HIGH CONFIDENCE SET REGULARIZATION IN SPARSE HIGH DIMENSIONAL LOGISTIC REGRESSION WITH MEASUREMENT ERROR

by

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ABSTRACT

MAORONG RAO. HIGH CONFIDENCE SET REGULARIZATION IN SPARSE HIGH DIMENSIONAL LOGISTIC REGRESSION WITH MEASUREMENT ERROR. (Under the direction of DR. JIANCHENG JIANG)

The complex nature of high dimensional data diminishes the efficacy of the classical statistics inference. Regularization technique has been actively developed in response to revolution inference.

l_1 based regularization such as Lasso [13] and Dantizg Selector [5] succeed in two aspects. First, the inherent sparsity of l_1 accords with the underlying nature of high dimensional data; second, the convexity essence paves the way to computational feasibility in high dimension. Based on the idea provided by Dantzig Selector [5], James, G. M. and P. Radchenko developed an algorithm [33] to solve Dantzig Selector for generalized linear model. Fan [8] abstracted this framework to the set of convex loss function as High Confidence Set. To fill the gap of theoretical support within this framework, we derive non-asymptotic error bound in aspects to prediction error and parameter error with logistic loss function. We termed this classifier as High Confidence Set Selector (HCS). An implicit assumption of high confidence set selector is that the data is collected precisely. However, the data is unavoidable to process with measurement error in reality. In response to this challenge, we introduce a new methodology abbreviated as MHCS, which accounts for measurement error. We derive the non-asymptotic error bound in aspects to prediction error and parameter error in theoretical study.

Our simulation study shows MHCS performs better than other competing clas-

sifiers especially when measurement error aggravates, which provides numerical support to our theory. Ascribe to embedded linearity instinct, HCS and MHCS are versatile to connect with various state of art technique such as word vectors, deep network, transfer learning, etc. We demonstrate the cutting-edge applications in two real examples.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1: INTRODUCTION

"High dimensional data are nowadays rule rather than exception." [4]

In high dimensional setting, the dimension d is larger than sample size n, sometimes even growing faster as the sample size increases. For example, in many contemporary applications, microarray data is collected with thousands in dimension, while the sample size n is typically in the order of tens. "The central conflict in high dimensional setup is that the model complexity is not supported by limited access to data." Fan points out the essential challenge in high dimensional statistics [8]. In other words, the "variance" of conventional models is high in such new settings, and even simple models such as LDA need to be regularized.

In the situation that the number of parameters is larger than the sample size, un-regularized empirical risk minimization approach would select a model with perfect performance in training simply by memorizing the training sample rather than generalize the trend of signal from population. In other words, it may severely fail to predict the unseen data.

In order to develop statistical inference with reasonable accuracy or asymptotic consistency in high dimension setting, it is crucial to pare down the high degree of complexity to its bare essentials.

A natural underlying form of simplicity in high dimension is sparsity, we hope that the nature of the world is not so complex as it appears. "It's possible to develop high dimensional statistical inference, if $log(p) \times (sparsity(\beta)) << n$."[4]

We refer to Hastie et al. [13] and Buhlmann et al. [4] for overviews of statistical challenges associated with high dimensionality.

In addition to the embedded simplicity, the other principle in high dimension statistics is efficiency in algorithm.

l_1 based regularization such as Lasso [13] and Dantizg Selector [5] succeed in two aspects. First, the inherent sparsity of l_1 accords with the underlying nature of high dimensional data; second, the convexity essence paves the way to computational feasibility in high dimension.

Fan[8] introduces a l_1 regularization framework in high dimension statistics. The fundamental idea is selecting the member with smallest l_1 norm in a set which carries the information of data, termed as high confidence set. We elaborate the main idea as follows:

Assume a random sample from the population (X, Y) is collected in the form $(X_1, Y_1), \ldots, (X_n, Y_n)$, the convex loss function $\rho_\beta(X, Y)$ has the form $\rho_\beta(X, Y) = \rho(X^T\beta, Y)$.

$\beta^* \in \mathbb{R}^d$ is the target parameter which minimizes the expected loss $E\rho(X^T\beta, Y)$, that is:

$$\beta^* = \arg\min_{\beta \in R^d} E\rho(X^T\beta, Y)$$

Our target is to find an estimate β^* through empirical risk minimization.

Denote the empirical loss as

$$L_n\rho(\beta) = \frac{1}{n}\sum_{i=1}^{n} \rho(X_i^T\beta, Y_i);$$

and the gradient with respect to β as $\nabla_\beta L_n\rho(\beta)$, the high confidence set is constructed as follow:

$$C_\lambda = \{\beta \in \mathbb{R}^d : \|\nabla L_n\rho(\beta)\|_\infty < \lambda\},$$

where the tuning parameter λ is chosen related to the confidence level, viz.

$$Pr(\beta^* \in C_\lambda) = Pr\{\|\nabla L_n\rho(\beta)\|_\infty < \lambda\} > 1 - \delta;$$

The high confidence set C_λ inherits the information about β^* from sample data. In addition, as we discuss before, if we impose the sparsity assumption on the underlying parameter β^*, a natural solution is selecting the sparsest solution in the high confidence set, viz.

$$\hat{\beta} = \arg\min_{\beta \in C_\lambda} \|\beta\|_1.$$

With this generalized framework, several works can be considered as examples of the high confidence set selection with specific loss measure. For instance, Dantzig Selector [5] can be viewed as high confidence set estimation for linear regression with quadratic loss; Cai and Liu [6] propose Linear Programming Discriminant rule (LPD) for two Multi-Gaussian distributed data, which apply the high confidence set selection approach maximum likelihood of Bayes rule. Barut [2] extends LPD to measurement error scenario.

Inspired by this idea, we apply this method to regularize high dimensional logistic regression. We term this method as High Confidence Set Selector (HCS).

An implicit assumption of HCS is that the data is collected precisely. However, in reality, the collection process is unavoidable to produce noise and missing value. In many real applications, such as image recognition, text mining and speech recognition, most problems are subject to measurement error.

There are various studies concerns on correction of measurement error. Within the context of estimate distribution of measurement error, estimators proposed in studies Liang and Li [34], Loh and Wainwright [35], Ma and Li [36], Rosenbaum and Tsybakov [41] yield sound asymptotic results by approaching maximum likelihood.

However, under a high dimensional setting, the distribution of measurement error is too complex to capture. Methods proposed in Rosenbaum and Tsybakov [40], Zhu, Leus and Giannakis [42], Chen and Caramanis[32] account for measurement error without requiring estimation of its distribution, standing out in practical application in high dimensional setting.

Inspired by the model proposed in Rosenbaum and Tsybakov [40], where an additional reguralization parameter is introduced to capture additive measurement error in linear regression, we derive classifier accounts for measurement error in generalized linear model with logistic loss. We name this classifier Modified High Confidence Set Selector (MHCS).

The remaining contents of this paper are laid out as follows: In Chapter 2, we introduce the model setup of High Confidence Set Selector (HCS) and derive its non-asymptotic error bounds in aspects to prediction error and parameter error with logistic loss function. In Chapter 3, we introduce the Modified High Confidence Set Selector (MHCS) which accounts for measurement error. We derive the non-asymptotic error bound in aspects to prediction error and parameter error in theoretical study as well. In Chapter 4, we eleborate our implementation algorithm and simulation experiment. The result shows MHCS performs better than other competing classifiers especially when measurement error aggravates, which provides numerical support to our theory derived in Chapter 3. In Chapter 5, we demonstrate how our methods connect to some cutting-edge techniques in sentiment analysis and image recognition.

CHAPTER 2: HIGH CONFIDENCE SET SELECTOR

2.1 Model Setup and Methodology

Consider a measurable space $\mathcal{M} = \mathcal{X} \times \mathcal{Y}$, where $\mathcal{Y} = \{0, 1\}$, and $(x_i, y_i)_{i=1}^n \in \mathcal{M}$ is a set of n i.i.d. random pairs of observations; $\phi(\cdot)$ is a set of bounded real value functions $\phi = (\phi_1, \ldots, \phi_d)$, $\|\phi(\cdot)\|_\infty < M_d$ [15], which maps original features from $\mathcal{X}$ to $\mathcal{Z} \in \mathbb{R}$, e.g., $\phi : \mathcal{X} \to \mathcal{Z} \in \mathbb{R}^d$.

Defined the parametric space $\Omega : (f, \phi)$, for a given ϕ, let $Z = \phi(X)$, then the generalized logistic regression model defined in parametric space $\Omega : (f, \phi)$ can be modeled as:

$$Pr(Y = 1|Z) = \frac{\exp f(Z)}{1 + \exp f(Z)} ,$$

where $f : \mathcal{Z} \to \mathbb{R}$, is the log odds ratio, i.e.,

$$f(Z) = \log \frac{Pr(Y = 1|Z)}{Pr(Y = 0|Z)},$$

Denote ρ_f as the loss function of generalized logistic regression given $Z = \phi(X)$, then,

$$\rho_f(Z, Y) = Y f\ (Z) - \log \Big\{ 1 + \exp\ \ f\ (Z) \Big\};$$

denote the corresponding empirical loss as L_n, then,

$$L_n\, \rho_f = \frac{1}{n} \sum_{i=1}^{n} \Big\{ Y_i f(Z_i) - \log\,[\, 1 + \exp\ \ f\ (Z_i)\,]\, \Big\} .$$

The expected risk $L\,\rho_f$ is the expectation of loss given f, which holds

$$L\,\rho_f = E\,(\,\rho_f\,) = E\,(\,L_n\,\rho_f\,).$$

Given $Z = \phi(X)$, denote f_0 as the best parameter in Ω which minimizes $L\,\rho_f$, e.g.:

$$f_0 = \arg\min_{f \in \Omega} L\,\rho_f.$$

For a set of given ϕ, consider the linear subspace $\Omega_\beta(\phi, f_\beta) \subset \Omega(\phi, f)$, such that:

$$f_\beta(Z) = \beta^T Z.$$

Correspondingly, in this linear subspace, the loss function is

$$\rho_\beta(Z, Y) = Y\beta^T Z - \log\left[\,1 + \exp\left(\,\beta^T Z\,\right)\right];$$

and the empirical loss is

$$L_n\,\rho_\beta\,(Z, Y) = \frac{1}{n}\sum_{i=1}^{n}\,\rho_\beta(Z, Y) = \frac{1}{n}\sum_{i=1}^{n}\,\left\{\,Y_i\,\beta^T Z_i - \log\left[\,1 + \exp\left(\beta^T Z_i\right)\right]\right\}.$$

Denote the expected loss as $L\,\rho_\beta$,

$$L\,\rho_\beta\,(Z, Y) = E\left[\,\rho_\beta\,(Z, Y)\,\right] = E\,\left[\,L_n\,\rho_\beta\,(Z, Y)\,\right]$$

The optimal parameter β^* in linear subspace is defined as the one that minimizes the expected loss, e.g.,

$$\beta^* = \arg\min_{\beta \in \Omega_\beta} L\,\rho_\beta\,(Z, Y); \tag{1}$$

It holds that:

$$\left.\frac{\partial\, L\, \rho_{\beta}\,(Z, Y)}{\partial \beta}\right|_{\beta^*} = 0 \tag{2}$$

In classical statistics setting, with fixed dimension of β, as $n \to \infty$, by asymptotic theory, we can achieve $\nabla_{\beta}\, L_n\, \rho_{\beta^*}(Z, Y) \to 0$ in probability. However, in high dimensional statistics, d is larger than n, sometimes it even grows faster than n, we cannot expect $\nabla_{\beta}\, L_n\, \rho_{\beta^*}(Z, Y) = 0$ to hold exactly. However, we would expect $\| \nabla_{\beta}\, L_n\, \rho_{\beta^*}(Z, Y) \|_{\infty} \leq \lambda$ with large probability when appropriate λ is chosen. Therefore, it is straightfoward to define the high confidence set as follow:

$$\mathcal{C}(\lambda) = \{\beta \in \mathbb{R}^d : \| \nabla_{\beta}\, L_n\, \rho_{\beta}\,(Z, Y) \|_{\infty} \leq \lambda\}; \tag{3}$$

where λ is chosen such that

$$Pr \left\{ \beta^* \in \mathcal{C}(\lambda) \right\} = Pr \left\{ \| \nabla_{\beta}\, L_n\, \rho_{\beta^*}(Z, Y) \|_{\infty} \leq \lambda \right\} \geq 1 - \delta \tag{4}$$

for a positive sequence $\delta \to 0$.

Then select the solution with minimum l_1 norm in $\mathcal{C}(\lambda)$ as a proxy of β^*, we name this estimatior as High Confidence Set Selector (HCS):

$$\hat{\beta}_{HCS} = \underset{\beta \in \mathcal{C}(\lambda)}{\arg\min}\ \| \beta \|_1 \tag{5}$$

2.2 Theoretic Property of High Confidence Set Selector

In this section, we investigate the theoretical properties of High Confidence Set Selector. First, we show that, with appropriate choice of λ, β^* falls in $C(\lambda)$ with high probability. Second, we derive the non-asymptotic error bound in terms of excess risk. With the assumptions of sparsity and restricted strong convexity [29], we derive the parameter error bound in the third result.

The following assumptions are used in theory derivation:

Assumption. A_1*:* $(Z_i, Y_i)_{i=1}^n$ *are i.i.d.*

Assumption. A_2*:* $\|\phi(\cdot)\|_\infty < M_d$*;*

Remark. Assumption A_1 and Assumption A_2 are general assumptions in the literature regarding generalized error bound in l_1 regularization and learning theory ([15], [16], [17], [18], [19]). In pratical, various data are collected bounded, such as the image data ranges from 0 to 255 in RGB; some base functions ϕ have bounded outputs in nature, such as sigmoid function, softmax function, wordvector, neural networks with certain activate functions, etc. The additional advantage of this setting is that X is distribution free, which avoids the complexity of density estimation in high dimension.

Assumption. A_3*:* $M_d\sqrt{log2d} \sim \mathcal{O}(\sqrt{n})$

Assumption. A_4*: Construct a sequence* $\{a_j\}_{j=0}^{J-1}, \quad a_j = 2a_{j-1}$*, for* $\forall a_0 > 0$*, there exists a positive integer* $J < \infty$*, such that,*

$$a_{J-1} = a_0\, 2^J \geq 2\|\beta^*\|_1$$

Remark. Assumption A_3 and Assumption A_4 are technique assumptions. A_4 restricts $\|\beta^*\|_1$ from growing too fast.

Assumption. A_5: $\| \beta^* \|_0 \leq s$

Remark. Assumption A_5 assumes the target parameter β^* is s-sparse, which means that the maximum number of nonzero components of β^* is s.

This assumption is widely used in the high dimensional setting, we refer to Hastie, Trevor, Tibshirani and Wainwright [43] for overviews.

Assumption (A_6 Restricted Strong Convexity)**.**

$$\delta L_n \rho_{(\Delta, \beta^*)} (Z, Y) \geq \kappa \| \Delta \|_2 \|\beta^*\|_1 < \infty.$$

Remark. The restricted strong convexity assumption is the key assumption in deriving parameter error bound. This assumption is introduced in Negahban and Wainwright [29], where they have proved that Restricted Strong Convexity holds for subgaussian random matrix with high probability.

Define the support set S by mapping nonzero components of β^* to a index set, i.e.:

$$S := \{j : \beta_j^* \neq 0\}, \quad |S| = s,$$

S^c is complementary set of S.

Denote Δ as deviation in the neighbor of β^*, $\Delta = \beta - \beta^*$;

The definition of Restricted Strong Convexity is:

$$\delta\, L_n\, \rho_{(\,\Delta,\,\beta^*\,)}\,(\,Z,\,Y\,)$$
$$= L_n\, \rho_{(\,\beta^*+\Delta\,)}(\,Z,\,Y\,) - L_n\, \rho_{\beta^*}(\,Z,\,Y\,) - \left\langle\, \nabla_\beta L_n\, \rho_{\beta^*}(\,Z,\,Y\,),\, \Delta \,\right\rangle$$
$$\geq \kappa \parallel \Delta \parallel_2 \tag{6}$$
$$for \quad \|\Delta_{S_c}\|_1 \leq \|\Delta_S\|_1.$$

The Restricted Strong Convexity in geometry is the curvature of loss function. We use the empirical loss to track the population performance. Once we have the estimator $\hat{\beta}$, we prefer that it is robust against the perturbation in empirical loss. If strong convexity exists, the solution to the parameter estimation will not change much to a small perturbation in empirical lost. It's therefore a stable solution. While in weak curvature, the opposite effect occurs, small perturbation in empirical loss would cause enormouse parameter shifts in parameter space.

From Theorem 2.2, we can see the excess risk is tracked by the l_1 norm of $\|\hat{\beta}-\beta^*\|$, in high dimension scenario, where $n << p$. There exists space with low curvature such that β^* is far away from $\hat{\beta}$, but it will not arouse fluctuation in empirical loss function. The main idea is to restrict the target parameter from lying in these directions. By l_1 regularization, $\|\hat{\beta}\|_1 \leq \|\beta^*\|_1$, apply the lemma from basis pursuit [45], we have following property for $\hat{\Delta}$: $\|\hat{\Delta}_{S_c}\|_1 \leq \|\hat{\Delta}_S\|_1$.

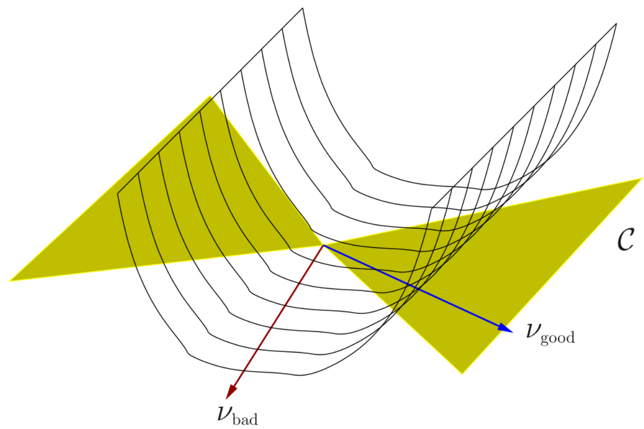


Figure 1: Illustration of Restricted Strong Convexity [43]

In high dimension setting, while we can't expect the strong convexity exists in every direction, we can expect it exists in the direction of $\hat{\Delta} : \|\hat{\Delta}_{S_c}\|_1 \leq \|\hat{\Delta}_S\|_1$. In Figure 1 we illustrate the 'restricted direction', where the shadow direction is desired.

The notations used in next section are listed below.

Notation:

$$\lambda^* \equiv \sqrt{2}\, M_d \sqrt{\frac{\log\,(\,2\,d\,) + \log n}{n}}\;;$$

$$\delta_1 \equiv \frac{2M_d}{n};$$

$$\delta_2 \equiv 2\, M_d \sqrt{\frac{2\,log\,2d}{n}}$$

$$\delta_0 = \delta_1 + \delta_2$$

2.2.1 High Confidence Set

Recall that the High Confidence Set defined in (3), as discussed in previous section, we expect the optimal linear solution β^* to fall in the high confidence set with high probability when appropriate λ is chosen. Define

$$Event \ \ A := \{\beta^* \in C_\lambda\},$$

then we have the following theorem;

Theorem 2.1 (Event A)**.** *With β^* defined in* (1)*, and C_λ defined in* (3)*, under Assumption $A_1 - A_3$, if $\ \lambda > \lambda^*,$ it holds that:*

$$P\ (\ \beta^* \in \mathcal{C}_\lambda\ ) > 1 - \frac{1}{n}.$$

2.2.2 Prediction Error

Define our solution set:

$$\mathcal{B}_\lambda := \left\{\ \beta\ \in\ \mathbb{R}^d:\ \beta = \underset{\beta \in \mathcal{C}(\lambda)}{\arg\min}\ \|\ \beta\ \|_1\ \right\} \tag{7}$$

The relationship between optimal linear solution β^*, solution to HCS ($\hat{\beta}_{HCS}$), linear parameter space (Ω_β), High Confidence Set (C_λ) and Solution Set of HCS ($\mathcal{B}_\lambda$) is illustrated in Figure 2.

