) \rangle ds\right) \alpha(u), \quad H(t, t) = \alpha(t).$$

Consequently,

$$dH(t, u) = \langle N(u), \alpha(u) \rangle H(t, u) du + \exp\left(\int_t^u \langle N(s), \alpha(s) \rangle ds\right) d\alpha(u).$$

Note that under \mathcal{P}^T ,

$$d\alpha(u) = Q' \alpha(u) du + dM^T(u),$$

and that

$$\langle N(u), \alpha(u) \rangle H(t, u) = \mathbf{diag}(N(u)) H(t, u)$$

for all $t \in \mathcal{T}$. Then

$$\begin{aligned} dH(t, u) &= \langle N(u), \alpha(u) \rangle H(t, u) du + \exp\left(\int_t^u \langle N(s), X(s) \rangle ds\right) Q' \alpha(u) du \\ &\quad + \exp\left(\int_t^u \langle N(s), \alpha(s) \rangle ds\right) dM^T(u) \\ &= \mathbf{diag}(N(u)) H(t, u) du + Q' H(t, u) du \\ &\quad + \exp\left(\int_t^u \langle N(s), \alpha(s) \rangle ds\right) dM^T(u) \\ &= \left[\mathbf{diag}(N(u)) + Q' \right] H(t, u) du + \exp\left(\int_t^u \langle N(s), \alpha(s) \rangle ds\right) dM^T(u). \end{aligned}$$

By taking integration from both sides, we have

$$\begin{aligned}
H(t, u) &= H(t, t) + \int_t^u \left[\mathbf{diag}(N(s)) + Q' \right] H(t, s) \, ds \\
&\quad + \int_t^u \exp \left(\int_t^\tau \langle N(s), \alpha(s) \rangle \, ds \right) \, dM^T(\tau) \\
&= \alpha(t) + \int_t^u \left[\mathbf{diag}(N(s)) + Q' \right] H(t, s) \, ds \\
&\quad + \int_t^u \exp \left(\int_t^\tau \langle N(s), \alpha(s) \rangle \, ds \right) \, dM^T(\tau).
\end{aligned}$$

The final integral is a martingale, so taking expectation under \mathcal{P}^T conditioning on $\mathcal{H}(t)$ gives:

$$\mathbb{E}^T \{H(t, u) | \mathcal{H}(t)\} = \alpha(t) + \int_t^u \left[\mathbf{diag}(N(s)) + Q' \right] \mathbb{E}^T \{H(t, s) | \mathcal{H}(t)\} \, ds.$$

Hence $\mathbb{E}^T \{H(t, u) | \mathcal{H}(t)\}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N :

$$\frac{d\mathbb{E}^T \{H(t, u) | \mathcal{H}(t)\}}{du} = \left[\mathbf{diag}(N(u)) + Q' \right] \mathbb{E}^T \{H(t, u) | \mathcal{H}(t)\} \quad (4.30)$$

with initial condition $\mathbb{E}^T \{H(t, t) | \mathcal{H}_t\} = \alpha(t)$. Suppose $\Psi(t, u)$ denotes the fundamental matrix solution of

$$\frac{d\Psi(t, u)}{du} = \left[\mathbf{diag}(N(u)) + Q' \right] \Psi(t, u), \quad \Psi(t, t) = \mathbf{I}.$$

If $\left[\mathbf{diag}(N(u)) + Q' \right] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Psi(t, u)$ is

$$\Psi(t, u) = \exp(\Delta (u - t)).$$

In general, there exists a unique fundamental matrix solution $\Psi(t, u)$ of the linear

matrix differential Eq. (4.30). Now, $\mathbb{E}^T\{H(t, u)|\mathcal{H}(t)\}$ can be represented in terms of the fundamental matrix solution $\Psi(t, u)$ as below:

$$\mathbb{E}^T\{H(t, u)|\mathcal{H}(t)\} = \Psi(t, u) \alpha(t). \quad (4.31)$$

Using (4.31) implies that

$$\begin{aligned} F(t, T) &= \mathbb{E}^T \left\{ \exp \left(e^{-\beta(T-t)} X(t) + \int_t^T \langle N(u), \alpha(u) \rangle du \right) \middle| \mathcal{H}(t) \right\} \\ &= \exp \left(e^{-\beta(T-t)} X(t) \right) \mathbb{E}^T \left\{ \exp \left(\int_0^T \langle N(t), \alpha(t) \rangle du \right) \middle| \mathcal{H}(t) \right\} \\ &= \exp \left(e^{-\beta(T-t)} X(t) \right) \mathbb{E}^T \left\{ \exp \left(\int_t^T \langle N(u), \alpha(u) \rangle du \right) \langle \alpha(T), \mathbf{1} \rangle \middle| \mathcal{H}(t) \right\} \\ &= \exp \left(e^{-\beta(T-t)} X(t) \right) \mathbb{E}^T \left\{ \left\langle \exp \left(\int_t^T \langle N(u), \alpha(u) \rangle du \right) \alpha(T), \mathbf{1} \right\rangle \middle| \mathcal{H}(t) \right\} \\ &= \exp \left(e^{-\beta(T-t)} X(t) \right) \left\langle \mathbb{E}^T \left\{ \exp \left(\int_t^T \langle N(u), \alpha(u) \rangle du \right) \alpha(T) \middle| \mathcal{H}(t) \right\}, \mathbf{1} \right\rangle \\ &= \exp \left(e^{-\beta(T-t)} X(t) \right) \langle \mathbb{E}^T\{H(t, T)|\mathcal{H}(t)\}, \mathbf{1} \rangle \\ &= \exp \left(e^{-\beta(T-t)} X(t) \right) \langle \Psi(t, T) \alpha(t), \mathbf{1} \rangle. \end{aligned}$$

