USES OF ARGUMENTATION IN AN UNDERGRADUATE CALCULUS CLASSROOM: MEDIATING STUDENT GENERALIZATION WITHIN RIEMANN SUMS AND INTEGRATION

by

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ABSTRACT

ELIZABETH BUMGARDNER. Uses of argumentation in an undergraduate calculus classroom: mediating student generalizations within Riemann sums and integration. (Under the direction of DR. ADALIRA SÁENZ-LUDLOW)

Many of the struggles students face in postsecondary mathematics courses arise from their rote memorization of mathematical concepts, rather than having developed a relational understanding of these concepts. A relational understanding of mathematical concepts could allow students to form generalizations and extend the ideas they understand to new similar concepts. This study investigated how argumentation in a calculus-based undergraduate mathematics classroom can be used as a cognitive tool to mediate mathematical generalization.

A classroom teaching experiment was conducted with twenty-nine students who were registered in two sections of a senior level undergraduate mathematics course. Formal instruction for the study took place during four teaching episodes in which the Teacher-Researcher (TR) modeled argumentation in her presentation of course content to emphasize why she made the steps she did when constructing Riemann sums and extending the ideas to other concepts. The concepts were approached first graphically. Then these ideas were extended by constructing informal equations, followed by constructing formal equations, and then followed by implementing these equations into Excel.

For each teaching session, the students participated in class activities and related homework tasks where they were asked to justify their work. Following the four

teaching sessions, each student participated in two one-to-one interviews where they worked through the concepts involved in Riemann sums and related topics. As in the class activities, students were asked to approach the topics graphically, using informal equations, using formal equations, implementing these equations into Excel, and, lastly, by applying these ideas to an area of the individual students' interest.

The interviews and written work of the students were analyzed under the lens of Vygotsky's zones of proximal development and levels of generalization, along with Toulmin's argumentation model. Three case studies were selected based on the types of learning demonstrated by the students: visual learners, visual and symbolic learners, and non-visual learners. Each of these cases was further broken into categories based on the similarities and differences in their interview tasks and the key strengths or weaknesses demonstrated in their coordination between visualization and conceptualization in the context of Riemann sums.

Results showed that the non-visual learners who either lacked symbolization or only demonstrated procedural symbolization did not attempt to verbalize argumentation and were unable to form meaningful generalizations regarding Riemann sums concepts. However, the other two cases each had students who were able to form meaningful generalizations. The distinction in these cases lies within the sub-categories of the cases with the most polarized results falling within the case of visual learners. The visual learner who used argumentation was able to form generalizations while the visual learner who did not use argumentation was unable to form the needed generalizations. Generalization formation among the visual and symbolic learners appeared more student

specific; while students who used argumentation did form generalizations, there were also students who formed generalizations without articulating the use of argumentation. Student feedback during the interviews indicated that many students believed the structure of the tasks and the emphasis on justifying their own thinking had a positive impact on their learning.

DEDICATION

This work is dedicated to my former and current mathematics and statistics students. You have inspired me to pursue my work in understanding how students learn. Because of you, I want to help others to fall in love with learning, and specifically, to fall in love with learning and utilizing mathematics and statistics.

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ZPD	Zone of proximal development
TR	Teacher-Researcher

CHAPTER 1: INTRODUCTION

Mathematics is an interconnected cultural system that was developed, in large part, out of the necessity for mathematical approaches that could help people solve problems within their everyday lives (Wilder, 1981). Generalizations were often formed throughout the development of these mathematical approaches. For example, counting takes the concept of the number of objects, for instance one person or one tree, and assigns this object the numeral 1. While one person is not the same as one tree, both are one of their own type of object. The very labeling of **1** to represent each of these objects is an act of generalization; the similarity between the two different objects as each being "one" of their own type has been identified. This is "one" type of generalization brought about by everyday necessity throughout different cultures.

Historically, generalizations within our mathematical cultural system arose naturally and often due to necessity, however, this "natural and necessary development" is far removed from the teaching-learning environment in which we expect students to form similar generalizations today. Often mathematics classrooms are rigid, culturally sterile environments in which students are expected to form conceptual connections and solve problems involving topics that are disconnected from their own life experiences. These types of classrooms are in direct opposition to those which would support the sociohistorical views of Vygotsky, who emphasized that intellectual development is the result of one's interactions with others and their social environments in the classroom, at home, or elsewhere (Moll, 1996). While it is important that strides be made by teachers, parents, and administrators to revamp today's mathematics classrooms to accept and utilize the cultural history of mathematics, this study approaches this topic only from the perspective of the teacher. How can teachers help students to form mathematical generalizations under their current time, experiences, and curricular restrictions? Regardless of these restrictions, what physical and mental tools can be used in a classroom to help students form mathematical generalizations? What kind of social environment can be motivated, supported, and sustained to form mathematical generalizations?

George Pólya (1957) proposed that there are two faces of mathematics: one showing mathematics as a systematic and deductive science, in the manner of Euclid, and the other showing the creative part of mathematical solutions, which appears to be an experimental and inductive science. Yes, teachers can certainly show students the steps to solve a problem, but are teachers helping students to create solutions? In his quest to understand how to help others learn and teach mathematics, Pólya (1957) explained the emergence of cognitive processes for creating and developing mathematical solutions rather than focusing solely on answers. In his books, he explicitly discussed the importance of forming generalizations, and then through examples, he gave the readers the opportunity to create their own solutions and to form generalizations on their own. It is this focus on helping students to develop cognitive processes to create solutions for mathematical problems, rather than focusing on answers alone, that distinguishes Pólya's approach to the teaching and learning of mathematics from many current teaching-learning practices.

It is a goal of this study, as it was the goal of Pólya's books *How to Solve It* (1957) and *Mathematical Discovery (*1962), to help students develop cognitive processes that can lead to the *creation* of mathematical solutions and the formation of mathematical generalizations. In forming generalizations, students must distinguish what is similar and what is different about the mathematical concepts. Categorizing these similarities and differences involves making comparisons and justifying to oneself and to others why the concepts should be categorized as they have. While justifying these categories, processes of argumentation emerge and come to play an important role in the formation of generalizations.

Argumentation is, according to Toulmin (1958), a systematic process of reasoning. In the context of this study, argumentation is taken as a progressive chain of connected justifications that leads to the construction of a persuading or a convincing mathematical argument for the solution of a mathematical task. While the Toulmin model provides the components of an argument that allow us to both develop and determine the quality of a constructed argument, Meany and Shuster (2002) extended the ideas of Toulmin to include a component of modality. This social component is based on the level of certainty of the arguer. Perelman and Olbrechts-Tyteca (1958) also extended the ideas of Toulmin to include the importance of the audience and their impact on the construction of each of Toulmin's components. The approach of this study takes into consideration these extensions of Toulmin's model, and these extensions are viewed as necessary under the framework of the study.

This study investigated how argumentation, under the lens of Toulmin and the extension of others, can be developed in the classroom to enable upper level undergraduate students to form generalizations in a calculus-based course. The content and context of argumentation was Riemann sums and their application in integration.

This study conducted a classroom teaching experiment to investigate this topic. The experiment involved classroom teaching sessions conducted by the Teacher-Researcher (TR), follow-up homework problems for the students, and one-to-one interviews of the students conducted by the TR. Throughout the semester, the emphasis of the teaching sessions was to model argumentation while developing the concepts of Riemann sums as approximations for area under the curve of a graph. Special care was given to provoke and mediate generalization beginning with a graphical analysis of finding the area under a curve between two specific x-values using left and right endpoints. The tasks then moved towards development of formal notation for these calculations that could be further generalized to any number of rectangles, functions, and any interval on which the function is defined. These ideas were then implemented using Excel software so that both argumentation and calculations could be made regarding accuracy of the approximations to integrals. It is important to notice that in this teaching experiment some students could more easily generate implicit arguments to persuade while others were able to generate somewhat more explicit arguments to convince their peers as well as the Teacher-Researcher (TR).

1.1 STATEMENT OF THE PROBLEM

Since a student's ability to generalize and relate different mathematical concepts is a key component to their success in a calculus-based course, the topic of generalization was the objective of this study. Generalization is rooted in most every aspect of mathematics, from the historical development of counting discussed above, to visual patterns, to the formation of an algebraic equation to solve a word problem. The hopeful outcome of instruction in a mathematics course, particularly at this level, is that students will be able to: (i) calculate a value as an answer to a problem; (ii) justify to themselves and to others why their calculations are correct; (iii) provide an interpretation of the numerical solution in the context of a problem; and (iv) form connections between the problem situation and other similar situations. The reality is that many students who take this course (i.e., mathematics majors with numerous prerequisite mathematics and statistics courses) still offer a calculated value with no justification instead of a full solution to an in-depth question that requires justification and a mathematical writing of their work. This indicates an underlying issue and disconnect between the goals of this calculus-based course and the cognitive processes developed by students in this and previous mathematics courses.

Students and, unfortunately, teachers often approach mathematics as the rote memorization of facts, rather than as a deep understanding of concepts and their relationship to each other in coordination to the students' prior cognitive experiences. Course after course of such methods has led students, and in particular graduating mathematics majors, to be unable to justify their solutions and to see little to no relationship in their solutions and life applications (Schoenfeld, 1991). Nonetheless, generalization takes shape in the mathematics classroom in which teaching-learning experiences are guided in a direction that prioritizes conviction over persuasion. Students may form connections, some correct and some incorrect, between mathematical concepts based on conviction alone as indicated by their teacher and peers. However, these same students who have little experience in utilizing persuasion in their learning process also lack a deep understanding of concepts and their connections to the real world. These students have little understanding of why those connections exists, how they know without a doubt that the connections formed are accurate, and how they can further extend these connections to other concepts outside of the direct instruction or influence of their teacher and peers.

In this study, generalization is defined as a student's distinguishing between similarities and differences among mathematical concepts. When students are able to make these connections between concepts, there is little need for rote memorization of facts about each concept, as concepts are simply extensions and generalizations of other known concepts and possible applications.

How can teachers mediate generalization in student thinking, and the role of silent or aloud verbal argumentation in the process? What methods or tools can be taught to students that would help them form these generalizations as they use progressive and cyclic processes of argumentation? By means of reflection, or thinking about their own thoughts, students should become aware of why each step in the solution of a problem is correct and necessary. This process of argumentation with self and with others facilitates the formation of generalizations that could be applied to other problems.

When particular mathematical steps are performed and justified, they will both yield the desired result and provide a deeper understanding of the concept involved. In a similar manner, argumentation will also provide a means for making decisions about the similarities and differences between concepts. Understanding why something is similar or different allows students to form generalizations by means of argumentation. When generalizations formed are appropriate and accurate, students will be able to see both the particular in the general and the general in the particular.

As simple as it could seem, little research has been done with respect to the teaching of argumentation techniques as a mediating cognitive tool to identify similarities and differences of mathematical concepts and to form mathematical generalizations.

1.2 RESEARCH QUESTIONS

This study aims to better understand the correlation associated with argumentation and students' formations of mathematical generalizations. This research project is guided by the following questions:

1. How do students in a calculus-based course justify their mathematical solutions, and how does argumentation evolve throughout the teaching experiment?
2. How do students in a calculus-based course argue their mathematical solutions after experiencing argumentation modeled by the Teacher-Researcher (TR) in the classroom?

3. How do students in a calculus-based course generalize within the topic of Riemann sums and integration when presented with sequential tasks?
4. How does the mathematical argumentation used by students in a calculus-based course prompt the formation of generalizations within the topic of Riemann sums and integration when presented with sequential tasks?

1.3 SIGNIFICANCE OF THE STUDY

The above research questions will expand the extant investigations on the understanding of argumentation and generalizations. Much research has been done on mathematical generalizations, for example the work of Davydov (1990), Buck (1995), and Becker and Rivera (2005). Some research has also been done on argumentation, for example that of Mariotti and Maracci (1999), Walton et al. (2008), and Arzarello et al. (2009). While the work of each of these authors will be discussed in the literature review, it should be noted that the direct connection between argumentation as a means toward generalization has not been the intention of the studies on generalization. Studies on argumentation of the above authors and others in the literature review have often focused on describing the process of argumentation itself rather than seeing it as a cognitive tool to facilitate students' formations of generalizations. Answering the research questions, this study provides a glimpse into the usefulness of argumentation as a mental cognitive tool to mediate students' processes of generalization.

The study of argumentation as a means of generalization is of high importance as students often struggle greatly in mathematics due to poor processes of generalization. Granted, relational understanding, in Skemp's sense (1987), is also an integral part of the process of generalization. Knowledge about how students construct relational understanding and how teachers encourage, promote, and facilitate such understanding may have a major impact in mathematics education, not just within the context of calculus. Argumentation contributes to students' development of conceptual schemes to help them to make connections within and between mathematical concepts and to reason rationally in different mathematical topics in the mathematics curriculum. The study of argumentation as a cognitive tool for generalization benefits teachers and students. Teachers can become aware of how and why they can model argumentation in the classroom to help students gain and increase their relational understanding of mathematical concepts, which will also induce the formation of mathematical generalizations. In turn, students will become aware of their capacity to increase their mathematical understanding and their own processes of generalization.

As an example of the importance of students forming these mathematical connections and generalizations, consider the current mathematics curriculum in North Carolina public schools. The Common Core, which was adopted by the North Carolina school system and others across the nation, focuses on the ability of students to connect and generalize mathematical concepts (Hirsch et al., 2012). This means that it is imperative that students be able to conceptualize these relations of similarities and differences among concepts to be able to generalize. The Common Core Standards are described as a method for "preparing students with the skills they need to succeed in college and work," "creating mathematically literate workers," and "encouraging life-long learning" (Hirsch et al., 2012). These authors promoted ideas such as "active learning” and “multiple ways of approaching problems.” The mathematics standard topics include operations and algebraic thinking, number operations in base ten and also with fractions, measurement and data, and geometry (Hirsch et al., 2012). Each of these topics involves the need for the process of generalization, which is referred to by the authors Hirsch et al. as “habits of thinking.”

For students to be successful in any mathematics course, they have to be able to gain both relational understanding of concepts and the application of those concepts to particular situations as well as to be able to generalize. For example, within the framework of the Common Core State Standards, students will be working with number, place value, base ten, and a slew of conceptually rich topics that demand students be able to generalize to master such concepts. At the university level, students are expected to further generalize the concepts learned under the Common Core State Standards, and apply these concepts to real world scenarios relative to their field of study. Students at the university level are also expected to be critical thinkers based on what they learned under the Common Core State Standards. Since developing students as critical thinkers is of importance at both the high school and university level, there is an absolute need to investigate the formation of mathematical connections and generalizations.

Even though many teachers have seen the need for every student to master mathematical concepts, the Common Core Standards bring this necessity to the forefront for teachers, students, and educational institutions. If students are going to be able to meet these standards, teachers must learn themselves to generalize in order to help students to do the same. Any means that would help students to do so, such as using argumentation to stimulate conceptual understanding and generalizations, should be of great value in the students' education. Although it is not known how much teaching can help the rate of development of students' generalization processes, it is important to investigate if it is possible to help students to make generalization processes habits of thinking.

The classroom teaching experiment in this study allowed participant students to see how argumentation is used to develop the steps that are needed to solve mathematical problems, rather than simply regurgitating and blindly following steps they are told to use. The study also provided students with the opportunity to use argumentation with the TR in the classroom setting, on their own in the homework tasks, and in one-to-one interviews with the TR where they could further develop their argumentation skills. Students were able to build confidence using argumentation to justify their work and to see how concepts that were previously viewed as separate ideas are in fact just a sensible application of a general concept to a particular problem-situation. These habits of thinking can help students to make connections within mathematics and to reason rationally within and between topics.

While mathematicians certainly use argumentation in the development of mathematical concepts, and while mathematics education researchers have discussed argumentation and student learning, very little work has addressed how argumentation can, and should be, modeled in the classroom. Furthermore, argumentation has been connected to student problem solving and problem modeling, but it has not been studied specifically in the context of mediating students' generalization of concepts. This study may benefit teachers and students through knowledge gained about how teachers can model argumentation and particularly how this can be used to help students use argumentation as a tool to form mathematical connections and generalizations.

CHAPTER 2: THEORETICAL FRAMEWORK

The theoretical framework for this study is based in Vygotsky's social theory of cognitive development. His notion of zones of proximal development (ZPD) is essential both to follow students' cognitive progression toward generalization by means of argumentation and to categorize the data during analysis. The tasks proposed to the students should trigger their progression through the personal zones of proximal development by means of argumentation to achieve generalizations. Within the overarching theme of sociocultural theory, consideration must also be given to the historical foundations of both argumentation and generalization for the context of this research.

2.1 SOCIOCULTURAL THEORY

The theoretical basis for this study is grounded in Vygotsky's sociocultural theory. This theory of human cognition involves both lower and higher order mental functions. Lower order mental functions, also referred to as natural mental functions, are considered as direct and innate behavioral responses to stimuli (Vygotsky, 1998). In contrast, higher order mental functions, or cultural mental functions, are formed during the process of cultural development (Vygotsky, 1998). These higher order functions include perception, emotions, verbal thought, logical memory, language, selective attention, and learned information (Moll, 1996). The evolution of natural mental processes into cultural mental processes is achieved through social interactions and symbolic tools (Kozulin, 1998).

The teaching of mathematics, and specifically the mathematics classroom setting for this study (see methodology), is the perfect arena for social interactions and the use of symbolic tools described by Vygotsky that, in turn, allow the psychological processes of individuals to evolve. These social interactions include those between teachers and student and among students themselves. The symbolic tools, such as signs of all sorts, mathematical symbols, text, formulas, and language, also extend to include beliefs, values, and abstract knowledge. Mediation is the process by which symbolic tools organize the activities of the individual and how the individual relates to their environment (Kozulin, 1990). The process of mediation connects what would otherwise be separate, and Vygotsky further suggested that the development of higher order functions stem from mediated activity (Kozulin, 1990). Mediation also triggers the formation of "zones of proximal development" that in turn triggers the formation of generalizations by means of argumentation.

Sáenz-Ludlow and Zellweger (2016) shed further light on how students construct and use symbolic tools and how tools transform their own understanding of mathematical concepts. Following Peirce's semiotic theory, they identified the Real Mathematical Object as it is seen and understood by mathematicians. Then, the goal of teachers is to move students' processes of intra-inter interpretation of mathematical objects from their own view or conception of the mathematical object towards the Real Mathematical or Concept Object (Sáenz-Ludlow & Zellweger, 2016). This is to say that the initial mathematical conceptions of the students progressively evolve and converge towards the Real Mathematical Object or mathematical concept (Sáenz-Ludlow & Zellweger, 2016).

It is through diagrammatic reasoning, discussed in Sáenz-Ludlow's article "Iconicity and Diagrammatic Reasoning in Meaning-Making," that such convergence occurs by means of an emergent progressive sequence of zones of proximal development (2018). Sáenz-Ludlow (2018) argued that mathematical diagrams facilitate inferential thinking that progressively develops into deductive thinking or symbolic level. The symbolic level is necessary for abstract mathematical thinking, a deep understanding of mathematical concepts, and the formation of generalizations. The students' prior knowledge and collateral knowledge, or knowledge gained through experience that was not intended by the teacher, affect how they interpret a diagram (as an icon, as an icon with added indexical traits, or as an icon with iconic-indexical-symbolic traits) (Sáenz-Ludlow, 2018). The students' mental transformation of mathematical diagrams (including graphs, tables, and mathematical formulas, etc.) stimulates their reasoning processes to achieve higher levels of understanding and therefore the convergence towards the Real Mathematical Object or mathematical concept (Sáenz-Ludlow, 2018). These progressive transformations appear to be processes of abstraction and generalization.

In this semiotic perspective, students' conceptions of mathematical objects as ever evolving in continuous processes of abstraction and generalization that tend towards the Real Mathematical Object or mathematical concept are mediated by the mental transformations of mathematical diagrams that happen on the students' zones of proximal development. These diagrams, when transformed, constitute themselves in the minds of the students to epistemological tools for understanding mathematical concepts. This is a complementary perspective that is used in the analysis of the data. This study investigated whether mathematical diagrams promote argumentation that mediated abstraction and generalization of Riemann sums moving students along a conceptual continuum of zones of proximal development.