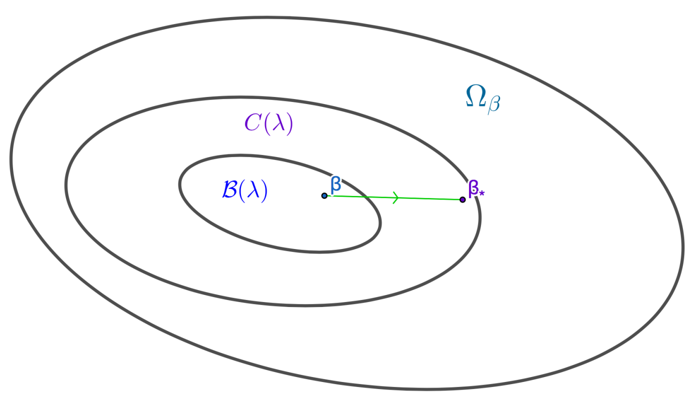


Figure 2: The relationship between β^*, $\hat{\beta}_{HCS}$, Ω_β, C_λ and $\mathcal{B}_\lambda$

Define the excess risk of $\hat{\beta} \in \Omega_\beta$ as:

$$\mathcal{E}(\hat{\beta}) = L\rho_{\hat{\beta}} - L\rho_{\beta^*}.$$

The prediction error bound in terms of excess risk is derived in Theorem 2.2.

Theorem 2.2 (Prediction Error Bound). *Denote the solution to HCS as $\hat{\beta}_{HCS}$, under Assumption $A_1 - A_4$, when $\lambda > \lambda^*$, with probability at least $1 - 2J\, e^{-2n} - \frac{1}{n}$, where J is a positive integer satisfies Assumption A_4, it holds that:*

$$\mathcal{E}(\hat{\beta}_{HCS}) \leq (\, \lambda + \delta_0 \,) \parallel \beta^* - \hat{\beta}_{HCS} \parallel_1 \, + \, \delta_0 \, a_0$$

2.2.3 Parameter Error

Theorem 2.3 (Parameter Error Bound). *Under Assumption $A_1 - A_6$, when $\lambda \geq \lambda^*$, with probability at least $1 - \frac{1}{n}$, it holds:*

$$(i) \quad \parallel \hat{\beta}_{HCS} \, - \, \beta^* \parallel_2 \, \leq \, \frac{4\,\lambda\,\sqrt{s}}{\kappa};$$

$$(ii) \quad \parallel \hat{\beta}_{HCS} \, - \, \beta^* \parallel_1 \, \leq \, \frac{8\,\lambda\,s}{\kappa}$$

Corollary (Prediction Error Bound). *Under Assumption $A_1 - A_6$, when $\lambda > \lambda^*$, with probability at least $1 - 2J\, e^{-2n} - \frac{1}{n}$, it holds that:*

$$\mathcal{E}(\hat{\beta}_{HCS}) \leq \, \frac{8\,\lambda\,s}{\kappa} \, (\, \lambda + \delta_0 \,) + \, \delta_0 \, a_0$$

CHAPTER 3: MODIFIED HIGH CONFIDENCE SET SELECTOR WITH MEASUREMENT ERROR

3.1 Model Setup and Methodology

As discussed in Chapter 1, the measurement error is unavoidable in reality.

Consider model with additive measurement error. Instead of (X, Y), we observe (U, Y), where $X \in \mathcal{X}, and \;\; U \in \mathcal{X}$.

Analogous to model setup in Chapter 2, $\phi(\cdot) : \mathcal{X} \to \mathcal{Z}$ is a set of base functions with $\|\phi(\cdot)\|_\infty \leq M_d$. After features transformation by $\phi(\cdot)$, we have (W, Y), where

$$W = \phi(U);$$

And the additive measurement error Ξ is defined as:

$$\Xi = \phi(U) - \phi(X)$$

For simplicity, denote $Z = \phi(X)$, thus,

$$W = Z + \Xi.$$

According to Theorem 2.1, β^* is feasible in C_λ if λ is chosen appropriately. However, the presence of measurement error causes the high confidence set to lost its efficacy.

To see this, if we roughly plug the achievable measure W into the high confidence set,

$$C_\lambda = \{\beta : \|\nabla_\beta L_n(W, Y, \beta)\|_\infty < \lambda\}$$

$E(\nabla_\beta L_n(X, Y, \beta^*)) = 0$ thus for $\forall \lambda > 0$, $\nabla_\beta L_n(X, Y, \beta^*) \to 0$ if $n \to \infty$ however, $E(\nabla_\beta L_n(W, Y, \beta^*))$ does not need to be 0, which means β^* may not in C_λ even when $n \to \infty$.

In the case of linear regression, Rosenbaum and Tsybakov [40] introduce an additional parameter γ to bound the magnitude of the measurement error in the matrix uncertainty selector (MUS), which yields the following two bounds:

$$\|W\epsilon\|_\infty < \lambda$$

and

$$\|\Xi\|_\infty < \gamma;$$

where W is the obeservation, Ξ is the measurement error and ϵ is the residual. These bounds are the sufficient conditions for β^* is feasible with high probability in following set:

$$\{\beta : \|W(Y - W\beta)\|_\infty < \lambda + \gamma\|\beta\|_1\}.$$

Inspired by this idea, we developed a modified high confidence set $C(\lambda, \gamma)$ for logistic loss. Note that logistic loss can be expressed in the form of mean function $\mu(Z\beta)$:

$$\|\nabla_\beta \, L_n \rho_\beta \, (W, Y)\|_\infty = \frac{1}{n} \; \| \, W^T \, [\, Y - \mu \, (\, W\beta \,) \,] \, \|_\infty;$$

where

$$\mu \, (\, W\beta \,) = \frac{\exp \, (\, W\beta \,)}{1 + \exp \, (\, W\beta \,)} \; \in \; (\, 0, 1 \,).$$

By model assumption,

$$W\beta = Z\beta + \Xi\beta;$$

Thus by Taylor expansion and Cauchy residual theorem,

$$\mu\,(\,W\beta\,) = \mu\,(\,Z\beta\,) + \mu'\,(\,\xi\,)\,(\,\Xi\beta\,)$$

where ξ lies in the segment between $W\beta$ and $Z\beta$.

Then by triangle inequality, $\frac{1}{n}\ \|\ W^T\ [\ Y - \mu\ (W\beta\,)\ ]\ \|_\infty$ is composed of two parts:

$$\begin{aligned}\frac{1}{n}\ \|\ W^T\ [\ Y - \mu\ (\ W\beta\ )\ ]\ \|_\infty =& \frac{1}{n}\ \|\ W^T\ [\ Y - \mu\ (\ Z\beta\ ) - \mu'\ (\ \xi\beta\ )\ (\ \Xi\beta\ )\ ]\ \|_\infty \\ \leq & \frac{1}{n}\ \|\ W^T\ [\ Y - \mu\ (\ Z\beta\ )\ ]\ \|_\infty + \frac{1}{n}\ \|\ W^T\ \mu'\ (\ \xi\ )\ (\ \Xi\beta\ )\ \|_\infty \\ \leq & \frac{1}{n}\ \|\ W^T\ [\ Y - \mu\ (\ Z\beta\ )\ ]\ \|_\infty + \frac{1}{n}\ \|\ W^T\ \mu'\ (\ \xi\ )\ \Xi\ \|_\infty \|\beta\|_1\end{aligned}$$

Therefore, it is intuitive to construct a modified high confidence set which accounts for measurment error as follow:

$$C(\lambda, \gamma)\ = \{\ \frac{1}{n}\ \|\ W^T\ [\ Y - \mu\ (\ W\beta\ )\ ]\ \|_\infty \leq \lambda\ +\ \gamma\ \|\ \beta\ \|_1\ \};$$

Where λ and γ are the high-confidence upper bound of $\frac{1}{n}\ \|\ W^T\ [\ Y - \mu\ (\ Z\beta\ )\ ]\ \|_\infty$ and $\frac{1}{n}\ \|\ W^T\ \mu'\ (\ \xi\ )\ \Xi\ \|_\infty$ respectively.

Like HCS, we select the member in $C(\lambda, \gamma)$ with minimal l_1 norm:

$$\hat{\beta} = \mathop{\arg\min}_{\beta \in \mathcal{C}\,(\,\lambda,\gamma\,)}\ \|\ \beta\ \|_1.$$

We term this estimator as Modified High Confidence Set Selector (MHCS).

3.2 Theoretic Property of MHCS

Analogous to HCS, we extended the study of high confidence set property, prediction error bound and parameter error bound to MHCS. The modified assumptions and notations used in this chapter are listed below.

Assumption and Notation

Assumption (C_1)**.** $(Z_i, Y_i)_{i=1}^n \;\; are \;\; i.i.d.$*, and* $(W_i, Y_i)_{i=1}^n$ *are i.i.d.;*

Assumption (C_2)**.** $W = Z + \Xi$*, and* $E(W) = 0$*.*

Assumption (C_3)**.** $\|\phi(\cdot)\|_\infty < M_d$*; i.e.,* $\| Z \|_\infty \leq M_d$*; and* $\| W \|_\infty \leq M_d$*;*

Assumption (C_4)**.** $M_d\sqrt{log2d^2} \sim \mathcal{O}(\sqrt{n})$*;*

Assumption (C_5)**.** *For* $\forall a_0 > 0, \exists J < \infty, \; such \;\; that, a_{J-1} = a_0\, 2^J \geq 2\|\beta^*\|_1$*;*

Assumption (C_6)**.** $\| \beta^* \|_0 \; \leq \; s$*;*

Assumption (C_7)**.** $\delta\, L_n\, \rho_{(\,\Delta,\, \beta^*\,)}\, (\,W,\, Y\,) \geq \; \kappa \,\|\, \Delta \,\|_2$

Remark. The model assumption of additive measurement error C_2 has been illustrated in Section 3.1, other assumptions are analogous to assumptions in Section 2.2.

Notation:

$$\lambda^* \equiv \sqrt{2}\, M_d \sqrt{\frac{\log(2d) + \log n}{n}}\,;$$

$$\gamma^* \equiv M_d^2 \sqrt{\frac{\log(2d^2) + \log n}{2n}}\,,$$

$$\delta_1 \equiv \frac{2M_d}{n};$$

$$\delta_2 \equiv 2\, M_d \sqrt{\frac{2\, log\, 2d}{n}}$$

$$\delta_0 = \delta_1 + \delta_2$$

We have the following properties for MHCS:

Theorem 3.1 (Event B)**.**

Under Assumption $C_1 - C_4$, when $\lambda > \lambda^*$, $\gamma > \gamma^*$,

$$P\,[\,\beta^* \in \mathcal{C}_{(\lambda,\gamma)}\,] > 1 - \frac{2}{n}.$$

Theorem 3.2 (Excess Risk)**.**

Under Assumption $C_1 - C_5$, when $\lambda > \lambda^*$, $\gamma > \gamma^*$, with probability at least $1 - \frac{2}{n}$, it holds:

$$\mathcal{E}(\hat{\beta}_{MHCS}) \leq \left(3\lambda + 2\gamma \|\beta^*\|_1 + \delta_0\right) \|\beta^* - \hat{\beta}_{MHCS}\|_1 + \delta_0\, a_0\,.$$

Theorem 3.3 (Parameter Error Bound)**.**

Under Assumption $C_1 - C_7$, when $\lambda \geq \lambda^*$ and $\gamma \geq \gamma^*$, with probability at least $1 - \dfrac{2}{n}$,

it holds:

$$(i) \quad \|\hat{\beta}_{MHCS} - \beta^*\|_2 \leq \frac{4\,(\lambda + \gamma \|\beta^*\|_1)\sqrt{s}}{\kappa};$$

$$(ii) \quad \|\hat{\beta}_{MHCS} - \beta^*\|_1 \leq \frac{8\,(\lambda + \gamma \|\beta^*\|_1)\, s}{\kappa}.$$

Corollary. *Under Assumption $C_1 - C_7$, when $\lambda > \lambda^*$, $\gamma > \gamma^*$, with probability at least $1 - 2J\,e^{-2n} - \frac{2}{n}$, it holds that:*

$$\mathcal{E}(\hat{\beta}_{MHCS}) \leq \frac{8s(\,\lambda + \gamma\|\beta^*\|_1\,)\,(3\,\lambda\, + 2\gamma\|\beta^*\|_1 + \delta_0)}{\kappa} + \;\delta_0\; a_0$$

CHAPTER 4: NUMERICAL STUDY OF HIGH CONFIDENCE SET ESTIMATION

4.1 Implementation

We propose an algorithm utilizing the Newton-Raphson method to solve this optimization problem, which involves a sequence of non-convexity approximations to the high confidence set. In the following, we introduce the main idea.

Notice that simple algebra leads to:

$$L'_n(Z,Y,\beta) \equiv \nabla_\beta \, L_n \, \rho_\beta(Z,Y) \; = n^{-1}\sum_{i=1}^{n}\Big\{-Y_iZ_i + \frac{\hat{Z}_i \exp(\beta^T Z_i)}{1+\exp(\beta^T Z_i)}\Big\}$$

and

$$L''_n(Z,Y,\beta) \equiv \frac{\partial^2 L_n \rho_\beta(Z,Y)}{\partial \beta^2} = n^{-1}\sum_{i=1}^{n}\frac{\hat{Z}_i\hat{Z}_i^T \exp(\beta^T Z_i)}{\{1+\exp(\beta^T Z_i)\}^2}.$$

Given an initial value $\hat{\beta}^{(0)}$, by Taylor's expansion, we have

$$L'_n(Z,Y,\beta) \approx L'_n(Z,Y,\hat{\beta}^{(0)}) + L''_n(Z,Y,\hat{\beta}^{(0)})(\beta - \hat{\beta}^{(0)}) \equiv \delta_0 + \Sigma_0\beta,$$

where $\delta_0 = L'_n(Z,Y,\hat{\beta}^{(0)}) - L''_n(Z,Y,\hat{\beta}^{(0)})\hat{\beta}^{(0)}$ and $\Sigma_0 = L''_n(Z,Y,\hat{\beta}^{(0)})$. Then $\mathcal{C}(\lambda)$ can be approximated by

$$\mathcal{C}(\lambda;\hat{\beta}^{(0)}) = \{\beta \in \mathbb{R}^d : \|\delta_0 + \Sigma_0\beta\|_\infty \le \lambda\}.$$

We then obtain the one-step approximation to $\hat{\beta}(\lambda)$:

$$\hat{\beta}^{(1)} = \arg\min_{\beta}\Big\{\|\beta\|_1 : \beta \in \mathcal{C}(\lambda;\hat{\beta}^{(0)})\Big\}. \qquad (8)$$

Using the above estimator as an updated initial value, we obtain a two-step approximation. Repeat this procedure up to convergence.

Finally, the remaining problem is to solve the optimization problem (8). This requires solving a non-convex program which can be written in the following form:

$$\min \mathbf{1}_p^T(\beta^+ + \beta^-)$$

$$s.t.\ \Sigma_0\beta^+ - \Sigma_0\beta^- + \delta \le \lambda$$

$$\Sigma_0\beta^+ - \Sigma_0\beta^- + \delta \ge -\lambda$$

$$\beta^+, \beta^- \ge 0$$

$$\beta_j^+\beta_j^- = 0,\ \text{for } j = 1, \ldots, d.$$

The convex relaxation of this problem can be obtained by dropping the final constraint. Furthermore, the relaxed problem is a linear program with $2d$ variables and $4d$ constraints. This linear program can be solved efficiently using a large set of methods, such as the interior point method or the dual simplex method.

It is straightforward to extend this algorithm to MHCS as follows:

$$L_n'(W, Y, \beta) \equiv n^{-1}\sum_{i=1}^n \Big\{-Y_iW_i + \frac{W_i\exp(\beta^TW_i)}{1+\exp(\beta^TW_i)}\Big\}$$

and

$$L_n''(W, Y, \beta) = n^{-1}\sum_{i=1}^n \frac{W_i\hat{Z}_i^T\exp(\beta^TW_i)}{\{1+\exp(\beta^TW_i)\}^2}.$$

Given an initial value $\hat{\beta}^{(0)}$, by Taylor's expansion, we have

$$L_n'(W, Y, \beta) \approx L_n'(W, Y, \hat{\beta}^{(0)}) + L_n''(W, Y, \hat{\beta}^{(0)})(\beta - \hat{\beta}^{(0)}) \equiv \delta_0 + \Sigma_0\beta,$$

where $\delta_0 = L'_n(W, Y, \hat{\beta}^{(0)}) - L''_n(Z, Y, \hat{\beta}^{(0)})\hat{\beta}^{(0)}$ and $\Sigma_0 = L''_n(Z, Y, \hat{\beta}^{(0)})$. Then $\mathcal{C}(\lambda, \gamma)$ can be approximated by

$$\mathcal{C}(\lambda, \gamma; \hat{\beta}^{(0)}) = \{\beta \in \mathbb{R}^d : \|\delta_0 + \Sigma_0\beta\|_\infty \leq \lambda + \gamma\|\beta\|_1\}.$$

Then we obtain the one-step approximation to $\hat{\beta}^{(1)}$ for next implementation:

$$\hat{\beta}^{(1)} = \arg\min_{\beta}\Big\{\|\beta\|_1 : \beta \in \mathcal{C}(\lambda, \gamma; \hat{\beta}^{(0)})\Big\}. \tag{9}$$

Using the above estimator as an updated initial value, we obtain a two-step approximation. Repeat this procedure up to convergence.

The remaining problem is to solve optimization problem (9). This requires solving a non-convex program which can be written in the following form:

$$\begin{aligned}
&\min \mathbf{1}_p^T(\beta^+ + \beta^-) \\
&s.t.\ (\Sigma_0 - \gamma)\beta^+ - (\Sigma_0 + \gamma)\beta^- + \delta \leq \lambda \\
&(\Sigma_0 + \gamma)\beta^+ - (\Sigma_0 - \gamma)\beta^- + \delta \geq -\lambda \\
&\beta^+, \beta^- \geq 0 \\
&\beta_j^+\beta_j^- = 0,\ \text{for } j = 1, \ldots, d.
\end{aligned}$$

4.2 Simulation Experiment

In this section, we introduce simulation experiments to investigate prediction error and parameter error of proposed methods (HCS and MHCS). Specifically, following the experiment setting in Barut, Bradic, Fan and Jiang [2], we evaluated the performance of our proposed classifiers in scopes of the following measures, and compared the results with other competitive l_1 and l_2 regularization approaches, i.e., LASSO and Ridge. The performance measures are listed as follows:

1. CE: Classification Error;

2. $Deviance$: Cross Entropy:

$$Deviance = -\frac{1}{n}\sum_{i}^{n}[y_i log(\hat{y_i}) + (1 - y_i) log(1 - \hat{y_i})];$$

   where $\hat{y} = 1/(1 + \exp(-x\hat{\beta}))$;

3. L_1: l_1 norm of the difference between standardized $\hat{\beta}$ and β^*;

$$L_1 = \left\|\frac{\hat{\beta}}{\|\hat{\beta}\|_2} - \frac{\beta^*}{\|\beta^*\|_2}\right\|_1;$$

4. L_2: l_2 norm of the difference between standardized $\hat{\beta}$ and β^*;

$$L_2 = \left\|\frac{\hat{\beta}}{\|\hat{\beta}\|_2} - \frac{\beta^*}{\|\beta^*\|_2}\right\|_2;$$

5. FN: False Negative Ratio, i.e., the number of zero coefficient of $\hat{\beta}$ for which β^* is non-zero

$$FN = \frac{s_0 - \|\hat{\beta}_S\|_0}{s_0}.$$

6. FP: False Positive Ratio, i.e., the number of non-zero coefficient of $\hat{\beta}$ for which β^* is zero

$$FP = \frac{\|\hat{\beta}_{S^c} - \beta^*_{S^c}\|_0}{p - s_0}$$

We generated the binomial distributed sample data as follows:

$$
\begin{aligned}
&GenerateX \sim MultiGaussian(\mathbf{0}^d, \mathbf{\Sigma}) \\
&\beta^* = [1^{s_0}, 0^{d-s_0}]^T, \ ; \\
&Pr = \frac{1}{1 + e^{-\mathbf{X}\beta^*}} \\
&Y = Binomial(n, 1, Pr)
\end{aligned}
\tag{10}
$$

In our experiment setting, dimension $d = 200$; sparsity parameter $s_0 = 10$; training sample size $n_{training} = 100$; and testing sample size $n_{testing} = 100$;

Three types of correlation matrix were taken into account:

Type 1: Identity Matrix: $\Sigma_{d\times d} = diag(d)$;

Type 2: Equal Correlation Matrix: $\Sigma : \Sigma_{i,j} = \rho^{1\{i\neq j\}}$;

Type 3: Toeplitz Matrix: $\Sigma : \Sigma_{i,j} = \rho^{|i-j|}$;

For each type of correlation matrix, we considered following measurement error scenarios:

Scenario 1. Missing Value: we randomly replaced a certain proportion (10%, 30%, 50%) of data entries with 0;

Scenario 2. Perturbation: standard Gaussian noise was randomly added to a certain proportion(10%, 30%, 50%) of the original data.

