A RANDOM HIERARCHICAL LAPLACIAN

by

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ABSTRACT

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The self-similar Hierarchical Laplacian, essentially proposed by Dyson [4] in his theory of one-dimensional ferromagnetic phase transitions, has a discrete spectrum with each eigenvalue having infinite multiplicity [14]. As a result, the integrated density of states is piecewise constant and the density of states is a sum of point-masses located on its spectrum.

To correct these "defects," we present a modification of the Hierarchical Laplacian obtained by allowing its deterministic coefficients to instead vary randomly, but without changing the eigenfunctions. The resulting spectrum is deterministic but the eigenvalues are now random with finite multiplicity and we obtain an absolutely continuous density of states. We will examine the eigenvalue statistics near an individual point of the spectrum and show that, locally, the spectrum is approximately a Poisson point process.

DEDICATION

To God who has given me more in life than I deserve and to Whom one can never be sufficiently grateful, to my wife Afsar and to my son who have been forced to sacrifice so much of the time and attention I owe to them, to my mother, father, and stepfather, who encouraged my interest in Mathematics as a child and continue giving me their love and support as I pursue my goals, and to my dear friend Art Gish who died in 2010 but continues to inspire me often.

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CHAPTER 1: INTRODUCTION

The self-similar Hierarchical Laplacian, essentially proposed by Dyson [4] in his theory of one-dimensional ferromagnetic phase transitions, has a discrete spectrum with each eigenvalue having infinite multiplicity [14]. As a result, the integrated density of states $N(\lambda)$ is piecewise constant and the density of states does not exist—or more precisely, it is a sum of point-masses located on the spectrum of $-\Delta$. When the probabilistic weights for the Hierarchical Laplacian are given by a geometric progression, the Hierarchical Laplacian can have an arbitrary spectral dimension s_h and as a result, it is similar to the classical fractals, e.g., the Sierpiński Lattice.

Usually in Mathematical Physics, after considering the Laplacian, we move on to consider the Schrödinger operator—in two different directions.

First in the classical spectral theory, the negative Laplacian typically has discrete non-negative spectrum which accumulates to the point zero. When we add a negative decreasing potential (potential well), the spectrum below zero will be discrete. The central questions are: under what conditions are there only finitely many negative eigenvalues and how can we estimate the number of negative eigenvalues [15, 16, 17]. Let us formulate several classical results. Consider in $\mathbb{R}^d$, $d \geq 3$, the Schrödinger operator $H = -\Delta - V(x)$, where $V(x) \geq 0$ and $V(x) \to 0$ as $|x| \to \infty$ in some sense. In this situation, the spectrum of H covers the half axis $[0, \infty)$ but for negative energies the spectrum is discrete. Letting $N_0(V) = \#\{\lambda_i < 0\}$, we have the *Lieb-Thirring Estimate*:

$$\sum_{i:\lambda_i<0} |\lambda_i|^\gamma \le C_{d,\gamma} \int_{\mathbb{R}^d} V^{d/2+\gamma}(x)\, dx \tag{1.1}$$

and taking $\gamma = 0$ in (1.1), we have the *Cwikel-Lieb-Rozenblum* (*CLR*) *Estimate*:

$$N_0(V) \le C_d \int_{\mathbb{R}^d} V^{d/2}(x)\, dx. \tag{1.2}$$

In particular, the CLR estimate implies that the operator $H = -\Delta + \sigma V(x)$ has non-negative spectrum whenever the coupling constant σ is small and $V \in L^{d/2}(\mathbb{R}^d)$. For small dimension, we have $N_0(\sigma V) > 0$ for any non-vanishing V and any $\sigma > 0$. Instead of (1.2), in the popular literature, one is often presented with Bargmann's estimate (see [18])

$$N_0(V) \le 1 + \int_{-\infty}^{\infty} |x| V(x) dx. \tag{1.3}$$

Another direction is the spectral theory of the random Schrödinger operator, i.e., $H = -\Delta + \sigma V_\omega(x)$, σ is a coupling constant, $V(x)$ i.i.d. One might conjecture that, in this case, classical Anderson phase-type transitions would be observed for small σ and $s_h > 2$, and that together with pure point spectrum, there exists some kind of continuous spectrum, i.e., Anderson delocalization [1]. Unfortunately, this natural conjecture appears to have been wrong [14]. In [9], it is shown that for more or less general distributions, for arbitrary spectral dimension s_h and arbitrary σ, the spectrum of the random Schrödinger operator is pure point. One can propose the following physical explanation of this fact. It is well known from the literature that the spectrum of the random Schrödinger operator on the lattice $\mathbb{Z}^d$ is pure point outside the spectrum of the Laplacian for arbitrarily small σ in any dimension [13]. Since the spectrum of the self-similar Hierarchical Laplacian consists of isolated points, all energies are outside the spectrum. Taking into account all these facts, it is important to modify the self-similar Hierarchical model in such a way that — instead of the isolated eigenvalues of infinite multiplicity — we will get spectrum which is dense on some interval and obtain a continuous density of states.