2.2 ZONES OF PROXIMAL DEVELOPMENT

Vygotsky's zone of proximal development (ZPD) is the difference between what a student can perform on his or her own, what the student can perform with help, and what the student is unable to perform (Vygotsky, 1978). These conceptual zones are identified in two ways: (i) the cognitive processes of the learner that are still being developed with assistance of others; and (ii) the fully developed cognitive processes of the learner in which no assistance is needed (Vygotsky, 1998). Vygotsky (1998) conceptualized these two categories as the "potential development level" and the "actual development level" of the learner.

This study used the notion ZPD to analyze students' progress toward generalizations. Because a student cannot be forced to understand something that they are not ready to understand, the one-to-one student interviews focused on the areas where students were able to work either without help or with the help of the Teacher-Researcher (TR).

Attempting to force all students to reach a particular zone of conceptual development during this study is contrary to Vygotsky's ZPD theory that supports the idea that a student is only able to learn what he or she is ready to learn. Rather, in working with Vygotsky's ZPD theory, this study focused on areas in which the individual student is confident and areas in which the individual student is able and ready to grow his or her understanding. In other words, the analysis focused on the actual development of the students and providing support and appropriate sequential tasks to move the students forward into their potential development level. The ZPD were also used to analyze and categorize the data collected in this study with respect to the levels of development of what a student cannot do, what a student can do on his or her own, and what a student can do with help.

2.3 ARGUMENTATION

The fundamental basis of argumentation in this study lies in the work of Stephen Toulmin. Toulmin (1958) defined the structure of an argument to consist of grounds for a claim, the claim itself, a warrant, backing, and rebuttal. The *grounds* for a claim are the evidence or data that support or justify the claim, and *the claim* is the statement of the argument that something is so (Toulmin, 1958). The *warrant*, the connection between the grounds and the claim, is the justification for how the grounds/evidence supports the claim. *Backing* is the support or justification of the warrant. Lastly, the *rebuttal* highlights any exceptions or restrictions of the claim. The rebuttal may be used to modify or qualify the initial claim into a final claim.

Meany and Shuster (2002) also addressed the component of modality in the structure of an argument to be the level of certainty of the arguer in the argument. This component, which is not in the original structure described by Toulmin, emphasized the importance of the individual presenting the argument. This highlights the social interactions that are involved in argumentation, whether the argument is written or verbal, that affect the conclusions of the argument. It is this social component of argumentation that, in this study, connects to the underlying theoretical framework of Vygotsky's sociocultural theory.

The idea that an audience is reacting to the argument and the emphasis that the audience should be influenced and convinced by the argument is grounded in the work of Perelman and Olbrechts-Tyteca (1958). While Toulmin's model of argumentation did not address the component of the audience, Perelman and Olbrechts-Tyteca proposed that the audience is a major component of the argument affecting every step of how the argument is constructed. The audience reacts to the argument, and the audience should be influenced and convinced by the argument (Perelman & Olbrechts-Tyteca, 1958). The audience of an argument in the classroom may be the students themselves, another student, or a group of individuals including the teacher.

Mathematical arguments are often viewed as having a universal audience; however, this study viewed the audience of the mathematical argument as a development from self-persuasion to convincing the TR or authority figure of the classroom. As the student argues, they follow a path of convergence from initial understanding of argumentation (self-argument) to a level of argument that may be understood and accepted by other students and the classroom teacher. Also, as each student moves along their own path of convergence, the arguments constructed become more formal, better structured, and more sophisticated because of necessity to convince the larger classroom audience (teacher and students). This study viewed the goal of the mathematical argument to move the students from a persuasive argument towards developing a convincing argument towards developing a mathematical proof for a proposition. However, due to the duration of the study and each student having his or her own ZPDs, a more practical goal was for the students to convince themselves and the TR of the validity of their ideas. Arriving to the level of mathematical proof would be more difficult to achieve. The evolution of the student arguments regarding propositions about particulars towards a general proposition and the possible emergence of a formal proof for such a proposition is in fact a generalization process.

2.4 GENERALIZATION

Vygotsky (1986) proposed that conscious use of concepts stems from systemization based on the relationship between concepts. The process of generalization is the transitioning from the description of the properties of particular objects to a description of a whole class of similar objects (Davydov, 1990); the creation of such classes stems from comparing particular objects and identifying the common elements. Generalization is directly connected to classification, which is the process of singling out what is common among objects and identifying relationships among particular objects as well as the contrasting characteristics of those particular objects (Davydov, 1990).

Vygotsky addressed the structure of generalization as comprised of a sequence or hierarchy of levels: syncretism, complex, preconcept, and concept proper (Vygotsky, 1986). Syncretism happens when one puts objects together in unorganized categories. Complex happens when one's relation between objects is concrete rather than abstract and logical; during this phase the scattered objects from the first level begin to be organized into groups. The third level towards concept formation, preconcept, allows for mental operations of the objects, but the construction of the preconcept is limited to a perceptual bond. The last level, concept proper, is reached when one is able to form a genuine concept as a result of advanced thinking and abstraction.

Vygotsky (1986) further stated that “each level, or generalization structure, has as its counterpart a specific level of generality, a specific relation of superordinate and subordinate concepts, a typical combination of the concrete and the abstract” (p. 198). It is these levels of generality that identify to those in mathematical generalizations, in particular, the mathematical concepts involved in this study. Within his discussion on generalization, Vygotsky also addressed “the law of equivalence of concepts, according to which any concept can be formulated in terms of other concepts in a countless number of ways” (1986, p.199). There are numerous ways for students to develop their initial conceptions of mathematical concepts that involve the structure of earlier mathematical concepts.

CHAPTER 3: LITERATURE REVIEW

This chapter presents previous research in the areas of argumentation and generalization, with focus given to those studies in the area of mathematics. It begins by connecting the theories presented in the theoretical framework chapter to the current research in the area of mathematical generalization and then in the area of argumentation.

3.1 MATHEMATICAL GENERALIZATION

Soviet Studies in Mathematics Education, Volume 2: Types of Generalization in Instruction: Logical and Psychological Problems in the Structuring of School Curricula, by Davydov (1990), is a historical source for any ideas or theories involving the generalization process or its products. The author investigated children's formations of generalizations under the framework of Vygotsky's sociocultural theory. Davydov (1990) presented social activities within the classroom as a means of bridging the divide between practical activity and abstract thinking. He emphasized the use of socially formed connections in developing what Skemp (1987) referred to as relational understanding of the concepts being taught rather than merely an instrumental understanding (i.e., pure rote memorization) of those concepts. Davydov (1990), like Vygotsky and Skemp, believed in the importance of dialogue, a sort of informal argumentation, in developing an understanding of scientific concepts and the need for the child to be aware of his or her own mental processes. In addition to these fundamental notions, this text provided the definition of generalization, stated in the theoretical framework, which was used as the foundation of this study. Davydov (1990) defined generalization when he said,

> If we mean the process of generalization, then the child's transition from a description of the properties of a particular object to finding and singling them out in a whole class of similar objects is usually indicated. Here the child finds and singles out certain stable, recurring properties of these objects. The following statement is typical of works in the psychology of education: "... a generalization is made – that is, similar qualities in all objects of the same type or class are acknowledged to be general." During generalization what occurs is, on the one hand, a search for a certain invariant in an assortment of objects and their properties, and a designation of that invariant by a word, and, on the other hand, the use of the variant that has been singled out to identify objects in a given assortment. (p. 5)

Research articles by Joanne Becker and Ferdinand Rivera (2005, 2007) also provide a background for the study of generalizations, specifically with visual patterns. In this study, Riemann sums was first presented as a graphical approach: a visualization of the problem and the use of argumentation to see what connections exist or do not exist between similar problems. By arguing which connections existed between the problems, students formed generalizations. Though Becker and Rivera do not specifically mention argumentation, the use of student argumentation is evidenced in their studies. In the article "Factors Affecting Seventh Graders' Cognitive Perceptions of Patterns Involving Constructive and Deconstructive Generalizations" (2007), the authors presented a two-year qualitative study on eight 7th graders' abilities to develop and justify both constructive generalizations and deconstructive generalizations involving patterns in algebra. It is these justifications provided by the students that can be related to a type of informal argumentation. Conclusions of their study included that students expressed generality in linear patterns as numerical or figural, and that those with figural ability were more apt to develop “algebraic generalizations” and justify them.

In their article "Generalization Strategies of Beginning High School Algebra Students" (2005), Becker and Rivera presented a qualitative study of twenty-two 9th graders’ beginning algebra class in an urban setting. Students performed generalizations on a task involving linear patterns and were asked to think out loud. Again, though the authors did not mention it, this “thinking aloud” helped the students to become aware of their mental processes as deemed necessary by Davydov. Conclusions of their study indicated that while students were successful in dealing with particular cases of patterns, they have difficulty in using algebra to express relationships and to generalize an explicit formula for a linear pattern. These two research articles by Becker and Rivera provide a glimpse into students' processes of forming generalizations and their struggles, both of which were useful when developing the tasks for this research project.

In “Fostering Connections Between Classes of Polynomial Functions,” Buck (1995) focused on building students’ concepts of higher degree polynomials from the concepts students already know about quadratic functions. Their research study showed that students made few attempts to connect higher degree polynomials to linear and quadratic functions they had previously studied. One particular and very interesting result was that given a new function that is created from two other functions, students totally ignored the provided graphs of the original functions and the original functions themselves. For example, students may consider that the multiplication of two linear functions produces a quadratic function, and they may relate the linear coefficients to the resulting quadratic coefficients, vertex, intercepts, or end behavior. In a similar manner, students may relate higher order polynomials to linear and quadratic functions. The students instead chose to randomly manipulate the new function that they did not know rather than relate it to the ones they did know. The study by Buck provides a great example of a situation in which mathematical generalization would be so very useful to the students, yet the students do not attempt to form the generalization. One has to wonder if this is a result of the students not being introduced to generalization on a regular basis in a manner that connects mathematical concepts, but rather being trained to memorize mathematical topics as entirely separate concepts.

A more classroom applicable text is that of Harold Jacobs. In his book *Mathematics, a Human Endeavor: A Book for Those Who Think They Don't Like the Subject* (1982), Jacobs used visual aids such as cartoons, comics, and puzzles to not only capture the interest of students, but to help them develop a greater and deeper understanding of mathematics and how it relates to the world around them. The paramount achievement of this textbook is the author’s use of everyday easily understandable language and the in-depth explanations of concepts for students in everyday language. Throughout the textbook, Jacobs opened the examples with an image or story, related that image or story to a mathematical question, stated the directions for the example visually using charts or tables, created equations and corresponding graphs, and then generalized the specifics of the problem at hand to problems involving similar concepts. This text serves as an excellent source for stimulating the development of problems involving argumentation and generalization.

George Pólya is another author who set forth a standard in the task development for this study. In *Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving, Volumes I and II (*1962, 1965), Pólya presented a step-by-step method for how to solve mathematical problems. He began his text by considering the geometric constructions of loci and walking the readers through the needed components of problem such as the unknown, the given, and the conditions (Pólya, 1962). After briefly discussing the solution to his straightforward first example, Pólya turned his attention to extending the particular case from the first example into a pattern. In other words, he generalized the particular solution of his first problem to other situations. Pólya followed this strategy of finding a problem that you can solve and then relating it to the one that you were originally unable to solve. Throughout all his books, which cover a variety of topics such as geometry, arithmetic, algebra, and numerous others, he promoted the process of generalization as an essential component to mathematical thinking. Pólya (1957) stated, "If you can't solve a problem, then there is an easier problem you can solve: find it." Essentially, Pólya used generalization as a strategy to solve more difficult and involved mathematical problems.

Pólya (1957) summarized his heuristics for solving problems (i.e., a guide to thinking through a solution) and went further into his explanation of steps to solving mathematical problems in his book *How to Solve It.* This method is broken down into four steps, each accompanied by statements and questions to help the reader progress through the steps. Pólya utilized these steps again and again as he tackled mathematical problems along with the readers. This allows the reader to become comfortable with the steps and to understand exactly how the steps can be applied to actual mathematical tasks. Pólya's heuristics for solving mathematical problems, presented below, were useful for the TR to guide students' thinking through questions either in the classroom or in one-to-one interviews. The following figure includes information taken from Pólya's book *How to Solve It* (1957):

Figure 1

Steps for Solving a Mathematical Problem

STEP 1: UNDERSTANDING THE PROBLEM

- **First.** You have to *understand* the problem.
- What is the unknown? What are the data? What is the condition?
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
- Draw a figure. Introduce suitable notation.
- Separate the various parts of the condition. Can you write them down?

STEP 2: DEVISING A PLAN

- **Second.** Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.
- Have you seen it before? Or have you seen the same problem in a slightly different form?
- *Do you know a related problem?* Do you know a theorem that could be useful?
- *Look at the unknown!* And try to think of a familiar problem having the same or a similar unknown.
- *Here is a problem related to yours and solved before. Could you use it?* Could you use its result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?
- Could you restate the problem? Could you restate it still differently? Go back to definitions.
- If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

STEP 3: CARRYING OUT THE PLAN

- **Third.** *Carry out* your plan.
- Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you prove that it is correct?

STEP 4: LOOKING BACK

- **Fourth.** *Examine* the solution obtained.
- Can you *check the result?* Can you check the argument?
- Can you derive the solution differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

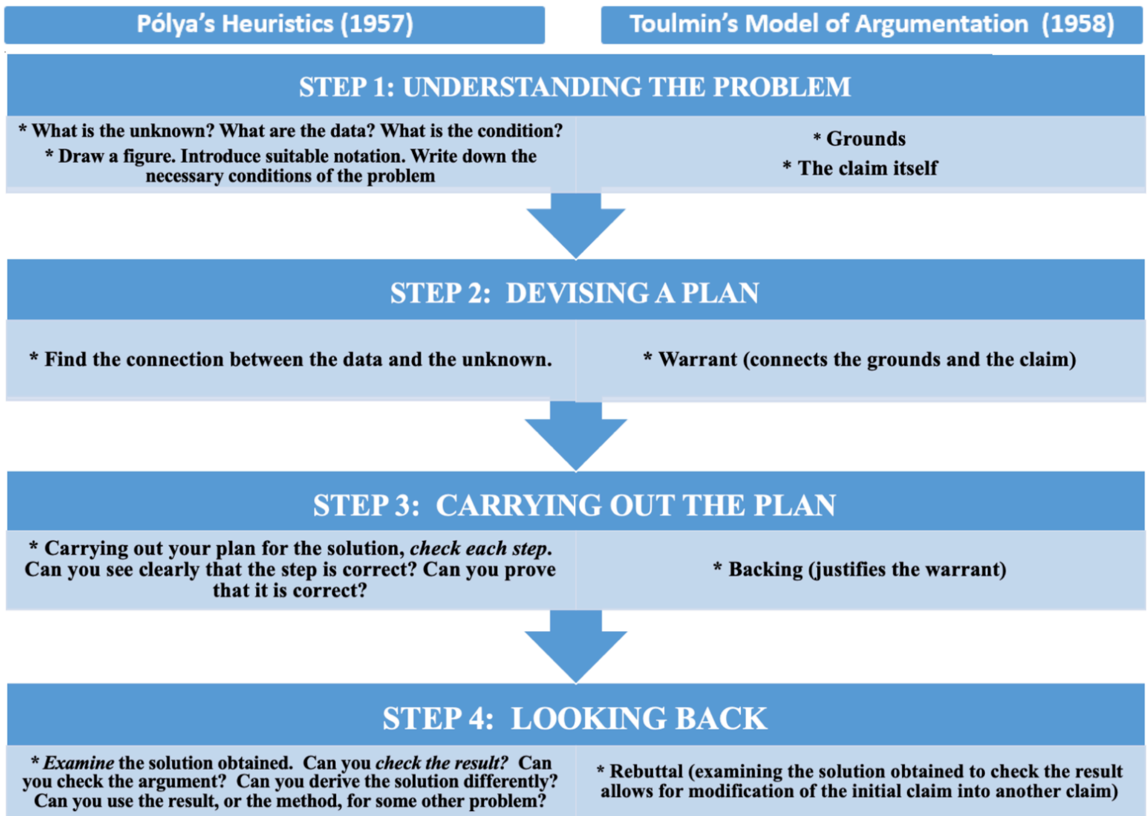
Note. From *How to Solve It* by G. Pólya, 1957.

Note that argumentation, discussed in the following section of this chapter, could easily be used when checking to see if each part of the plan has been correctly carried out in Step 3 and in Step 4 to encourage readers to generalize the problem to other situations.

A clear parallel can be pointed out between Pólya's heuristics for solving mathematical problems and Toulmin's model for argumentation. In Step 1, Understanding the Problem, deciding what is unknown, what is the data, and what are the conditions for formulating the *grounds* for the claim in Toulmin's model; drawing a figure, introducing suitable notation, and writing down the problem correspond to *the claim* itself. In Step 2, Devising a Plan, finding the connection between the data and the unknown connects the grounds and the claim, and so it parallels Toulmin's *warrant*. In Step 3, Carrying out the Plan, checking that each step is correct *justifies the warrant*, and relates to *backing* in Toulmin's model. In Step 4, Looking Back, examining the solution obtained to check the result and argument and to see if the result can be applied to another problem allows for modification of the initial claim into another claim, and thus aligns with *rebuttal* in the Toulmin model. The following figure summarizes the connections discussed above between Pólya's heuristics and Toulmin's argumentation model:

Figure 2

Comparison of Pólya's Heuristics and Toulmin's Argumentation Model



Giaquinto's text *Visual Thinking in Mathematics: An Epistemological Study* (2011) aimed to bridge the gap between experimental psychology, philosophy, and mathematics education. This text focused on visual thinking as the visual imagination emerging from the perception of diagrams and symbols and the mental operations on these diagrams and symbols. He viewed visual thinking not simply as an aid to understanding mathematics, but as a means of discovery, understanding, and possibly proof. More specifically, he stated that visualization can induce mathematical knowledge. He also believed that general truths can be derived from specific images and that these visual images can help individuals to gain insight of abstract structures. As the study for this dissertation addressed visualization as the first component of the tasks regarding Riemann sums and the generalization of graphical techniques to formal notation, this aspect in the solution of the tasks was analyzed. Furthermore, Giaquinto presented examples of different types of visual mathematical reasoning as used in geometry, arithmetic, and even analysis. The other focus of his text was the importance of using approaches from the multiple disciplines of psychology, philosophy, and mathematics in order to understand visual thinking within mathematics education. In fact, this is one of the underlying arguments of his book.

The empirical works of Williams (1991, 2011) shed further light on generalization. In "Models of Limits in College Calculus Students" (1991), Williams investigated factors that affected students' understanding of the concept of limits. This study, which involved ten college students participating in five individual interviews, documented students' development of their concept of limits from their initial thought models to the formation of a more formal model of limits (Williams, 1991). This development process from informal to formal concepts of limits is a generalization of the concept. Williams (1991) noted that students were resistant to change their initial models of the concept of limits, and that this resistance was impacted by the students' previous beliefs and experiences with the graphs of simple functions. This study showed the difficulty in moving students from an informal towards a more formal and generalized understanding of a mathematical concept such as limits.

In “Predications of the Limit Concept: An Application of Repertory Grids” (2001), Williams further studied the thinking of two of the college students from the aforementioned study as they developed their own models of the concept of limits. One of the selected students for this study had a strongly held model for the concept of limit, while the other student had a less articulated model for the concept of limit. Both students participated in five experimental sessions over the course of seven weeks in which they were exposed to examples that challenged their current models of the concept of limits. Conclusions of the study highlighted the differences in defining the concept of limits versus finding limits. Williams (2001) emphasized that approaching limits mathematically and approaching limits cognitively are two very different things, and that the latter may involve psychological issues such as reconciling physical experiences with formal notions. Both students in this study continued to cling to their own informal models of the concept of limits even when confronted with opposing examples (Williams, 2001). This again emphasizes the difficulty in eliciting change and growth in student understanding of limits.

3.2 ARGUMENTATION

The Uses of Argument by Toulmin (1958) focused on the uses of practical rather than formal proof. Toulmin believed that truth is not a universal concept and that there are two types of arguments: substantial and analytical. Substantial arguments can be evaluated according to their context and require an inference to move from the evidence to the conclusion or final claim; in contrast, analytical arguments are universal and their conclusions are in their evidence (Toulmin, 1958). For example, Toulmin (1958) considered an argument that if it is known the height of a wall is six feet high, and the sun is at an angle of 30 degrees, a physicist will say that the shadow much have a depth of 10.5 feet. He considered this a substantial argument because the conclusion is based in context rather than being an abstract number, meaning that in this situation, the 10.5 feet may be a prediction and not a fact. In the abstract, the conclusion of 10.5 feet from solving the equation tangent of 30 degrees is equal to x divided by 6 would be analytical.