Denote the contaminated training dataset as W_{train}, testing dataset as W_{test}, and the original training dataset as Z_{train}, testing dataset as Z_{test}. In measurement error experiments, classifiers were trained on (W_{train}, Y_{train}) and performance measures (1-6) were tested on (W_{test}, Y_{test}) and (Z_{test}, Y_{test}) respectively. The corresponding

Classification Error and Deviance measures are denoted as $CE(Z_{test})$, $CE(W_{test})$, $Deviance(Z_{test})$ and $Deviance(W_{test})$ in result table.

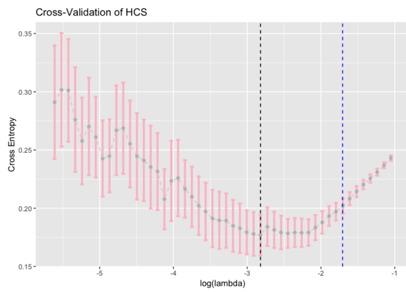
For regularization in the parameter selection, we sampled the tuning parameter from a grid, and conducted a 5-fold cross validation on the training set to select the best tuning parameter. The effect of regularization parameters on β and cross validation are illustrated in Experiment 1.

Experiment 1: Regularization Approach on Different Level of Perturbation

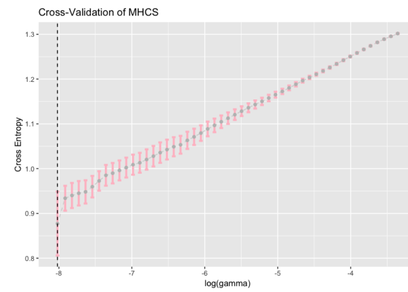
Following the process of general simulation setup, we generated Type 1 data with different levels of perturbation (10%, 30%, 50%). Figure 3 summarizes the 5-fold cross validation error varying with tuning parameter from 0% to 30% of perturbation error. Graphs in the left column illustrate cross validation error of HCS varying along with λ. The black dash reference line on the left column indicates the optimal λ, denoted as λ^*, which minimizes the cross-validation error. The blue reference line denotes λ^* plus standard error. For tuning process of MHCS, fixed the $\lambda = \lambda^*$, where λ^* was attained from HCS cross-validation, then conducted a 5-fold cross validation on a sequence of γ. The black dash reference line on the right column indicates the optimal $\gamma = \gamma^*$ which minimizes the cross-validation error. Figure 3 presents that, as perturbation level increasing, the reference lines of λ^* and γ^* slide to the right, which implies regularization power intenses as measurement error aggravates. In the right column, in order to illustrate how cross validation error and tuning parameter γ differ as measurement error increases, graphs (b), (d), (f), (h) starts with ($\lambda = \lambda^*$, $\gamma = 0$), which is the solution to HCS, the corresponding cross validation error is plotted at the most beginning of x-axis (e^{-7}) instead of $\gamma = 0$, since the x-axis is log scale. From (b), (d) in Figure 3, it is seen that, for data without measurement error or with low perturbation level (10%), $\gamma^* = 0$, which implies the tuning parameter λ is competent in these situations to some extent. However, as the measurement error aggravates in (f) and (g), γ^* increases in response. This result strongly supports our theory introduced in chapter 3.

In Figure 4, we traced β route varying with regularization parameters, where the colored lines indicate β_j for $j \in S$ ($S = \{j : \beta_j^* \neq 0\}$), while for $j \in S^c$, β_j lines in light grey. In our experiments, only the first ten elements are colored. The figures on the left column trace β route move along with λ. The black dash line denotes the position where λ^* is. The figures on the right column trace β routes regard to γ tuning process with fixed λ^*. It is seen that as the regularization parameters (λ, γ) increase, the magnitude of all β_j shrinks to some degree. As the perturbation level increased, the route of β exhibits the longer trumble effect.

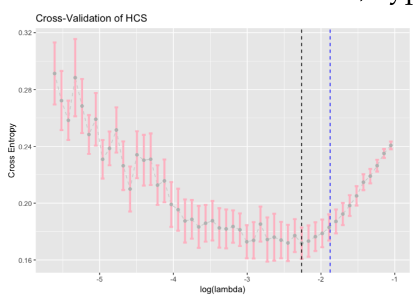
For graphs in the right column, the black dash reference line denotes $\lambda = \lambda^*$, $\gamma = 0$; while blue dash line denotes $\lambda = \lambda^*$, $\gamma = \gamma^*$. The figures show how both HCS and MHCS demonstrate the capability in feature selection. However, MHCS selected less features in a more effective yet critical way in our experiments, especially when measurement error aggravates.



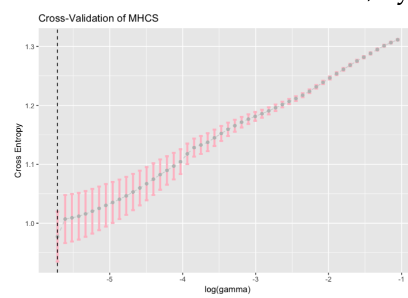
(a) Without measurement error, Type 1



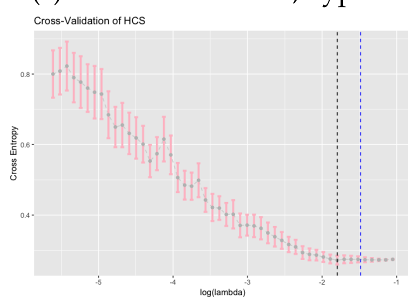
(b) Without measurement error, Type 1



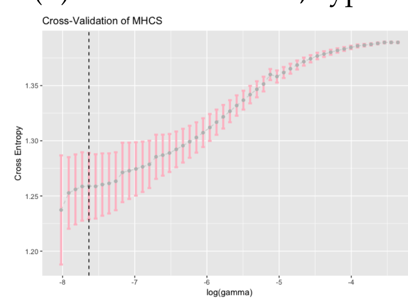
(c) 10% Perturbation, Type 1



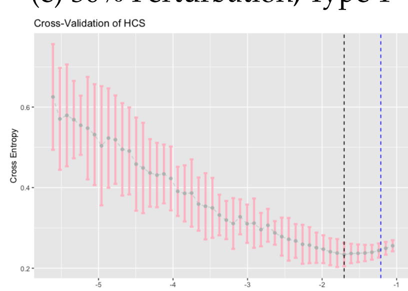
(d) 10% Perturbation, Type 1



(e) 30% Perturbation, Type 1



(f) 30% Perturbation, Type 1



(g) 50% Perturbation, Type 1

(h) 50% Perturbation, Type 1

Figure 3: Tuning Parameter Selection Illustration: Cross Validation Error with different level of Perturbation, Type 1

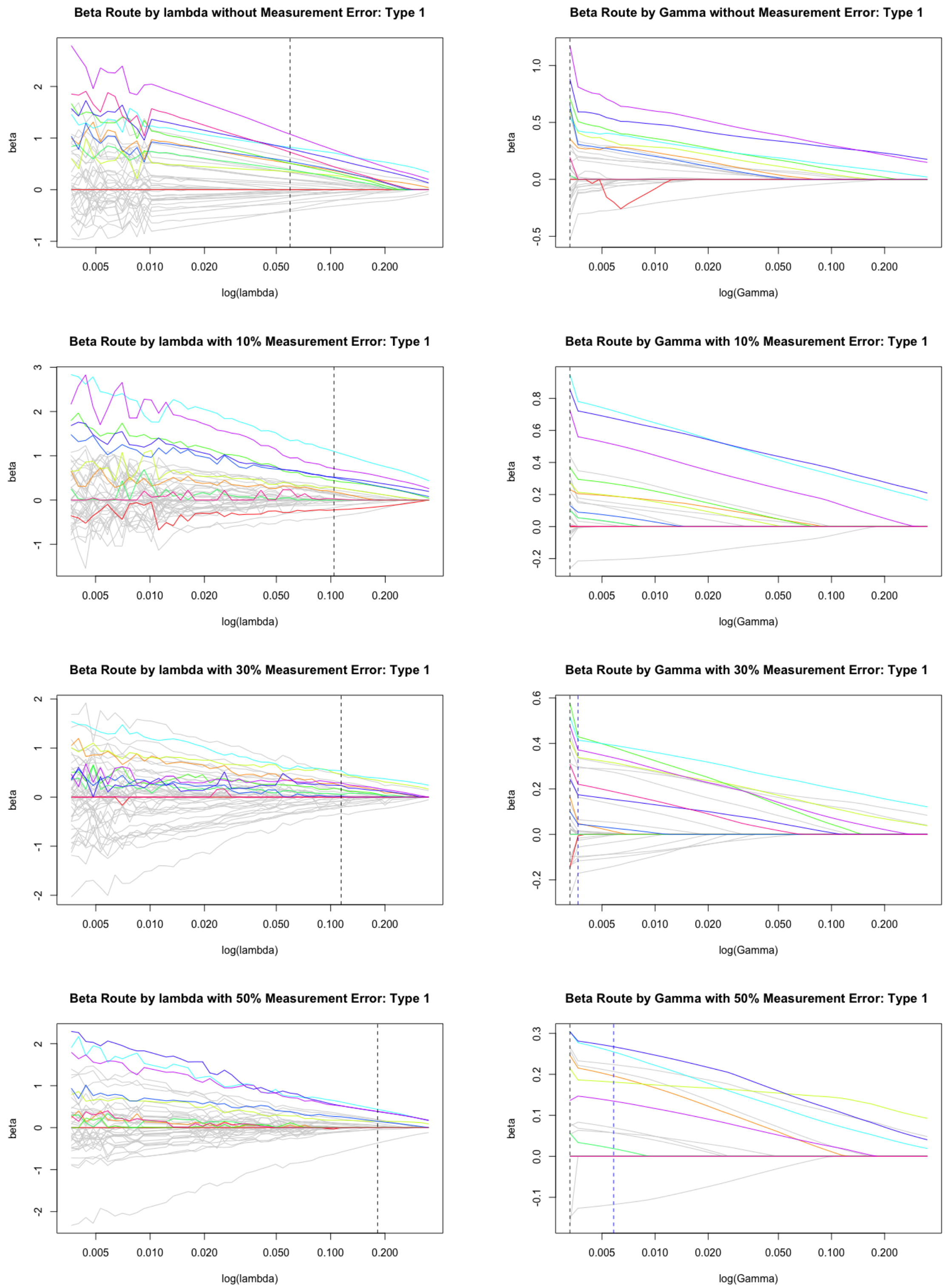


Figure 4: β route with different level of Perturbation

Experiment 2: Type 1 data in different scenarios

In this experiment, we compared performances (1-6) of four regularized classifiers on the dataset generated from Type 1 correlation matrix, i.e., identity correlation matrix. Table 1- Table 7 summarize the results of 7 different scenarios respectively. From Table 1 , 2 and 5, where no measurement error or mild measurement error(10%) exists, LASSO, HCS and MHCS perform comparably in terms of prediction error (CE, Deviance) and parameter error (L1 and L2).

Ridge regression has fair capacity in capturing the classification error, however, it is not designed for sparse setting, which led to the large l1 norm and failed to conduct the feature selection. With respect to the features selection, LASSO tends to reduce the False Positive number at the cost of increasing False Negative number; while HCS acts on the opposite, MHCS plays a moderate role in between. As the measurement error aggravates, which is shown in Table 3, Table 4 Table 6, Table 7, all the performance measures worsen to some degree. However, it is seen that MHCS performs relatively more robustly against measurement error than other classifiers. To see this, the margins of performance measure (CE, Deviance) between MHCS and LASSO, MHCS and HCS increase while measurement error rises.

Table 1: Result without Measurement Error, Type 1

	LASSO		RIDGE		HCS		MHCS	
CE	0.34	(0.05)	0.38	(0.03)	0.32	(0.04)	0.31	(0.05)
$Deviance$	1.24	(0.13)	1.33	(0.02)	1.34	(0.24)	1.24	(0.2)
L_1	3.45	(0.55)	11.12	(0.36)	4.52	(0.63)	4.14	(0.58)
$L2$	0.83	(0.2)	1.09	(0.08)	0.75	(0.17)	0.75	(0.17)
FN	0.3	(0.19)	0	(0)	0.13	(0.13)	0.14	(0.13)
FP	0.08	(0.04)	1	(0)	0.2	(0.03)	0.17	(0.03)

Table 2: Result of 10% Missing Value, Type 1

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.35	(0.06)	0.4	(0.03)	0.34	(0.06)	0.34	(0.04)
$Deviance(Z_{test})$	1.25	(0.1)	1.34	(0.02)	1.44	(0.22)	1.32	(0.18)
$CE(W_{test})$	0.37	(0.05)	0.41	(0.05)	0.34	(0.06)	0.34	(0.05)
$Deviance(W_{test})$	1.29	(0.11)	1.34	(0.02)	1.44	(0.23)	1.33	(0.19)
L_1	3.68	(0.53)	11.33	(0.33)	4.87	(0.75)	4.57	(0.65)
L_2	0.88	(0.14)	1.13	(0.08)	0.84	(0.2)	0.84	(0.2)
FP	0.34	(0.11)	0	(0)	0.18	(0.15)	0.19	(0.14)
FN	0.08	(0.04)	1	(0)	0.22	(0.03)	0.18	(0.02)

Table 3: Result of 30% Missing Value, Type 1

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.37	(0.07)	0.4	(0.05)	0.37	(0.05)	0.35	(0.05)
$Deviance(Z_{test})$	1.38	(0.1)	1.33	(0.03)	1.85	(0.36)	1.42	(0.15)
$CE(W_{test})$	0.45	(0.05)	0.41	(0.04)	0.41	(0.04)	0.41	(0.04)
$Deviance(W_{test})$	1.43	(0.11)	1.35	(0.02)	1.81	(0.23)	1.43	(0.1)
L_1	4.25	(0.71)	11.72	(0.46)	4.37	(0.73)	4.25	(0.62)
$L2$	1.18	(0.19)	1.23	(0.09)	1.21	(0.2)	1.21	(0.2)
FP	0.59	(0.23)	0	(0)	0.29	(0.15)	0.35	(0.12)
FN	0.07	(0.05)	1	(0)	0.25	(0.02)	0.16	(0.02)

Table 4: Result of 50% Missing Value, Type 1

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.41	(0.05)	0.41	(0.04)	0.40	(0.06)	0.37	(0.05)
$Deviance(Z_{test})$	1.35	(0.08)	1.35	(0.03)	1.94	(0.36)	1.42	(0.18)
$CE(W_{test})$	0.42	(0.05)	0.44	(0.04)	0.42	(0.04)	0.41	(0.04)
$Deviance(W_{test})$	1.43	(0.16)	1.36	(0.02)	1.76	(0.23)	1.39	(0.08)
L_1	4.55	(0.62)	10.36	(3.81)	5.57	(0.43)	4.33	(0.52)
$L2$	1.22	(0.25)	1.25	(0.16)	1.21	(0.16)	1.21	(0.16)
FP	0.62	(0.2)	0	(0)	0.31	(0.1)	0.37	(0.07)
FN	0.05	(0.05)	1	(0)	0.27	(0.02)	0.16	(0.01)

Table 5: Result of 10% Measurement Error, Type 1

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.34	(0.04)	0.39	(0.03)	0.35	(0.05)	0.34	(0.04)
$Deviance(Z_{test})$	1.28	(0.11)	1.34	(0.02)	1.44	(0.2)	1.28	(0.14)
$CE(W_{test})$	0.36	(0.03)	0.38	(0.03)	0.36	(0.04)	0.36	(0.05)
$Deviance(W_{test})$	1.31	(0.12)	1.34	(0.02)	1.51	(0.19)	1.31	(0.14)
L_1	3.73	(0.74)	11.27	(0.46)	4.93	(0.89)	3.98	(0.73)
$L2$	0.88	(0.22)	1.12	(0.1)	0.87	(0.23)	0.87	(0.23)
FP	0.29	(0.17)	0	(0)	0.18	(0.15)	0.25	(0.12)
FN	0.09	(0.04)	1	(0)	0.21	(0.03)	0.12	(0.03)

Table 6: Result of 30% Measurement Error, Type 1

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.38	(0.06)	0.38	(0.04)	0.37	(0.06)	0.35	(0.06)
$Deviance(Z_{test})$	1.3	(0.08)	1.34	(0.03)	1.43	(0.18)	1.26	(0.12)
$CE(W_{test})$	0.38	(0.07)	0.41	(0.05)	0.38	(0.05)	0.36	(0.05)
$Deviance(W_{test})$	1.36	(0.11)	1.35	(0.02)	1.64	(0.28)	1.35	(0.17)
L_1	4.01	(0.71)	11.58	(0.34)	5.36	(0.53)	4.48	(0.52)
$L2$	1.01	(0.23)	1.2	(0.08)	0.95	(0.19)	0.95	(0.19)
FP	0.41	(0.25)	0	(0)	0.23	(0.11)	0.27	(0.08)
FN	0.09	(0.07)	1	(0)	0.23	(0.03)	0.14	(0.02)

Table 7: Result of 50% Measurement Error, Type 1

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.40	(0.05)	0.4	(0.05)	0.40	(0.05)	0.37	(0.06)
$Deviance(Z_{test})$	1.39	(0.18)	1.35	(0.03)	1.56	(0.2)	1.35	(0.11)
$CE(W_{test})$	0.42	(0.04)	0.43	(0.04)	0.41	(0.03)	0.42	(0.03)
$Deviance(W_{test})$	1.5	(0.21)	1.36	(0.03)	1.85	(0.2)	1.47	(0.12)
L_1	4.56	(0.75)	10.83	(2.73)	5.92	(0.67)	4.91	(0.62)
$L2$	1.08	(0.21)	1.19	(0.1)	1.14	(0.15)	1.14	(0.15)
FP	0.48	(0.25)	0	(0)	0.33	(0.13)	0.38	(0.17)
FN	0.12	(0.07)	1	(0)	0.23	(0.03)	0.14	(0.03)

Experiment 3: Type 2 data with different scenarios

In this experiment, we compared performance measures (1-6) of four regularized classifiers on the dataset generated from Type 2 correlation matrix, i.e., equal correlation matrix with $\rho = 0.5$. Table 8-Table 14 summarize results of 7 different scenarios, respectively.