The goal of the thesis is the analysis of a random Hierarchical Laplacian obtained by allowing the deterministic eigenvalues of the Hierarchical Laplacian to instead vary randomly. The way in which we allow the eigenvalues to be random does not change which functions are eigenfunctions but it does have the effect of breaking each isolated (deterministic) eigenvalue of infinite multiplicity into a countable dense set of eigenvalues each having (the same) finite multiplicity. The spectrum remains deterministic but the isolated points of spectrum become widened into spectral bands supporting a continuous density of states. These spectral bands may or may not overlap depending on the value of a parameter $0 < \sigma < 1$ — for values of σ closer to one, the spectrum will be an interval while for $\sigma = 0$ we obtain the original (deterministic) Hierarchical Laplacian. In the last section, we examine the eigenvalue statistics near an individual point of the spectrum and show that, locally, the spectrum is approximately a Poisson point process.

CHAPTER 2: HIERARCHICAL LATTICE

2.1 Definitions

A *hierarchical lattice* is an ultrametric space (X, d_h) where X is an infinite set and the *hierarchical distance* d_h is an integer-valued ultrametric with the property that for each integer $r \geq 1$, there exists an integer $\nu_r \geq 2$ such that every closed metric ball of radius r (which we refer to as a *cube of rank* r)

$$Q^{(r)}(x) = \overline{B}(x, r) = \{y \in X : d_h(x, y) \leq r\} \tag{2.1}$$

contains exactly ν_r balls of radius $r - 1$. We call a hierarchical lattice *self-similar* if each $\nu_r = \nu$ for some integer $\nu \geq 2$. To say d_h is an *ultrametric* means, instead of just the triangle inequality, d_h satisfies the stronger condition that for all $x, y, z \in X$,

$$d_h(x, y) \leq \max\left\{d_h(x, z), d_h(y, z)\right\}. \tag{2.2}$$

Because d_h is an ultrametric, each element of a cube can serve as its center. As a result, two cubes are either disjoint or one is a subset of the other. In particular, because two different cubes of the same rank/radius must be disjoint, the hierarchical distance can be expressed as

$$\begin{aligned} d_h(x, y) &= \min\left\{r : Q^{(r)}(x) = Q^{(r)}(y)\right\} \\ &= \max\left\{r : Q^{(r-1)}(x) \cap Q^{(r-1)}(y) = \varnothing\right\}. \end{aligned} \tag{2.3}$$

Observe that the sequence $\{Q^{(r)}(x)\}_{r\geq 0}$ increases to X, i.e.,

$$x \in Q^{(0)}(x) \subseteq Q^{(1)}(x) \subseteq Q^{(2)}(x) \subseteq \cdots \subseteq \bigcup_{r=0}^{\infty} Q^{(r)}(x) = X. \tag{2.4}$$

It follows that for each $r \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}$, the collection of all cubes of rank r

$$\Pi_r = \{Q^{(r)}(x) : x \in X\} \tag{2.5}$$

forms a partition of X into finite subsets where every cube belonging Π_r is a disjoint union of ν_r cubes belonging to Π_{r-1}. Then the cardinality or *volume* of a cube is given by

$$\left|Q^{(r)}(x)\right| = \nu_1 \nu_2 \cdots \nu_r.$$

Since each $\nu_r \geq 2$, it follows that each inclusion in (2.4) is strict and $|Q^{(r)}(x)|$ is of at least exponential order as $r \to \infty$.

The requirement that d_h be integer-valued implies X is discrete as a topological space. In fact, the definition implies the set X—being a countable union (2.4) of finite sets, must itself be countable. More generally, we could have simply required d_h to take as its values the terms of some strictly increasing sequence, $0 = t_0 < t_1 < t_2 < \cdots$. For any such sequence, we can define a *renormalized* hierarchical distance by taking $\rho_h(x, y) = t_{d_h(x,y)}$. In this case, the cubes remain the same but the d_h-balls of radius r become ρ_h-balls of radius t_r. In a self-similar hierarchical lattice, taking $t_r = \beta^r$ for some $\beta > 1$ and all $r \geq 1$, the volume of a renormalized metric ball becomes, essentially as in $\mathbb{R}^d$, a power function of its radius, i.e., if $R = \beta^r$, we have

$$|\{y \in X : \rho_h(x, y) \leq R\}| = |Q^{(r)}(x)| = \nu^r = R^{\log_\beta \nu}.$$