Furthermore, Perelman and Tyteca (1958), in their text *The New Rhetoric: A Treatise on Argumentation,* highlighted the differences between explicit and implicit argumentation and how direct and specific an argument must be to either persuade or convince others. Implicit argumentation may be more informal and yet persuade an audience, while explicit argumentation is more formal and directly states a claim and reasoning to persuade or to convince. These authors saw implicit argumentation or "persuasion," which is emotional, as somewhere between arbitrary reason-giving and absolute proof. Because students should be able to give details to support their work, but since they also may not have a deep enough understanding to offer a formal proof, it is Toulmin's practical use of argumentation that serves as the foundation for the types of argumentation, somewhere between disorganized reasoning and proof, which would facilitate students' progress through zones of proximal development and the formation of mathematical generalizations.

As discussed previously in the theoretical framework (see Chapter 2), Vygotsky's zone of proximal development is the difference between what a learner can do without help and what he or she can do with help (Moll, 1996). A student develops intellectually as he or she progresses through their own zones of proximal development. Vygotsky believed that interactions with other children, possibly during cooperative learning opportunities, help students to progress from one ZPD to a subsequent one. Providing justification through argumentation of one's own thoughts can be a part of these student interactions, and a defining indicator of their progress as they progress in their zones of proximal development. From the beginning of the teaching experiment, the TR hypothesized the ZPD of the students based on these criteria: their performance on particular tasks; their overall performance in class; the kind of questions they ask in class; and their cognitive behavior in the course. The TR then followed their progress through their ZPDs through a sequence of tasks and one-to-one interviews as well as their overall progress in the course. To further support this connection between a student's progress through zones of proximal development and the formation of mathematical generalizations by means of argumentation, we continue with additional discussion of the literature.

Kinard and Kosulin's text, *Rigorous Mathematical Thinking,* is based on Vygotsky's sociocultural theory which views psychological tools as mediators of cognitive processes (2008). The authors emphasized the importance of higher-order mental processes and psychological tools in mathematical thinking. Though not stated by the authors, argumentation may be one such tool which can be used as those described in this text. In this study, the process of argumentation was used to help students form generalizations, meaning argumentation was viewed as a psychosocial tool that mediates higher order process of generalization.

Walton et al. (2008), in their text *Argumentation Schemes*, presented ninety-six argumentation schemes and gave a classification and an analysis of some of the schemes which were most commonly used in informal arguments. The schemes are essentially patterns that the authors and other researchers have seen while studying human reasoning (Walton et al., 2008). The authors of this text took a stand against most works on argumentation which have used an approach based on fallacy (Walton et al., 2008). They acknowledged that some arguments based on fallacies are acceptable. However, this should not be the sole basis of how to teach others to argue, especially students who are just learning how to argue (Walton et al., 2008). Instead, the authors created argumentation schemes by empirically collecting, analyzing, and evaluating arguments of all types to create a standard form of an argument (Walton et al., 2008). They proposed argumentation schemes that encouraged good reasoning be used in place of traditional fallacy approaches, and that these schemes should also be used, along with critical questions on the part of the listener or evaluator, to assess informal arguments (Walton et al., 2008). *Argument from a rule*, *practical reasoning*, *lack of knowledge arguments*, *pleas for help and excuses*, *arguments from inconsistency*, and *argument from expert opinion* are a few examples of argumentation schemes in this text that are applicable in the analysis of the student arguments in this study. In particular, some students followed the steps given to them by experts, the teachers, and used these memorized facts as their initial argument. This text was very useful in assessing the strengths and weaknesses of arguments provided by students for this research project as it allowed focus to be placed on good reasoning as well as on fallacy approaches. Such arguments indicated the depth of level of understanding of a concept that was not new to the students' experience although the expectations for justifications were new to them.

In "Argumentative Aspects of Proving: Analysis of Some Undergraduate Mathematics Students' Performances" (1999), Douek examined undergraduate students' abilities and methods of proving a proposition about natural numbers that they were first asked to generalize. In this study, there was a struggle for the students to form, construct, and then prove their conjectures. The main conclusions drawn from this study involve the difference between formal proofs and the method of having students make their own conjectures and then prove them. This research article is one of the few works that relates argumentation with generalization. The authors were not seeking such a relationship intentionally, nor were they using argumentation as a tool for generalization. Instead, they asked students to first arrive at a mathematical generalization and then to provide an argument justifying their (the students') generalization afterward.

In their article "Logical and Semiotic Levels in Argumentation" (2009), Arzarello et al. used the Toulmin model of argumentation as a basis for formal analysis of an argument and also expanded the ideas of this type of analysis. In this work, the authors presented a case study of two episodes from a group activity in 10^{th} grade calculus that focused on the argumentative processes of these calculus students. The activity was part of a long teaching experiment, which focused on the use of different semiotic resources for argumentation. In this research study, the Toulmin model was specifically used to analyze "the dialectic role of examples, counterexamples, and refutations in the evolution of students' argumentations" (Arzarello et al., 2009). Conclusions of the article included exposing the limits of the Toulmin argumentation model as a tool for analysis of arguments and emphasizing the importance of the semiotic resources, such as speech, inscriptions, artifacts, and gestures used by students to represent and make meaning of the problems they are asked to solve (Arzarello et al., 2009). The ideas presented in this article tied in well with both the topic of argumentation and also the theoretical framework by providing background knowledge on the semiotic means of argumentation in regards to speech and gestures.

In the article "Justification, Enlightenment and the Explanatory Nature of Proof," Kidron and Dreyfus (2009) focused on how students viewed and used justification in order to both expand their knowledge and construct new knowledge. While this article does not directly use the notion of argumentation, I consider that argumentation is evident in the justifications provided by the students during their task completion. Their research paper highlighted three examples that showed that justification is often associated with combining constructions of knowledge. "When constructions combine, we observe a new degree of enlightenment in which students not only see that it is true, but why it is true" (Kidron & Dreyfus, 2009). If argumentation can stimulate the formation of generalizations, or the combining and recognizing of similarities, then argumentation can be seen as a process and generalization as the result of such a process. This is to say that argumentation and generalization are a "combing of constructions" which the authors believed created a relational understanding of both how and why something is true.

In "Conjecturing and Proving in Problem Solving Situations" (1999), Mariotti and Maracci presented a work focused on conjecturing and proving. Interestingly, the participants of their research study were not undergraduate college students but rather what the authors called "brilliant" high school students studying geometry (Mariotti & Maracci, 1999). Despite the giftedness of these students, the results of this study showed a lack of clarity in student conjectures (Mariotti & Maracci, 1999). Additional results of the study included positive strides made by the students that supported the teacher interventions and questioning as a means to help students in the proving process. This "questioning" can be linked to the triggering of informal argumentation, and the TR interventions of questioning provided an example of possible intervention strategies for this research study.

In "How Can the Relationship Between Argumentation and Proof be Analysed?" (2007), Pedemonte based his work on the Toulmin model of argumentation and the idea that proof is a special case of argumentation. The author conjectured that this model can be used to analyze the structure of an argument and also the structure of a proof. This article was interesting because it highlighted the relationship between an abductive argumentation with its deductive proof and also the relationship between an inductive argumentation with its mathematical inductive proof (Pedemonte, 2007). This article included, described, and related the topics of argumentation and the formal proof. It is soundly based on the Toulmin model of argumentation discussed in the Theoretical Framework of this paper, yet it also discussed the analysis of the structure of an argument, which tied back to the text *Argumentation Schemes* (2008) by Walton et al. While this article did not use specifically stated argumentation schemes, it did provide ideas of how to gauge or analyze the quality of an argument, which is of importance in the analysis of my data.

The foundational ideas of argumentation and generalization discussed in the literature review have certainly provided a starting place for both task development and the data analysis of this study. In particular, I: (i) developed tasks so that students follow the steps presented by Pólya; (ii) sequenced the tasks to help students generate generalizations by means of formal and informal argumentation; (iii) incorporated opportunities for visual mathematical reasoning into my teaching sessions and the tasks for the students; (iv) identified and analyzed critical components needed in my teaching sessions to help students make argumentation a habit of thinking; (v) used questioning to help students trigger informal or formal arguments; and (vi) identified and analyzed the formation and type of student generalizations through the process of different kinds of argumentation.

CHAPTER 4: METHODOLOGY

This chapter presents the methodology of the classroom teaching experiment that was used to investigate argumentation as a cognitive tool for mediating student generalization in Riemann sums and integration. The study took place over the Spring 2019 semester that included 14 weeks from January to May. Two weeks of the course were devoted to the teaching sessions regarding Riemann sums, and the following four weeks, while the course went on with other topics, were devoted to student one-to-one interviews. The data was analyzed using both thematic and content analysis, and careful measures were taken to ensure validity and trustworthiness.

In the following sections of this chapter, additional details will be provided about the design of the study, the context of the study, the tasks involved, and the data collection methods that were applied. This chapter ends with a description of the data analysis approach that was implemented and the possible limitations of the study.

4.1 TEACHING EXPERIMENT

Qualitative research lends itself to the study of how people experience an event and the development of a deep understanding or awareness of this experience (Saldana, 2011). Qualitative research involves the collection of data which includes but is not limited to interview transcripts, fieldnotes, and other documents (Saldana, 2011). Under the umbrella of qualitative research, action research allows practitioners, in this case the Teacher-Researcher (TR), to systemically investigate and improve their practice (Efron & Ravid, 2013). In action research, practitioners explore their own methods and techniques in a critical and reflective manner “using strategies that are appropriate for their practice” (Efron & Ravid, 2013, p. 4).

This study focused on understanding *how* teachers can model argumentation and *how* students develop argumentation to achieve generalizations from Riemann sums to integration. Qualitative methods were needed in order to gather a rich description of students’ ways of developing argumentation as they tackled mathematical tasks. When knowledge is gained about these student thought processes, then perhaps teachers will be able to help students to form mathematical generalizations and to provide guidance and tools that promote and stimulate this process. Thus, this study used a qualitative teaching experiment methodology.

The classroom teaching experiment conducted for this study is rooted in the constructivist methodology designed by Steffe (1980) and extended to the classroom teaching experiment by Cobb and Yackel (1995). Teaching experiments allow the investigator to experience the mathematical learning and reasoning of the students (Steffe & Thompson, 2000). Furthermore, courses that emphasize learning mathematical concepts with focus on application and meaning allow for the extension of the teaching experiment methodology to the classroom teaching experiment (Wood et al., 1991). Sáenz-Ludlow (1997) expounded on the classroom teaching experiment of Cobb and Yackel by the addition of: (i) the pilot-teacher experiment; (ii) the active role of the researcher as teacher and co-teacher throughout the duration of the teaching experiment facilitated one-to-one interviews with students; and (iii) the post teaching experiment one-to-one interviews. In a classroom teaching experiment, the interview is a focused observation of students solving mathematical tasks in which they are instructed to think aloud as they create solutions. During these interviews, the TR may ask guided questions in order to gather richer descriptions from the students regarding their problem solving processes. This classroom teaching experiments yields continuity between content and methodology of the course with content and methodology of the one-to-one interviews. This type of teaching experiment in mathematics education is what is called the intervention in action research.

4.2 DESIGN OF THE STUDY

Cobb and Yackel (1995) indicated that theory grows out of practice and feeds back to inform and guide practice. This sheds light on the teaching experiment for this research project. The TR has gathered her research ideas from prior practice teaching the same course several semesters. Note a description of the course is given in the paragraph below. The teaching experiment for this study stemmed from the Teacher-Researcher's (TR's) recognition of students' difficulties in justifying and explaining their calculations within the context of Riemann sums and integration in the particular mathematics course under consideration. In an attempt to understand the struggles faced by the students, theory regarding argumentation and generalization was studied to identify potential ways to help students overcome this difficulty, and now the study has been conducted to confirm and expand these ideas so that teaching practices might be improved to advance the students' levels of understanding and generalization.

The mathematical content for this study was provided by an upper level undergraduate mathematics course named *Computer Exploration and Generation of*

Data. The overarching goal of this course is to help students deepen their understanding of particular concepts and master them through applications. In an advanced mathematics course like this, students, though having the same prerequisites, come into the course with very different levels of understanding of mathematical concepts and with different beliefs about what it means to understand and master a mathematical concept. Thus when conducting one-to-one interviews, it was important to focus on the difference in what each student was able to do or not do on their own and with help of the TR in order to assess the progression in each student's own ZPD. A student cannot be forced to understand something that they are not ready to understand, hence spending time in the areas where students are able to work with or without help produced much richer data for the study than attempting to somehow make all students reach a particular level of understanding.

The students' discourse and engagement that the TR was able to trigger throughout the semester was a key factor in the success or failure in their processes of argumentation to form generalizations. How can cognitive processes be analyzed if the students do not express and present their thoughts and ideas? For this reason, it was of utmost importance that the TR modelled argumentation in the presentation of the course material during the teaching sessions and also that she encouraged argumentation on the part of the students throughout the course and the interviews.

The teaching experiment consisted of: (i) teaching sessions throughout the semester in which argumentation was essential in all topics but four sessions specifically focused on Riemann sums; (ii) student homework tasks; (iii) a pair of one-to-one interviews where students worked through tasks while sharing their justifications aloud and in writing with the TR; and (iv) one post-interview so that students had time to use an application of their knowledge of Riemann sums and integration.

It was of the utmost importance that the TR (i.e., the instructor of the course) focused on students' relational understanding of the curricular concepts of the course, including Riemann sums and their relationship to integration and applications. Awareness of one's own thinking and understanding when teaching is essential to direct students' thought processes and their processes of argumentation. To this effect, the TR: (i) made written observation of her own practice in a journal; (ii) constantly reflected on her teaching practice and students' interventions; (iii) provided students with written feedback to homework tasks; (iv) took notes on the feedback for each student; and (v) interviewed the students taking into account, as much as possible, the inferred or hypothesized cognitive behaviors.

This teaching experiment consisted of in-classroom teaching sessions and one-to-one individual interviews. During the classroom teaching sessions, the TR modeled argumentation in her presentations of course material and in her interactions with the students for the duration of the semester. The modeling, specific to Riemann sums, took place during two weeks of classes, comprising four separate seventy-five minute class sessions. The Riemann sum concepts were taught, developed from graphical to symbolic forms, and extended from small numbers of partitions to larger number of partitions and from endpoints to midpoints to limits and lastly to integrals. During this group intervention, the TR modeled argumentation, both in written and verbal forms. As she solved several problems, she described her thoughts verbally and in writing to the students while working out problems in class.

Finally, during four weeks following the Riemann sum teaching sessions, students were interviewed twice on a one-to-one basis regarding this topic. Each interview lasted approximately thirty minutes and took place on campus in the office of the TR. During these interviews the students talked through their thought processes and provided verbal and written justification for their ideas as they worked through a series of tasks on Riemann sums and their relationship to integration and applications. These interviews allowed the TR to observe the students' verbal and written mediations, and to distinguish patterns and processes within the verbal data of the students (Ericsson & Simon, 1993). The interviews were semi-structured so that each interviewee centered on the same concepts but allowed for students' explanations of ideas in their own way; during the post-interview, students expressed their conceptualization through application in areas of their interest.

The mathematical tasks of this study aimed to not only allow the TR the opportunity to observe students making generalizations through the process of argumentation but also through the TR's observations of the students in class participation, homework, and one-to-one interviews. Such tasks allowed the Teacher-Research (TR) to obtain a snapshot of students' thinking during their process of argumentation before, during, and after a generalization took place.

4.3 CONTEXT OF THE STUDY

This study took place in a four-year university located in North Carolina. This institution's population is 29,317 students, of which 23,914 are undergraduates. Students at the university represent 49 states and 85 different countries. This study took place in an upper level undergraduate mathematics course *Computer Exploration and Generations of Data* that is highly focused on calculus and differential equation concepts and their applications to the real world. Prerequisites for this course include math calculus courses (Calculus I, Calculus II, and Calculus III), as well as multiple statistics courses, including an advanced statistics course. This course utilizes Excel to explore mathematical concepts and their applications. Areas of application include: Dynamical Models (epidemics, harvesting models, population dynamics, predator-prey models); Optimization (inventory control, apportionment algorithms); Financial Mathematics (stock price simulation, pricing of derivatives); Business Simulations (net present value comparisons, risk evaluation, sensitivity analyses).

The class size for the course on which this study was based typically includes between fifteen to thirty students. The majority of participants in the course are pure mathematics and actuarial mathematics majors. Each semester, approximately three of the thirty students are computer science majors. Regardless of their major, all students in this course have taken and successfully completed several advanced undergraduate mathematics and statistics courses. Most of these students will graduate with a mathematics degree within a year and a half (or less) of the semester in which they take this course. This information is important as it sets the stage for levels of conceptual understanding that one would, correctly or incorrectly, expect from students with this background, mathematics history, and experience.

4.4 CONTENT OF THE STUDY

This particular course was chosen for the study because students in the course are required to have taken many introductory and advanced level math courses. This math background allows for the study of how students connect ideas between mathematical topics, and the sophistication of the course concepts makes possible analysis of students abstract conceptual development. As many of these students are soon-to-be graduating math majors, it is also assumed by many that these students would have an in-depth understanding of the topics involved in this study, however, that is not the case as shown in their performance in this course each semester. This provides an opportunity for this study to shed light on student understanding of these important mathematical concepts, generalizations, and misconceptions in the study of Riemann sums and integration. In addition, the TR has direct access to the students as the instructor for this course.

The TR is a full time lecturer who holds a Master of Science degree in Mathematics with ten years of teaching experience teaching at the same institution and who had taught this course for four years prior to conducting this teaching experiment. This course meets a general education requirement for oral communication for undergraduate students, so interviews and presentations are a required component of the course. During the interviews, students are expected to present and justify their work and conclusions about various concepts from the course material. Because students are already completing one-to-one interviews with the instructor for the course requirements, this study did not create any additional work or hardship for students if they participated. For this reason, a high rate of participation in the study was anticipated and met with thirty-two students choosing to participate in the study and twenty-nine students completing the study. Conscious effort was made to ensure all students were given the same level of attention and instruction whether they participated in the study or not.

As their instructor, the TR was highly invested in developing a deeper understanding of the thought processes of all students so that they can be helped to both better comprehend the material involved in this study and, more importantly, to develop critical thinking that will serve them in other courses and in the workplace. With graduation around the corner for most of these students, it was imperative to help them be prepared for not only giving correct numerical calculations, but to also ensure that they are able to justify why their calculations were correct. Furthermore, each student needed to be able to accurately interpret the meanings of their calculations. For instance, what good is the value 3.2565 with no context? How does one know that the value 3.2565 is actually the value that the employer needed? What unit of measurement is involved? It is this type of question that is so important, yet, based on prior classroom observations, students seem unprepared to deal with.

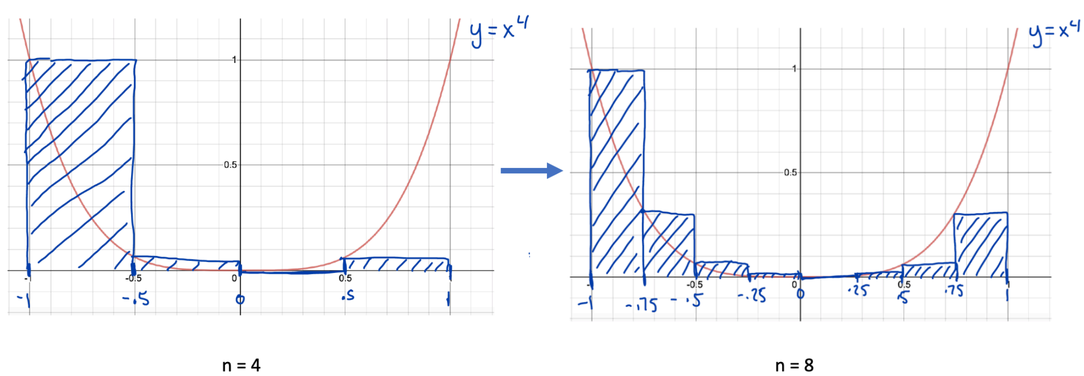
4.5 TASKS

This study took place over the entire Spring 2019 semester and during teaching sessions that routinely included argumentation with the goal to mediate generalization. Specifically, four of the course teaching sessions were dedicated to the study of Riemann sums. All students of the course experienced the same teaching practices and course work.