The result presents that classification error (i.e., $CE(Z_{test})$, Deviance) from all four classifiers are close to each other among all scenarios. In terms of classification error (CE), Ridge regression surpasses other classifiers with a small lead, however, the performance of L1 and feature selection (FN, PN) in this dataset failed to exceed l_1 regularization. With respect to the feature selection, the performance of LASSO, HCS and MHCS are consistently close to each other in every setting. The corresponding results exhibit that False Negative ratio, among all the l_1 regularized classifiers (LASSO, HCS and MHCS), exceeds 50%, and False Positive is relatively high compared to Type 1 and Type 3 dataset, each of which goes beyond 11%. As the measurement error increases, parameter error (L1, L2) and feature selection measure (FP, FN) worsen to some extent, however, the classification risk appears to consistently drift around 11% to 13%. The reason is Type 2 data is generated with the equal correlation matrix, which all features are correlated to each other. With this inherent structure, the l_1 based classifier is more robust with respect to prediction error as the measurement error increases, though pays the price of increment of parameter error.

Table 8: Result without Measurement Error, Type 2

	LASSO		RIDGE		HCS		MHCS	
CE	0.12	(0.04)	0.11	(0.03)	0.12	(0.04)	0.12	(0.03)
$Deviance$	0.56	(0.08)	0.64	(0.05)	0.56	(0.12)	0.56	(0.1)
L_1	5.08	(0.54)	14.23	(0.45)	5.5	(0.3)	5.43	(0.31)
$L2$	1.31	(0.17)	1.41	(0.05)	1.4	(0.15)	1.4	(0.15)
FP	0.56	(0.08)	0	(0)	0.53	(0.12)	0.55	(0.13)
FN	0.1	(0.02)	1	(0)	0.12	(0.02)	0.11	(0.02)

Table 9: Result of 10% Missing Value, Type 2

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.12	(0.04)	0.11	(0.03)	0.12	(0.04)	0.12	(0.03)
$Deviance(Z_{test})$	0.56	(0.12)	0.62	(0.05)	0.59	(0.14)	0.58	(0.13)
$CE(W_{test})$	0.13	(0.04)	0.11	(0.03)	0.13	(0.04)	0.12	(0.03)
$Deviance(W_{test})$	0.59	(0.15)	0.65	(0.05)	0.59	(0.14)	0.59	(0.13)
L_1	5.25	(0.42)	14.24	(0.44)	5.49	(0.67)	5.42	(0.58)
$L2$	1.38	(0.18)	1.42	(0.06)	1.41	(0.23)	1.41	(0.23)
FP	0.57	(0.11)	0	(0)	0.56	(0.12)	0.57	(0.08)
FN	0.11	(0.02)	1	(0)	0.11	(0.02)	0.11	(0.02)

Table 10: Result of 30% Missing Value, Type 2

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.12	(0.03)	0.11	(0.03)	0.11	(0.03)	0.11	(0.03)
$Deviance(Z_{test})$	0.54	(0.12)	0.57	(0.06)	0.57	(0.14)	0.57	(0.13)
$CE(W_{test})$	0.13	(0.04)	0.11	(0.03)	0.13	(0.04)	0.12	(0.03)
$Deviance(W_{test})$	0.59	(0.16)	0.67	(0.05)	0.59	(0.12)	0.58	(0.1)
L_1	5.66	(0.5)	14.3	(0.41)	5.82	(0.37)	5.88	(0.44)
$L2$	1.43	(0.16)	1.44	(0.05)	1.45	(0.13)	1.45	(0.13)
FP	0.53	(0.09)	0	(0)	0.52	(0.1)	0.6	(0.08)
FN	0.12	(0.01)	1	(0)	0.13	(0.02)	0.13	(0.02)

Table 11: Result of 50% Missing Value, Type 2

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.13	(0.04)	0.11	(0.03)	0.12	(0.03)	0.12	(0.04)
$Deviance(Z_{test})$	0.59	(0.18)	0.52	(0.07)	0.57	(0.24)	0.59	(0.23)
$CE(W_{test})$	0.15	(0.05)	0.12	(0.04)	0.15	(0.04)	0.16	(0.02)
$Deviance(W_{test})$	0.74	(0.23)	0.7	(0.04)	0.66	(0.15)	0.67	(0.12)
L_1	6.18	(0.81)	14.28	(0.39)	6.11	(0.58)	6.35	(0.66)
$L2$	1.57	(0.25)	1.48	(0.06)	1.56	(0.22)	1.56	(0.22)
FP	0.66	(0.13)	0	(0)	0.62	(0.14)	0.65	(0.16)
FN	0.15	(0.02)	1	(0)	0.14	(0.02)	0.14	(0.02)

Table 12: Result of 10% Measurement Error, Type 2

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.12	(0.03)	0.11	(0.03)	0.13	(0.03)	0.13	(0.03)
$Deviance(Z_{test})$	0.57	(0.1)	0.65	(0.05)	0.56	(0.12)	0.56	(0.12)
$CE(W_{test})$	0.12	(0.04)	0.11	(0.03)	0.12	(0.03)	0.12	(0.03)
$Deviance(W_{test})$	0.59	(0.11)	0.65	(0.05)	0.58	(0.13)	0.58	(0.13)
L_1	5.38	(0.5)	14.19	(0.41)	5.78	(0.54)	5.74	(0.57)
$L2$	1.43	(0.22)	1.42	(0.05)	1.46	(0.21)	1.46	(0.21)
FP	0.63	(0.17)	0	(0)	0.61	(0.15)	0.63	(0.13)
FN	0.11	(0.01)	1	(0)	0.13	(0.01)	0.13	(0.02)

Table 13: Result of 30% Measurement Error, Type 2

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.12	(0.02)	0.1	(0.03)	0.12	(0.02)	0.12	(0.03)
$Deviance(Z_{test})$	0.58	(0.1)	0.66	(0.05)	0.59	(0.11)	0.57	(0.12)
$CE(W_{test})$	0.14	(0.03)	0.11	(0.03)	0.15	(0.02)	0.14	(0.02)
$Deviance(W_{test})$	0.62	(0.1)	0.67	(0.05)	0.65	(0.13)	0.63	(0.13)
L_1	5.8	(0.94)	14.26	(0.39)	6.21	(0.76)	6.07	(0.76)
$L2$	1.54	(0.27)	1.45	(0.06)	1.6	(0.25)	1.6	(0.25)
FP	0.61	(0.14)	0	(0)	0.64	(0.2)	0.63	(0.19)
FN	0.11	(0.02)	1	(0)	0.13	(0.02)	0.13	(0.02)

Table 14: Result of 50% Measurement Error, Type 2

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.13	(0.03)	0.11	(0.03)	0.13	(0.04)	0.12	(0.02)
$Deviance(Z_{test})$	0.59	(0.07)	0.68	(0.04)	0.56	(0.1)	0.55	(0.09)
$CE(W_{test})$	0.15	(0.03)	0.11	(0.03)	0.15	(0.04)	0.13	(0.03)
$Deviance(W_{test})$	0.67	(0.09)	0.69	(0.04)	0.68	(0.12)	0.66	(0.11)
L_1	6.23	(0.64)	14.21	(0.29)	6.55	(0.76)	6.52	(0.8)
$L2$	1.69	(0.19)	1.46	(0.05)	1.67	(0.2)	1.67	(0.2)
FP	0.7	(0.17)	0	(0)	0.72	(0.13)	0.72	(0.16)
FN	0.12	(0.02)	1	(0)	0.14	(0.02)	0.14	(0.02)

Experiment 4: Type 3 data with different scenarios

In this experiment, we compared performance measures (1-6) of four regularized classifiers on the dataset generated from Type 3 correlation matrix, i.e., Toeplitz correlation matrix with $\rho = 0.5$. Table 15 - Table 21 summarize results of 7 different scenarios respectively.

The results present that LASSO, HCS and MHCS perform comparably in terms of prediction error (CE, Deviance) in all settings, though MHCS appears to have a small lead when the measurement error is imposed.

With respect to parameter error, MHCS has the lowest L1 error, while LASSO has the lowest L2 error. The margin of L1 between MHCS and LASSO decreases while the margin of L2 between MHCS and LASSO increases as the measurement error increases. As we discuss before, HCS is a specific solution to MHCS when $\gamma = 0$. With the appropriate γ, MHCS is apt to approach the solution in the direction of L1 decaying, which dramatically improves the performance of FP with the trade off in a small increment in FN, especially in the case of measurement error presents. To see this, compare FN and FP of MHCS with HCS in Table 17, Table 18, where the missing value reaches 30% and 50% respectively, MHCS amends to reduce the FN by 13% by only bringing up 1% increment to FP. As the measurement error leverages, all the performance margins between MHCS and HCS increase, which suggests that MHCS performs more robust against measurement error.

Table 15: Result without Measurement Error, Type 3

	LASSO		RIDGE		HCS		MHCS	
CE	0.17	(0.04)	0.24	(0.04)	0.18	(0.03)	0.17	(0.04)
$Deviance$	0.78	(0.11)	1.21	(0.03)	0.85	(0.19)	0.79	(0.08)
L_1	2.46	(0.29)	9.09	(0.34)	3.49	(0.27)	2.26	(0.32)
$L2$	0.48	(0.1)	0.67	(0.04)	0.55	(0.07)	0.55	(0.07)
FP	0.13	(0.08)	0	(0)	0.12	(0.1)	0.15	(0.08)
FN	0.06	(0.02)	1	(0)	0.16	(0.03)	0.04	(0.01)

Table 16: Result of 10% Missing Value, Type 3

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.19	(0.03)	0.24	(0.05)	0.2	(0.05)	0.19	(0.04)
$Deviance(Z_{test})$	0.81	(0.11)	1.21	(0.03)	0.93	(0.2)	0.8	(0.08)
$CE(W_{test})$	0.2	(0.03)	0.25	(0.05)	0.21	(0.05)	0.19	(0.04)
$Deviance(W_{test})$	0.84	(0.1)	1.22	(0.03)	0.95	(0.2)	0.84	(0.09)
L_1	2.87	(0.76)	9.38	(0.33)	3.84	(0.47)	2.42	(0.42)
$L2$	0.54	(0.14)	0.72	(0.05)	0.63	(0.12)	0.63	(0.12)
FP	0.16	(0.11)	0	(0)	0.14	(0.11)	0.16	(0.11)
FN	0.09	(0.05)	1	(0)	0.17	(0.02)	0.05	(0.02)

Table 17: Result of 30% Missing Value, Type 3

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.21	(0.03)	0.27	(0.05)	0.23	(0.03)	0.21	(0.04)
$Deviance(Z_{test})$	0.9	(0.1)	1.21	(0.03)	1.22	(0.27)	0.85	(0.1)
$CE(W_{test})$	0.25	(0.04)	0.29	(0.04)	0.29	(0.05)	0.22	(0.05)
$Deviance(W_{test})$	0.99	(0.12)	1.25	(0.03)	1.34	(0.23)	0.97	(0.09)
L_1	2.96	(0.68)	10.17	(0.35)	4.79	(0.58)	2.81	(0.51)
$L2$	0.58	(0.13)	0.84	(0.06)	0.78	(0.14)	0.78	(0.14)
FP	0.17	(0.13)	0	(0)	0.12	(0.06)	0.13	(0.09)
FN	0.08	(0.06)	1	(0)	0.21	(0.03)	0.08	(0.01)

Table 18: Result of 50% Missing Value, Type 3

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.19	(0.04)	0.26	(0.07)	0.23	(0.07)	0.2	(0.05)
$Deviance(Z_{test})$	0.85	(0.12)	1.19	(0.06)	1.35	(0.54)	0.83	(0.16)
$CE(W_{test})$	0.24	(0.05)	0.33	(0.05)	0.3	(0.04)	0.27	(0.05)
$Deviance(W_{test})$	1.02	(0.18)	1.29	(0.03)	1.43	(0.26)	1.07	(0.09)
L_1	3.23	(0.73)	10.61	(0.32)	5.26	(0.72)	3.23	(0.62)
$L2$	0.65	(0.13)	0.96	(0.09)	0.88	(0.17)	0.88	(0.17)
FP	0.2	(0.12)	0	(0)	0.14	(0.08)	0.15	(0.1)
FN	0.09	(0.05)	1	(0)	0.23	(0.02)	0.1	(0.02)

Table 19: Result of 10% Measurement Error, Type 3

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.17	(0.04)	0.23	(0.06)	0.18	(0.04)	0.17	(0.04)
$Deviance(Z_{test})$	0.82	(0.12)	1.22	(0.03)	0.86	(0.19)	0.82	(0.08)
$CE(W_{test})$	0.19	(0.04)	0.25	(0.06)	0.19	(0.04)	0.18	(0.04)
$Deviance(W_{test})$	0.85	(0.12)	1.23	(0.03)	0.95	(0.18)	0.85	(0.08)
L_1	2.67	(0.32)	9.27	(0.29)	3.9	(0.57)	2.47	(0.41)
$L2$	0.53	(0.12)	0.71	(0.04)	0.63	(0.13)	0.63	(0.13)
FP	0.17	(0.08)	0	(0)	0.16	(0.11)	0.18	(0.1)
FN	0.07	(0.02)	1	(0)	0.17	(0.03)	0.05	(0.02)

Table 20: Result of 30% Measurement Error, Type 3

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.2	(0.05)	0.24	(0.06)	0.22	(0.07)	0.19	(0.06)
$Deviance(Z_{test})$	0.89	(0.13)	1.24	(0.03)	1	(0.26)	0.87	(0.12)
$CE(W_{test})$	0.24	(0.04)	0.26	(0.04)	0.26	(0.06)	0.23	(0.04)
$Deviance(W_{test})$	0.94	(0.1)	1.24	(0.03)	1.2	(0.23)	0.93	(0.1)
L_1	2.75	(0.57)	9.78	(0.42)	4.57	(0.81)	2.82	(0.62)
$L2$	0.56	(0.17)	0.79	(0.07)	0.73	(0.18)	0.73	(0.18)
FP	0.19	(0.14)	0	(0)	0.13	(0.09)	0.14	(0.11)
FN	0.06	(0.03)	1	(0)	0.2	(0.03)	0.07	(0.02)

Table 21: Result of 50% Measurement Error, Type 3

	LASSO		RIDGE		HCS		MHCS	
$CE(Z_{test})$	0.19	(0.04)	0.27	(0.04)	0.21	(0.05)	0.2	(0.04)
$Deviance(Z_{test})$	0.91	(0.09)	1.26	(0.02)	0.96	(0.21)	0.91	(0.06)
$CE(W_{test})$	0.25	(0.04)	0.31	(0.06)	0.27	(0.04)	0.24	(0.05)
$Deviance(W_{test})$	1.03	(0.12)	1.27	(0.03)	1.29	(0.2)	1.01	(0.09)
L_1	2.85	(0.59)	10	(0.4)	4.36	(0.46)	2.75	(0.45)
$L2$	0.58	(0.13)	0.82	(0.07)	0.73	(0.14)	0.73	(0.14)
FP	0.17	(0.13)	0	(0)	0.11	(0.07)	0.17	(0.12)
FN	0.08	(0.04)	1	(0)	0.2	(0.03)	0.07	(0.02)

CHAPTER 5: REAL DATA ANALYSIS

Real Data Example 1: Sentiment Analysis of IMDb Movie Review

This example presents the proposed techniques (HCS, MHCS) to perform sentiment analysis in IMDB movie reviews, we compared the results with competing methods: penalized logistic regression(PLR) and support vector machine (SVM).

Experiment setup

To begin the experiment, we downloaded a sample data set developed by [27], the training data set contains 2000 movie reviews from IMDb, where 1000 reviews are labeled as positive (1), 1000 reviews are labeled as negative (0); the testing data set contained 1000 reviews, with 500 reviews are labeled as negative, 500 reviews are labeled as positive. We processed the following techniques for text preparation: removed all the links and punctuations in text, converted all words into lower case, and tokenize the text into a sequence of single word (unigrams).

Bag of Words representation was applied to this example, where each unique word served as a feature. We also applied a rough dimension reduction technique by simply removing stopword according to NLTK stopword list and dropping features which appear less than 10 times over all samples, which resulted in 12,932 dimensions in total number of features.

Denote $n_w\{i, j\}$ as the number of occurrences of word j in review i and $n_d\{i\}$

as the total number of words in review i. then we denote the value of feature j in review i as:

$$z_{ij} = \frac{n_w\{i,j\}}{n_d\{i\}}$$

$y_i = 1$ if the review is positive, $y_i = 0$ if the review is negative.

The full data set was randomly separated to 70% for training, the remaining 30% was for testing. Next we fit the different classifiers on training set, and recorded the numbers of non-zero coefficients ($\|\hat{\beta}\|_0$) and testing classification error (CE). The tuning parameter λ in HCS was selected by 5-fold cross validation on training set. For MHCS, λ and γ were sampled from a grid search, then similar to HCS, we selected the one that produced best result by 5-fold cross validation on training set. This process was repeated 50 times, and the mean value of $\|\hat{\beta}\|_0$ and CE are summarized in Table 22:

The results show that among all classifiers, SVM achieves lowest mean classification error, which is 0.1. HCS and MHCS perform comparatively with the mean classification error being 0.12 and 0.11 respectively. Although SVM performs slightly better in terms of mean classification error, from Table 22, it is seen MHCS performs more stable than SVM with standard error 0.01 while for SVM is 0.08.

In the aspect to the capability of feature selection, the proposed classifiers show prominent advantage of l_1 regularization in high dimensional setting. According to the number of non-zero, HCS and MHCS select 82 and 47 features among 12,932 features, while other classifiers do not present the power of feature selection.

Table 22: Performance measures of IMDB movie review

	HCS		MHCS		SVM		PLR	
CE	0.12	(0.02)	0.11	(0.01)	0.1	(0.08)	0.13	(0.03)
$\|\|\hat{\beta}\|\|_0$	82	(1.5)	47	(1.1)	12,932	(0)	12,932	(0)

We exhibit the features with top 10 positive and negative coefficients selected by MHCS in a test trail. According to the result shows in Table 23 , the positive and negative terms demonstrate a close match to a human's emotional sentiment.

Table 23: Demo: Top 10 Positive and Negative Features Selected by MHCS

coef	word	coef	word
5.253178e-03	wonderful	4.628492e-03	poor
4.500304e-03	favorite	-2.106760e-03	worst
4.462270e-03	loved	-1.062546e-03	disappointing
4.112702e-03	excellent	-5.603305e-04	terrible
2.744693e-03	amazing	-5.047209e-04	waste
8.060263e-04	worth	-2.728851e-04	awful
7.088666e-04	enjoy	-2.314834e-04	boring
6.042885e-04	perfect	-9.118960e-05	save
4.579394e-04	best	-2.501499e-05	horrible
4.315558e-04	holiday	-5.956561e-06	disappointment

Missing Value Scenarios

In order to investigate the proposed classifiers' capability of dealing with measurement error, we randomly deleted a certain proportion of word sequence which generated the original data set Z. Denote this new data set as W, we then sampled 70% data from W as training set, denote as W_{train}. The training process was the same as previous example. We then applied the fitted model and tuning parameter selected by 5-fold cross validation to conduct prediction test on remaining testing data W_{test}, we tested on Z which had the same index as W_{test} as well, denoted as Z_{test}. Repeated this process 50 times, the mean and standard error are recorded in Table 24: the result presents that HCS, MHCS and SVM perform better than PLR over all settings, although the standard error of SVM is higher than PLR. As the missing proportion increases, the performance of all the classifiers worsen to some degree. Nevertheless, the result shows that MHCS performs better than other classifiers, demonstrates its robustness against missing value.