Then under the forward measure \mathcal{P}^T , the time- T price of a U -maturity futures contract is given by:

$$\begin{aligned} F(T, U) &= \mathbb{E}^T\{S(U)|\mathcal{H}(T)\} = \exp \left(e^{-\beta(U-T)} X(T) \right) \times \langle \Psi(T, U) \alpha(T), \mathbf{1} \rangle \\ &= \exp \left(e^{-\beta(U-T)} \left[e^{-\beta(T)} X(0) + \int_0^T e^{-\beta(T-u)} (\beta\theta(u) - \rho(u)\gamma(u)\sigma(u)B(u, T)) du \right. \right. \\ &\quad \left. \left. + \int_0^T \sigma(u) e^{-\beta(T-u)} dW_S^T(u) \right] \right) \times \langle \Psi(T, U) \alpha(T), \mathbf{1} \rangle \\ &= \exp \left(e^{-\beta U} X(0) + e^{-\beta U} \int_0^T (\beta\theta(u) - \rho(u)\gamma(u)\sigma(u)B(u, T)) e^{\beta u} du \right. \\ &\quad \left. + e^{-\beta U} \int_0^T \sigma(u) e^{\beta u} dW_S^T(u) \right) \times \langle \Psi(T, U) \alpha(T), \mathbf{1} \rangle. \end{aligned}$$

where $T < U < T^*$.

Note that given $\mathcal{F}^\alpha(T)$, $Y(T) = \ln(F(T, U))$ has Gaussian distribution with mean

$$\begin{aligned} \mathbb{E}^T\{Y(T)|\mathcal{F}^\alpha(T)\} &= e^{-\beta U} X(0) + e^{-\beta U} \int_0^T (\beta \theta(u) - \rho(u)\gamma(u)\sigma(u)B(u, T))e^{\beta u} du \\ &\quad + e^{-\beta U} \int_0^T \sigma(u)e^{\beta u} dW_S^T(u) + \ln\langle \Psi(T, U) \alpha(T), \mathbf{1} \rangle \end{aligned} \quad (4.32)$$

and variance

$$\text{Var}(Y(T)|\mathcal{F}^\alpha(T)) = \int_0^T \sigma^2(u) e^{-2\beta(U-u)} du. \quad (4.33)$$

It follows from (4.33) and (4.34) that

$$\begin{aligned} \psi_{\mathcal{F}^\alpha(T)}(\nu) &= \exp\left(i\nu \mathbb{E}\{Y(T)|\mathcal{F}^\alpha(T)\} - \frac{1}{2}\nu^2 \text{Var}(Y(T)|\mathcal{F}^\alpha(T))\right) \\ &= \exp\left(i\nu e^{-\beta U} X(0) + i\nu e^{-\beta U} \int_0^T (\beta \theta(u) - \rho(u)\gamma(u)\sigma(u)B(u, T))e^{\beta u} du \right. \\ &\quad \left. + i\nu e^{-\beta U} \int_0^T \sigma(u)e^{\beta u} dW_S^T(u)\right) \times \langle \Psi(T, U) \alpha(T), \mathbf{1} \rangle^{i\nu} \\ &\quad \times \exp\left(-\frac{1}{2}\nu^2 \int_0^T \sigma^2(u) e^{-2\beta(U-u)} du\right). \end{aligned}$$

Define $Z(\nu) = (Z_1(\nu), Z_2(\nu), \dots, Z_N(\nu))$ and $z(t, \nu) = (z_1(t, \nu), z_2(t, \nu), \dots, z_N(t, \nu))$ for each $\nu \in \mathbb{R}$ and $t \in \tau$, where $Z_j(\nu)$ and $z_j(t, \nu)$ for each $j = 1, 2, \dots, N$ are

$$Z_j(\nu) = Z(e_j, \nu) = \exp\left(i\nu e^{-\beta U} x(0)\right) \times \langle \Psi(T, U) e_j, \mathbf{1} \rangle^{i\nu} \quad (4.34)$$

and

$$z_j(t, \nu) = z(t, e_j, \nu) = i\nu e^{-\beta(U-t)}(\beta \theta_j - \rho_j \gamma_j \sigma_j B(t, T)) - \frac{1}{2}\nu^2 e^{-2\beta(U-t)} \sigma_j^2. \quad (4.35)$$

Therefore,

$$\psi_{\mathcal{F}^\alpha(T)}(\nu) = Z(X(T), \nu) \exp\left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt\right). \quad (4.36)$$