To begin the study, students were given a problem involving Riemann sums from a prerequisite class and asked to work out the problem with sufficient details. The study continued with the teaching sessions for Riemann sums by the TR. The TR began with a graphical representation of approximating the area under a curve. This was done with different shapes, and ultimately led to the formation of the left, right, and midpoint Riemann sums approximations for the area. Figure 3 shows a graphical representation for a left endpoint Riemann sum approximation with four rectangles in the first diagram and then with eight rectangles in the second diagram.

Figure 3

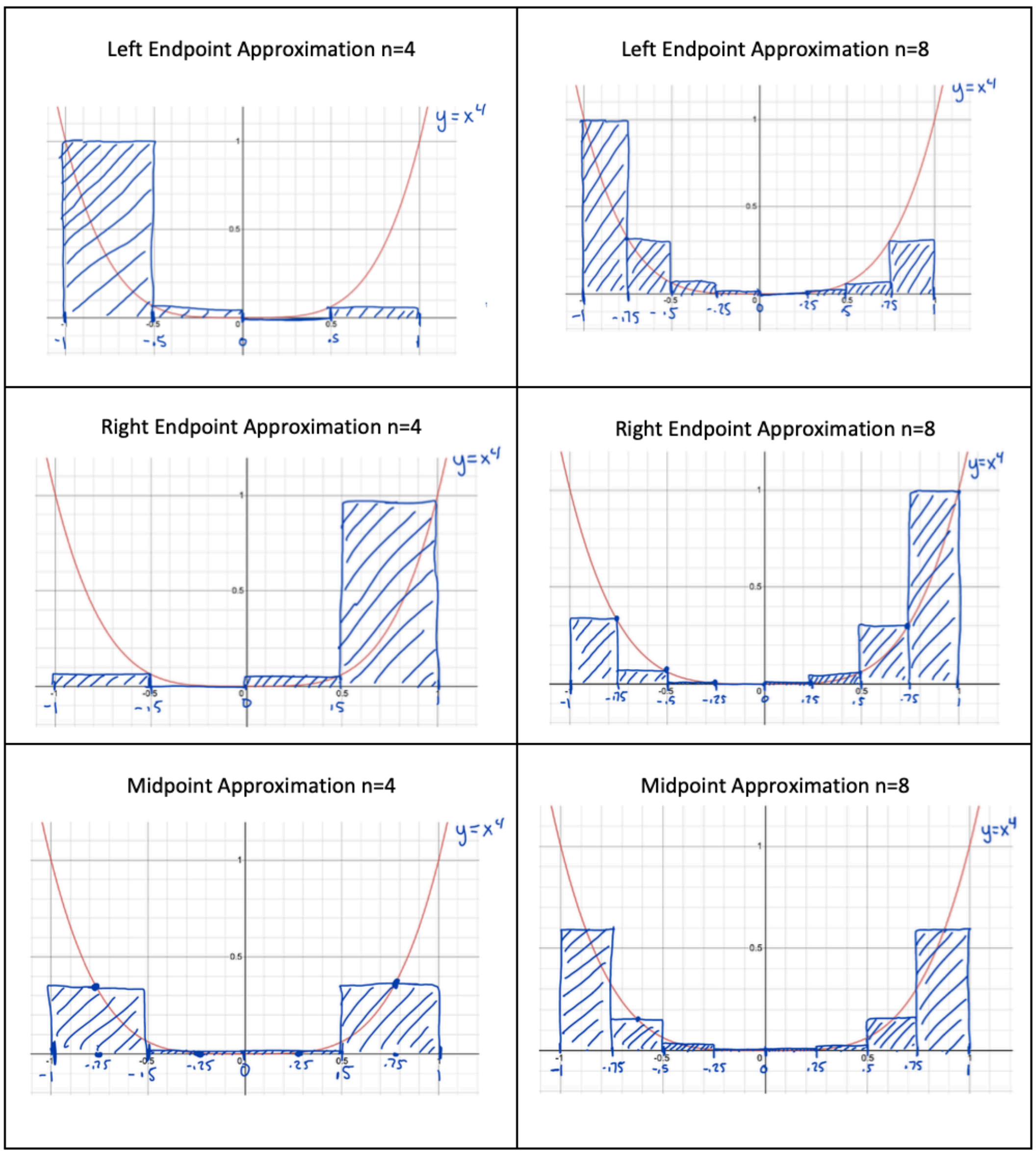
Solution for Similar Exercise: Four to Eight Rectangles



This exercise is similar to what the students were given in their tasks. Care was taken to emphasize, through argumentation, why each step was needed, and how each sum finds an approximation (upper, lower, and midpoint) of the area under the curve. Figure 4 shows a graphical representation for a left endpoint Riemann sum approximation using four rectangles and then eight rectangles, a right endpoint Riemann sum approximation using four rectangles and then eight rectangles, and a midpoint Riemann sum approximation using four rectangles and then eight rectangles. Care was also given to generalize each step and tasks from the previous steps, rather than approaching the tasks in a separate and disconnected manner.

Figure 4

Solution for Similar Exercise: Left Endpoints, Right Endpoints, Midpoints



These tasks moved from a graphical approach to the area, to basic analytical calculations, to the creation of informal formulas, to more sophisticated structures involving formal notation, and finally, to the implementation of these ideas using Excel software. Figure 5 shows a solution to a mock Excel sheet similar to the students' interview task. The mock Excel sheets allowed students to the opportunity to think through their work to decide if their functions would truly generate answers to what they wanted to solve. Though not part of the interviews, Figure 6 shows the Excel output that would be generated based on the mock Excel sheet equations from Figure 5.

Figure 5

Solution for Similar Exercise: Mock Excel Sheet

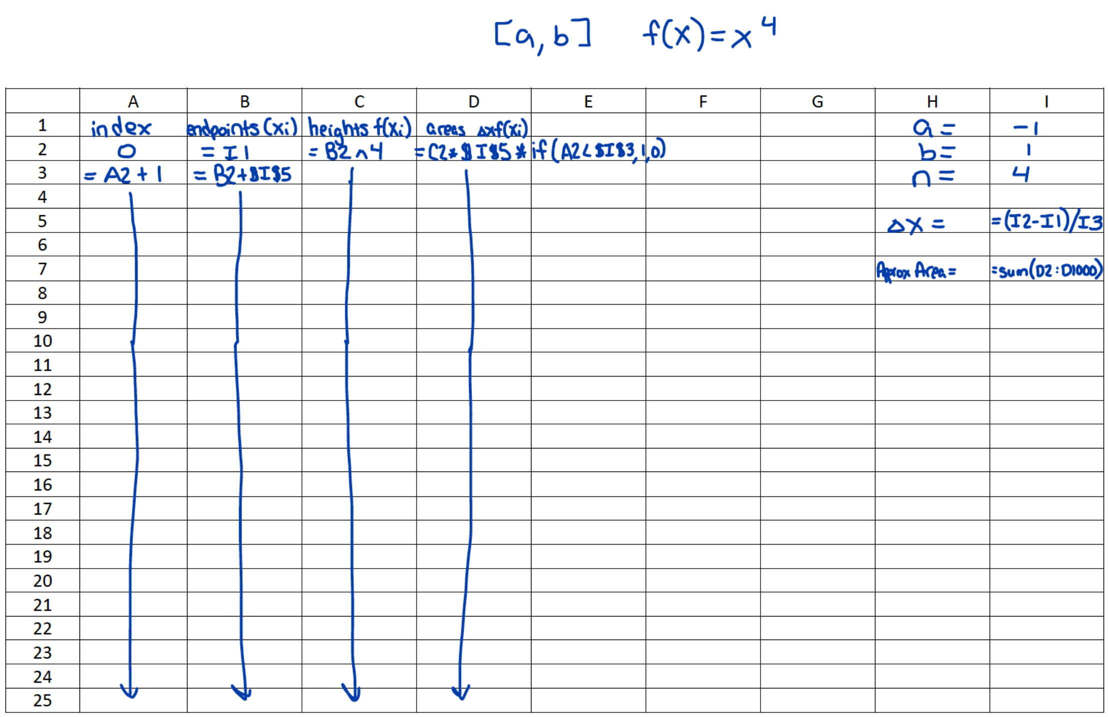


Figure 6

Solution for Similar Exercise: Excel Output

	A	B	C	D	E	F	G	H	I
1	**index**	**endpoints (xi)**	**heights f(xi)**	**areas deltax*f(xi)**				a=	-1
2	0	-1	1	0.5				b=	1
3	1	-0.5	0.0625	0.03125				n=	4
4	2	0	0	0					
5	3	0.5	0.0625	0.03125				deltax=	0.5
6	4	1	1	0					
7	5	1.5	5.0625	0				**Approximate area=**	0.5625
8	6	2	16	0					
9	7	2.5	39.0625	0					
10	8	3	81	0					
11	9	3.5	150.0625	0					
12	10	4	256	0					
13	11	4.5	410.0625	0					
14	12	5	625	0					
15	13	5.5	915.0625	0					
16	14	6	1296	0					
17	15	6.5	1785.0625	0					
18	16	7	2401	0					
19	17	7.5	3164.0625	0					
20	18	8	4096	0					
21	19	8.5	5220.0625	0					
22	20	9	6561	0					
23	21	9.5	8145.0625	0					
24	22	10	10000	0					
25	23	10.5	12155.0625	0					

In general, the class participating in the teaching experiment worked through the tasks, answered questions of the TR regarding Riemann sums during the tasks, and asked their own questions while solving the tasks. In particular, when studying Riemann sums, they also completed a separate task at the end of the corresponding teaching session as a home exercise. The home exercises were very similar to the tasks presented during the teaching sessions, and they allowed students to attempt the problems on their own and to receive feedback. This is important as it sheds light on the ZPD of each student and provides a glimpse into his or her own level of understanding. The four teaching sessions that were dedicated to Riemann sums were complemented with the Trapezoidal and Simpson's approximations for area under a curve.

After the teaching sessions, submissions of the homework tasks, and receiving feedback, students signed up for each of the two one-to-one interviews. The interviews were of great importance as it is crucial to collect a verbal report to be able to evaluate the sequences of thoughts and behaviors of the students (Ericsson & Simon, 1993). The interviews involved the students working through a task similar to those presented in the teaching sessions. Asking students to work through a task while thinking aloud is an ideal method of collecting information about the cognitive processes of the students (Ericsson & Simon, 1993). The tasks, as displayed in Figure 4 and Figure 5, were composed of problems which were sequenced such that concepts could be built and extended from one problem to the next should the student chose to do so.

The student began marking in a graph of the function x^2 what is meant by the area under the curve. They then proceeded to approximate the area under the curve using left endpoints and four rectangles, and then using left endpoints and eight rectangles. Students were asked to develop a formula for the steps they took that would work for any number of rectangles. The students were then be asked to use sigma notation to condense their formula.

The interview continued with the student considering the area under the curve using right endpoints with four rectangles and then eight rectangles. Similarly, the student was asked to develop a formula for the steps they took for any number of rectangles, and then to use sigma notation to condense this formula. Students were then given a table that mimicked a blank Excel sheet. They were asked to sketch into the

Excel sheet how they would use the software to make the calculations they performed for left endpoints, and then later for right endpoints for *n* rectangles.

A similar task was used during the second interview that focused on midpoint approximations of the area, and, depending on the time remaining for the individual student, possibly more advanced techniques such as the Trapezoidal Rule or Simpson's Rule. After the approximations, the students were asked to calculate the true area under the curve and then to describe how to make the approximations they used earlier in the task more accurate and closer to this true value. The mathematical task ended with the student choosing their favorite subject of application that involved integrals and describing an application by means of integration.

The last portion of the interview asked the students to write a reflection on their two interviews. The last portion of the last interview was kept fairly open as to allow the TR to direct the last question to the area of interest of each student that arose in the interview process. Depending on time available for a particular student, other questions asked the student about how well they felt they expressed their knowledge to the TR, how they prepared for the interviews, or how they felt the interviews had affected their understanding.

4.6 DATA COLLECTION

Prior to carrying out the classroom teaching experiment, approval was required by the Institutional Review Board (IRB). A proposal was prepared in August of 2018, and was submitted to IRB following the date of approval of the proposal by the dissertation committee. In Spring 2019, students who wanted to participate in the study signed the approved Informed Consent papers before the study began. From two sections of the course, thirty-two chose to participate in the study with twenty-nine students completing all components of the study. The first section of the course had seventeen students, and the second section of the course had fifteen students. It should be noted that there were only two sections of the course in Spring 2019, and that both sections were taught with the same topics and the same methodology by the TR. These twenty-nine participants from the two sections included twenty-two males and seven females with four students being of junior class standing and twenty-five students being of senior class standing.

This study collected four types of data: Teacher-Researcher self-observations, a TR reflective journal, student written work, and audio recordings of the student interviews. Detailed notes and observations from the classroom teaching and modeling of argumentation were kept. It is noted that the initial data collection, as observations of a typical classroom activity, did not require recruitment. The TR also kept a self-reflective journal throughout the semester about observations of her own teaching and students' learning processes. This allowed the TR to critically evaluate the progress of the research study and her own actions and role within the study (Denzin, 1997).

The TR conducted every interview for all participants in the study. The written work samples of the students during the interviews were kept for analysis. The TR also made field notes during the student interviews. To record the audio data from the interviews, audio recordings were taken with a tablet and the recordings were labeled with the student's randomly assigned number to help protect the anonymity of the participants. The recordings were transferred to an external hard drive for storage, and the data were removed from the tablet after the data transfer immediately following the interview. The tablet was not connected to any networks throughout the duration of this study and served only as a recording tool.

In order to more deeply investigate the cognitive processes of the students, several descriptive case studies were prepared after a review of all the students' written work samples and notes made by the TR during the interviews. The TR identified different types of students based on where students were initially and ultimately in the interview process as indicated by their homework and class participation. Students were then categorized based on the similarities and differences in their processes of understanding during their interview tasks and the key strengths and/or weaknesses demonstrated in their conceptual understanding of Riemann sums. The information for all the students in each of the case studies was saved by their randomly assigned number, but some students whose direct work is included in the upcoming chapter also received a pseudonym for the analysis of their work. The work for the case study students was coded thematically, and a file was kept for each student.

4.7 QUALITATIVE DATA ANALYSIS

The data was analyzed qualitatively using both content analysis and thematic analysis. Content analysis uses text, pictures, and audio data to find meaningful patterns to better understand context (Ezzy, 2002). Glaser and Strauss (1967) described content analysis as an abductive process of constructing working hypotheses that allow the researcher to connect, in this case, the students' work to the research question through inference. Based on these working hypotheses, the TR created categories, founded in the literature regarding argumentation and generalization, which were used to analyze the data. Vygotsky's levels of generalization and zones of proximal development, along with Toulmin's argumentation model, shed light on the validity of such categories. The data was then analyzed according to these categories.

Thematic analysis, another method of qualitative data analysis, is an inductive process of hypothesis verification, contrary to content analysis that is based on an abductive process. Thematic analysis begins with student work as the center of the analysis, and then looks to find patterns in the data (Ezzy, 2002). The students' work was the focus of this study as indicators of their processes and understanding. Once initial categories were formed using content analysis, then the TR created additional categories or levels based on the student responses and work. Rather than using preconceived categories of what labels were given to the work, as was done during the content analysis, the TR allowed the data to guide the development of the categories during this second part of the data analysis. This is useful, as student work may not fit perfectly into deductively preconceived categories. The thematic analysis method allowed for the TR to capture relationships in the data that may otherwise have been missed with content analysis alone.

4.8 CONSTRAINTS AND LIMITATIONS OF THE STUDY

It is important to be aware of difficulties that could be present when working with students. Steffe and Thompson (2000) highlighted two difficulties to be considered when gathering data. One is the constraints experienced by researchers to engage students and to elicit verbal explanations of how they are thinking. The other is the students' extant misconceptions that could persist regardless of the researcher's intentions to guide students in their progressive conceptualizations.

The students' preconceived notions about what it means to solve a mathematical problem and provide justification were the first stepping stone of the study. It is the job of the TR to model argumentation and justification of mathematical ideas; however, student beliefs that they have fully solved a problem simply because they have generated a numerical value constituted the first stumbling block to overcome. This is in no way the fault of the student, but rather of the educational system, in particular at the university level, which has given the student a stamp of approval and passing grade for numerical calculation without justification. One goal of this study was to lessen these types of surface level understandings, but, realistically, this is one semester of teaching argumentation as compared to years and years of regurgitating calculations and memorized steps. In this sense, this was a constraint of the study.

In all qualitative research, we must consider the personal bias of the researcher as another constraint. The TR does have much invested in the students involved in this study, however, it is to the benefit of all involved in the study that she remained as unbiased as possible so that the results of this study will be accurate and, in return, helpful to the students as well as to the refinement of her own teaching practices. Member checks, detailed notes, and recordings helped to maintain the integrity of this study, as well as the reflective journal which documented her thoughts throughout the study.

A limitation of this study could be considered the small number of students who were selected for each case study. While the total number of students who completed the study was twenty-nine, and while all information was collected for these twenty-nine students, each case study only contained a small number of students. Qualitative studies for large numbers of students are not practical, but qualitative studies do offer a glimpse of student thought processes that are so very valuable to understanding how to better lead and serve our students.

Another limitation common to qualitative research and case studies is a lack of generalizability of the results. Just as students often find false connections when forming generalizations, researchers must be careful not to overgeneralize the results of their work. While this study will certainly impact the teaching practices of the TR, and while the results may provide direction for how to help students use argumentation to form generalizations, the results of the study cannot be used to make a general conclusion about a larger population. Despite not being able to make a conclusion about the larger population, the impact of this study on the TR feeds into and guides the future practice of her as a teacher and, hopefully, of many other teachers.

4.9 SUMMARY

This study followed the methodology of the classroom teaching experiment of Cobb and Yackel and expounded upon by Sáenz-Ludlow. The aim of this study was to investigate argumentation as a cognitive tool of mediating student generalization in Riemann sums and integration in a senior level undergraduate calculus based course. Special care was taken to analyze students' relational understanding of Riemann sums and their relationship to integration and applications. Data, in the form of teaching session observation notes, the instructor's reflective journal, written assignment feedback, and interview notes was collected over the Spring 2019 semester. The data will be analyzed in the upcoming chapter using both thematic and content analysis under the lens of Vygotsky's levels of generalization and zones of proximal development, along with a modified Toulmin's argumentation model.

CHAPTER 5: DATA ANALYSIS

This chapter will present a qualitative analysis of the data, which includes three case studies that arose in this study.

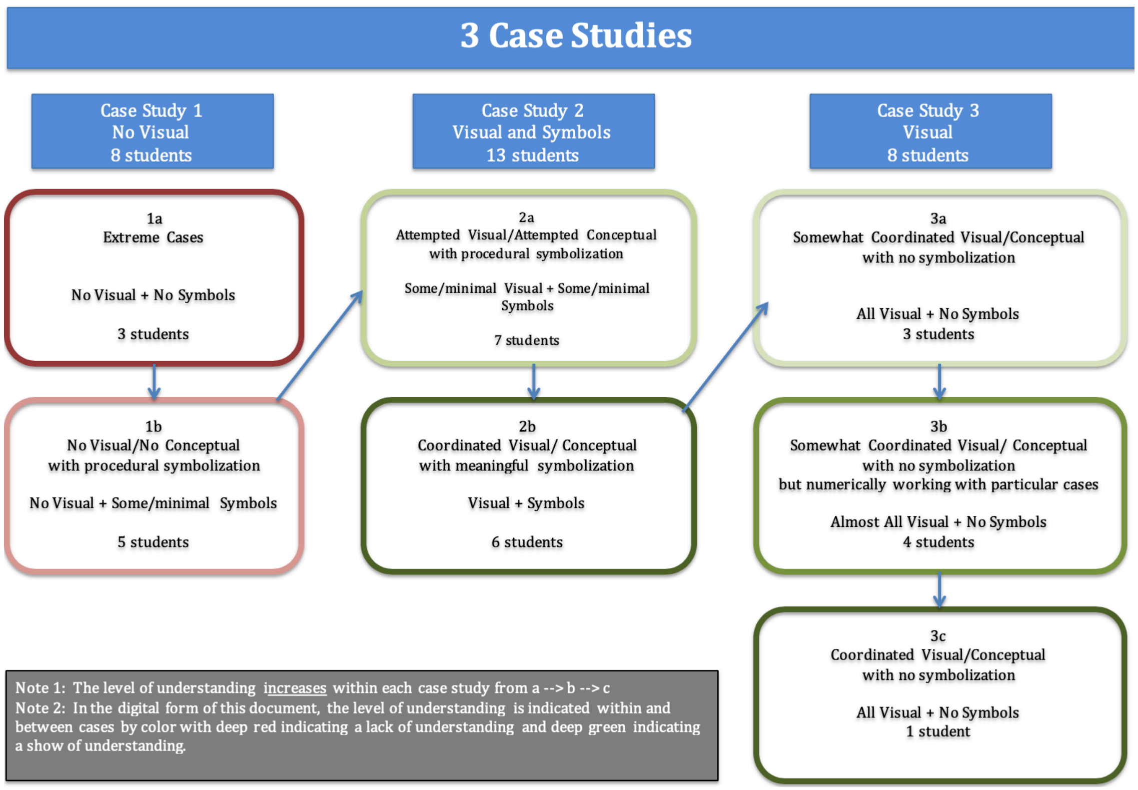
Using content analysis, the de-identified student work was first grouped into categories based on the level of generalization shown in the work of each student. There were students who were: (i) unable to solve problems with particular values; (ii) unable to generalize from particular values to general equations; (iii) able to generalize from particular values to general equations but not to the Excel work; and (iv) able to generalize from particular values to general equations and to the Excel work. It should be noted that Vygotsky's zones of proximal development which are the difference between what a student is unable to do, what a student is able to do with help, and what a student is able to do on their own played a role in the development of these categories. The ZPD of the students highlighted the differences in the generalization categories from a student being unable, to somewhat able, to able.