Table 24: Performance measures of IMDb movie review Missing Value Scenario

		HCS		MHCS		SVM		PLR	
		mean	sd	mean	sd	mean	sd	mean	sd
10%	W_{test}	0.12	(0.02)	0.11	(0.02)	0.11	(0.07)	0.15	(0.02)
	Z_{test}	0.13	(0.02)	0.12	(0.02)	0.10	(0.11)	0.14	(0.02)
30%	W_{test}	0.15	(0.03)	0.12	(0.01)	0.14	(0.1)	0.16	(0.02)
	Z_{test}	0.15	(0.02)	0.13	(0.02)	0.13	(0.09)	0.16	(0.01)
50%	W_{test}	0.16	(0.02)	0.15	(0.02)	0.17	(0.1)	0.18	(0.01)
	Z_{test}	0.15	(0.02)	0.14	(0.02)	0.17	(0.7)	0.2	(0.01)

Real Data Example 2: Cat vs. Dog Image Recognition

Image data is remarkably high dimensional and frequently process with noise. In this example, we used a small sample of labeled images of a dog and a cat from kaggle [26]. Our aim is to build a classifier automatically distinguish whether images contain either a dog or a cat. The original data set contains 25,000 images of dogs and cats in training fold, and 12,500 images in test folds. In order to demonstrate proposed classifier in $d > n$ setting, we only use a small sample in this example. We use 1,200 images from training fold, 600 are labeled as cat and 600 are labeled as dog, then randomly split the data to 1000 images for training, 200 images for testing. The main idea is input the image data ($3 \times 224 \times 224$) to a pre-trained network (VGG-16 [28]) for feature extraction, then foward the extracted features to the linear classifiers concerned. VGG-16[28] is a CNN network pre-trained on ImageNet data set, its architecture is illustrated in Figure 5. ImageNet[44] is large data set contains

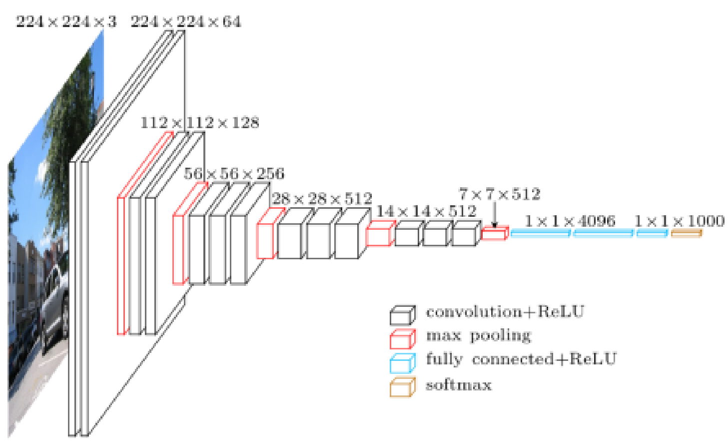


Figure 5: VGG-16 Architature [28]

1.2 M labeled images from 1000 categories.

In image recognition, pre-trained networks demonstrates strong capability to conduct new deep learning tasks via transfer learning. Besides computationally efficiency due to pre-trained weights, the first few layers of CNN in image recognition training usually capture universal features such as lines, edges, curves that related to other task.

To conduct the feature extraction, we freezed all weights, utilizing the entire network as feature extractor, then forwarded the extracted features to the HCS, MHCS, SVM and PLR.

The first block in Table 25 illustrates top 5 features extracted by VGG-16 with the a sample cat image 'original Murphy' (Figure 6).

Perturbation Scenarios:

In order to investigate proposed classifiers' capability to cope with noise contaminated data, we added Gaussian noise to image data set on purpose. Figure 6 elaborates the effect with different proportion of noise, the corresponding top 5 features extracted by VGG-16 are listed in Table 25.

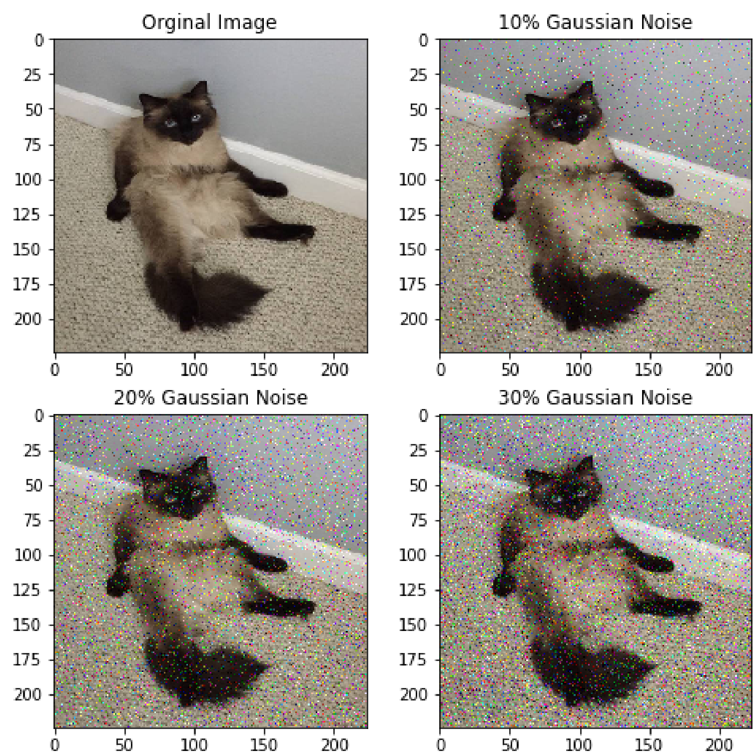


Figure 6: Murphy with different proportion Gaussian Noise

Table 25: Top 5 Features Extracted by VGG-16

Orignal Murphy		10% Noise	
Feature	Value	Feature	Value
Siamese cat	0.998657	Siamese cat	0.972666
paper towel	0.000367	cairn	0.014863
tub	0.000216	West Highland white terrier	0.001727
toilet tissue	0.000196	Scotch terrier	0.001670
lynx	0.000178	paper towel	0.001231
20% Noise		30% Noise	
Feature	Value	Feature	Value
Siamese cat	0.982971	West Highland white terrier	0.505062
cairn	0.004131	Scotch terrier	0.141582
West Highland white terrier	0.002864	cairn	0.134488
Scotch terrier	0.000798	Siamese cat	0.073840
giant panda	0.000571	Norwich terrier	0.037540

In the second step, we trained the classifiers on extracted features, this process is the same as previous examples. Table 26 and Table 27 summarize the results of performance. It is seen that MHCS performs best among four classifiers, with lowest classification error (19%) and smallest number of selected features (17 out of 1,000). HCS also demonstrates the capability of feature selections which the number is 32 out of 1,000, while SVM and PLR selected all the features.

Table 26: Performance measures of Cat vs Dog Image Recognition

	HCS	MHCS	SVM	PLR
CE	0.21	0.19	0.2	0.23
$\|\|\hat{\beta}\|\|_0$	32	17	1,000	1,000

From Table 27, it's shown that HCS and MHCS perform more stable than SVM and PLR as the proportion of perturbation increases. In aspects of prediction error and robustness against noise, MHCS surpasses other classifiers.

Table 27: Performance Measures of Cat vs Dog with Noise

		HCS		MHCS		SVM		PLR	
		Mean	SE	Mean	SE	Mean	SE	Mean	SE
10%	Z_{test}	22.8	0.2	20.5	0.1	22.2	0.3	25.6	0.1
	W_{test}	22.3	0.2	21.4	0.1	23.1	0.3	25.1	0.1
20%	Z_{test}	23.5	0.1	21.2	0.2	23.2	0.3	25.8	0.2
	W_{test}	22.8	0.2	21.9	0.2	23.5	0.3	25.3	0.1
30%	Z_{test}	24.1	0.1	22.4	0.1	25.3	0.3	26.7	0.1
	W_{test}	24.5	0.1	22.7	0.1	25.4	0.2	26.4	0.1

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APPENDIX A: PROOFS OF THE THEOREMS IN CHAPTER 2

Assumption and Notation used in proofs of Chapter 2

Assumption.

$$A_1 : (Z_i, Y_i)_{i=1}^{n} \;\; are \;\; i.i.d.$$

$$A_2 : \|\phi(\cdot)\|_\infty < M_d$$

$$A_3 : M_d \sqrt{log 2d} \sim \mathcal{O}(\sqrt{n})$$

$$A_4 : For \;\; \forall a_0 > 0, \exists J < \infty, \;\; such \;\; that, a_{J-1} = a_0\, 2^J \geq 2\|\beta^*\|_1$$

$$A_5 : \|\, \beta^* \,\|_0 \;\leq\; s$$

$$A_6 : \delta\; L_n\, \rho_{(\,\Delta,\, \beta^*\,)}\; (\,Z,\, Y\,) \geq\; \kappa\; \|\; \Delta\; \|_2$$

Notation:

$$\lambda^* \;\equiv\; \sqrt{2}\; M_d \sqrt{\frac{\log\,(\,2\,d\,) + \log n}{n}}\;;$$

$$\delta_1 \;\equiv\; \frac{2M_d}{n};$$

$$\delta_2 \;\equiv\; 2\, M_d\, \sqrt{\frac{2\, log\, 2d}{n}}$$

$$\delta_0 = \delta_1 + \delta_2$$

A1. Proof of Theorem 2.1

proof of Theorem 2.1.

$$L_n\,\rho_{\beta^*}\,(Z,Y) = \frac{1}{n}\sum_{i=1}^{n}\left\{\,Y_i\,Z_i\beta^* - \log\left[\,1+\exp\left(Z_i\beta^*\right)\right]\right\}.$$

Let $\nabla_{\beta_j}\,L_n\,\rho_{\beta^*}\,(Z,Y)$ denote the gradient of $L_n\,\rho_{\beta^*}\,(Z,Y)$ with respect to β_j, then for each j,

$$\nabla_{\beta_j}\,L_n\,\rho_{\beta^*}\,(Z,Y) = \frac{1}{n}\sum_{i=1}^{n}\left\{\,Z_{ij}\,[\,Y_i - \mu\,(\,Z_i\beta^*\,)\,]\right\}$$

where

$$\mu\,(\,Z_i\beta^*\,) = \frac{\exp\left(Z_i\beta^*\right)}{1+\exp\left(Z_i\beta^*\right)} \;\in\; (\,0,1\,)$$

let $\epsilon_i = Y_i - \mu\,(\,Z_i\beta\,)$, then $\epsilon_i \in (\,-1,\ 1\,)$

$$\nabla_{\beta_j}\,L_n\,\rho_{\beta^*}\,(Z,Y) = \frac{1}{n}\sum_{i=1}^{n}Z_{ij}\,\epsilon_i$$

Since $\{\,Z_i,\,Y_i\,\}_{i=1}^{n}$ are $i.i.d$, then for each j, $\{\,Z_{ij}\epsilon_i\,\}_{i=1}^{n}$ is a set of n independent random variables.

We have following properties for $Z_{ij}\epsilon_i$:

According to Lemma A.1, we have

$$E\,(\,Z_{ij}\epsilon_i\,) = 0; \tag{A.1}$$

According to Assumption A_2, $\|Z\|_\infty \le M_d$ and $\epsilon_i \in (\,-1,\ 1\,)$, we have

$$Z_{ij}\,\epsilon_i \;\in\; (\,-M_d,\ M_d\,) \tag{A.2}$$

Then, combine (A.1) and (A.2), we can apply Hoeffding's Inequality to $\{ \, Z_{ij}\epsilon_i \, \}_{i=1}^{n}$,

$$\begin{aligned} & P \left\{ \left| \, \nabla_{\beta_j} \, L_n \, \rho_{\beta^*} \, (Z, Y) \, \right| > \lambda \right\} \\ = & P \left\{ \left| \, \frac{1}{n} \sum_{i=1}^{n} \, Z_{ij}\epsilon_i \, \right| > \lambda \right\} \leq \, 2 \, \exp \, \left[- \frac{2 \, n^2 \, \lambda^2}{\sum_{i=1}^{n} \, (\, 2 \, M_d \,)^2} \, \right] \\ = & \, 2 \, \exp \, \left(\, - \frac{n \, \lambda^2}{2 \, M_d^2} \right) \end{aligned} \tag{A.3}$$

Then by De Morgan's Law, we abtain the union bound:

$$\begin{aligned} P \left\{ \, \beta^* \in \mathcal{C}_\lambda \, \right\} = & P \left\{ \, \left\| \, \nabla_\beta \, L_n \, \rho_{\beta^*} \, (Z, Y) \, \right\|_\infty \leq \lambda \, \right\} \\ = & P \left(\, \cap \, {}_{j=1}^{d} \left\{ \, \left| \, \nabla_{\beta_j} \, L_n \, \rho_{\beta^*} \, (Z, Y) \, \right| \leq \lambda \, \right\} \right) \\ = & 1 - P \Big(\, \cup \, {}_{j=1}^{d} \left\{ \, \left| \, \nabla_{\beta_j} \, L_n \, \rho_{\beta^*} \, (Z, Y) \, \right| \, > \, \lambda \, \right\} \Big) \\ \geq & 1 - \sum_{j=1}^{d} \, P \left\{ \, \left| \, \nabla_{\beta_j} \, L_n \, \rho_{\beta^*} \, (Z, Y) \, \right| \, > \, \lambda \, \right\} \end{aligned} \tag{A.4}$$

Plug the result of (A.3) into (A.4), it holds,

$$\begin{aligned} (A.4) \geq & \, 1 - 2d \, \exp \, \left(\, - \frac{n \, \lambda^2}{2 \, {M_d}^2} \right) \\ = & \, 1 - \exp \left[\left(\, - \frac{n \, \lambda^2}{2 \, {M_d}^2} \right) + \log \, (\, 2 \, d \,) \, \right] \end{aligned} \tag{A.5}$$

therefore,

$$\begin{aligned} & P \left\{ \, \left\| \, \nabla_\beta \, L_n \, \rho_{\beta^*} \, (Z, Y) \, \right\|_\infty \right. \\ & \left. \leq \lambda \, \right\} \geq 1 - \exp \, \left[\left(\, - \frac{n \, \lambda^2}{2 \, {M_d}^2} \right) + \log \, (\, 2 \, d \,) \, \right] \end{aligned}$$

let

$$\lambda \, \geq \, \sqrt{2} \, M_d \sqrt{\frac{\log \, (\, 2 \, d \,) + \tau}{n}} \; ;$$

then

$$P\,(\,\beta^* \,\in \mathcal{C}_\lambda\,) = P\left\{ \left\| \,\nabla_\beta\, L_n\, \rho_{\beta^*}\,(Z,Y)\, \right\|_\infty \leq \sqrt{2}\;M_d \sqrt{\frac{\log\,(\,2\,d\,)+\tau}{n}} \right\} > 1 - e^{-\tau}.$$

To be specific, set $\tau = \log n$, then with

$$\lambda \;\geq\; \sqrt{2}\;M_d \sqrt{\frac{\log\,(\,2\,d\,)+\log n}{n}} \;\equiv \lambda^*,$$

it holds that:

$$P\,(\,\beta^* \,\in \mathcal{C}_\lambda\,) \geq 1 - \frac{1}{n}.$$

□

Lemma A.1. *With same notations in Theorem 2.1,*

$$E\,(\;Z_{ij}\epsilon_i\;) = 0$$

Proof.

By definition of $L_n\, \rho_{\beta^*}\,(Z,Y)$,

$$L_n\, \rho_{\beta^*}\,(Z,Y)\;= \frac{1}{n}\sum_{i=1}^{n}\;\rho_{\beta^*}\,(\,Z_i, Y_i\,)$$

Thus for $1 \leq j \leq d$, the gradient w.r.t. β^* is:

$$\nabla_{\beta_j}\, L_n\, \rho_{\beta^*}\,(Z,Y)\;= \frac{\partial\, \frac{1}{n}\;\sum_{i=1}^{n}\;\rho_\beta(\,Z_i, Y_i\,)}{\partial\,\beta_j}\;\Big|_{\beta^*} = \frac{1}{n}\sum_{i=1}^{n}\;\frac{\partial\,\rho_\beta(\,Z_i, Y_i\,)}{\partial\,\beta_j}\;\Big|_{\beta^*}$$

Take the expectation of the gradient, combine with the definition of β^* and (2), we have:

$$E\,[\,\nabla_{\beta_j}\,L_n\,\rho_{\beta^*}\,(Z,Y)\,] = E\left\{\,\frac{1}{n}\,\sum_{i=1}^{n}\,\frac{\partial\,\rho_\beta(\,Z_i,Y_i\,)}{\partial\,\beta_j}\,\Big|_{\beta^*}\,\right\}$$
$$= E\left\{\,\frac{\partial\,\rho_\beta(\,Z,Y\,)}{\partial\,\beta_j}\,\Big|_{\beta^*}\,\right\} = \frac{\partial\,E\,[\,\rho_\beta\,(Z,Y)\,]}{\partial\,\beta_j}\,\Big|_{\beta^*}$$
$$= \frac{\partial\;L\,\rho_{\beta^*}\,(Z,Y)}{\partial\beta_j}\bigg|_{\beta^*} = 0. \tag{A.6}$$

Therefore,

$$E\,(\,Z_{ij}\epsilon_i\,) = E\left[\,\frac{1}{n}\,\sum_{i=1}^{n}\,Z_{ij}\epsilon_i\,\right] = E\left[\,\nabla_{\beta_j}\;L_n\,\rho_{\beta^*}\,(Z,Y)\,\right] = 0;$$

□

A2. Parameter Error Bound

A2.1 Preliminary

A2.1.1 Concentration Inequality

Lemma A.2 (Hoeffding's Inequality)**.**

If $Z_1, Z_2, \ldots, Z_n$ *are independent with* $P\ (\ a_i \leq Z_i \leq b_i\ ) = 1$*, then for any* $t > 0$*,*

$$P\ (\ |\ \frac{1}{n}\ \sum_{i=1}^{n}\ Z_i - E(Z)\ |\ > \lambda\ ) \leq 2\, e^{-2n\lambda^2/c};$$

where $c = \frac{1}{n}\sum_{i=1}^{n}(b_i - a_i)^2.$

Lemma A.3 (McDiarmid Inequality[23])**.** *Let* $Z_1, \ldots, Z_n \in \mathcal{Z}$ *be independent random variables, a mapping* $G : \mathcal{Z} \rightarrow R$*, and there exist nonnegative numbers* $c_1, \ldots, c_n$ *such that* $\forall i \in \{1, , n\}$*, and* $\forall Z_1, \ldots, Z_n, Z'_k \in \mathcal{Z}$*, the function G satisfies*

$$\sup_{Z_1,\ldots,Z_n,Z'_i} \Big| G(Z_1, \ldots, Z_i, \ldots, Z_n) - G(Z_1, \ldots, Z'_i, \ldots, Z_n) \Big| \leq c_i \tag{A.7}$$

then,

$$P\Bigg(\ \Big|\ G(Z_1, \ldots, Z_n) - E\big[G(Z_1, \ldots, Z_n)\big] \Big| \geq \delta \Bigg) \leq 2\exp\Bigg(- \frac{2\delta^2}{\sum_{i=1}^{n}\ c_i^2} \Bigg) \tag{A.8}$$

Lemma A.4. *[7]*

Let Z be a random variable with mean 0, and $a \leq\ Z \leq b$*. Then, for any t,*

$$E\ (\ e^{tZ}\ ) \leq e^{t^2(b-a)^2/8}.$$

A2.1.2 Measure of Complexity

To develop uniform bound, it's necessary to introduce a way to measure how complex the hypothesis class is. There are several approaches to measure the complexity such as VC Dimension, Covering, Rademacher Complexity, etc. In this theoretical study, we utilize Rademacher Complexity to measure the complexity of function class for high confidence set selection.