Substituting (4.36) in (4.26) implies

$$\hat{c}_f(u) = \frac{P(0, T) \mathbb{E}\left\{Z(\alpha(T), \nu) \exp\left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt\right)\right\}}{(\varrho_f - iu)(1 + \varrho_f - iu)} \quad (4.37)$$

where $\nu = -(u + i(1 + \varrho_f))$. As we have seen before, in order to derive an explicit formula for $\hat{c}_f(u)$, It's necessary to calculate the expectation given in (4.37). Let $\mathbf{diag}(z(t, \nu))$ denote the diagonal matrix with diagonal elements given by the components of $z(t, \nu)$. Let's define

$$\Gamma(t, \nu) = \exp\left(\int_0^t \langle z(s, \nu), \alpha(s) \rangle ds\right) \alpha(t), \quad \Gamma(0, \nu) = \alpha(0). \quad (4.38)$$

Applying Itô differentiation rule to $\Gamma(t, \nu)$ gives

$$d\Gamma(t, \nu) = \langle z(t, \nu), \alpha(t) \rangle \Gamma(t, \nu) dt + \exp\left(\int_0^t \langle z(s, \nu), \alpha(s) \rangle ds\right) d\alpha(t).$$

Note that under \mathcal{P}^T ,

$$d\alpha(t) = Q' \alpha(t) dt + dM^T(t),$$

and that

$$\langle z(t, \nu), \alpha(t) \rangle \Gamma(t, \nu) = \mathbf{diag}(z(t, \nu)) \Gamma(t, \nu)$$

for all $t \in \mathcal{T}$. Then

$$\begin{aligned}
d\Gamma(t, \nu) &= \left\langle z(t, \nu), \alpha(t) \right\rangle \Gamma(t, \nu) dt \\
&\quad + \exp \left(\int_0^t \left\langle z(s, \nu), \alpha(s) \right\rangle ds \right) Q' \alpha(t) dt \\
&\quad + \exp \left(\int_0^t \left\langle z(s, \nu), \alpha(s) \right\rangle ds \right) dM^T(t) \\
&= \mathbf{diag}(z(t, \nu)) \Gamma(t, \nu) dt + Q' \Gamma(t, \nu) dt \\
&\quad + \exp \left(\int_0^t \left\langle z(s, \nu), \alpha(s) \right\rangle ds \right) dM^T(t) \\
&= \left[\mathbf{diag}(z(t, \nu)) + Q' \right] \Gamma(t, \nu) dt \\
&\quad + \exp \left(\int_0^t \left\langle z(s, \nu), \alpha(s) \right\rangle ds \right) dM^T(t).
\end{aligned}$$

By taking integration from both sides, we have

$$\begin{aligned}
\Gamma(t, \nu) &= \Gamma(0, \nu) + \int_0^t \left[\mathbf{diag}(z(s, \nu)) + Q' \right] \Gamma(s, \nu) ds \\
&\quad + \int_0^t \exp \left(\int_0^\tau \left\langle z(s, \nu), \alpha(s) \right\rangle ds \right) dM^T(\tau) \\
&= \alpha(0) + \int_0^t \left[\mathbf{diag}(z(s, \nu)) + Q' \right] \Gamma(s, \nu) ds \\
&\quad + \int_0^t \exp \left(\int_0^\tau \left\langle G(s, \nu), \alpha(s) \right\rangle ds \right) dM^T(\tau).
\end{aligned}$$

Taking expectation under \mathcal{P}^T gives:

$$\mathbb{E}^T \{ \Gamma(t, \nu) \} = \alpha(0) + \int_0^t \left[\mathbf{diag}(z(s, \nu)) + Q' \right] \mathbb{E}^T \{ \Gamma(s, \nu) \} ds.$$

Hence $\mathbb{E}^T \{ \Gamma(t, \nu) \}$ satisfies the following homogeneous system of linear ODEs of order one and dimension N :

$$\frac{d\mathbb{E}^T \{ \Gamma(t, \nu) \}}{dt} = \left[\mathbf{diag}(z(t, \nu)) + Q' \right] \mathbb{E}^T \{ \Gamma(t, \nu) \} \quad (4.39)$$

with initial condition $\mathbb{E}^T\{\Gamma(0, \nu)\} = \alpha(0)$. Suppose $\Upsilon(t, \nu)$ denotes the fundamental matrix solution of

$$\frac{d\Upsilon(t, \nu)}{dt} = [\mathbf{diag}(z(t, \nu)) + Q'] \Upsilon(t, \nu), \quad \Upsilon(0, \nu) = \mathbf{I}.$$

If $[\mathbf{diag}(z(t, \nu)) + Q^T] = \Delta$ (i.e. a constant matrix), the fundamental matrix solution $\Phi(t, \nu)$ is

$$\Upsilon(t, \nu) = \exp(\Delta t).$$

In general, there exists a unique fundamental matrix solution $\Upsilon(t, \nu)$ of the linear matrix differential Eq. (4.39). Now, $\mathbb{E}^T\{\Gamma(t, \nu)\}$ can be represented in terms of the fundamental matrix solution $\Upsilon(t, \nu)$ as below:

$$\mathbb{E}\{\Gamma(t, \nu)\} = \Upsilon(t, \nu) \alpha(0).$$

Using $\langle \alpha(T), \mathbf{1} \rangle = 1$ implies:

$$\begin{aligned} & \mathbb{E}\left\{Z(\alpha(T), \nu) \exp\left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt\right)\right\} \\ &= \mathbb{E}\left\{Z(\alpha(T), \nu) \exp\left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt\right) \langle \alpha(T), \mathbf{1} \rangle\right\} \\ &= \mathbb{E}\left\{\left\langle \exp\left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt\right) \alpha(T), Z(\nu) \right\rangle\right\} \\ &= \left\langle \mathbb{E}\left\{\exp\left(\int_0^T \langle z(t, \nu), \alpha(t) \rangle dt\right) \alpha(T)\right\}, Z(\nu) \right\rangle \\ &= \langle \mathbb{E}\{\Gamma(T, \nu)\}, Z(\nu) \rangle \\ &= \langle \Upsilon(T, \nu) \alpha(0), Z(\nu) \rangle. \end{aligned}$$