From these four categories related to generalization, each student's work was then considered again. This time, a thematic analysis, which began with student work as the center of consideration, led to the realization of patterns in the data. By comparing the work each student did in their graphs, value calculations, equations, and Excel file, the TR noticed different ways in which students were solving the interview tasks. The TR noticed some students used visualization and that some students used different levels of symbolization in their work. Some students were able to conceptualize what a rectangle represented in the problem and what the height of the rectangle has to do with the function height. In addition, some students demonstrated an understanding of the x-axis, the y-axis, and coordinate systems. With the students' work as the focus, these patterns served as indicators of student processes and the students' levels of understanding. The thematic analysis first took place within each of the generalization categories, but later across generalization categories. The TR allowed the data to guide the development of new categories during this second part of the data analysis. Themes that arose dealt with the students' processes of visualization, conceptualization, and symbolization. Because these three distinctions seemed prominent in the work of each student, these processes and their relationship to one another formed three new categories and a basis for subcategories. Also due to the prominence of the distinction of the processes in student work, these categories became the case studies of focus for this study. In contrast to other case studies which take the cases in consideration to be those of individual students, this study took the cases in consideration to be the three prominent categories that arose during the thematic analysis, and that mediated argumentation in the students' processes of generalization. These cases are: (i) students who did not demonstrate nor attempt visualization, demonstrated no conceptualization, and demonstrated no to minimal symbolization; (ii) students who demonstrated or attempted visualization, demonstrated conceptualization, and demonstrated symbolization; and (iii) students who demonstrated visualization, demonstrated conceptualization, and did not demonstrate symbolization.

The three case studies and each of their subcategories are summarized in Figure 7 below. This figure is helpful to visualize the relationship between the subcategories of each case relative to the level of understanding demonstrated by student in each subcategory. Each of these case studies and subcategories will now be described in further detail.

Figure 7

Case Study Summary



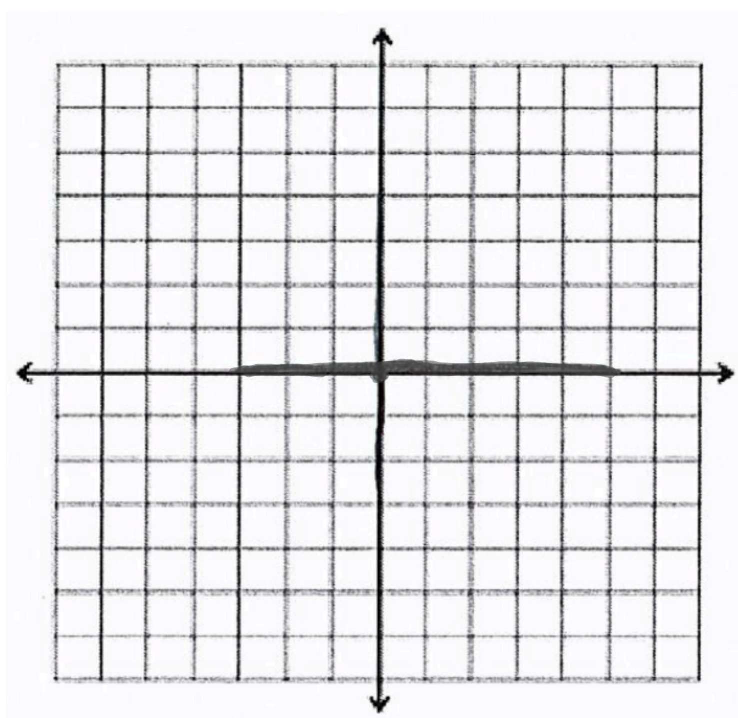
5.1 CASE STUDY 1: MINIMAL VISUALIZATION OR SYMBOLIZATION

This case study included eight students who either did not attempt visualization even when prompted, or they attempted but did not demonstrate accurate visualization processes. In addition, these students demonstrated minimal conceptual understanding of the topics. During the thematic analysis, these students showed differences in their work that prompted the creation of two subcategories: those who did not work with symbols and those who worked with symbols but lacked an understanding of the symbols. Three of the eight students fell into this first subcategory and did not demonstrate symbolization in their work. The other five students fell into the second subcategory and showed procedural symbolization in their work though they lacked meaningful symbolization.

The students in this case study shared the common characteristics in their work of being unable to generate a graph of a basic function such as a parabola centered at zero, being unable to shade the area under the curve corresponding to an integral of a given function and a particular interval, and being unable sketch rectangles into the graph. For instance, in Figure 8, one student attempted a sketch of the function $y = x^3$on the interval [-2,2] by plotting a point at the origin and then darkening in the x and y axes for a number of units unrelated to the function or the interval.

Figure 8

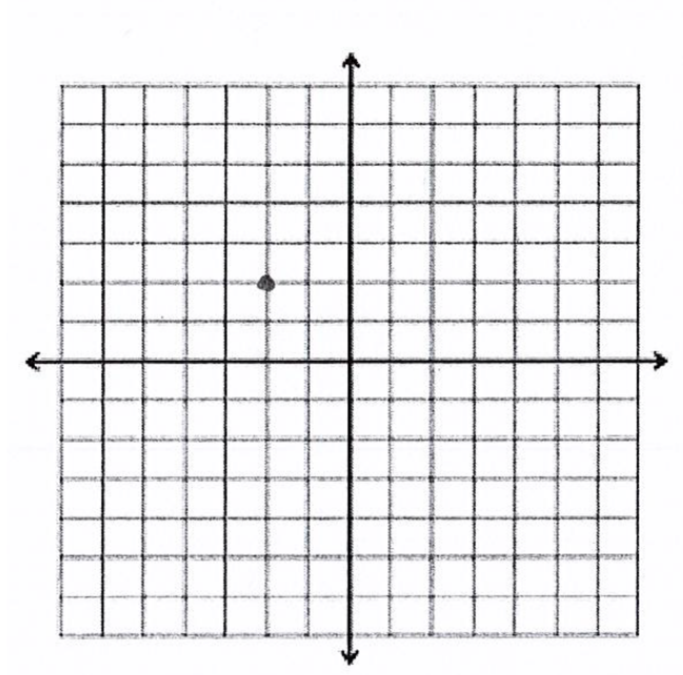
Case Study 1 Student Work Example a



In Figure 9, another student attempting the same task plotted the point (-2,2) rather than graphing the function or marking the interval [-2,2].

Figure 9

Case Study 1 Student Work Example b

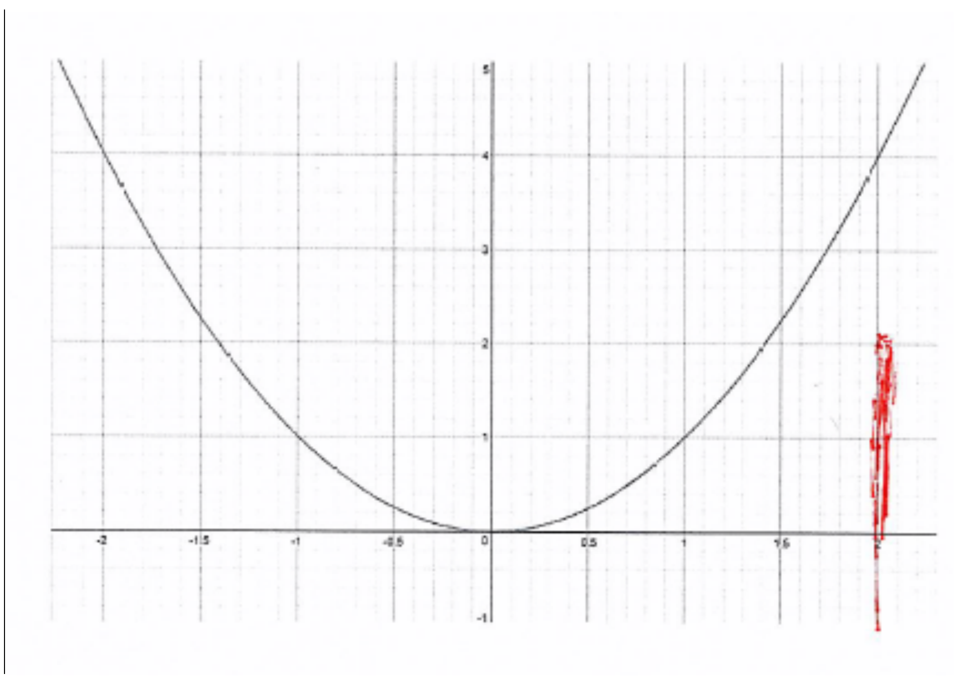


The work of these students as indicated by but not limited to the examples above suggested a lack of visualization processes and a misunderstanding of the coordinate plane, points, intervals, functions, and their visual representations.

Another common and interesting result from this case study that ties to a lack of visualization processes was the work samples in which some students were unable to accurately represent a rectangle. In the following examples of work, students were presented with the task of using four rectangles to estimate the area under the function $y = x^2$on the interval [-2,2]. Students were provided with the graph of the function, and asked to show all steps and provide justification for their work. In Figure 10, the student attempted to shade the area of a rectangle that he would use in his Riemann sums estimation. Notice that the shaded region is not a rectangle, and it is also not inside of the requested interval. The shaded region does not touch any part of the function he was estimating the area under, and the student did not provide any justification for why he drew what he did when prompted by the TR. He asked the TR if his sketch was correct, and when redirected to share his thoughts on why he selected this region, he said he was unsure.

Figure 10

Case Study 1 Student Work Example c

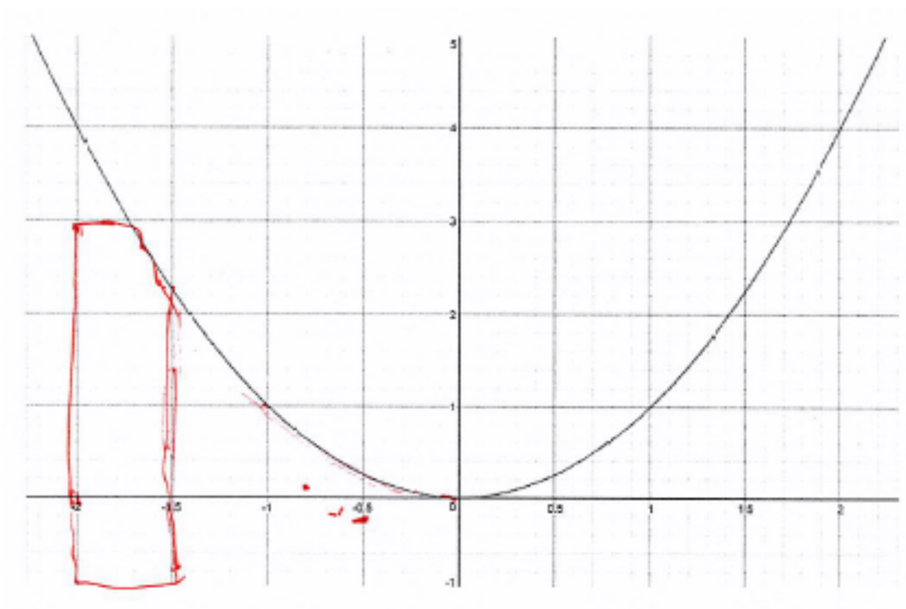


The TR continued to work with the student to try to help move him from his current level of understanding to a clearer level of understanding. The student was asked to point out in the graph what interval he was using. He could not do this. He then found the interval listed in the problem statement, pointed to this, and asked the TR if this was the interval.

The TR confirmed this was the interval he needed to use and again asked the student to indicate that interval in the graph. He found the left bound in the graph and pointed at it. After many questions prompting the student to consider the width of the needed rectangles, and having pointed to the beginning of the interval, the student constructed the “rectangle” in Figure 11. The region is not a rectangle as the top right of the shape was kept below the graph and does not form the needed forth corner. The region also went below the x-axis though the curve and was not above the x-axis. This again suggested a lack of visualization processes and a misunderstanding of the coordinate plane, points, intervals, functions, and their visual representations just as the work samples from the cubic example.

Figure 11

Case Study 1 Student Work Example d



The first subcategory of Case 1 included students who did not work with symbols. These students, who were also unable to sketch or visualize the problem, were either unable to generate any correct values that would lead to the requested estimation, or they were able to write the initial estimate in numbers such as -2, -1, 0, 1 followed by writing the square of the values as a sum such as 4, 1, 0, 1. It should be noted that two of the three students in this subcategory were unable to construct the correct values for the heights, widths, and areas of the rectangles, and the student who was able to construct the correct values did so after several attempts. The student with the correct value was very unsure of her answer, and could not explain how the values she wrote related to the area.

The second subcategory of Case 1 included students who attempted to work with symbols. These students were unable to sketch or visualize the problem, but they did attempt to write out the calculation for the estimate. These students were able to work with symbols, but they lacked an understanding of the symbols represented. For instance, some of these students were able to write the initial estimate as

$$1(-2)^2 + 1(-1)^2 + 1(0)^2 + 1(1)^2 = 6.$$

It should be noted that two of the five students in this subcategory indicated that their final answer value represented the area under the curve of the function. None of the five were able to relate the individual values in their work to the area of a rectangle. Students had memorized the equations and plugged in numbers with no understanding of what those symbols or numbers were calculating or representing. This indicated that the students in this category, though they had a procedural symbolization that allowed some students to arrive at the correct value, lacked a meaningful visualization of what their work represented or why it was justified.

None of the Case 1 students were able to successfully generalize their work on the initial value-based problems to a general interval [a,b] with justification. Interestingly, none of the Case 1 students attempted to utilize argumentation strategies during their interview to help them solve the problems. Some were not able to answer the TR's prompts meant to encourage argumentation strategies; others were able to answer a very direct prompt, but then did not proceed any further with the technique to carry out the rest of the problem. Some of the Case 1 students had memorized the format of the Excel sheet, but, after making a small error, could not correct the issue since they had no understanding of the meaning of what they had written. None of the Case 1 students related what they attempted in the Excel file to their early work with particular functions and intervals. This could be due to lack of understanding in earlier parts. If a student did not know what they had calculated in the earlier task, then it would be difficult to relate a later task to it. It is important to point out that little to no attempt was made to form these connections.

Some comments made by students in written homework tasks or verbally in the interviews included "Hearing Dr. Telusca's voice in my head 'Top minus bottom' wasn't helping me fill in the missing gaps," "Sorry, I just don't remember," and "I have no clue what to do." These statements emphasized memorizing over working through and trying to understand a problem. This is consistent with the work presented and the conclusions above. The Case 1 minimal to no visualization learners who either lacked symbolization or only demonstrated procedural symbolization did not attempt to verbalize argumentation and were unable to form meaningful generalizations regarding Riemann sums concepts.

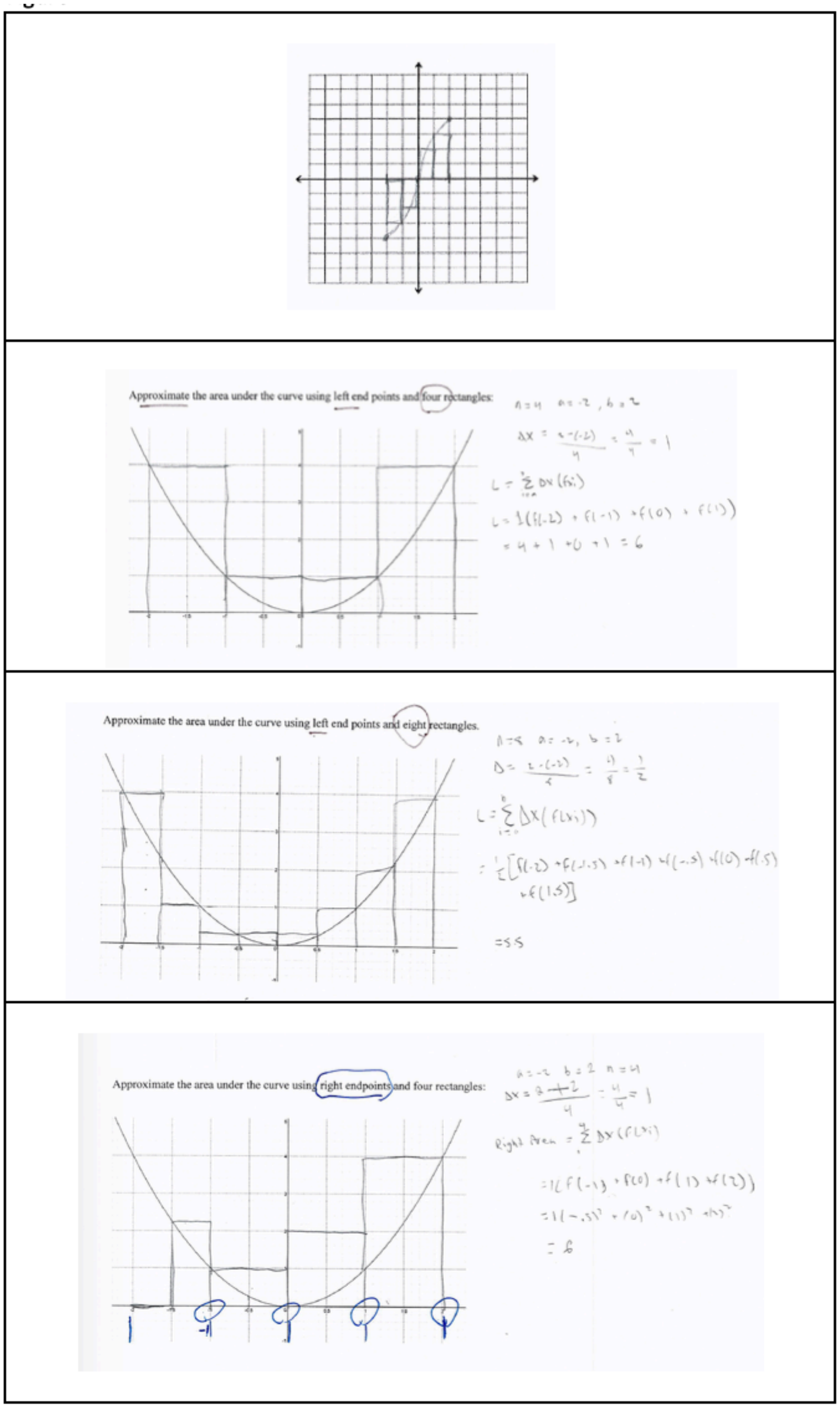
5.2 CASE STUDY 2: VISUALIZATION AND SYMBOLIZATION

This case study included thirteen students who either attempted visualization and attempted conceptualization or they demonstrated coordinated visualization and conceptualization. During the thematic analysis, the differences in student work suggested the need for two subcategories: those with procedural symbolization and those with meaningful symbolization. Seven of the thirteen students fell into this first subcategory of students who demonstrated some symbolization in their work, but who could not justify the meaning or need for the symbolization. The other six students fell into the second subcategory of students who demonstrated much more symbolization in their work, and who were able to connect the symbols used to the visual components of their work and to the reason why their symbols and sketches were representative of what the task requested.