Definition 5.1 (Rademacher Random Variable)**.**

Rademacher Random Variable $\{r_1, r_2, \ldots, r_n\}$is a set of independent and identical random variables, with $P(r_i = 1) = P(r_i = -1) = 0.5$.

Definition 5.2 (Rademacher Complexity)**.** [7]

Rademacher Complexity of $\mathcal{F}$ is

$$Rad\ (\ \mathcal{F}\ ) = E\left[\ \sup_{f\in\mathcal{F}} \frac{1}{n}\sum_{i=1}^{n}\ r_i\ f(\ Z_i\ )\ \right]$$

The more complex the function class is , the larger the $Rad\ (\ \mathcal{F}\ )$ would be. Intuitively, if the function class is complex enough, it's possible to pick some $f \in \mathcal{F}$, which match the sign of Rademacher Random Variable, to make the $Rad(\mathcal{F})$ large. There are a lot of important properties of Rademacher Complexity. we introduce one useful Lemma below, which apply symmetrization technique.

Lemma A.5 (Symmetrization Theorem [24])**.** *Let $Z_1, ..., Z_n$ be indepedent random variables with values in $\mathcal{Z}$, $r_1, ..., r_n$ be a Rademacher sequence independent of $Z_1, ..., Z_n$; f is* *a real valued functions on $\mathcal{Z}$, Then*

$$E\left(\sup_{f\in\mathcal{F}} \left| \, (\, L_n - L \,) \, f(Z_i) \right| \right) \; \leq \; 2\, E\left(\sup_{f\,\in\,\mathcal{F}} \left| \, \frac{1}{n} \sum_{i=1}^{n} r_i \, f\,(\,Z_i\,) \, \right| \right).$$

Lemma A.6 (Contraction Theorem [14])**.** *Let $Z_1, ..., Z_n$ be non-random elements of some space $\mathcal{Z}$ and let $\mathcal{F}$ be a class of real valued functions on $\mathcal{Z}$, Consider Lipschitz functions $\rho_i : R \to R$, i.e.*

$$\left| \, \rho_i(x) - \rho_i(x') \, \right| \leq | \, x - x' \, |, \forall x, x' \in R,$$

Let $r_1, ..., r_n$ be a Rademacher sequence. Then for any function $\phi : \mathcal{Z} \to R$, we have:

$$E\left(\sup_{f\in\mathcal{F}} \left| \, \sum_{i=1}^{n} r_i \left[\, \rho_i(\, \phi(x_i) \,) - \rho_i(\, \phi'(x_i) \,) \, \right] \, \right| \right) \leq 2\, E\left(\sup_{f\in\mathcal{F}} \left| \, \sum_{i=1}^{n} r_i \left[\, \phi(x_i) - \phi'(x_i) \, \right] \, \right| \right).$$

A2.2 Proof of Theorem 2.2

proof of Theorem 2.2. Define the solution set of HCS:

$$\mathcal{B}_\lambda := \left\{ \hat{\beta} \in R^d : \hat{\beta} = \underset{\beta \in \mathcal{C}_\lambda)}{\arg\min} \| \beta \|_1 \right\} \tag{A.9}$$

Define a quantity V_λ:

$$V_\lambda = \sup_{\hat{\beta} \in \mathcal{B}_\lambda} \frac{(L_n - L) (\rho_{\beta^*} - \rho_{\hat{\beta}})}{a_0 + \|\hat{\beta} - \beta^*\|_1} \tag{A.10}$$

Where L_n is the emperical loss operator, L is the expected loss operator;

$\rho_{\hat{\beta}}$, ρ_{β^*} are logistic loss with respect to $\hat{\beta}$ and β^* respectively;

a_0 is a small quantity which by assumption A_4 satisfies: $a_0 > \frac{\|\beta^*\|_1}{2^J}$.

Construct a partition set of $\mathcal{B}_\lambda$ according to the distance between β^* and $\hat{\beta}$:

$$\begin{aligned}
&\mathcal{B}_0 = \{\hat{\beta} : \hat{\beta} \in \mathcal{B}_\lambda, \|\hat{\beta} - \beta^*\|_1 \le a_0\} \\
&\mathcal{B}_j = \{\hat{\beta} : \hat{\beta} \in \mathcal{B}_\lambda, a_{j-1} < \|\hat{\beta} - \beta^*\|_1 \le a_j\}; (1 \le j \le J-1) \\
&\mathcal{B}_J = \{\hat{\beta} : \hat{\beta} \in \mathcal{B}_\lambda, \|\hat{\beta} - \beta^*\|_1 > a_{J-1}\}
\end{aligned} \tag{A.11}$$

For $1 \le j \le J-1$: $a_j = 2a_{j-1}$, by Assumption A_4, it holds $a_{J-1} \ge 2\|\beta^*\|_1$.

Then, we can derive the bound according to this partition $\mathcal{B}_\lambda$ as follow:

$$\begin{aligned}
P (V_\lambda > \delta_0) &= P \left(\sup_{\hat{\beta} \in \mathcal{B}_\lambda} \frac{(L_n - L) (\rho_{\beta^*} - \rho_{\hat{\beta}})}{a_0 + \|\hat{\beta} - \beta^*\|_1} > \delta_0 \right) \\
&\le \sum_{j=0}^{J} P \left(\sup_{\hat{\beta} \in \mathcal{B}_j} \frac{(L_n - L) (\rho_{\beta^*} - \rho_{\hat{\beta}})}{a_0 + \|\hat{\beta} - \beta^*\|_1} > \delta_0 \right).
\end{aligned} \tag{A.12}$$

to be simplified, let

$$V_j = \sup_{\hat{\beta} \in \mathcal{B}_j} \frac{(L_n - L)(\rho_{\beta^*} - \rho_{\hat{\beta}})}{a_0 + \|\hat{\beta} - \beta^*\|_1} \tag{A.13}$$

then (A.12) is equivalent to

$$P(V_\lambda > \delta_0) \le \sum_{j=0}^{J} P\Big(V_j > \delta_0\Big).$$

According to Lemma A.7: For $0 \le j \le J - 1$

$$P\Big(V_j > \delta_0\Big) < 2\, e^{-2n}$$

By Theorem 2.1, when $\lambda > \lambda^*$, it holds

$$\begin{aligned} P\Big(V_J > \delta_0\Big) &\le P\Big(\|\hat{\beta} - \beta^*\|_1 > a_{J-1}\Big) \\ &\le P\Big(\|\hat{\beta} - \beta^*\|_1 > 2\,\|\beta^*\|_1\Big) \\ &\le P(\beta^* \notin \mathcal{C}_\lambda) \le \frac{1}{n} \end{aligned}$$

Thus,

$$P\Big(V_\lambda > \delta_0\Big) < 2J\, e^{-2n} + \frac{1}{n}.$$

It comes to conclude that, with probability at least $1 - 2J\, e^{-2n} - \frac{1}{n}$, it holds:

$$V_\lambda := \sup_{\hat{\beta} \in \mathcal{B}_\lambda} \frac{|(L_n - L)(\rho_{\beta^*} - \rho_{\hat{\beta}})|}{a_0 + \|\beta^* - \hat{\beta}\|_1} < \delta_0.$$

Since our estimator $\hat{\beta}_{HCS} \in \mathcal{B}_\lambda$, it holds

$$\frac{|(L_n - L)(\rho_{\beta^*} - \rho_{\hat{\beta}_{HCS}})|}{a_0 + \|\beta^* - \hat{\beta}_{HCS}\|_1} \le \sup_{\hat{\beta} \in \mathcal{B}_\lambda} \frac{|(L_n - L)(\rho_{\beta^*} - \rho_{\hat{\beta}})|}{a_0 + \|\beta^* - \hat{\beta}_{HCS}\|_1} \le \delta_0;$$

thus,

$$L_n(\rho_{\beta^*} - \rho_{\hat{\beta}_{HCS}}) - L(\rho_{\beta^*} - \rho_{\hat{\beta}_{HCS}}) \le \delta_0\, a_0 + \delta\, \|\beta^* - \hat{\beta}_{HCS}\|_1 .$$

rearrange the orders, then

$$\mathcal{E}(\hat{\beta}_{HCS}) = L\,(\,\rho_{\hat{\beta}_{HCS}} - \rho_{\beta^*}\,)$$

$$\leq L_n\,(\,\rho_{\hat{\beta}_{HCS}} - \rho_{\beta^*}\,) + \,\delta_0 \,\|\,\beta^* - \hat{\beta}_{HCS}\,\|_1 \,+\, \delta\, a_0$$

by (3): $L_n(\,\rho_{\hat{\beta}_{HCS}} - \rho_{\beta^*}\,) \;\leq\; \lambda \,\|\,\hat{\beta}_{HCS} - \beta^*\,\|_1$, thus

$$\mathcal{E}(\hat{\beta}_{HCS}) \leq (\,\lambda + \delta\,)\,\|\,\beta^* - \hat{\beta}_{HCS}\,\|_1 \,+\, \delta_0\, a_0$$

□

Lemma A.7. *As $\mathcal{B}_j$, V_j, defined in (A.11), (A.13), for $0 \leq j \leq J-1$, it holds:*

$$P\,(\,V_j > \delta_0\,) < 2\,e^{-2n}.$$

Proof. The process to prove V_j is bouned contains following two steps: Step 1, prove V_j is concerntrated around its mean $E(\,V_j\,)$; Step 2, prove the mean $E(V_j)$ is bounded above.

Step 1: Concerntraction around mean:

$$P\Big(\,|\,V_j - E\,(\,V_j\,)\,| > \delta_1\Big) < 2\,e^{-2n}$$

Denote $\{D_i\}_{i=1}^n = (Z_i, Y_i)_{i=1}^n$. It suffices to apply McDiarmid Inequality (Theorem A.3) to derive the concentration bound if it satisfies:

$$\sup_{D_i}\Big|\,V_j\,(\,D_1, \ldots, D_k, \ldots, D_n\,) - V_j\,(\,D_1, \ldots, D_k', \ldots, D_n\,)\,\Big| \leq c_i.$$

$$\text{Let }\; h_\beta = \frac{\rho^*_\beta - \rho_\beta}{a_0 + \|\hat{\beta} - \beta^*\|_1} \quad \text{and }\; \bar{h}_\beta = h_\beta - E(h_\beta) \tag{A.14}$$

then,

$$V_j(\, D_1, \ldots, D_n \,) = \sup_{\beta \in \mathcal{B}_j} \frac{(L_n - L)(\rho^*_\beta - \rho_\beta)\left\{\, D_1, \ldots, D_n \,\right\}}{a_0 + \|\, \hat{\beta} - \beta^* \|_1}$$

$$= \sup_{\beta \in \mathcal{B}_j} \frac{1}{n} \sum_{i=1}^{n} \bar{h}_\beta\,(\, D_i\,) \tag{A.15}$$

Construct a set $\left\{\, D_i' \,\right\}_{i=1}^{n}$, such that:

$$D_i' = \begin{cases} (\, Z_i',\, Y_i'\,) & \text{when } i = k \\ (\, Z_i,\, Y_i\,) & \text{when } i \neq k \end{cases}$$

then,

$$V_j\ (\, D_1', \ldots, D_k', \ldots, D_n' \,) = \sup_{\beta \in \mathcal{B}_j} \frac{1}{n} \sum_{i=1}^{n} \bar{h}_\beta\,(\, D_i'\,)$$

By definition of V_j in (A.15), for $\forall \beta_1 \in \mathcal{B}_j$ we have:

$$\frac{1}{n} \sum_{i=1}^{n} \bar{h}_{\beta_1}(\, D_i\,) - V_j\ (\, D_1, \ldots, D_k', \ldots, D_n \,) \tag{A.16}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \bar{h}_{\beta_1}(\, D_i\,) - \sup_{\beta \in \mathcal{B}_j} \frac{1}{n} \sum_{i=1}^{n} \bar{h}_\beta\,(\, D_i'\,)\ ; \tag{A.17}$$

For $\forall \beta_1 \in \mathcal{B}_j$, it holds $\sup_{\beta \in \mathcal{B}_j} \frac{1}{n} \sum_{i=1}^{n} \bar{h}_\beta\,(\, D_i'\,) > \frac{1}{n} \sum_{i=1}^{n} \bar{h}_{\beta_1}(\, D_i'\,)$, thus,

$$\text{(A.16)} \leq \frac{1}{n} \sum_{i=1}^{n} \bar{h}_{\beta_1}(\, D_i\,) - \frac{1}{n} \sum_{i=1}^{n} \bar{h}_{\beta_1}(\, D_i'\,)$$

$$\leq \frac{1}{n} \left[\, \bar{h}_{\beta_1}(\, D_k\,) - \bar{h}_{\beta_1}(\, D_k'\,) \,\right] \tag{A.18}$$

Since D_k and D_k' are from same distribution, we have

$$E\left[\, h_{\beta_1}(\, D_k\,) \,\right] = E\left[\, h_{\beta_1}(\, D_k'\,) \,\right] \tag{A.19}$$

Therefore, by definiton of $\bar{h}_\beta$, (A.14):

$$(A.18) = \frac{1}{n}\left\{\Big(h_{\beta_1}(D_k) - E\left[\, h_{\beta_1}(D_k)\right]\Big) - \Big(h_{\beta_1}(D'_k) - E\left[\, h_{\beta_1}(D'_k)\,\right]\Big)\right\}$$

$$= \frac{1}{n}\left[\, h_{\beta_1}(D_k) - h_{\beta_1}(D'_k)\,\right]$$

by definition of h_β (A.14),

$$= \frac{1}{n}\left\{\frac{(\rho^*_\beta - \rho_{\beta_1})(D_k) - (\rho^*_\beta - \rho_{\beta_1})(D'_k)}{a_0 + \|\beta^* - \beta_1\|_1}\right\}$$

by the triangle inequality,

$$\leq \frac{1}{n}\left\{\frac{\left|\,(\rho^*_\beta - \rho_{\beta_1})(Z_k, Y_k)\,\right| + \left|\,(\rho^*_\beta - \rho_{\beta_1})(Z'_k, Y'_k)\,\right|}{a_0 + \|\beta^* - \beta_1\|_1}\right\}$$

by Lipschitz property of ρ_β,

$$\leq \frac{1}{n}\left\{\frac{\left|\, Z_k^T\beta^* - Z_k^T\beta_1\right| + \left|\, Z_k'^T\beta^* - Z_k'^T\beta_1\right|}{a_0 + \|\beta^* - \beta_1\|_1}\right\}$$

by Holder's inequality,

$$\leq \frac{1}{n}\left\{\frac{\|\, Z_k^T\,\|_\infty\|\,\beta^* - \beta_1\,\|_1 + \|\, Z_k'^T\,\|_\infty\|\,\beta^* - \beta_1\,\|_1}{a_0 + \|\beta^* - \beta_1\|_1}\right\}$$

by Assumption:

$$\leq \frac{2M_d}{n}\,.$$

Since for $\forall\beta_1 \in \mathcal{B}_j$, it holds

$$\frac{1}{n}\sum_{i=1}^{n}\bar{h}_{\beta_1}(D_i) - V_j(D_1, \ldots, D'_k, \ldots, D_n) \leq \frac{2M_d}{n}$$

thus,

$$\sup_{\beta\in\mathcal{B}_j}\frac{1}{n}\sum_{i=1}^{n}\bar{h}_\beta(D_i) - V_j(D_1, \ldots, D'_k, \ldots, D_n) \leq \frac{2M_d}{n}\,,$$

by (A.15),

$$V_j(D_1, \ldots, D_k, \ldots, D_n) = \sup_{\beta\in\mathcal{B}_j}\frac{1}{n}\sum_{i=1}^{n}\bar{h}_\beta(D_i)\,,$$

thus,

$$V_j\ (\ D_1, \ldots, D_k, \ldots, D_n\ ) - V_j\ (\ D_1, \ldots, D_k', \ldots, D_n\ ) \leq \frac{2M_d}{n}\ .$$

Analogously, it can be proved

$$V_j\ (\ D_1, \ldots, D_k', \ldots, D_n\ ) - V_j\ (\ D_1, \ldots, D_k, \ldots, D_n\ ) \leq \frac{2M_d}{n}\ ,$$

thus,

$$\left|\ V_j\ (\ D_1, \ldots, D_k, \ldots, D_n\ ) - V_j\ (\ D_1, \ldots, D_k', \ldots, D_n\ )\ \right| \leq \frac{2M_d}{n}\ .$$

Since $D_1, \ldots, D_n$ are i.i.d,

$$\sup_{D_1,\ldots,D_k,\ldots,D_n,D_{k'}} \left|\ V_j\ (\ D_1, \ldots, D_k, \ldots, D_n\ ) - V_j\ (\ D_1, \ldots, D_k', \ldots, D_n\ )\ \right| \leq \frac{2M_d}{n}\ .$$

Thus, the condition $(A.7)$ in McDiarmid Inequality (Theorem A.3) meets with

$$c_i = \frac{2M_d}{n}\ .$$

By setting $\delta = \dfrac{2M_d}{n}$, it conclude that:

$$P\left(\ |\ V_j\ -\ E\,(\,V_j\ )\,|\ >\ \frac{2M_d}{n}\ \right)\ <\ 2e^{-2n} \tag{A.20}$$

Step 2: Upper Bounded $E(V_j)$

$$E\ (\ V_j\ )\ \leq\ \delta_2$$

$$E\,(\ V_j\ ) = E\left(\ \sup_{\hat{\beta}\in\mathcal{B}_j}\ \frac{|\ (\ L_n - L\ )\ (\rho_{\beta^*} - \rho_{\hat{\beta}})\ |}{a_0 + \|\ \beta^* - \hat{\beta}\ \|_1}\right)$$

recall the definition of $\mathcal{B}_j$:

$$\mathcal{B}_j = \{\hat{\beta} : \hat{\beta} \in \mathcal{B}_\lambda, a_{j-1} < \| \beta^* - \hat{\beta} \|_1 < a_j\};$$

thus,

for $\quad j = 0$:

$$a_0 + \| \beta^* - \hat{\beta} \|_1 \geq a_0,$$

for $\quad 1 \leq j \leq J-1$:

$$a_0 + \| \beta^* - \hat{\beta} \|_1 \geq a_{j-1}.$$

therefore,

for $\quad j = 0$:

$$E\,(\,V_j\,) \leq \frac{1}{a_0}\,E\left(\sup_{\hat{\beta}\in\mathcal{B}_j} \left|\,(\,L_n - L\,)\,(\rho_{\beta^*} - \rho_{\hat{\beta}})\,\right|\right)$$

for $\quad 1 \leq j \leq J-1$:

$$E\,(\,V_j\,) \leq \frac{1}{a_{j-1}}\,E\left(\sup_{\hat{\beta}\in\mathcal{B}_j} \left|\,(\,L_n - L\,)\,(\rho_{\beta^*} - \rho_{\hat{\beta}})\,\right|\right)$$

Denote $\tilde{h}_{\hat{\beta}} = \rho_{\beta^*} - \rho_{\hat{\beta}}$, by Symmetrization Lemma (Lemma A.5), it holds:

$$E\left(\sup_{\hat{\beta}\in\mathcal{B}_j} |\,(\,L_n - L\,)\,\tilde{h}_{\hat{\beta}}\,|\right) \leq 2\,Rad\,\{\,\tilde{h}_{\hat{\beta}},\ \hat{\beta}\in\mathcal{B}_j\,\}; \tag{A.21}$$

Where $Rad\,\{\,\tilde{h}_{\hat{\beta}},\ \hat{\beta}\in\mathcal{B}_j\,\}$ is Rademacher Complexity of $\{\,\tilde{h}_{\hat{\beta}},\ \hat{\beta}\in\mathcal{B}_j\,\}$:

$$Rad\,\{\,\tilde{h}_{\hat{\beta}},\ \hat{\beta}\in\mathcal{B}_j\,\} = E\left(\sup_{\hat{\beta}\,\in\,\mathcal{B}_j} \left|\,\frac{1}{n}\sum_{i=1}^{n} r_i\,\tilde{h}_{\hat{\beta}}\,(\,Z_i,\,Y_i\,)\,\right|\right);$$

and $\{r_i\}_{i=1}^n$ is a set of i.i.d Rademacher random variable.