Therefore under Markovian regime-switching Ornstein-Uhlenbeck model with stochas-

tic interest rate, the price of the futures option is given by:

$$\hat{c}_f(u) = \frac{P(0, T) \langle \Upsilon(T, \nu) \alpha(0), Z(\nu) \rangle}{(\varrho_f - iu)(1 + \varrho_f - iu)}, \quad \nu = -(u + i(1 + \varrho_f)). \quad (4.40)$$

CHAPTER 5: CONCLUSION AND FUTURE DIRECTIONS

This dissertation is concerned with pricing of European-style derivatives such as call, commodity, and futures options under different regime-switching models. Only regime-switching models are considered in this thesis. We assume the Markov chain is observable.

Possible future research directions are as follows:

One can demonstrate the practicality of the model via numerical examples. It is also possible to apply the techniques developed here to American options. It is also possible to consider the hedging of these products under regime-switching models. Static hedging and dynamic hedging are two main types of hedging. We could investigate the static hedging and the dynamic hedging of standard options, exotic options and insurance products with embedded option features and provide comparisons of these hedging strategies.

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APPENDIX A: PROOF FOR THE PAIR OF EQUATIONS (2.40)

Recalling that for the initial state $\alpha(0) = j_0$ we define

$$\phi_{j_0}(\theta, T) = \mathbb{E}(e^{i\theta T_1} | \alpha(0) = j_0), \quad j_0 = 1, 2.$$

Let $J \sim \exp(\lambda_1)$ be the first time jumping from state 1 to state 2. Then

$$\begin{aligned} \phi_1(\theta, T) &= \mathbb{E} \left[e^{i\theta T_1} \middle| \alpha(0) = 1 \right] \\ &= \mathbb{E} \left[e^{i\theta \left(J + \int_J^T \mathbb{I}_{\{\alpha(s)=1\}} ds \right)} \mathbb{I}_{\{J < T\}} + e^{i\theta T} \mathbb{I}_{\{J \geq T\}} \middle| \alpha(0) = 1 \right] \\ &= \mathbb{E} \left[e^{i\theta \left(J + \int_J^T \mathbb{I}_{\{\alpha(s)=1\}} ds \right)} \mathbb{I}_{\{J < T\}} \middle| \alpha(0) = 1 \right] + e^{i\theta T} \mathbb{P}(J \geq T) \\ &= \int_0^T \mathbb{E} \left[e^{i\theta \left(t + \int_t^T \mathbb{I}_{\{\alpha(s)=1\}} ds \right)} \middle| \alpha(t) = 2 \right] \lambda_1 e^{-\lambda_1 t} dt + e^{i\theta T} e^{-\lambda_1 T} \\ &= \int_0^T e^{i\theta T} \lambda_1 e^{-\lambda_1 t} \mathbb{E} \left[e^{i\theta \int_t^T \mathbb{I}_{\{\alpha(s)=1\}} ds} \middle| \alpha(t) = 2 \right] dt + e^{i\theta T} e^{-\lambda_1 T} \\ &= \int_0^T e^{i\theta T} \lambda_1 e^{-\lambda_1 t} \phi_2(\theta, T - t) dt + e^{i\theta T} e^{-\lambda_1 T} \end{aligned}$$

Note that since J is exponentially distributed with rate λ_1 , hence

$$\mathbb{E} \left[\mathbb{I}_{\{J \geq T\}} \middle| \alpha(0) = 1 \right] = \mathbb{P}(J \geq T) = e^{-\lambda_1 T}.$$

The fourth equality holds by well know property of expectations.

$$\begin{aligned} &\mathbb{E} \left[e^{i\theta \left(J + \int_J^T \mathbb{I}_{\{\alpha(s)=1\}} ds \right)} \mathbb{I}_{\{J < T\}} \middle| \alpha(0) = 1 \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[e^{i\theta \left(J + \int_J^T \mathbb{I}_{\{\alpha(s)=1\}} ds \right)} \mathbb{I}_{\{J < T\}} \middle| \alpha(J) = 2 \right] \middle| \alpha(0) = 1 \right] \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \left[\mathbb{E} \left[e^{i\theta \left(J + \int_J^T \mathbb{I}_{\{\alpha(s)=1\}} ds \right)} \mathbb{I}_{\{J < T\}} \middle| \alpha(J) = 2 \right] \right] \\
&= \int_0^T \mathbb{E} \left[e^{i\theta \left(t + \int_t^T \mathbb{I}_{\{\alpha(s)=1\}} ds \right)} \middle| \alpha(t) = 2 \right] \lambda_1 e^{-\lambda_1 t} dt
\end{aligned}$$