The students in this case study shared the common characteristics in their work of being somewhat able up to fully able to generate a graph of a basic function such as a parabola centered at zero, shade the area under the curve corresponding to an integral or given a function and an interval, sketch rectangles into the graph, and calculate the appropriate values. For instance, in the first diagram of Figure 12, one student sketched the function $y = x^3$on the interval [-2,2] correctly, and attempted to sketch the rectangles corresponding to a right end point Riemann sum using four subinterval. The graph of the function is correct, but the rectangles sketched are not representative of a right endpoint approximation. The student mixed the left endpoint and right endpoint approximating rectangles. As seen in the remaining Figure 12 diagrams, the student made this same type of graphical error in each subsequent task. Note that the student's calculations were accurate despite the rectangles not being entirely correct. This suggested that the student has demonstrated some visualization processes and some understanding of the coordinate plane, points, intervals, functions, and their visual representations. This discrepancy between the incorrect rectangles and the correct values suggested that the student, though he had an attempted conceptualization, lacked some coordination between the visual, conceptual, and symbolic components of the task. Note that this student belonged to subcategory 1 of Case 2 which had shown a procedural symbolization.

Figure 12

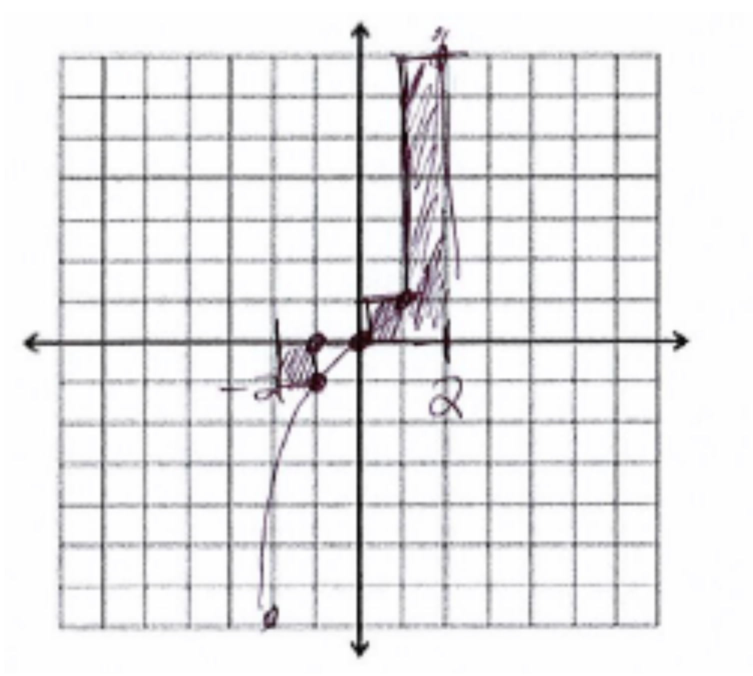
Case Study 2 Student Work Example a



In Figure 13, another student working on the cubic task graphed the function, labeled the axes, labeled the endpoints, and correctly sketched the rectangles for a right endpoint Riemann sum approximation of the function on the interval [-2,2]. This student utilized more in-depth symbolization in his work, and also generated the correct values for the task. Note that this student belonged to subcategory 2 of Case 2 which had demonstrated meaningful symbolization.

Figure 13

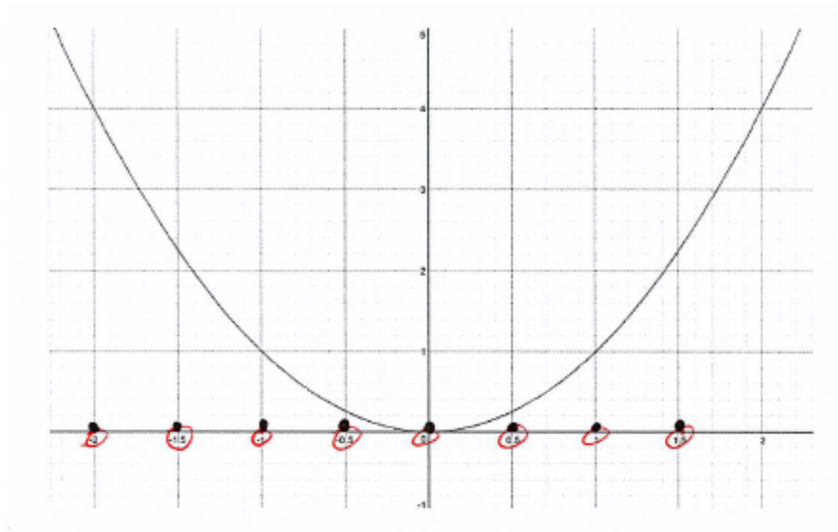
Case Study 2 Student Work Example b



Students were presented, as described in the Case 1 results, with the task of using four rectangles to estimate the area under the function $y = x^2$ on the interval [-2,2]. All students received the same directions to show all steps and provide justification for their work. In Figure 14, one student marked where the edges of the rectangles would be, but did not fully sketch in the rectangles that would be used to form a left endpoint approximation.

Figure 14

Case Study 2 Student Work Example c



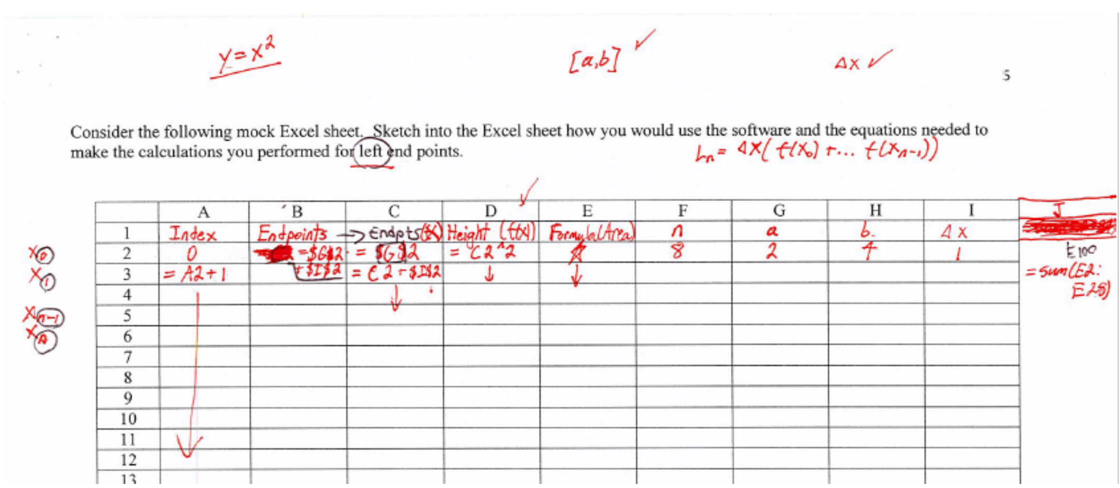
The student used some symbolization, and she arrived at the correct value. She was also able to write out the more general equation for a left endpoint approximation of the area, but she was not able to explain what each part of the equation meant or where it came from. Note that this student belonged to subcategory 1 of Case 2 which had shown a procedural symbolization. The student was able to perform similarly when the number of rectangles was increased, and when a right endpoint approximation was requested. In each task, the student only sketched the edges of the rectangles, demonstrated procedural symbolization, and arrived at the correct value.

The student experienced some difficulties when she arrived at the task requesting a midpoint approximation of the area. She marked the edges of the rectangles correctly, but she miscalculated the midpoints in her written work. The student was unaware that she had made a mistake as she was not comparing her written work to full rectangles in a sketch. The TR had to ask the student to check her work, at which time the student went back through her calculations and found the error. The student also made a mistake in writing her more general equation for the midpoint approximation of the area. The student was initially unable to correct the error, but, after being asked by the TR why she had written some of the things she did in comparison to the values from the previous parts, the student was able to find the mistake and justify her new answer. This showed that the student was able to utilize argumentation, under the direction of the TR to find the connection between the particular value-based problems and the more general equation. This suggested that argumentation was utilized by the TR as a tool for mediating generalization in this instance.

In her Excel work, see Figure 15, the student again reverted to writing and stating memorized equations.

Figure 15

Case Study 2 Student Work Example d

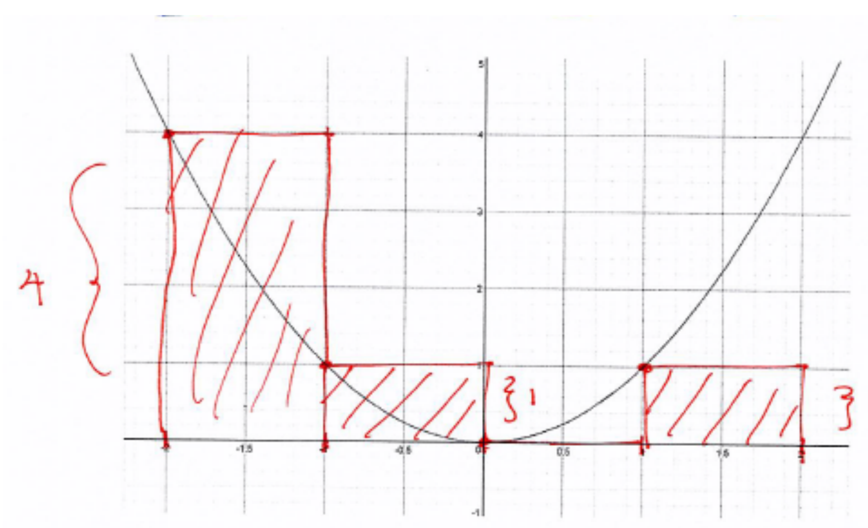


However, when the student was reminded to justify her solution and steps to the TR, she flipped back and forth between the particular value-based problems and her Excel sheet to emphasize the connections. In this manner, she was able to find two of her own mistakes in what she had initially written out from memory. The student remained in the first subcategory of procedural symbolization as she related her symbols to her earlier values but not to the concept or the relationship to the graph and rectangles. However, she was still able to utilize argumentation as a means to move from the particular to the general.

Figure 16 shows the work of another Case 2 student. This student is in subcategory 2 which is comprised of students who demonstrated meaningful symbolization. The student fully sketched the rectangles appropriate for a left endpoint approximation.

Figure 16

Case Study 2 Student Work Example e

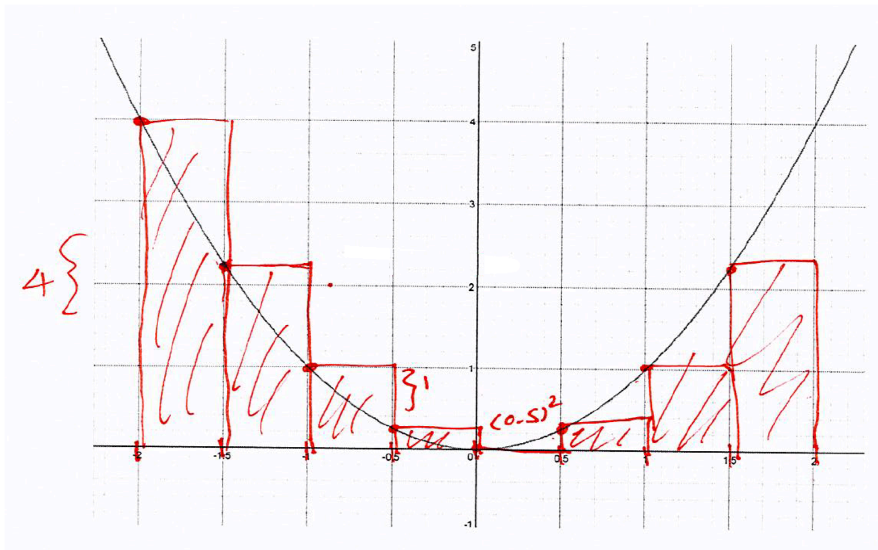


The student used appropriate symbolization and arrived at the correct value. The student talked through her work as she wrote it out, and she connected each symbol that she wrote to the rectangles that she had drawn. For instance, the student said "x_0 is the first left endpoint," and she pointed with her finger at the left endpoint in the graph. Then she said "f(x_0) is the height of the first rectangle; this comes from plugging the first left endpoint into the function," and she pointed to the height corresponding to the function evaluated at the left endpoint in the graph. The student was able to write out the more general equation for a left endpoint approximation of the area, and she explained what each part of the equation meant relative to endpoints and areas and why it was calculating the requested information. In this way, the student demonstrated meaningful symbolization and her work suggested a coordinated visualization and conceptualization.

The student performed similarly throughout her interview tasks when the number of rectangles was increased as in Figure 17, when a right endpoint approximation was requested, when a midpoint approximation was requested, and in the more general equations of each of these tasks. In her left endpoint approximation with eight rectangles, the student said, "This is the same idea as the last one," and then proceeded to draw and explain the rectangles in Figure 17. This is an example of how the student generalized the four-rectangle approximation to the eight-rectangle approximation. In the general midpoint equation, the student double-checked her formula by asking herself what a particular part of the equation meant, and then justified it visually by sketching out the problem for a general interval and n.

Figure 17

Case Study 2 Student Work Example f



In her Excel work, the student again justified each step as she went, and while she was doing this, had arguments with herself to make sure she was thinking about the steps correctly. For instance, the student said, “We need a condition; we can multiply by a statement that compares the index to n.” She then proceeded to write out an if-statement while saying, “If the index is less than n multiply by one; otherwise multiply by zero.” In this manner, she was able to generalize from the particular value-based problems to the general equations to an Excel sheet that would incorporate these equations and integrate Excel if statements to automate the calculations. The student checked her work and said to herself, “Is this doing what we want it to do?” She then made the calculations she had indicated in the Excel sheet with a few values she selected to verify the sheet would do what she had intended. The student was still able to utilize argumentation as a means to move from the particular to the general and verify her results as she progressed.

Several Case 2 students were able to use argumentation to connect the particular to the general. Differences in subcategories tended to be that students with procedural symbolization relied more on teacher-led argumentation, and these students connected the general equations to particular values they had previously found. Students with meaningful symbolization also responded well to any teacher-led argumentation, but many of these students were able to use self-argumentation, as seen in the dialogue of the student discussed above, to form generalizations that connected not only the values to the general equations but also to the visual representations and Excel extensions. There were also students in this subcategory who formed generalizations without articulating the use of argumentation.

Comments made by students from Case 2 in written homework tasks or verbally in the interviews included, "I've learned more in this class than any other class I have taken," "I like thinking about the 'why' of something," and "You are the first person who has made me explain my work." These statements suggest that students in Case 2 recognized that the course structure was different from others they have taken, and that several recognized the emphasis of the course on understanding why something is true and why it is connected.

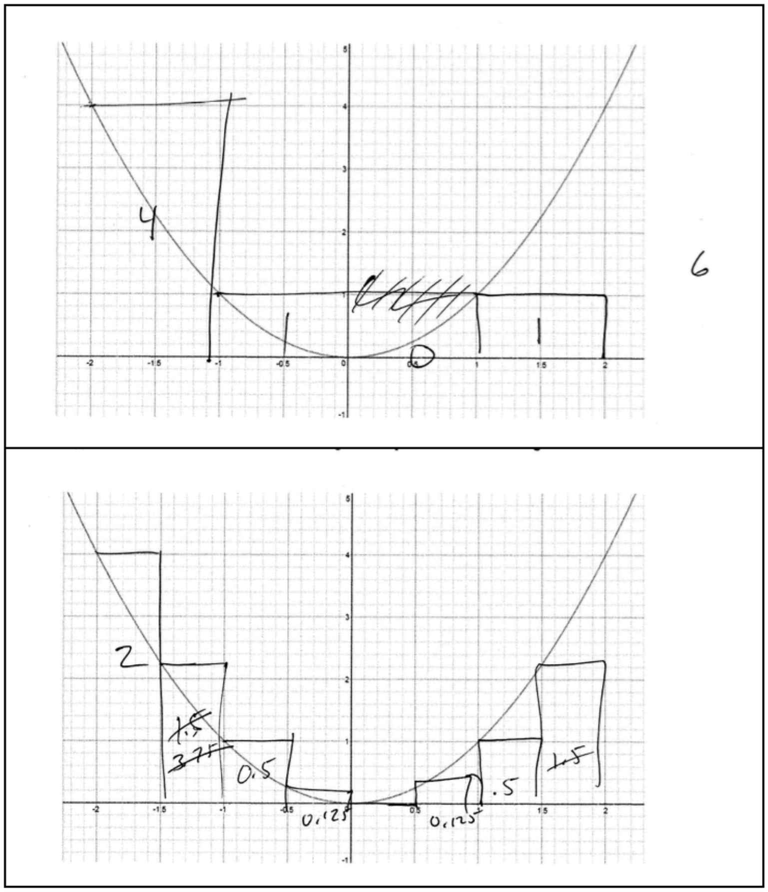
5.3 CASE STUDY 3: VISUALIZATION NO SYMBOLIZATION

This case study included eight students who demonstrated visualization and either attempted conceptualization or demonstrated conceptualization. During the thematic analysis, differences in student work suggested the need for three subcategories: (i) those with somewhat coordinated visualization and conceptualization and with no symbolization; (ii) those with somewhat coordinated visualization and conceptualization and with no symbolization but working with particular cases, and (iii) those with coordinated visualization and conceptualization and with no symbolization.

Three of the eight students fell into the first subcategory of students who demonstrated visualization in their work and who could somewhat justify the meaning of the visual representations. As seen in Figure 18, these students were able to generate a number value for earlier tasks based solely on their graphical representations; they were unable to use symbolization to capture any of the ideas or calculations, and they were unable to complete all tasks.

Figure 18

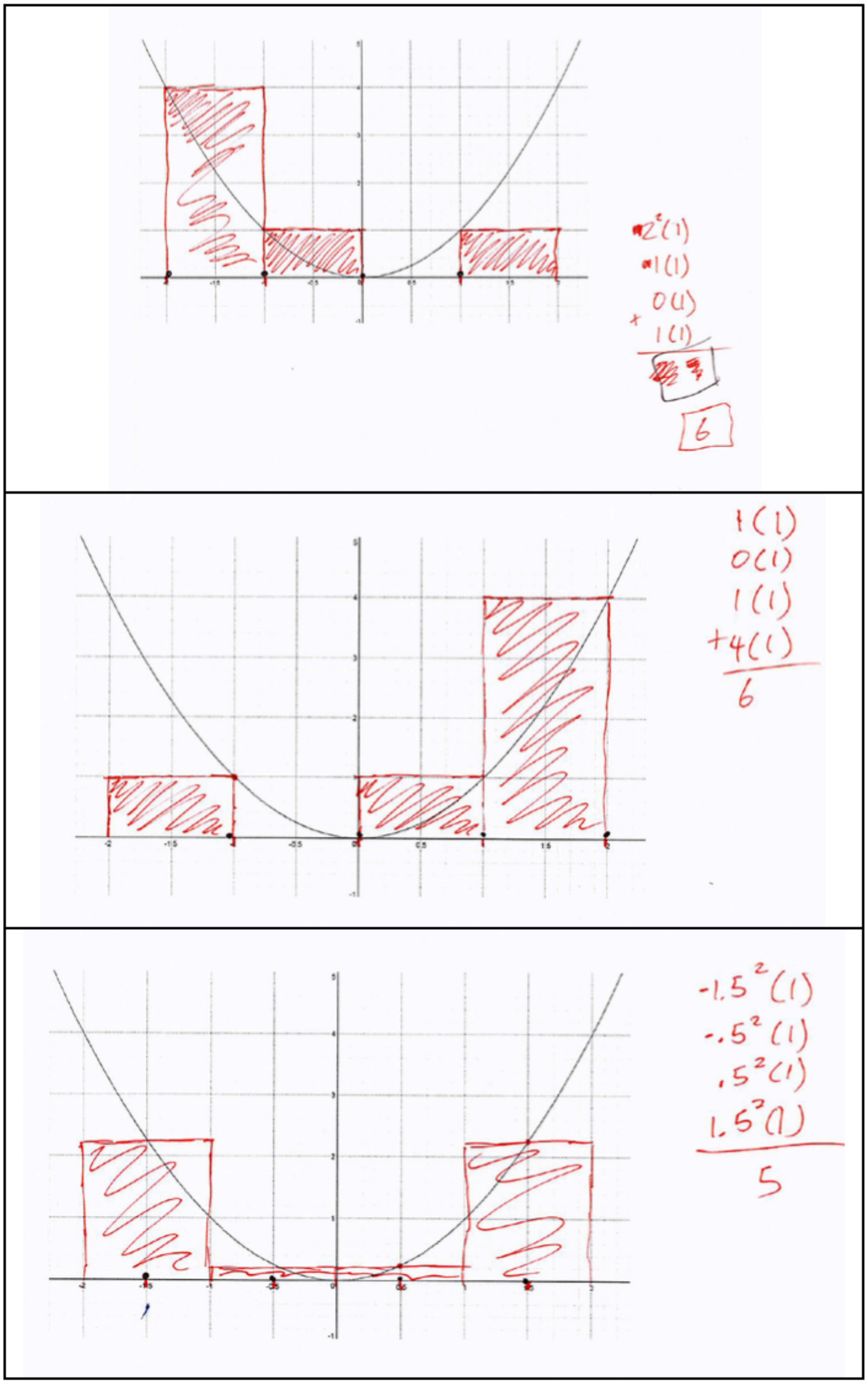
Case Study 3 Student Work Example a



Four of the eight students fell into the second subcategory of students. These students, like the first subcategory demonstrated visualization in their work, somewhat justified the meaning of the visual representations, and lacked symbolization of either type, procedural or meaningful. However, as seen in Figure 19, these students were able to use particular numbers written in known patterns to reach a particular answer. They were able to work these particular cases into more advanced tasks and in the Excel sheet. The third sub-category contained one student. The justification for such a small group is that the work of this student was very different than the work of any other student. While this student demonstrated visualization in his work like the first two categories, this student fully justified the meaning of the visual representations. He was able to generate not only correct values from his visual representations alone, but also was able to fully complete the more complex of the tasks with no symbolization.