Thus,

$$E\left(\sup_{\hat\beta\in\mathcal{B}_j} |\,(\,L_n - L\,)\,\tilde h_{\hat\beta}\,|\right) \le 2\,E\left(\sup_{\hat\beta\,\in\,\mathcal{B}_j}\left|\,\frac{1}{n}\sum_{i=1}^{n} r_i\,\tilde h_\beta\,(\,Z_i,\,Y_i\,)\right|\right)$$

By Contraction Theorem (Lemma A.6),

$$\le 2\,E\left(\sup_{\hat\beta\,\in\,\mathcal{B}_j}\left|\,\frac{1}{n}\sum_{i=1}^{n} r_i\,Z_i\,(\,\beta^* - \hat\beta\,)\right|\right)$$

By Holders Inequality,

$$\le 2\,E\left(\sup_{\hat\beta\,\in\,\mathcal{B}_j}\left[\frac{1}{n}\;\;\|Z^T\,r\|_\infty\,\|\,\beta^* - \hat\beta\,\|_1\,\right]\right)$$

since $\|\beta^* - \hat\beta\|_1 < a_j$

$$\le 2\,a_j\,E\;\left[\;\frac{1}{n}\;\;\|Z^T\,r\|_\infty\;\;\right]$$

By Lemma A.8:

$$\le 2a_j\,M_d\sqrt{\frac{2\,log\,2d}{n}}$$

Thus, for $\;j = 0:$

$$E\,(\,V_0\,) \le \frac{2a_0}{a_0}\;M_d\sqrt{\frac{2\,log\,2d}{n}} \le 2\,M_d\sqrt{\frac{2\,log\,2d}{n}};$$

for $\;1 \le j \le J-1:$

$$E\,(\,V_j\,) \le \frac{2a_j}{a_{j-1}}\;M_d\sqrt{\frac{2\,log\,2d}{n}} \le 4\,M_d\sqrt{\frac{2\,log\,2d}{n}};$$

which concludes that, for $\;0 \le j \le J-1$:

$$E\,(\,V_j\,) \le 4\,M_d\sqrt{\frac{2\,log\,2d}{n}} \equiv \delta_2. \tag{A.22}$$

Set $\delta_0 = \delta_1 + \delta_2$, combine (A.20) and (A.22), it concludes:

$$P\,(\,V_j > \delta_0\,) < 2Je^{-2n} + \frac{1}{n} \tag{A.23}$$

□

Lemma A.8. *With same notations of Lemma A.7, it holds:*

$$E\left(\max_{1 \leq j \leq d} \frac{1}{n} \mid Z_j^T \, r \mid \right) \leq M_d \sqrt{\frac{2\,log\,2d}{n}}.$$

Proof.

Let $T_{ij} = \dfrac{1}{n} Z_{ij}\, r_i,$ then $E\ (\ T_{ij}\ ) = 0,$ and $|T_{ij}| \ \leq \ \dfrac{M_d}{n}.$

By Lemma A.4,

$$E\ [\ exp\ (\ t\ T_{ij}\ )\ ]\ \leq exp\ (\ \frac{t^2\ {M_d}^2}{2\ n^2}\ ). \tag{A.24}$$

Let $T_j = \sum_{i=1}^n T_{ij}$, then,

$$E\ [\ exp\ (\ t\ T_j\ )\ ] = E\ [\ exp\ (\ t\ \sum_{i=1}^{n}\ T_{ij}\ )\ ]$$

by independency of $\{T_{ij}\}_{i=1}^n$,

$$= \prod_{i=1}^{n} E\ [\ exp\ (\ t\ T_{ij}\ )\ ]$$

by the result of (A.24),

$$\leq \prod_{i=1}^{n} exp\ (\ \frac{t^2 {M_d}^2}{2n^2}\ ) = exp\ (\ \frac{t^2 {M_d}^2}{2n}\ ) \tag{A.25}$$

Thus, T_j is a subgaussian random variable with $\sigma = \frac{M_d}{\sqrt{n}}$.

Create a set $\{\ T'_j\ \}_{j=1}^{2d}$, with $2d$ elements, where

$$\{\ T'_j\ \}_{j=1}^{d} = \{\ T_j\ \}_{j=1}^{d};$$

$$\{\ T'_j\ \}_{j=d+1}^{2d} = \{\ -T_j\ \}_{j=1}^{d}.$$

Notice that,

$$\max_{1 \le j \le d} | T_j | = \max_{1 \le j \le 2d} T'_j;$$

thus,

$$exp\,[\; t\; E\; (\max_{1 \le j \le d} | T_j |\;)\;] = exp\,[\; t\; E\; (\max_{1 \le j \le 2d} T'_j\;)\;].$$

Then, by Jensen's Inequality and convexity of e^x, it holds:

$$exp\,[\; t\; E\; (\max_{1 \le j \le 2d} T'_j\;)\;] \le E\;[\; exp\;(\; t\; \max_{1 \le j \le 2d} T'_j\;)\;]$$

$$= E\;[\; \max_{1 \le j \le 2d} exp\;(\; t\, T'_j\;)\;]$$

$$\le \sum_{j=1}^{2d} E\;[\; exp\;(\; t\, T'_j\;)\;]$$

by the result of subgaussian tails in (A.25):

$$\le 2d\; exp\;(\; \frac{t^2\, {M_d}^2}{2\,n}\;).$$

take log for both sides, we have

$$E\;(\max_{1 \le j \le d} | T_j |\;) \le \frac{\log\, 2d}{t} + \frac{t\, {M_d}^2}{2\,n}.$$

setting $t = \sqrt{2n\, log\, 2d}\,/M_d$,

$$E\;(\max_{1 \le j \le d} | T_j |\;) \le M_d \sqrt{\frac{2\, log\, 2d}{n}}.$$

□

A3. Proof of Theorem 2.3

proof of Theorem 2.3. Since $\hat{\beta}_{HCS}$ is the solution from $\mathcal{C}_\lambda$, by definition of $\mathcal{C}_\lambda$,

$$\| \nabla_\beta L_n \rho_{\hat{\beta}_{HCS}} (Z, Y) \|_\infty \leq \lambda$$

Under Event A, we have $\beta^* \in \mathcal{C}_\lambda$, therefore

$$\| \nabla_\beta L_n \rho_{\beta^*} (Z, Y) \|_\infty \leq \lambda;$$

then by the triangle inequality,

$$\| \nabla_\beta L_n \rho_{\hat{\beta}_{HCS}} (Z, Y) - \nabla_\beta L_n \rho_{\beta^*} (Z, Y) \|_\infty$$

$$\leq \| \nabla_\beta L_n \rho_{\hat{\beta}_{HCS}} (Z, Y) \|_\infty + \| \nabla_\beta L_n \rho_{\beta^*} (Z, Y) \|_\infty \leq 2 \lambda; \tag{A.26}$$

let $\hat{\Delta} = \hat{\beta} - \beta^*$, then the first order Taylor error is

$$\delta L_n \rho_{(\hat{\Delta}, \beta^*)}(Z, Y)$$

$$:= L_n \rho_{(\beta^* + \hat{\Delta})}(Z, Y) - L_n \rho_{\beta^*}(Z, Y) - \left\langle \nabla_\beta L_n \rho_{\beta^*}(Z, Y), \hat{\Delta} \right\rangle$$

by first order derivative property of convexity function,

$$\leq \left\langle \nabla_\beta L_n \rho_{\hat{\beta}_{HCS}} (Z, Y), \hat{\Delta} \right\rangle - \left\langle \nabla_\beta L_n \rho_{\beta^*} (Z, Y), \hat{\Delta} \right\rangle$$

rearrange inner product,

$$= \left\langle \left[\nabla_\beta L_n \rho_{\hat{\beta}_{HCS}} (Z, Y) - \nabla_\beta L_n \rho_{\beta^*} (Z, Y) \right], \hat{\Delta} \right\rangle$$

by Holder's inequality,

$$\leq \| \nabla_\beta L_n \rho_{\hat{\beta}_{HCS}} (Z, Y) - \nabla_\beta L_n \rho_{\beta^*} (Z, Y) \|_\infty \| \hat{\Delta} \|_1$$

by the result of (A.26), it holds:

$$\leq 2\lambda \parallel \hat{\Delta} \parallel_1 \tag{A.27}$$

Since $\hat{\beta}_{HCS} = \arg\min_{\beta \in \mathcal{C}_\lambda} \parallel \beta \parallel_1$, thus $\parallel \hat{\beta}_{HCS} \parallel_1 \leq \parallel \beta^* \parallel_1$, similar to basis pursuit [45], we have following two properties for $\hat{\Delta}$:

$$\parallel \hat{\Delta}_{J_c} \parallel_1 \leq \parallel \hat{\Delta}_J \parallel_1 \tag{A.28}$$

$$\parallel \hat{\Delta} \parallel_1 \leq 2\sqrt{s} \parallel \hat{\Delta} \parallel_2 \tag{A.29}$$

By $Assumption\ A_6$, $\hat{\Delta}$ satisfies restricted strong convexity assumption, that is,

$$\delta\, L_n\, \rho_{(\hat{\Delta}, \beta^*)}(Z, Y) \geq \kappa \parallel \hat{\Delta} \parallel_2^2;$$

combine with (A.27) and (A.29) we have,

$$\kappa \parallel \hat{\Delta} \parallel_2^2 \leq 2\,\lambda \parallel \hat{\Delta} \parallel_1 \leq 4\,\lambda \sqrt{s} \parallel \hat{\Delta} \parallel_2;$$

therefore,

$$\parallel \hat{\Delta} \parallel_2 \leq \frac{4\,\lambda\sqrt{s}}{\kappa}; \tag{A.30}$$

plug (A.30) into (A.29), we have

$$\parallel \hat{\Delta} \parallel_1 \leq \frac{8\,\lambda\, s}{\kappa}. \tag{A.31}$$

□

[Corollary] Under Assumption $A_1 - A_6$, when $\lambda > \lambda^*$, with probability at least

$1 - 2J\, e^{-2n} - \frac{1}{n}$, it holds that:

$$\mathcal{E}(\hat{\beta}_{HCS}) \leq \frac{8\,\lambda\, s}{\kappa}\,(\,\lambda + \delta_0\,) + \delta_0\, a_0$$

Proof. Take the result of (A.29) into Theorem 2.2, the result can be achieved. □

APPENDIX B: PROOFS OF THE THEOREMS IN CHAPTER 3

Assumptions and Notations in Chapter 3

Assumption (C_1)**.** $(Z_i, Y_i)_{i=1}^n$ *are i.i.d.*, *and* $(W_i, Y_i)_{i=1}^n$ *are i.i.d.;*

Assumption (C_2)**.** $W = Z + \Xi$, *and* $E(W) = 0$.

Assumption (C_3)**.** $\|\phi(\cdot)\|_\infty < M_d$; *i.e.,* $\| Z \|_\infty \le M_d$; *and* $\| W \|_\infty \le M_d$;

Assumption (C_4)**.** $M_d\sqrt{log 2d^2} \sim \mathcal{O}(\sqrt{n})$;

Assumption (C_5)**.** *For* $\forall a_0 > 0, \exists J < \infty,\ such\ \ that, a_{J-1} = a_0\, 2^J \ge 2\|\beta^*\|_1$;

Assumption (C_6)**.** $\| \beta^* \|_0 \ \le \ s$;

Assumption (C_7)**.** $\delta\, L_n\, \rho_{(\,\Delta,\ \beta^*\,)}\ (\,W,\ Y\ ) \ge \ \kappa \ \| \ \Delta \ \|_2$

Notation:

$$\lambda^* \ \equiv \ \sqrt{2}\ M_d \sqrt{\frac{\log\,(\,2\,d\,) + \log n}{n}}\ ;$$

$$\gamma^* \equiv \ M_d^2 \sqrt{\frac{\log\,(\,2\,d^2\,) + \log n}{2n}}\ ,$$

$$\delta_1 \ \equiv \ \frac{2M_d}{n};$$

$$\delta_2 \ \equiv \ 2\,M_d\sqrt{\frac{2\,log\,2d}{n}}$$

$$\delta_0 = \delta_1 + \delta_2$$

B1. Proof of Theorem 3.1

proof of Theorem 3.1.

Since $L_n \, \rho_\beta \, (W, \, Y \,) = \dfrac{1}{n} \sum_{i=1}^{n} \, \Big\{ \, Y_i \, W_i \, \beta - \log \, [\, 1 + \exp \, (\, W_i \, \beta \,) \,] \, \Big\} \, ;$

it holds, $\nabla_\beta \, L_n \, \rho_\beta \, (W, Y \,) = \dfrac{1}{n} \, \sum_{i=1}^{n} \, \Big\{ \, W_i \, [\, Y_i - \mu \, (\, W_i \, \beta \,) \,] \, \Big\};$

Thus, (**??**) is equivalent to

$$\mathcal{C}_{(\, \lambda, \, \gamma \,)} = \Big\{ \, \beta \, \in \, R^d : \, \frac{1}{n} \, \| \, W^T \, [\, Y - \mu \, (\, W \beta \,) \,] \, \|_\infty \, \leq \, \lambda + \, \gamma \, \| \, \beta \, \|_1 \, \Big\}, \tag{B.1}$$

where

$$\mu \, (\, W \beta \,) = \frac{W \exp \, (\, W \beta \,)}{1 + \exp \, (\, W \beta \,)} \, \in \, (\, 0, 1 \,).$$

By Assumption C_2,

$$W \beta = Z \beta + \Xi \beta$$

thus by Cauchy Remainder Theorem,

$$\mu \, (\, W \beta \,) = \mu \, (\, Z \beta \,) + \mu' \, (\, \xi \, \beta \,) \, (\, \Xi \beta \,)$$

where $\xi \beta$ lies in the segment between $W \beta$ and $Z \beta$.

Therefore,

$$\begin{aligned}
& P \, \Big(\, \beta^* \in \mathcal{C}_{\, (\, \lambda, \gamma \,)} \Big) \\
=& P \, \Big\{ \, \frac{1}{n} \, \| \, W^T \, [\, Y - \mu \, (\, W \beta^* \,) \,] \, \|_\infty \, \leq \, \lambda + \, \gamma \, \| \, \beta^* \, \|_1 \, \Big\} \\
=& P \, \Big\{ \, \frac{1}{n} \, \| \, W^T \, [\, Y - \mu \, (\, Z \beta^* \,) - \mu' \, (\, \xi \beta^* \,) \, (\, \Xi \beta^* \,) \,] \, \|_\infty \, \leq \, \lambda + \, \gamma \, \| \, \beta^* \, \|_1 \, \Big\}
\end{aligned}$$

By triangle inequality,

$$\frac{1}{n} \parallel W^T [\, Y - \mu\,(\, Z\beta^* \,) - \mu'\,(\, \xi\beta^* \,)\,(\, \Xi\beta^* \,) \,] \parallel_\infty$$
$$\leq \frac{1}{n} \parallel W^T [\, Y - \mu\,(\, Z\beta^* \,) \,] \parallel_\infty + \frac{1}{n} \parallel W^T \mu'\,(\, \xi \,)\,(\, \Xi\beta^* \,) \parallel_\infty$$

Define

$$Event \;\; B := \{\, \beta^* \;\in\; \mathcal{C}\,(\, \lambda, \gamma \,) \,\}$$
$$:= \{\, \frac{1}{n} \parallel W^T [\, Y - \mu\,(\, Z\beta^* \,) - \mu'\,(\, \xi\, \beta^* \,)\,(\, \Xi\beta^* \,) \,] \parallel_\infty \leq \lambda \;+\; \gamma \parallel \beta^* \parallel_1 \,\};$$

Define

$$Event \;\; B_1 := \{\, \frac{1}{n} \parallel W^T [\, Y - \mu\,(\, Z\beta^* \,) \,] \parallel_\infty \leq \lambda \,\};$$
$$Event \;\; B_2 := \{\, \frac{1}{n} \parallel W^T \mu'\,(\, \xi\, \beta^* \,)\,(\, \Xi\beta^* \,) \parallel_\infty \leq \gamma \parallel \beta^* \parallel_1 \,\}.$$

Notice that, $Event \;\; B_1$ and $Event \;\; B_2$ implies $Event \;\; B$. Now we investigate the probability of each event respectively.

$(i) \;\; Event \;\; B_1$:

$$P\,(\, B_1 \,) = P\,\{\, \frac{1}{n} \parallel W^T [\, Y - \mu\,(\, Z\beta^* \,) \,] \parallel_\infty \leq \lambda \,\}$$

let $\epsilon = Y - \mu\,(\, Z\beta^* \,)$, then since $E(\epsilon) = 0$ and $\|W\|_\infty < M_d$, analogous to Theorem 2.1, it holds:

$$\text{when} \;\; \lambda \;\geq\; \sqrt{2}\, M_d \sqrt{\frac{\log\,(\, 2\, d \,) + \log n}{n}} \;\equiv \lambda^*,$$
$$P\,(\, B_1 \,) \geq 1 - \frac{1}{n}.$$

$(ii) \;\; Event \; B_2$

Notice that

$$\mu\,(\,\xi_i\,\beta\,) = \frac{\exp\,(\,\xi_i\,\beta\,)}{1+\exp\,(\,\xi_i\,\beta\,)} \;\in\; (\,0,\;1\,),$$

then

$$\mu'\,(\,\xi_i\,\beta\,) = \frac{\exp\,(\,\xi_i\,\beta\,)}{[\,1+\exp\,(\,\xi_i\,\beta\,)\,]^2} = \mu\,(\,\xi_i\,\beta\,)\,[\,1-\mu\,(\,\xi_i\,\beta\,)\,] \;\in\; (\,0,\;\frac{1}{4}\;);$$

Thus,

$$\frac{1}{n}\;\|\;W^T\,\mu'\,(\,\xi\,\beta\,)\,(\,\Xi\beta^*\,)\;\|_\infty \le \frac{1}{4\,n}\;\|\;W^T\,(\,\Xi\beta^*\,)\;\|_\infty \le \frac{1}{4\,n}\;\|\;\Xi^T\,W\;\|_\infty\|\;\beta^*\;\|_1.$$

Define

$$Event\;\;B_2' := \;\{\;\frac{1}{4\,n}\;\|\;\Xi^T\,W\;\|_\infty \le \gamma\;\};$$

then,

$$P\;(\;B_2'\;) = P\left(\;\frac{1}{4\,n}\;\|\;\Xi^T\,W\;\|_\infty \le \gamma\;\right) = P\left(\;\max_k\;\max_j\;\frac{1}{n}\;\left|\;\sum_{i=1}^{n}\;\frac{1}{4}\;\Xi_{ik}\;W_{ij}\;\right| \le \gamma\;\right).$$

Denote

$$T_{i,k,j} = \frac{1}{4}\;\Xi_{ik}\,W_{ij},$$

then

$$\sum_{i=1}^{n}\;T_{i,k,j} = \sum_{i=1}^{n}\;\frac{1}{4}\;\Xi_{ik}\,W_{ij};$$

Since $E\;(\,W_{ij}\,) = 0$,

$$E\left(\;\frac{1}{n}\;\sum_{i=1}^{n}\;T_{i,k,j}\right) = E\left(\;\frac{1}{4\,n}\;\sum_{i=1}^{n}\;\Xi_{ik}\,W_{ij}\;\right) = \frac{1}{4\,n}\;E\left[\;E\left(\;W_{ij}\;\right)\;\sum_{i=1}^{n}\;\Xi_{ik}\;\right] = 0$$

and

$$|\;T_{i,j,k}\;| = \frac{1}{4}\,|\,\Xi_{ik}\,W_{ij}\,| \le \frac{1}{4}\;\|\,\Xi\,\|_\infty\,\|\,W\,\|_\infty \le \frac{1}{2}\;M_d^2$$

Then apply Hoeffding's Inequality,

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} T_{ijk}\right| > \gamma\right) \;\leq\; 2\;\exp\left[-\frac{2\,n^2\,\gamma^2}{\sum_{i=1}^{n}\,[\,2\,(\frac{1}{2}\,M_d^2\,)\,]^2}\;\right]$$

$$=\;2\;\exp\left[-\frac{2\,n\,\gamma^2}{{M_d}^{\,4}}\right] \tag{B.2}$$

The union bounds can be achieved as following:

$$\begin{aligned}
P\,(B_2') \;&= P\left\{\left\|\frac{1}{4n}\,\Xi^T\,W\right\|_\infty \leq \gamma\right\} \\
&= P\left(\max_k\;\max_j\;\frac{1}{n}\,|\sum_{i=1}^{n}\;T_{ijk}\,|\;\leq\;\gamma\right) \\
&= P\left(\cap_{k=1}^{d}\;\cap_{j=1}^{d}\;\left\{\frac{1}{n}\,|\sum_{i=1}^{n}\;T_{ijk}\,|\;\leq\;\gamma\right\}\right) \\
&= 1 - P\left(\cup_{k=1}^{d}\;\cup_{j=1}^{d}\;\left\{\frac{1}{n}\,|\sum_{i=1}^{n}\;T_{ijk}\,|\;>\;\gamma\right\}\right) \\
&\geq 1 - \sum_{k=1}^{d}\;\sum_{j=1}^{d}\;P\left(\frac{1}{n}\,|\sum_{i=1}^{n}\;T_{ijk}\,|\;>\;\gamma\right)
\end{aligned}$$

by the result of (B.2),

$$\geq 1 - \exp\left[\;-\frac{2\,n\,\gamma^2}{M_d^2}\;+\log\,(\,2\,d^2\,)\,\right] \tag{B.3}$$

With

$$\gamma\;\geq\;M_d^2\,\sqrt{\frac{\log\,(\,2\,d^2\,)+\tau}{2n}}\;,$$

it holds,

$$P\,(\,{B_2}'\,)\;\geq 1 - e^{-\tau}.$$

let

$$\gamma \;\geq\; M_d^2 \sqrt{\frac{\log\,(\,2\,d^2\,) + \log n}{2n}} \;\equiv \gamma^*,$$

then,

$$P\,(\,B_2'\,) \;\geq\; 1 - \frac{1}{n}.$$

Therefore, when $\lambda > \lambda^*$, $\gamma > \gamma^*$,

$$P\,(\,B\,) \;>\; 1 - \frac{2}{n}.$$

□

B2. Proof of Theorem 3.2

Proof of Theorem 3.2.