For each $r \geq 0$, we denote the collection of all cubes of rank $\geq r$ by

$$\mathcal{V}_r = \bigcup_{k=r}^{\infty} \Pi_k.$$

For each $r \geq 0$, $\mathcal{V}_r$ forms a simple connected graph with edges

$$\mathcal{E}_r = \left\{ \{Q, Q^+\} : Q \in \mathcal{V}_r \right\}$$

where we write $Q^+ = Q^{(r+1)}(x)$ whenever $Q = Q^{(r)}(x)$. The graph distance d_g between two cubes $Q \in \Pi_m$ and $Q' \in \Pi_n$ is given by

$$d_g(Q, Q') = \begin{cases} n - m & \text{if } Q \subseteq Q' \\ 2r - m - n & \text{if } d_h(Q, Q') = r > 0 \end{cases} . \tag{2.6}$$

Note that for $y \notin Q$, the mapping $Q \ni x \mapsto d_h(x, y)$ is constant. Therefore, whenever Q and Q' are disjoint we have $d_h(Q, Q') = d_h(x, x')$ for all $x \in Q$ and $x' \in Q'$.

Equation (2.3) shows that the hierarchical distance can be recovered from a knowledge of the partitions (2.5). To see this, let's start from scratch and suppose we are given an abstract countably infinite set X and a sequence $\{\Pi_r\}_{r \geq 1}$ of partitions of X into finite subsets where every set belonging to Π_r is contained in some set belonging to Π_{r+1} and contains at exactly $\nu_r \geq 2$ subsets belonging to Π_{r-1}. Assume further that for each $x \in X$,

$$\bigcup_{r=0}^{\infty} Q^{(r)}(x) = X \tag{2.7}$$

where $Q^{(r)}(x)$ is the unique set from Π_r containing x and $Q^{(0)}(x) = \{x\}$. If $d_h(x, y)$ is defined by (2.3) then (X, d_h) is a hierarchical lattice. We assume (2.7) in order to ensure that $d_h(x, y) < \infty$ for all $x, y \in X$.

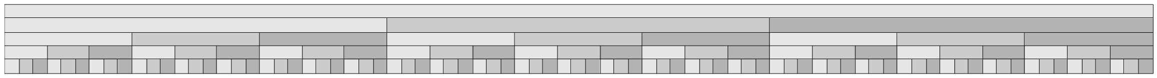

Figure 2.1: Cube of rank 4 in a self-similar hierarchical lattice where $\nu = 3$.

The simplest example of a self-similar hierarchical lattice is given by $X = \mathbb{N}$ with

$$Q_i^{(r)} = \{n \in \mathbb{N} : i\nu^r \leq n < (i+1)\nu^r\} \text{ for all } r \geq 0 \text{ and } i \in \mathbb{N} \tag{2.8}$$

We denote the hierarchical distance on $\mathbb{N}$ by $\hat{d}_h(m,n)$.

2.2 Enumeration of Self-similar Hierarchical Lattice

PROPOSITION 2.1. *In a self-similar hierarchical lattice, we can enumerate the points $X = \{x_0, x_1, \ldots\}$ in such a way that for all $m, n \in \mathbb{N}$,*

$$d_h(x_m, x_n) = \hat{d}_h(m,n).$$

As a result, we can enumerate the cubes of rank r

$$\Pi_r = \{Q_0^{(r)}, Q_1^{(r)}, Q_2^{(r)}, \ldots\}, \tag{2.9}$$

by defining for each $i = 0, 1, 2, 3, \ldots$,

$$Q_i^{(r)} = \{x_n : i\nu^r \leq n < (i+1)\nu^r\} \tag{2.10}$$

First, we need a lemma.

LEMMA 2.2. *In a self-similar hierarchical lattice, every cube can be enumerated*

$$Q = \{x_n : 0 \leq n < |Q|\}$$

in such a way that $d_h(x_m, x_n) = \hat{d}_h(m,n)$ for all $m, n < |Q|$.

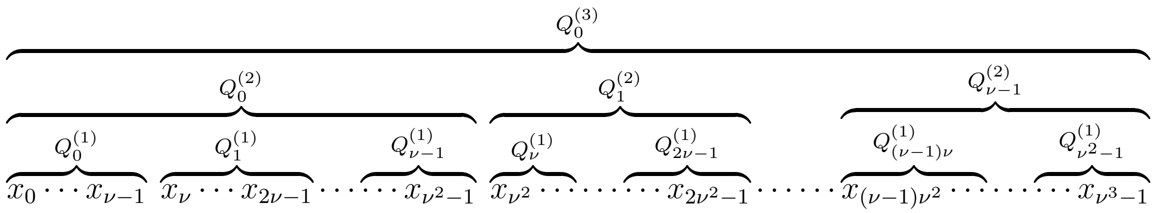

Figure 2.2: Cube of rank 3 in a self-similar hierarchical lattice.