Figure 19

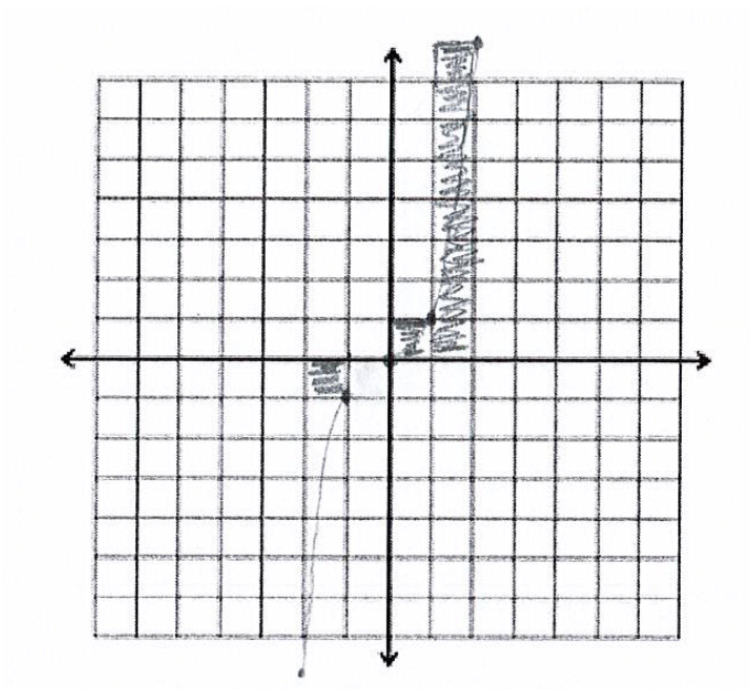
Case Study 3 Student Work Example b



The students in this case study shared the common characteristics in their work of being somewhat able up to fully able to generate a graph of a basic function such as a parabola centered at zero, shade the area under the curve corresponding to an integral or given a function and an interval, and sketch rectangles into the graph. The students in this case all lacked symbolization, though one category mimicked the equation pattern with particular values. For instance, in Figure 20, one student sketched the function $y = x^3$ on the interval [-2,2] correctly, and sketched the rectangles corresponding to a right endpoint Riemann sum using four subintervals.

Figure 20

Case Study 3 Student Work Example c



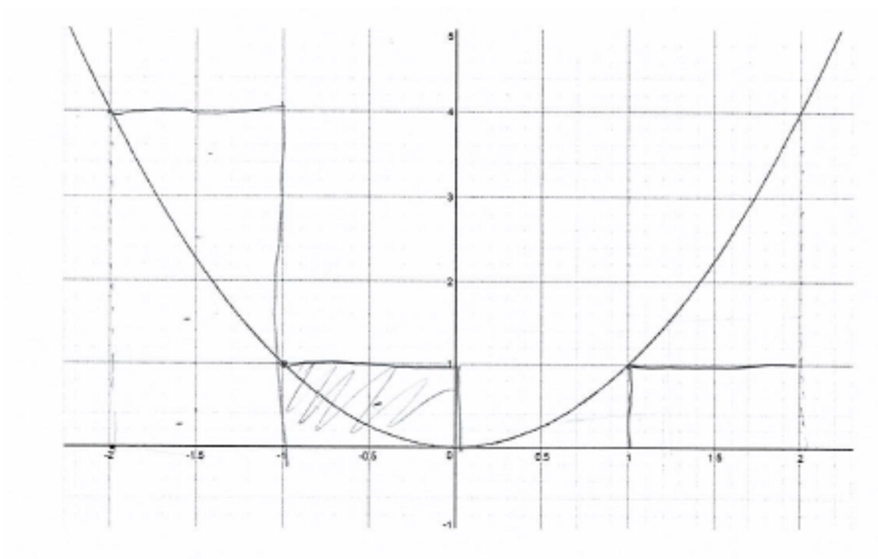
This student was unable to calculate the appropriate value. This student fell into the first subcategory of students who demonstrated visualization and could somewhat justify the meaning of their visual representations. This student was sometimes able to obtain the correct value from his graphs, and other times was not able to do so. This student did not demonstrate argumentation or provide justifications for his answers, and he most often did not follow the prompting of the TR to think about why something had to be true or how he knew that it was correct. As the tasks progressed, this student was unable to complete them accurately. The student made an attempt at the Excel task, but put in little effort to complete the task accurately despite the suggestions and promptings of the TR. The work of this student and the discrepancy shown between the student being able to correctly graph the problem but being unable to form steps or equations to solve the problem suggested less coordinated visualization and conceptualization. The student demonstrated no argumentation and struggled to form generalizations.

Another student in this case study had the same correct work as that of Figure 20, but this student was able to calculate the appropriate value. This student did not use symbolization, but he did write out the particular values that would correspond to each rectangle. This student fell into the second subcategory of students who demonstrated visualization, who could somewhat justify the meaning of their visual representations, and who used particular values within known patterns. This student responded to the prompting of the TR to think about why something had to be true, and he responded in some instances by pointing to his visual representation and connecting it to the particular number he had written. As the tasks progressed, this student was unable to complete them all accurately. The student demonstrated a procedural or memorized knowledge in the Excel task and was able to implement a particular case of four rectangles on the particular interval [-2,2]. The student was unable to generalize the particular case in Excel to a general case.

In Figure 21, another student from this case study provided a similar graphical representation of an interview task.

Figure 21

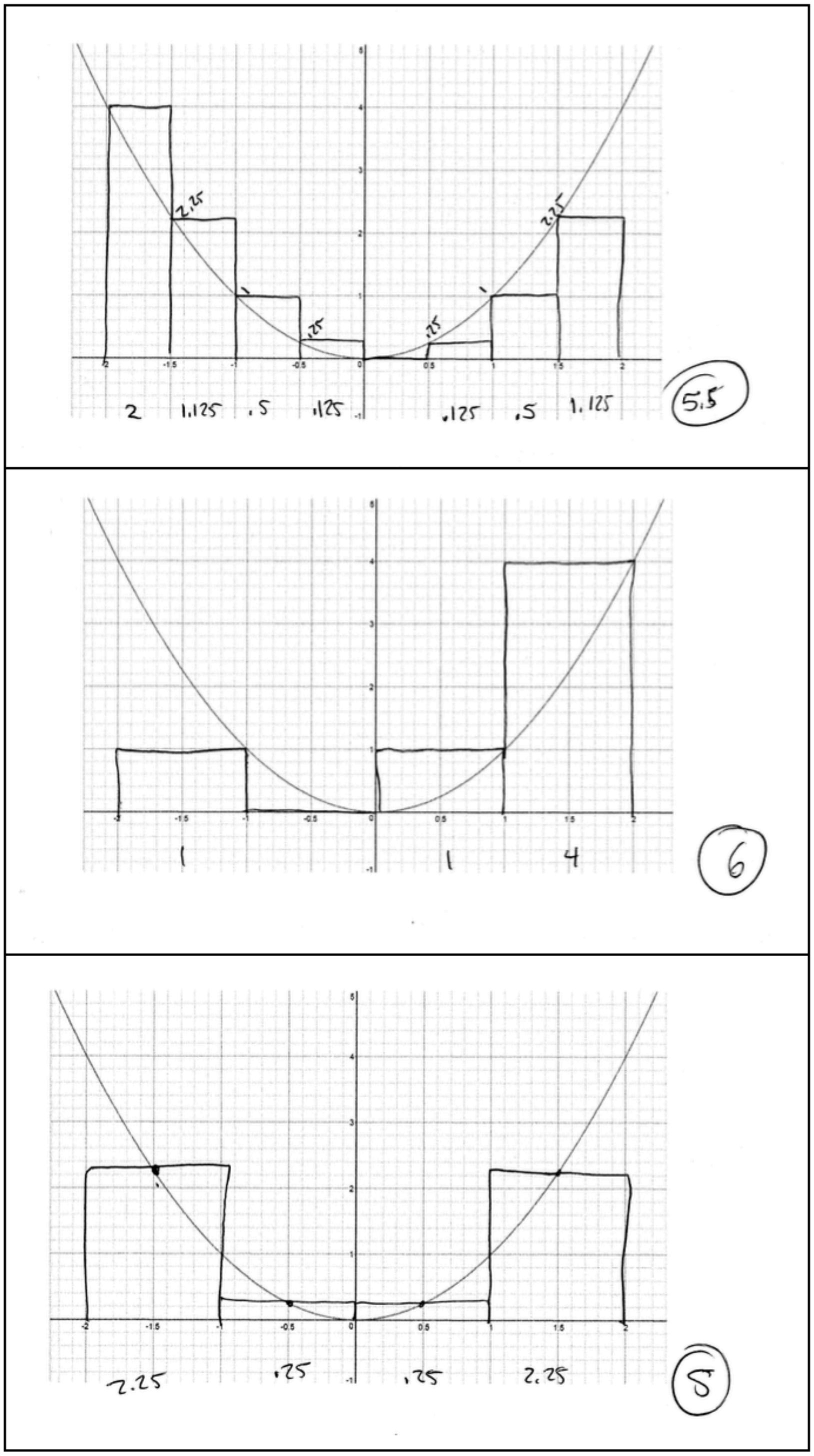
Case Study 3 Student Work Example d



Note that Figure 21 is similar to Figure 20 in that it contained the correctly sketched rectangles for a left endpoint Riemann sum approximation of the function on the interval [-2,2]. Note that this is the work of the student belonging to subcategory 3 of Case 3 which had demonstrated no symbolization but yet coordinated visualization and conceptualization. There was no discrepancy between the graphs of rectangles and the calculated values and explanations of value meanings during the student interview. In this manner, the student showed coordination between the visual and conceptual components of the task. This student was able to correctly calculate the appropriate value with no symbolization and without writing particular patters or values. When asked how he arrived at his answer, the student was able to describe to the TR what the picture was doing relative to the problem statement and the area of each individual rectangle he had sketched. He then labeled the areas of each rectangle inside the rectangle itself to emphasize to the TR what he was explaining. He then said that he just added the individual areas of the rectangles to form his approximation of the area under the curve on that interval. The student performed a similar process on the tasks involving more rectangles, a right endpoint approximation, and the midpoint approximation. As seen in Figure 22, the student was able to visualize each task, sketch the appropriate work, calculate the correct answer, and explain his process. However, the student was unable to use symbols or equations to capture any of the ideas or calculations.

Figure 22

Case Study 3 Student Work Example e



For the Excel portion, the student paused when he saw the sheet, and then told the TR he had not looked over any of this but that he would give it a try. He began talking himself through the steps he had explained to in the picture. At some points the student would point to his picture and say, “I want to take this,” and then he would write what he wanted to do to it in Excel leaving the parentheses set empty and carrying on with the Excel calculation. At other times he assigned a name to something to try to capture it; these were not names that the TR had used in her teaching sessions, nor were they common names for the problem components.

The steps that the student applied to the components he was having a hard time with labeling were correct. The student utilized self-argumentation as he completed the Excel sheet, asking himself if this would work or why it would not. He convinced himself that his process was correct before moving to the next portion. His entire Excel sheet would work if those empty parentheses he left as holding places for the interval endpoints and the number of rectangles were just linked to the appropriate a, b, or n for the problem. In fact, his spreadsheet would work for the most general of cases if these parameter cells were created, meaning his spreadsheet would work for any interval [a,b] and any number of rectangles n. Throughout the Excel tasks, the student continued to talk through his thought process and justify his work. Despite no symbolization in his earlier tasks, he was able to use argumentation to form meaningful generalizations in the Excel task.

In summary, Case 3 students were able to demonstrate visualization in their work. This case had the most polarized results, with some student not being able to complete the tasks up to a student who completed the task to its most general form. A potential distinction in the success or failure of the students in this case seems to be tied to the students' argumentation strategies or lack thereof. The visual learners who did not use argumentation were unable to form the needed generalizations. Some of the visual learners who were able to following the prompting of the TR to utilize argumentation were able to form generalizations. The visual learner who used self-argumentation was able to form meaningful generalizations. Case 2, which had students with both visualization and symbolization, had some students who formed generalizations without argumentation. That was not the results of the Case 3 students. These students who demonstrated visualization and no symbolization did not form meaningfully generalizations without the use of argumentation.

Comments made by students from Case 3 verbally in the interviews included "I've never thought about what this could be used for before," and "This has made me thing about things differently; I wish I had worked this hard in other classes." These statements suggested that students in Case 3 also recognized that the course structure was different from others they have taken. It is also important to note that providing justifications for steps does put more responsibility on the students to think and to work. One student pointed out that, "To be honest, I am graduating with two degrees and I basically have no idea how to apply anything that I've learned. No one talks about the point of anything. I like this class. You really made me think about what is useful."

CHAPTER 6: FINDINGS AND RECOMMENDATIONS

This chapter will present the findings from the data analysis in light of the research questions, recommendations for instruction, and recommendations for future research.

6.1 DISCUSSION OF FINDINGS

The first research question proposed in this study was, "How do students in a calculus-based course justify their mathematical solutions, and how does justification evolve throughout the teaching experiment?" Based on the three case studies, students justified their mathematical solutions in different ways or lacked justifications of them.

In Case Study 1, many students did not attempt to justify and/or argue their mathematical solutions. Despite course emphasis on justifying one's own work, despite the TR's demonstrating this type of justification, and also despite the TR's expecting and reminding the student to provide justification during the interview tasks, some students presented only the numerical value of their answers as their complete solutions. This suggests that habits of thinking created in prior courses are very powerful and difficult to overcome. In this case, the habit of thinking involved rote memorization, procedural knowledge, and a conflation of the meaning of answer and the justification of the solution process.

In contrast, some students in Case Study 2 and Case Study 3 were able to expand their thinking by refining their justifications and even constructing an argument of a particular solution given before to the task at hand. For example, almost every student began with the idea that a short answer or an equation was more than enough to justify their particular solutions. As the study progressed, many students were able to construct better and more sophisticated justifications of their work. Such refinement occurred especially during the one-to-one interviews guided by the TR (the interviewer). Students i) added in another component such as a graph to help justify their equations; ii) verbally explained what was meant by either pointing at a component of the graph or by pointing at mathematical notation they had already used; or iii) verbally derived the equations relative to the graphs. This suggests that students can utilize argumentation as a means to further evolve their current justifications.

The second research question proposed in this study was, "How do students in a calculus-based course argue their mathematical solutions after experiencing argumentation modeled by the TR in the classroom?" Case Study 1 showed that some students do not use argumentations let alone simple justifications of their mathematical solutions even after experiencing argumentation in the classroom as modeled by the TR. This is likely tied to the beliefs of these students as to what it means to justify and argue a solution. In contrast, Case Study 2 and Case Study 3 showed that some students constructed some type of argumentation when specifically requested to justify their answers, while other students used, on their own, self-argumentation to convince themselves and others of the accuracy of their solutions.

The third research question proposed in this study was, "How do students in a calculus-based course generalize within the topic of Riemann sums and integration when presented with sequential tasks?" Case Study 1 showed that some students do not generalize within the topic of Riemann sums and integration. For instance, when presented with sequential tasks, many of the Case 1 students chose to provide memorized components of solutions that were collectively generated during teaching episodes instead of looking for connections between the tasks and constructing a logical argument. This is consistent with the results of Buck (1995) along with Becker and Rivera (2005) whose studies showed that students had difficulties in generalizing or chose not to generalize from known topics to extensions of these topics. In contrast, some students in Case Study 2 used argumentation, either teacher-prompted or self-generated, to form generalizations, while others formed generalizations by observing the repetitive pattern of a particular task. Moreover, in Case Study 3, the students who formed generalizations within the topic of Riemann sums did so through argumentation, either teacher-prompted, self-generated, or a combination of these two.

This suggests that some students formed evolving processes of generalization even though they may have not expressed explicitly their processes of argumentation, in the sense that they may not have given explicit justifications to the TR either in writing or in the one-to-one interviews. This also suggests that students can be taught argumentation, and that argumentation can be used to mediate the formation of generalizations in the context of Riemann sums whether it is explicit or not in the mind of the student. These results extend the works of Buck (1995) and Becker and Rivera (2005) in that while students may struggle to generalize as they did in these previous studies, argumentation was used as a tool by some students in this study to mediate the formation of generalizations and overcome these difficulties.

The above discussion of Research Question Three leads directly to the fourth research question proposed in this study which was, "How does the mathematical argumentation used by students in a calculus-based course prompt the formation of generalizations within the topic of Riemann sums and integration when students are presented with sequential tasks?" Case Study 1 and Case Study 2 suggested that when students construct argumentations, they are consequently able to form progressive generalizations using sequential tasks posed to them. The meaningfulness of these generalizations is rooted in the students' conceptualization of earlier tasks, and the sophistication of the students' justifications to construct mathematical argumentations. While Williams (1991) showed that students often resist change to their initial conceptual models and have difficulties moving from informal to more formal and generalized understanding of concepts, and while Buck (1995) showed that students often do not attempt to form generalizations, this study suggests that argumentation can be used to move students through a formation of progressive generalizations. In addition, the analysis of coordination of visualization and conceptualization demonstrated by some students in this study extends the work of Williams (1991) and Buck (1995) in that this attention to coordination provides another avenue in which to analyze student generalizations between related topics.

6.2 IMPLICATIONS FROM THE STUDY

Several implications arose from this study. There is a large discrepancy between what mathematics students are expected to know when they enter a mathematics course and what many students know. Rote memorization and procedural knowledge without relational understanding are how many students have survived their mathematics courses. Rote memorization and procedural knowledge without relational understanding is not enough to help students understand mathematics and its applications. Knowledge about how students are able to construct relational understanding and how teachers encourage, promote, and facilitate this type of understanding may have a major impact in mathematics education, not just within the context of calculus. Argumentation may help students to make connections within and between math concepts. If so, teachers can become aware of how they can and why they should model argumentation in the classroom to help students gain and increase their relational understanding of mathematical concepts. In addition, as teachers place emphasis on utilizing argumentation to impact student learning, students can become aware of their capacity to increase their own mathematical understanding and learning processes.

In regard to implications in the area of argumentation research, this study suggests that argumentation can be used as a mediating tool for generalizations and potentially for other conceptual processes. While many authors and researchers such as Toulmin (1958) and Perelman and Tyteca (1958) have discussed processes of argumentation, and while others such as Walton et al. (2008) have discussed argumentation schemes and classifications of argumentation, these works considered argumentation for the sake of argumentation. This study expands these current works of literature by utilizing argumentation as a tool for mediation rather than as a product or outcome in itself. Arzarello et al. (2009) considered the argumentative processes of calculus students, but again the focus was placed on the evolution of student arguments whereas this study triggered the part of the student argumentations by imitating the argumentation methodology, the basis of the teaching methodology in the study. The results of this study imply that argumentation can be used as a process to stimulate conceptual understanding.