Define

$$V_{\lambda,\gamma} = \sup_{\hat{\beta} \in \mathcal{B}(\lambda,\gamma)} \frac{(L_n - L)(\rho_{\beta^*} - \rho_{\hat{\beta}})(Z, Y)}{a_0 + \| \hat{\beta} - \beta^* \|_1} \tag{B.4}$$

where

$$\mathcal{B}(\lambda, \gamma) := \left\{ \hat{\beta} \in R^d : \hat{\beta} = \underset{\hat{\beta} \in \mathcal{C}(\lambda,\gamma)}{\arg\min} \| \hat{\beta} \|_1 \right\} \tag{B.5}$$

Similar to Theorem 2.2, we patition $\mathcal{B}_{(\lambda,\gamma)}$ into $\{\mathcal{B}_j\}_{j=0}^{J}$:

$$\mathcal{B}_0 = \{\hat{\beta} : \hat{\beta} \in \mathcal{B}_{(\lambda,\gamma)}, \|\hat{\beta} - \beta^*\|_1 \leq a_0\}$$

$$\mathcal{B}_j = \{\hat{\beta} : \hat{\beta} \in \mathcal{B}_{(\lambda,\gamma)}, a_{j-1} < \|\hat{\beta} - \beta^*\|_1 \leq a_j\}; (1 \leq j \leq J-1)$$

$$\mathcal{B}_J = \{\hat{\beta} : \hat{\beta} \in \mathcal{B}_{(\lambda,\gamma)}, \|\hat{\beta} - \beta^*\|_1 > a_{J-1}\} \tag{B.6}$$

For $1 \leq j \leq J-1$:

$$a_j = 2a_{j-1};$$

by Assumption C_5, it holds:

$$a_{J-1} \geq 2\|\beta^*\|_1 \qquad and \qquad a_0 \geq \frac{\|\beta^*\|_1}{2^J};$$

Then, we can derive the bound according to this partition $\mathcal{B}_{(\lambda,\gamma)}$ as follow:

$$P(V_{\lambda,\gamma} > \delta_0) = P\Big(\sup_{\hat{\beta} \in \mathcal{B}_{(\lambda,\gamma)}} \frac{(L_n - L)(\rho_{\beta^*} - \rho_{\hat{\beta}})}{a_0 + \|\hat{\beta} - \beta^*\|_1} > \delta_0 \Big)$$

$$\leq \sum_{j=0}^{J} P\Big(\sup_{\hat{\beta} \in \mathcal{B}_j} \frac{(L_n - L)(\rho_{\beta^*} - \rho_{\hat{\beta}})}{a_0 + \|\hat{\beta} - \beta^*\|_1} > \delta_0 \Big). \tag{B.7}$$

to be simplified, let

$$V_j = \sup_{\hat\beta \in \mathcal{B}_j} \frac{(\, L_n - L\,)\,(\,\rho_{\beta^*} - \rho_{\hat\beta}\,)}{a_0 + \|\hat\beta - \beta^*\|_1}(Z, Y) \tag{B.8}$$

then (B.7) is equivalent to

$$P\,(\,V_{\lambda,\gamma} > \delta_0\,) \le \sum_{j=0}^{J} P\,\Big(V_j > \delta_0\,\Big).$$

According to Lemma B.1: For $0 \le j \le J-1$

$$P\,\Big(V_j > \delta_0\,\Big) < 2\,e^{-2n}$$

By Theorem 3.1, when $\lambda > \lambda^*$ and $\gamma > \gamma^*$, it holds

$$\begin{aligned} P\,\Big(V_J > \delta_0\,\Big) &\le P\Big(\,\|\hat\beta - \beta^*\|_1 > a_{J-1}\Big) \\ &\le P\Big(\,\|\hat\beta - \beta^*\|_1 > 2\,\|\beta^*\|_1\Big) \\ &\le P\,(\,\beta^* \notin \mathcal{C}_{\lambda,\gamma}\,)\ \le\ \frac{2}{n} \end{aligned}$$

Thus,

$$P\,\Big(V_{\lambda,\gamma} > \delta_0\,\Big) < 2J\,e^{-2n} + \frac{2}{n}.$$

It comes to conclude that, with probability at least $1 - 2J\,e^{-2n} - \frac{2}{n}$, it holds:

$$V_{\lambda,\gamma} := \sup_{\hat\beta\in\mathcal{B}_{(\lambda,\gamma)}} \frac{|(\,L_n - L\,)\,(\,\rho_{\beta^*} - \rho_{\hat\beta}\,)(Z, Y)|}{a_0 + \|\beta^* - \hat\beta\|_1}\ < \delta_0.$$

Since our estimator $\hat\beta_{MHCS} \in \mathcal{B}_{(\lambda,\gamma)}$, it holds

$$\begin{aligned} &\frac{|\,(L_n - L\,)\,(\,\rho_{\beta^*} - \rho_{\hat\beta_{MHCS}}\,)\,(\,Z,\,Y\,)\,|}{a_0 + \|\hat\beta_{MHCS} - \beta^*\|_1} \\ &\le \sup_{\hat\beta\,\in\,\mathcal{B}(\lambda,\gamma)} \frac{|(L_n - L)\,(\,\rho_{\beta^*} - \rho_{\hat\beta}\,)\,(\,Z,\,Y\,)\,|}{a_0 + \|\hat\beta - \beta^*\|_1} \le \delta_0 \end{aligned}$$

thus,

$$L_n\ (\ \rho_{\beta^*} - \rho_{\hat{\beta}_{MHCS}}\ ) - L\ (\ \rho_{\beta^*} - \rho_{\hat{\beta}_{MHCS}}\ ) \leq\ \delta_0\ a_0\ + \delta_0 \parallel \beta^* - \hat{\beta}_{MHCS} \parallel_1 .$$

rearrange the orders, then

$$\mathcal{E}(\hat{\beta}_{MHCS}) = L\ (\ \rho_{\hat{\beta}_{MHCS}} - \rho_{\beta^*}\ )$$

$$\leq L_n\ (\ \rho_{\hat{\beta}_{MHCS}} - \rho_{\beta^*}\ ) +\ \delta_0 \parallel \beta^* - \hat{\beta}_{MHCS} \parallel_1\ +\ \delta_0\ a_0$$

since:

$$\left|\ L_n \rho_{\hat{\beta}_{MHCS}}\ (\ Z,\ Y\ ) - L_n \rho_{\beta^*}\ (\ Z,\ Y\ )\ \right|$$

$$\leq\ \parallel \nabla_\beta\ L_n\ \rho_{\hat{\beta}_{MHCS}}(\ Z, Y\ ) \parallel_\infty \parallel \hat{\beta}_{MHCS} - \beta^* \parallel_1$$

by Lemma B.2

$$\leq (\ 3\,\lambda + 2\,\gamma \parallel \hat{\beta}_{MHCS} \parallel_1\ ) \parallel \hat{\beta}_{MHCS} - \beta^* \parallel_1;$$

then,

$$\mathcal{E}(\hat{\beta}_{MHCS})\ = L\rho_{\hat{\beta}_{MHCS}}(\ Z,\ Y\ ) - L\rho_{\beta^*}(\ Z,\ Y\ )$$

$$\leq \left|L_n\rho_{\hat{\beta}_{MHCS}}(\ Z,\ Y\ ) - L_n\rho_{\beta^*}(\ Z,\ Y\ )\right| + \delta_0 \parallel \beta^* - \hat{\beta}_{MHCS} \parallel_1 + \delta_0\ a_0$$

$$\leq (\ 3\,\lambda + 2\,\gamma \parallel \hat{\beta}_{MHCS} \parallel_1\ ) \parallel \beta^* - \hat{\beta}_{MHCS} \parallel_1$$

$$+\ \delta_0 \parallel \beta^* - \hat{\beta}_{MHCS} \parallel_1 + \delta_0\ a_0\,;$$

Under $\beta^* \in \mathcal{C}_{\lambda,\gamma},\ \parallel \hat{\beta}_{MHCS} \parallel_1 \leq \parallel \beta^* \parallel_1;$

$$\mathcal{E}(\hat{\beta}_{MHCS}) \leq\ \left(\ 3\,\lambda + 2\,\gamma \parallel \beta^* \parallel_1 + \delta_0\ \right) \parallel \beta^* - \hat{\beta}_{MHCS} \parallel_1\ +\ \delta_0\, a_0\,.$$

□

Lemma B.1. *As $\mathcal{B}_j$, V_j, defined in (B.6), (B.8), for* $0 \leq j \leq J-1$, *it holds:*

$$P\,(\,V_j > \delta_0\,) < 2\,e^{-2n}.$$

Proof. Analogous to Theorem 2, we have following results for V_j:

$$P(\,|\,V_j - E\,(\,V_j\,)\,|\ > \delta_1\,) < 2\,e^{-2n};$$

and

$$E(\,V_j\,)\ \leq\ \delta_2;$$

set $\delta_0 = \delta_1 + \delta_2$, then

$$P(\,V_j >\ \delta_0\,) < 2\,e^{-2n}.$$

□

Lemma B.2.

With same notations in Theorem 3.2, when $\lambda > \lambda^*$ and $\gamma > \gamma^*$, then with probability at least $1 - \frac{2}{n}$, it holds:

$$\frac{1}{n}\left\|\,Z^T\,[\,Y - \mu\,(\,Z\hat{\beta}_{MHCS}\,)\,]\,\right\|_\infty\ \leq\ 3\,\lambda + 2\,\gamma\,\|\,\hat{\beta}_{MHCS}\,\|_1\,.$$

Proof.

By triangle inequality,

$$\begin{aligned}
\frac{1}{n} \left\| Z^T [\, Y - \mu (Z\hat{\beta}_{MHCS}) \,] \right\|_\infty &- \frac{1}{n} \left\| \Xi^T [\, Y - \mu (Z\hat{\beta}_{MHCS}) \,] \right\|_\infty \\
&- \frac{1}{n} \left\| W^T \mu' (\xi \, \hat{\beta}_{MHCS})(\Xi\hat{\beta}_{MHCS}) \right\|_\infty \\
\leq \frac{1}{n} \Big\| Z^T [\, Y - \mu (Z\hat{\beta}_{MHCS}) \,] \, &+ \, \Xi^T [\, Y - \mu (Z\hat{\beta}_{MHCS}) \,] \\
&- \, W^T \mu' (\xi \, \hat{\beta}_{MHCS}) (\Xi\hat{\beta}_{MHCS}) \Big\|_\infty \\
= \frac{1}{n} \left\| W^T [\, Y - \mu \, (W\hat{\beta}_{MHCS}) \,] \right\|_\infty .
\end{aligned}$$

Thus, after rearranging orders, the gradient of target population through high confidence set estimation, would be bounded by the following three parts.

$$\begin{aligned}
\frac{1}{n} \left\| Z^T [\, Y - \mu (Z\hat{\beta}_{MHCS}) \,] \right\|_\infty &\leq \frac{1}{n} \left\| W^T [\, Y - \mu \, (W\hat{\beta}_{MHCS}) \,] \right\|_\infty \\
+ \frac{1}{n} \left\| \Xi^T [\, Y - \mu (Z\hat{\beta}_{MHCS}) \,] \right\|_\infty &+ \frac{1}{n} \left\| W^T \mu' (\xi \, \hat{\beta}_{MHCS}) (\Xi\hat{\beta}_{MHCS}) \right\|_\infty
\end{aligned}$$

First, since $\hat{\beta}_{MHCS} \in \mathcal{C}_{\,(\,\lambda,\,\gamma\,)}$, by definition of $C_{(\lambda,\gamma)}$, it holds,

$$\frac{1}{n} \, \| \, W^T [\, Y - \mu \, (W\hat{\beta}_{MHCS}) \,] \, \|_\infty \, \leq \, \lambda + \, \gamma \, \| \, \hat{\beta}_{MHCS} \, \|_1; \tag{B.9}$$

Similar to the process in proving previous theorem $Event\ B_1$ and $Event\ B_2$, under the condition $\lambda \geq \lambda^*$ and $\gamma \geq \gamma^*$, the second part and third part will be bounded with high probability:

$$P \left(\frac{1}{n} \left\| \Xi^T [\, Y - \mu (Z\hat{\beta}_{MHCS}) \,] \right\|_\infty \leq 2 \, \lambda \right) \, > 1 - \frac{1}{n}; \tag{B.10}$$

$$P \left(\frac{1}{n} \left\| W^T \mu' (\xi \, \hat{\beta}_{MHCS}) (\Xi\hat{\beta}_{MHCS}) \right\|_\infty \leq \gamma \, \| \, \hat{\beta}_{MHCS} \, \|_1 \right) > 1 - \frac{1}{n}; \tag{B.11}$$

Thus by De Morgan's Law again,

$$P\left(\frac{1}{n}\left\|Z^T\left[Y-\mu\left(Z\hat{\beta}_{MHCS}\right)\right]\right\|_\infty \leq 3\,\lambda + 2\,\gamma\,\|\,\hat{\beta}_{MHCS}\,\|_1\right) \geq 1-\frac{2}{n}.$$

□

B3. Proof of Theorem 3.3

proof of Theorem 3.3.

By the definition of $\mathcal{C}_{(\lambda,\gamma)}$, it holds:

$$\| \nabla_\beta L_n \rho_{\hat{\beta}_{MHCS}} (W, Y) \|_\infty \le \lambda + \gamma \| \hat{\beta}_{MHCS} \|_1;$$

Condition on $\beta^* \in \mathcal{C}_{(\lambda,\gamma)}$,

$$\| \nabla_\beta L_n \rho_{\beta^*} (W, Y) \|_\infty \le \lambda + \gamma \| \beta^* \|_1;$$

then by the triangle inequality,

$$\begin{aligned} & \| \nabla_\beta L_n \rho_{\hat{\beta}_{MHCS}} (W, Y) - \nabla_\beta L_n \rho_{\beta^*} (W, Y) \|_\infty \\ & \le \| \nabla_\beta L_n \rho_{\hat{\beta}_{MHCS}} (W, Y) \|_\infty + \| \nabla_\beta L_n \rho_{\beta^*} (W, Y) \|_\infty \\ & \le 2\lambda + \gamma \| \hat{\beta}_{MHCS} \|_1 + \gamma \| \beta^* \|_1 \le 2\lambda + 2\gamma \| \beta^* \|_1 \end{aligned} \tag{B.12}$$

Thus, the corresponding result of (A.27) is:

$$\begin{aligned} & \delta L_n \rho_{(\hat{\Delta}, \beta^*)}(W, Y) \\ & := L_n \rho_{(\beta^* + \hat{\Delta})}(W, Y) - L_n \rho_{\beta^*}(W, Y) - \left\langle \nabla_\beta L_n \rho_{\beta^*}(W, Y), \hat{\Delta} \right\rangle \\ & \le \| \nabla_\beta L_n \rho_{\hat{\beta}_{MHCS}} (W, Y) - \nabla_\beta L_n \rho_{\beta^*} (W, Y) \|_\infty \| \hat{\Delta} \|_1 \\ & \le 2 (\lambda + \gamma \| \beta^* \|_1) \| \hat{\Delta} \|_1 \end{aligned} \tag{B.13}$$

Under $Event\ B$, $\|\hat{\beta}_{MHCS}\|_1 \le \|\beta^*\|_1$, thus, (A.28) and (A.29) still valid.

By $Assumption\ C_7$,

$$\delta L_n \rho_{(\hat{\Delta}, \beta^*)} (W, Y) \ge \kappa \| \hat{\Delta} \|_2^2;$$

combine with (B.13) and (A.29) we have,

$$\kappa \parallel \hat{\Delta} \parallel_2^2 \;\leq\; 2\,(\,\lambda + \gamma \parallel \beta^* \parallel_1\,) \parallel \hat{\Delta} \parallel_1 \;\leq\; 4\,(\,\lambda + \gamma \parallel \beta^* \parallel_1\,)\,\sqrt{s}\,\parallel \hat{\Delta} \parallel_2;$$

therefore,

$$\parallel \hat{\Delta} \parallel_2 \;\leq\; \frac{4\,(\,\lambda + \gamma \parallel \beta^* \parallel_1\,)\,\sqrt{s}}{\kappa}; \tag{B.14}$$

plug (B.14) into (A.29), we have

$$\parallel \hat{\Delta} \parallel_1 \;\leq\; \frac{8\,(\,\lambda + \gamma \parallel \beta^* \parallel_1\,)\,s}{\kappa}. \tag{B.15}$$

□

[Corollary] Under Assumption C_1-C_7, when $\lambda > \lambda^*$, $\gamma > \gamma^*$, with probability at least $1 - 2J\,e^{-2n} - \frac{2}{n}$, it holds that:

$$\mathcal{E}(\hat{\beta}_{MHCS}) \leq \frac{8s(\,\lambda + \gamma\|\beta^*\|_1\,)\,(3\,\lambda\, + 2\gamma\|\beta^*\|_1 + \delta_0)}{\kappa} \;+\; \delta_0\; a_0$$

Proof. Take the result of (B.15) into Theorem 3.2, the result can be achieved. □