Following the same methodology we will get the second equality for $\phi_2(\theta, T)$. Now let's assume $J \sim \exp(\lambda_2)$ be the first time jumping from state 2 to state 1. Then

$$\begin{aligned}
\phi_2(\theta, T) &= \mathbb{E} \left[e^{i\theta T_1} \middle| \alpha(0) = 2 \right] \\
&= \mathbb{E} \left[e^{i\theta \left(\int_J^T \mathbb{I}_{\{\alpha(s)=1\}} ds \right)} \mathbb{I}_{\{J < T\}} + e^{i\theta 0} \mathbb{I}_{\{J \geq T\}} \middle| \alpha(0) = 2 \right] \\
&= \mathbb{E} \left[e^{i\theta \left(\int_J^T \mathbb{I}_{\{\alpha(s)=1\}} ds \right)} \mathbb{I}_{\{J < T\}} \middle| \alpha(0) = 2 \right] + \mathbb{P}(J \geq T) \\
&= \int_0^T \mathbb{E} \left[e^{i\theta \left(\int_t^T \mathbb{I}_{\{\alpha(s)=1\}} ds \right)} \middle| \alpha(t) = 1 \right] \lambda_2 e^{-\lambda_2 t} dt + e^{-\lambda_2 T} \\
&= \int_0^T \lambda_2 e^{-\lambda_2 t} \mathbb{E} \left[e^{i\theta \int_t^T \mathbb{I}_{\{\alpha(s)=1\}} ds} \middle| \alpha(t) = 1 \right] dt + e^{-\lambda_2 T} \\
&= \int_0^T \lambda_2 e^{-\lambda_2 t} \phi_1(\theta, T - t) dt + e^{-\lambda_2 T}
\end{aligned}$$

Note that since J is exponentially distributed with rate λ_2 , hence

$$\mathbb{E}[\mathbb{I}_{\{J \geq T\}} | \alpha(0) = 2] = \mathbb{P}(J \geq T) = e^{-\lambda_2 T}.$$

APPENDIX B: PYTHON CODE FOR FFT METHOD IN CHAPTER 2

The Python code used throughout was generated using Python 3.5.2 64bits, Qt 5.6.0, PyQt5 5.6 on Darwin through the Anaconda Navigator 1.3.1 distribution, available at <https://anaconda.org/anaconda/anaconda-navigator/files?version=1.3.1>. I hope this information will help any reader to replicate our result without any trouble.

The following is the code that generated the data seen in Table 2.3 and Tabel 2.4.

```
import numpy as np

from math import pi

from scipy.interpolate import interp1d

from time import time

t0 = time()

class FFT_Euro:

    def __init__(self, m1, m2, r1, r2, sig1, sig2, l1, l2, a, w, T):

        self.m1    = m1

        self.m2    = m2

        self.r1    = r1

        self.r2    = r2

        self.sig1  = sig1

        self.sig2  = sig2

        self.l1    = l1

        self.l2    = l2

        self.a     = a

        self.w     = w

        self.T     = T

    def A(self):

        m1, m2, r1, r2, sig1, sig2, a, w = self.m1, self.m2, \

        self.r1, self.r2, self.sig1, self.sig2, self.a, self.w
```

```

value = ((m1 - m2) + (0.5 + a) * (sig1**2 - sig2**2)) * -w \
        + 0.5 * 1j * w**2 * (sig1**2 - sig2**2) \
        + ((r1 - r2) - (1 + a) * (m1 - m2) \
        - 0.5 * a * (1 + a) * (sig1**2 - sig2**2)) * 1j
return value

def B(self):
    m2, r2, sig2, a, w = self.m2, self.r2, \
        self.sig2, self.a, self.w
    value = 1j * -w * (m2 + (0.5 + a) * sig2**2) - \
        0.5 * w**2 * sig2**2 + (1 + a) * m2 - \
        r2 + 0.5 * a * (1 + a) * sig2**2
    return value

def price(self):
    l1, l2, T, a, w = self.l1, self.l2, self.T, self.a, self.w
    A = self.A
    B = self.B
    #-----
    # Prepare phi function according to
    # equation 2.22-2.23 paper 2006
    #-----
    s1 = 0.5 * ((1j * A() - l1 - l2) - \
        np.sqrt((l1 + l2 - 1j * A())**2 \
        + 4 * 1j * A() * l2))
    s2 = 0.5 * ((1j * A() - l1 - l2) + \
        np.sqrt((l1 + l2 - 1j * A())**2 \
        + 4 * 1j * A() * l2))
    phi0 = (1/(s1-s2)) * ((s1 + l1 + l2) * np.exp(s1 * T)\

```

```

        - (s2 + l1 + l2) * np.exp(s2 * T))
phi1 = (1/(s1-s2)) * ((s1 + l1 + l2 - 1j * A())\
    * np.exp(s1 * T) - (s2 + l1 + l2 - 1j * A())\
    * np.exp(s2 * T))
#-----
# Prepare characteristic function of
# modified price equation 2.24 paper 2006
#-----
D = np.exp(B()*T) / (a**2 + a - w**2 - (1j * (1 + 2 * a) * w))
q0 = D * phi0
q1 = D * phi1
#-----
# Prepare a mesh in the frequency (w)
# & space (k) domain
#-----
S0 = 100.0
N = int(2**12)
h = 0.1534
dk = (2.0 * pi) / (h * N)
kmin = (-N/2) * dk
kmax = ((N/2)-1) * dk
wmax = (N-1) * h
k = np.linspace(kmin, kmax, N)
w = np.linspace(0, wmax, N)
dw = np.zeros(N)
dw[0] = 1/3
for j in range(1,N):