In regard to implications in the areas of argumentation and generalization within mathematics education, this study indicates that argumentation as a tool to mediate progressive generalization can further deepen student understanding of mathematical concepts such as Riemann sums. This expands the work of Douek (1999) who studied students that were asked to arrive at a mathematical generalization and then to justify their generalization. However, Douek did not utilize argumentation as a tool to mediate such generalizations. While Douek (1999) analyzed the argumentations of students after the formation of a generalizations, in contrast, this study used argumentation with the objective of helping students to generalize. As the generalizations within Riemann sums are similar to generalizations studied by Buck (1995), this suggests that argumentation may mediate progressive generalizations in the context of quadratics and polynomials. The coordination of visualization and conceptualization or lack thereof demonstrated by some students in this study and the relative success or failure of those students to form generalizations indicates that understanding students' coordination of visualization and conceptualization may shed light on how students will progress through generalizations of mathematical concepts including but not limited to Riemann sums, quadratics, and polynomials.

6.3 RECOMMENDATIONS FOR FUTURE RESEARCH

Several recommendations on research about argumentation and student formation of mathematical generalizations arose from the study. Additional research is needed on topics that were brought to light in this study including progressive generalizations by means of argumentation and student coordination of visualization and conceptualization. While this study indicates that argumentation mediates progressive generalizations, this idea is a new extension to previous works on generalization. This extension would be further strengthened by future studies on argumentation to mediate generalization within topics of mathematics similar to and different from the concept of Riemann sums. Likewise, the relationships between student coordination of visualization, conceptualization, and symbolization that arose during this study would be strengthened by further investigation in future studies.

This study was conducted in an upper level mathematics course. While this provided an opportunity to highlight the differences between what students are expected to know and what many students do know, the study of argumentation in introductory mathematics and statistics courses would allow students to utilize their argumentation processes from the beginning of their college courses through all courses that they take. This progress could be tracked and studied to shed further light on how students develop these processes and the impact utilizing argumentation has on their conceptual understanding. If introductory courses are considered for study, it should be noted that large sections may impact either the type of study or the number of tasks to be analyzed.

This study involved one TR's attempt to teach students how to argue within a calculus-based course. An interesting area of research may be instructional techniques for how teachers can teach students to argue within mathematics. What are the most effective ways of teaching argumentation in small classes versus in large classes? What impact does course structure (face-to-face, hybrid, or online) have on teaching argumentation?

6.4 RECOMMENDATIONS FOR INSTRUCTION

Several recommendations for mediating student generalizations in mathematics arose over the course of the study. This study indicates that modeling argumentation in the classroom could be viewed as a preliminary phase of helping students to construct generalizations. Students can become aware of their processes of argumentation and the importance of explicitly utilizing steps for argumentation with themselves and others, and that students should be given tasks to explore their own arguments. In order for argumentation to be used as a tool for mediating the development of mathematical generalizations, specific task sequences must be thoughtfully considered and designed. Students can be guided through task sequencing and the utilization of argumentation to form mathematical generalizations that show an understanding of conceptual relationships rather than simply memorizing what appears to be disconnected facts or formulas. Teaching students to argue, in general and specifically within mathematics, should be considered a necessary and useful practice at any level of mathematical education from elementary school to college level undergraduate and graduate education.

6.5 CONCLUSION

Many studies have investigated students' processes of generalization within mathematical concepts, but this study extended these ideas by considering students' formation of generalizations as mediated by self-argumentation or argumentation with others. While the results of this study are consistent with previous works of literature, such as Williams (1991), Buck (1995), and Becker and Rivera (2005), in that some students in this study struggled to form or did not form generalizations, this study also included students who were able to form generalizations by means of argumentation. Many students in this study demonstrated that argumentation could be used by students as a tool to mediate progressive generalizations within the topic of Riemann sums. Several students also indicated in their interviews that the course emphasis on justifying their own thinking positively affected their learning of mathematical concepts and their understanding of real world applications. The results of this study suggest that teaching argumentation as a tool to mediate generalizations may help students to overcome difficulties in constructing generalizations, help students to deepen their understanding of mathematical concepts, and help students to connect mathematical concepts to real world applications.

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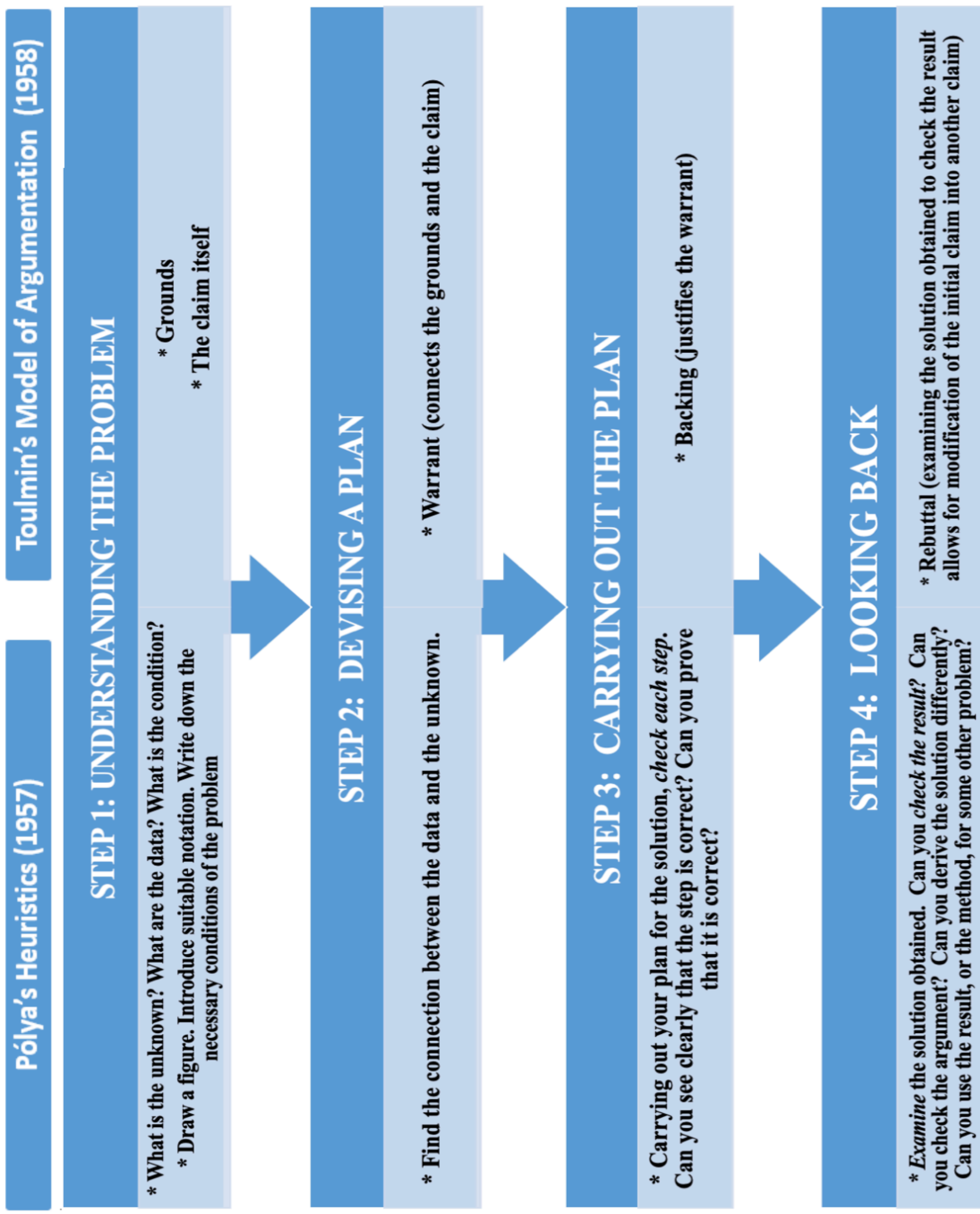
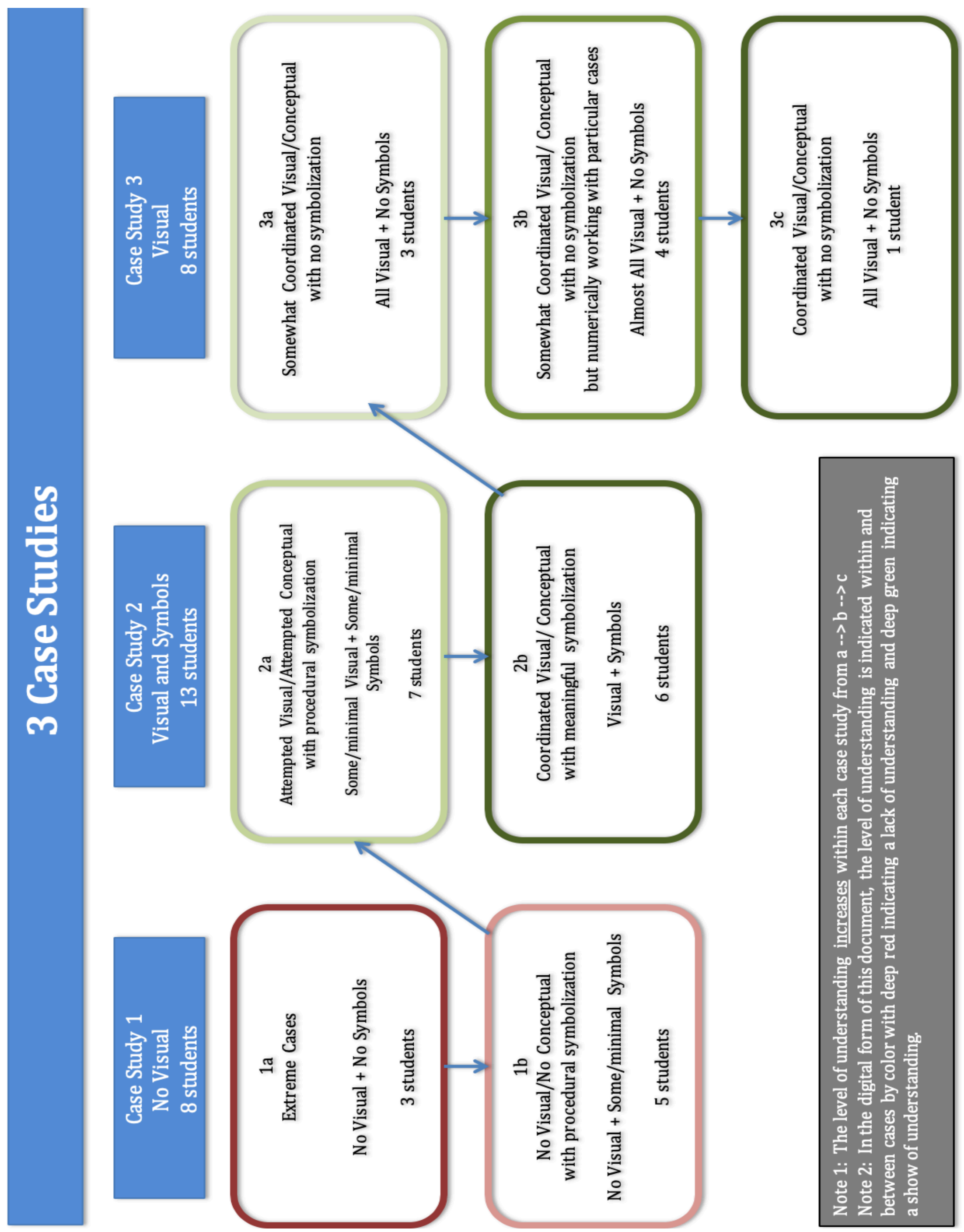
APPENDIX A: COMPARISON GRAPHIC

Comparison of Pólya's heuristics and Toulmin's argumentation model



APPENDIX B: CASE STUDY SUMMARY GRAPHIC

Enlarged Case Study Summary Graphic



APPENDIX C: INTERVIEW GUIDE

Interview Guide

Interview Tasks and Questions

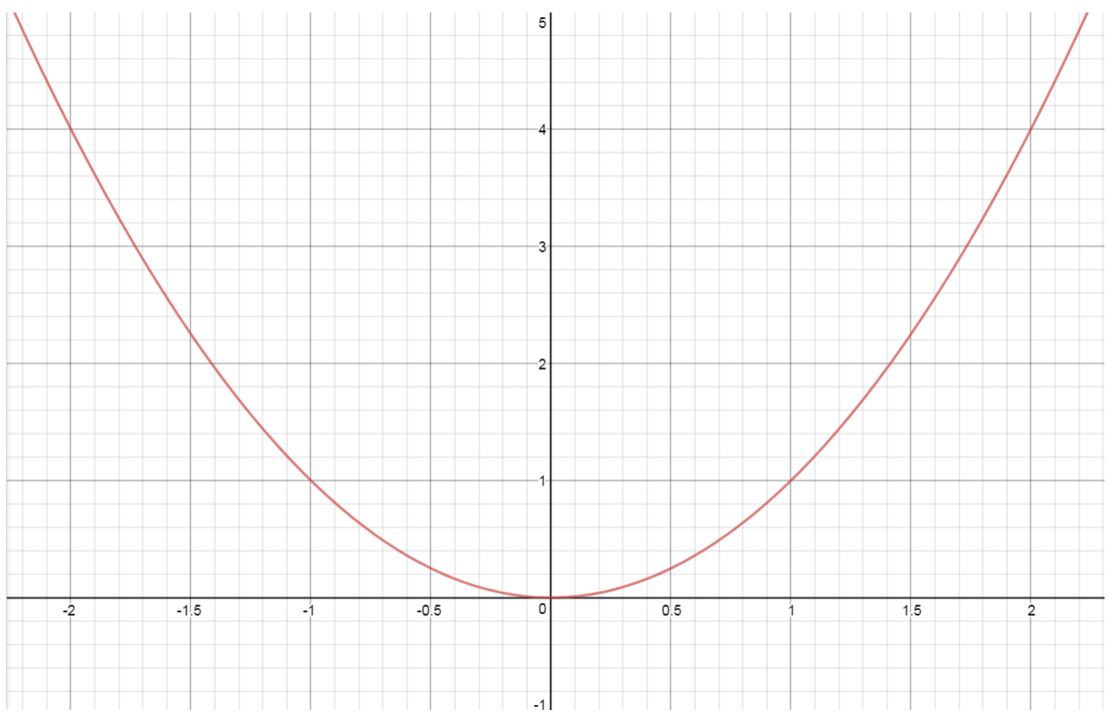
The students will be interviewed by the researcher on two occasions during the semester (weeks 8-9 and weeks 10-11). During the interviews, students will solve a variety of Riemann sums problems while “thinking aloud.” Students will work individually in solving the problems and will be given as much time as they wish to complete the tasks. Interviews will audio recorded for subsequent analysis.

The interviews will follow the constructivist methodology designed by Steffe (1980) and extended to the classroom-teaching experiment by Cobb and Yackel (1995). The interviewer’s (or Teacher-Researcher’s) questions will range from questions that ask the solver to graphically indicate what is meant by area under a curve to asking the solver to approach the Riemann Sums problems analytically to asking the solver for clarification or further explanation of actions that he or she performed. In cases of an extended period of silence is accompanied by an absences of paper and pencil activity, the interviewer may ask the solver what he or she is thinking. Research data on the use of self-reports, such as those of the solvers in these interviews, suggests that these types of questions from the interviewer cause only minor interruption of the solver and do not threaten the overall validity of the data (Schwarz, 1999).

<u>Interview 1 Tasks (Weeks 8-9)</u>

In Interview 1, students work through a task similar to those presented in the teaching sessions. The students are instructed to solve the problems by "thinking aloud," or self-reporting their solutions steps. They are provided paper that includes a graph of the function x-squared, a pencil to record their work, and can use a calculator if they choose. Computers will not be used.

The instructor will give students the following printed graph of the function x^2:



The students begin the interview by marking in a graph of the function x^2 what is meant by the area under the curve. They proceed to approximate the area under the curve using left endpoints and four rectangles and then using left endpoints and eight rectangles. Students are asked to develop a formula for the steps they are taking that will work for any number of rectangles. The students are then asked to use sigma notation to condense their formula. The students then consider the area under the curve using right endpoints with four rectangles and then eight rectangles. Similarly, students are asked to develop a formula for the steps they are taking for any number of rectangles. The students are asked to use sigma notation to condense this formula. Students are given a printed table that mimics a blank Excel sheet as follows:

	A	B	C	D	E	F	G	H	I
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They are asked to sketch into the Excel sheet how they would use the software and equations to make the calculations they performed for left endpoints, and then later for right endpoints for n rectangles.

Interview 2 Task (Weeks 10-11)

In Interview 2, students work through a task similar that of Interview I about the function x^2, but focusing on midpoint approximations of the area, and, depending on the time remaining for the individual student, possibly more advanced techniques such as the Trapezoidal Rule or Simpson's Rule. The students are instructed to solve the problems by "thinking aloud," or self-reporting their solutions steps. They are provided paper that includes a graph of the function x-squared, a pencil to record their work, and can use a calculator if they choose. Computers will not be used. After similar approximations to Interview 1, using the same graph and mock excel file to find the new approximations, the student will be asked to calculate the true area under the curve and then to describe how to make the approximations they used earlier in the task more accurate and closer to this true value.

APPENDIX D: OBSERVATION GUIDE

Observation Guide

Subjects will be observed during Interview 1 as they complete Riemann sums problems in Interviews 1 and 2.

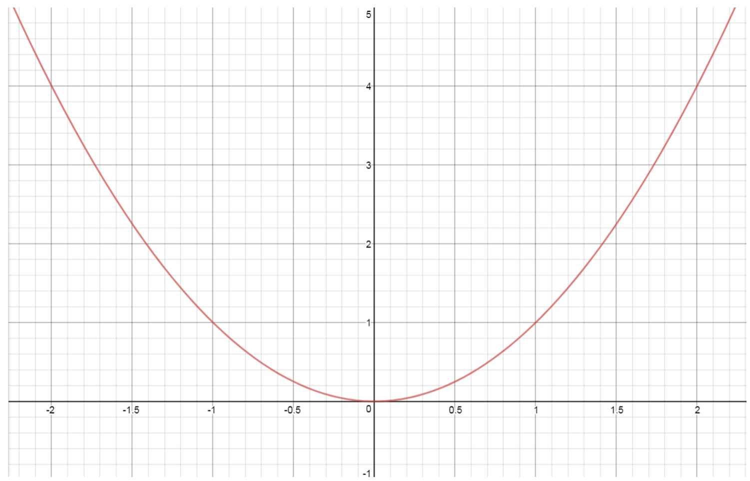
Students will work the problems using paper and pencil. The researcher is interested in the following kinds of solution actions:

1. The students' informal mathematical argumentation strategies they use to solve problems;
2. The students' generalizations formed to solve the problems;
3. The students' verbal statements of how they "see" the problem and the connections between the solution steps.

Interview 1 Problem

In Interview 1, students work through a task similar to those presented in the teaching sessions. The students are instructed to solve the problems by "thinking aloud," or self-reporting their solutions steps. They are provided paper that includes a graph of the function x-squared, a pencil to record their work, and can use a calculator if they choose. Computers will not be used.

The instructor will give students the following printed graph of the function x^2:



The students begin the interview by marking in a graph of the function x^2 what is meant by the area under the curve. They proceed to approximate the area under the curve using left endpoints and four rectangles and then using left endpoints and eight rectangles. Students are asked to develop a formula for the steps they are taking that will work for any number of rectangles. The students are then asked to use sigma notation to condense their formula. The students then consider the area under the curve using right endpoints with four rectangles and then eight rectangles. Similarly, students are asked to develop a formula for the steps they are taking for any number of rectangles. The students are asked to use sigma notation to condense this formula. Students are given a printed table that mimics a blank Excel sheet as follows:

	A	B	C	D	E	F	G	H	I
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They are asked to sketch into the Excel sheet how they would use the software and equations to make the calculations they performed for left endpoints, and then later for right endpoints for n rectangles.

<u>Interview 2 Problem</u>

In Interview 2, students work through a task similar that of Interview I about the function x^2, but focusing on midpoint approximations of the area, and, depending on the time remaining for the individual student, possibly more advanced techniques such as the Trapezoidal Rule or Simpson's Rule. The students are instructed to solve the problems by "thinking aloud," or self-reporting their solutions steps. They are provided paper that includes a graph of the function x-squared, a pencil to record their work, and can use a calculator if they choose. Computers will not be used. After the approximations, the student will be asked to calculate the true area under the curve and then to describe how to make the approximations they used earlier in the task more accurate and closer to this true value.

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