```

```

        if (j % 2 == 0):
            dw[j] = 2/3
        else:
            dw[j] = 4/3

#-----
# Prepare the A vector for
# Python's FFT implementation
#-----
I = np.zeros(N)
for i in range(N):
    I[i] = i

A_vector0 = np.exp(-1j * I * pi) * q0 * dw * N
A_vector1 = np.exp(-1j * I * pi) * q1 * dw * N
#-----

# Compute the DFT of A_vector
# and retrieve its real part
#-----

a_vector0 = np.fft.ifft(A_vector0)
a_vector1 = np.fft.ifft(A_vector1)
a_vector0 = np.real(a_vector0)
a_vector1 = np.real(a_vector1)
#-----

# Convert the a_vector into a
# value vector of European option
#-----

V_vector0 = (h * S0 / pi) * np.exp(-a * k) * a_vector0
V_vector1 = (h * S0 / pi) * np.exp(-a * k) * a_vector1

```

```

#-----
# Obtain a continuous value f
# unctio using interpolation
#-----

# This is linear interpolation.
K_vector = S0 * np.exp(k)
V0 = interp1d(K_vector, V_vector0)
V1 = interp1d(K_vector, V_vector1)
return V0, V1, K_vector

if __name__=='__main__':
    l1 = 20.0
    l2 = 30.0
    m1 = r1 = 0.05
    m2 = r2 = 0.1
    sig1 = 0.5
    sig2 = 0.3
    T = 1.0
    S0 = 100.0
    K = [74.082, 81.873, 90.484, 100.0, 110.517, 122.140, 134.986]
    N = int(2**12)
    h = 0.1534
    dk = (2.0 * pi) / (h * N)
    kmin = (-N/2) * dk
    kmax = ((N/2)-1) * dk
    wmax = (N-1)*h
    k = np.linspace(kmin, kmax, N)
    w = np.linspace(0, wmax, N)

```

```

a = 1.0

print ('\n   European Call Option Price via FFT method ')
print ('\n   Strike Price          STATE 1          STATE 2 \n')
for n in range(7):
    callprice = FFT_Euro(m1, m2, r1, r2, sig1, sig2,\
        l1, l2, a, w, T)
    V0, V1, K_vector = callprice.price()
    print ('%20f %20.10f %20.10f \n' % (K[n] ,V0(K[n]), V1(K[n])))
tn = time()-t0
print ("\n Duration in seconds %7.3f \n" %tn)

```

APPENDIX C: PYTHON CODE FOR MCS IN CHAPTER 2

The following is the codes that generated the data seen in Table 2.3, Tabel 2.4, Table 2.1 and Tabel 2.2. Monte Carlo algorithm relies on repeated random sampling to obtain numerical results. Its essential idea is using randomness to solve problems that might be deterministic in principle. As we said before, it's very time consuming versus FFT is less than a second and it's not feasible in most practical use in real time. Other reason makes FFT more useful is that simulating of random variable in MC makes slightly different answers each try, while we always get the same results in FFT.

This code is created for MC simulations with different strike prices in Table 2.3 and Tabel 2.4.

```
import numpy as np
import random
from time import time
t0 = time()
# Given parameters
T = 1.0
S0 = 100.0
Initial_State = 1.0
sig1 = 0.5
sig2 = 0.3
r1 = mu1 = 0.05
r2 = mu2 = 0.1
n = 100000
Strikes = [74.082, 81.873, 90.484, 100.0, 110.517, 122.140, 134.986]
l1 = 20.0
l2 = 30.0
```

```

#np.random.seed(19)

print ('\n  European Call Option Price under RSGBM via MC method ')

print ('\n          Strike Price          STATE 1  \n')

for j in range(7):
    K = Strikes[j]
    SumofTermVals = 0.0
    TermValOneRun = 0.0
    for x in range(n):
        LogStock = float(np.log(S0))
        Curr_Time = 0.0
        tau1 = 0.0
        Curr_State = Initial_State
        #Determine our occupation time of state 1
        while Curr_Time < T:
            #Determine time until next change of state
            p = random.uniform(0, 1)
            if Curr_State==1:
                ExpRV = -1*np.log(p)/l1
            else:
                ExpRV = -1*np.log(p)/l2
            #If the next state change is before maturity,
            # increment tau
            if Curr_Time + ExpRV < T and Curr_State==1:
                tau1 = tau1 + ExpRV
            # Else there is no state change between now and maturity
            else:
                if Curr_State==1:

```

```

        tau1 = tau1 + T - Curr_Time

#Increment to next switch time

Curr_Time = Curr_Time + ExpRV

#Switch State

if Curr_State==1:

    Curr_State = 2

else:

    Curr_State = 1

# Obtain a pseudo-random sample from

# standard normal distribution

SimRand = float(np.random.standard_normal(1))

#Calculate our terminal log stock price

LogStock = LogStock + (mu1 - 0.5 * sig1**2) * tau1 + \

    (mu2 - 0.5 * sig2**2) * (T-tau1) \

    + SimRand*np.sqrt(tau1*sig1**2 \

    + (T-tau1)*sig2**2)

#Calculate terminal option value

TermValOneRun = np.maximum(0, np.exp(LogStock) - K)

#Add terminal option value running total

SumofTermVals = SumofTermVals + TermValOneRun

callprice = (SumofTermVals / n) * np.exp(-1 * ((tau1 * r1 )+...

    r2*(T-tau1)))

print ('%20f %20.10f \n' % (K ,callprice))

tn = time()-t0

print ("\n Duration in seconds %7.3f \n" %tn)

```

This code is created for MC simulations with different maturities in Table 2.1 and Tabel 2.2.

```
import numpy as np
import random
from time import time
t0 = time()

# Given parameters
Maturities = [0.1, 0.2, 0.5, 1.0, 2.0, 3.0]
S0 = 100.0
Initial_State = 2.0
sig1 = 0.2
sig2 = 0.3
r1 = mu1 = 0.1
r2 = mu2 = 0.1
n = 100000
K = 90.0
l1 = 1.0
l2 = 1.0
#np.random.seed(19)

print ('\n  European Call Option Price under RSGBM via MC method ')
print ('\n          Maturity      STATE 2   \n')
for j in range(6):
    T = Maturities[j]
    SumofTermVals = 0.0
    TermValOneRun = 0.0
    for x in range(n):
        LogStock = float(np.log(S0))
```

```

Curr_Time = 0.0

tau1 = 0.0

Curr_State = Initial_State

#Determine our occupation time of state 1
while Curr_Time < T:

    #Determine time until next change of state
    p = random.uniform(0, 1)

    if Curr_State==1:
        ExpRV = -1*np.log(p)/l1
    else:
        ExpRV = -1*np.log(p)/l2

    #If the next state change is before maturity,
    # increment tau

    if Curr_Time + ExpRV < T and Curr_State==1:
        tau1 = tau1 + ExpRV

    # Else there is no state change between now and maturity
    else:
        if Curr_State==1:
            tau1 = tau1 + T - Curr_Time

    #Increment to next switch time
    Curr_Time = Curr_Time + ExpRV

    #Switch State
    if Curr_State==1:
        Curr_State = 2
    else:

```

```

        Curr_State = 1

    # Obtain a pseudo-random sample from
    # standard normal distribution
    SimRand = float(np.random.standard_normal(1))

    #Calculate our terminal log stock price
    LogStock = LogStock + (mu1 - 0.5 * sig1**2) * tau1 + \
        (mu2 - 0.5 * sig2**2) * (T-tau1) \
        + SimRand*np.sqrt(tau1*sig1**2 \
        + (T-tau1)*sig2**2)

    #Calculate terminal option value
    TermValOneRun = np.maximum(0, np.exp(LogStock) - K)

    #Add terminal option value running total
    SumofTermVals = SumofTermVals + TermValOneRun

    callprice = (SumofTermVals / n) * np.exp(-1 * ((tau1 * r1 )+...
        r2*(T-tau1)))

    print ('%20f %20.10f \n' % (T ,callprice))

tn = time()-t0

print ("\n Duration in seconds %7.3f \n" %tn)

```

APPENDIX D: PYTHON CODE FOR ANALYTICAL PRICES IN CHAPTER 2

The following is the code that generated the data seen in Table 2.1 and Tabel 2.2.

```
# first page

# mu(100,0 , 0, 0.1, 0.2, 0.3, 1, 1/3) = 4.668503519321425
# mu(100,0 , 0, 0.1, 0.2, 0.3, 1, 1)   = 4.685170185988092
# mu(100,0 , 0, 0.1, 0.2, 0.3, 1, 0)   = 4.6601701859880915
# var(0.2, 0.3, 1, 1/3) = 0.07333333333333333

import numpy as np
from math import log, pi
def mu(S0, d0, d1, r, sig0, sig1, T, t):
    value = log(S0)+(d1 - d0 - 0.5 * (sig0**2 - sig1**2)) * t \
            + (r - d1 - 0.5 * sig1**2)*T
    return value
def var(sig0, sig1, T, t):
    value = (sig0**2 - sig1**2) * t + sig1**2 * T
    return value

# second page

# call0(100,0 , 0, 0.1, 0.2, 0.3, 1, 1, 1, 90) =21.075037242396174
# call1(100,0 , 0, 0.1, 0.2, 0.3, 1, 1, 1, 90) = 22.002645343515372

import numpy as np
from math import pi, log
from scipy.integrate import quad
from parameters import mu, var
def integrand0(y, S0, d0, d1, r, sig0, sig1, T, l0, l1, K):
    m0T = mu(S0, d0, d1, r, sig0, sig1, T, T/3)
    mT  = mu(S0, d0, d1, r, sig0, sig1, T, T)
    v0T = var(sig0, sig1, T, T/3)
```

```

vT = var(sig0, sig1, T, T)
a = (1/np.sqrt(2*pi*v0T)) * np.exp(-(log(y+K)-m0T)**2 / (2*v0T))
b = (1/np.sqrt(2*pi*vT)) * np.exp(-(log(y+K)-mT)**2 / (2*vT))
value = (y/(y+K)) * (a * (1-np.exp(-l0*T)) + b * np.exp(-l0*T))
return value

def call0(S0, d0, d1, r, sig0, sig1, T, l0, l1, K):
    return np.exp(-r*T) * quad(integrand0, 0, np.inf, \
                                args=(S0, d0, d1, r, sig0, sig1, T, l0, l1, K))[0]

def integrand1(y, S0, d0, d1, r, sig0, sig1, T, l0, l1, K):
    m0T = mu(S0, d0, d1, r, sig0, sig1, T, T/3)
    m0 = mu(S0, d0, d1, r, sig0, sig1, T, 0)
    v0T = var(sig0, sig1, T, T/3)
    v0 = var(sig0, sig1, T, 0)
    a = (1/np.sqrt(2*pi*v0T)) * np.exp(-(log(y+K)-m0T)**2 / (2*v0T))
    b = (1/np.sqrt(2*pi*v0)) * np.exp(-(log(y+K)-m0)**2 / (2*v0))
    value = (y/(y+K)) * (a * (1-np.exp(-l1*T)) + b * np.exp(-l1*T))
    return value

def call1(S0, d0, d1, r, sig0, sig1, T, l0, l1, K):
    return np.exp(-r*T) * quad(integrand1, 0, np.inf, \
                                args=(S0, d0, d1, r, sig0, sig1, T, l0, l1, K))[0]

# third page

import numpy as np

from price import call0, call1

from time import time

t0 = time()

def main():
    d0 = d1 = 0

```

```

l0    = l1    = 1.0

sig0  = 0.2

sig1  = 0.3

T     = [0.1, 0.2, 0.5, 1.0, 2.0, 3.0]

S0    = 100.0

K     = 90.0

r     = 0.1

V0    = np.zeros((6))

V1    = np.zeros((6))

print ('\n European Call Option Price under RS
GBM-Analytical Solutions ')

print ('\n   Strike Price           STATE 1           STATE 2 \n')

for i in range(6):
    V0[i] = call0(S0, d0, d1, r, sig0, sig1, T[i], l0, l1, K)
    V1[i] = call1(S0, d0, d1, r, sig0, sig1, T[i], l0, l1, K)
    print ('%20f %20.10f %20.10f \n' % (T[i] ,V0[i], V1[i]))

tn = time()-t0

print ("\n Duration in seconds %7.3f \n" %tn)

main()

```

APPENDIX E: PYTHON CODE FOR SEMI-MC SIMULATIONS IN CHAPTER 2

This approach only takes random sampling of the Markov chain and then takes advantage of the availability of analytical formula (therefore exact) of the conditional price. Thus semi-Monte Carlo simulation outperforms Monte Carlo method. As we can see the obtained results from FFT and semi-MC simulation are so closed to each other than the results obtained from MC simulations.

The following is the code that generated the data seen in Table 2.1 and Tabel 2.2 for for semi-MC simulations with different maturities. One can simply obtain the data generated in Table 2.3 and Tabel 2.4 for semi-MC simulations with different strike price.

```
from scipy import log, sqrt, exp
import random
from time import time
from scipy.stats import norm
t0 = time()
# Given parameters
Maturities = [0.1, 0.2, 0.5, 1.0, 2.0, 3.0]
S0 = 100.0
Initial_State = 2.0
sig1 = 0.2
sig2 = 0.3
r1 = mu1 = 0.1
r2 = mu2 = 0.1
n = 100000
K = 90.0
l1 = 1.0
l2 = 1.0
```

```

#np.random.seed(19)

print ('\n  European Call Option Price under
RSGBM via Semi-MC method ')

print ('\n          Maturity      STATE 2   \n')

for j in range(6):
    T = Maturities[j]
    SumofTermVals = 0.0
    TermValOneRun = 0.0
    for x in range(n):
        LogStock = float(log(S0))
        Curr_Time = 0.0
        tau1 = 0.0
        Curr_State = Initial_State
        #Determine our occupation time of state 1
        while Curr_Time < T:
            #Determine time until next change of state
            p = random.uniform(0, 1)
            if Curr_State==1:
                ExpRV = -1*log(p)/l1
            else:
                ExpRV = -1*log(p)/l2
            #If the next state change is before maturity,
            # increment tau
            if Curr_Time + ExpRV < T and Curr_State==1:
                tau1 = tau1 + ExpRV
            # Else there is no state change between now and maturity
            else:

```

```

        if Curr_State==1:

            tau1 = tau1 + T - Curr_Time

            #Increment to next switch time

            Curr_Time = Curr_Time + ExpRV

        #Switch State

        if Curr_State==1:

            Curr_State = 2

        else:

            Curr_State = 1

    LT = mu1 * tau1 + mu2 * (T-tau1)

    VT = sig1**2 * tau1 + sig2**2 * (T-tau1)

    RT = tau1 * r1 + r2 * (T-tau1)

    d1 = (log(S0/K) + LT + (0.5 * VT)) / sqrt(VT)

    d2 = d1 - sqrt(VT)

    TermValOneRun = S0 * exp(-(RT - LT)) * norm.cdf(d1) \
- K * exp(-RT) * norm.cdf(d2)

    SumofTermVals = SumofTermVals + TermValOneRun

callprice = (SumofTermVals / n)

print ('%20f %20.10f \n' % (T ,callprice))

tn = time()-t0

print ("\n Duration in seconds %7.3f \n" %tn)

```