Proof. For $m < n$, we have $\hat{d}_h(m,n) = r$ if and only if there exist $i, j \in \mathbb{N}$ with

$$i\nu^r \leq m < j\nu^{r-1} \leq n < (i+1)\nu^r. \tag{2.11}$$

But if (2.11) holds we will also have

$$(i+1)\nu^r \leq m + \nu^r < (j+\nu)\nu^{r-1} \leq n + \nu^r < (i+2)\nu^r$$

hence $\hat{d}_h(m+\nu^r, n+\nu^r) = \hat{d}_h(m,n)$. It follows by induction that

$$\hat{d}_h\big(m + k\nu^{\hat{d}_h(m,n)}, n + k\nu^{\hat{d}_h(m,n)}\big) = \hat{d}_h(m,n) \quad \text{for all} \quad k \geq 1. \tag{2.12}$$

Now, let $Q \in \Pi_{r+1}$ and assume the result holds for all cubes of smaller rank contained in Q. Then there exist $Q_1, \ldots, Q_\nu \in \Pi_r$ with $Q = \bigcup_{i=1}^{\nu} Q_i$ hence for $1 \leq i \leq \nu$, we have

$$Q_i = \left\{x_n^{(i)} : 0 \leq n < \nu^r\right\} \quad \text{with} \quad d_h(x_m^{(i)}, x_n^{(i)}) = \hat{d}_h(m,n).$$

We define $x_N = x_n^{(i)}$ for

$$N = (i-1)\nu^r + n \in \left\{0, 1, 2, \ldots, \nu^{r+1} - 1\right\}.$$

Then we have

$$Q = \left\{x_N : 0 \leq N < \nu^{r+1}\right\} \qquad \text{and} \qquad Q_i = \{x_N : (i-1)\nu^r \leq N < i\nu^r\}.$$

For $M = (i-1)\nu^r + m$ and $N = (j-1)\nu^r + n$ with $1 \le i, j \le \nu$ and $0 \le m, n < \nu^r$,

$$d_h(x_M, x_N) = d_h(x_m^{(i)}, x_n^{(j)}) = \begin{cases} \hat{d}_h(m,n) & \text{if } i = j \\ r+1 & \text{if } i \neq j \end{cases}.$$

If $i = j$ then by (2.12), we have

$$\hat{d}_h(M,N) = \hat{d}_h(m,n) = d_h(x_M, x_N).$$

If $i < j$ then $0 \le M < i\nu^r \le N < \nu^{r+1}$ hence

$$\hat{d}_h(M,N) = r+1 = d_h(x_M, x_N).$$ □

Proof of Proposition 2.1. It is clear from the construction in Lemma 2.2 that we may recursively construct an infinite sequence $\{x_n\}_{n\ge 0}$ with $d_h(x_m, x_n) = \hat{d}_h(m,n)$ for all $m, n \ge 0$ and with the first ν^r terms of this sequence enumerating $Q^{(r)}(x_0)$

$$Q^{(r)}(x_0) = \{x_n : 0 \le n < \nu^r\} \quad \text{for each} \quad r.$$

For the first step of the recursion, we may choose $x_0 \in X$ (*the origin*) arbitrarily. At the r^{th} step, we generate the next $\nu^r - \nu^{r-1}$ terms of the sequence which enumerate $Q^{(r)}(x_0) \diagdown Q^{(r-1)}(x_0)$. By (2.7), this sequence must enumerate all of X. □

2.3 Hierarchical Addition

Enumerating each partition Π_r as in (2.9–2.10) we have

$$Q_i^{(m+r)} = \bigcup_{k=0}^{\nu^r - 1} Q_{i\nu^r + k}^{(m)} = \bigcup_{k=0}^{\nu^m - 1} Q_{i\nu^m + k}^{(r)} \tag{2.13}$$

and in particular, taking $m = 0$, we have $Q^{(r)}(x_n) = Q_i^{(r)}$ where $i = \lfloor n/\nu^r \rfloor$.

Now let's define a mapping $n : X \to \mathbb{N}$ by putting $n(x) = n$ if $x = x_n$ in the enumeration of X. Let's further define, for each $r \geq 0$, the r^{th}-*coordinate mapping*

$$n_r : X \to \{0, 1, \ldots, \nu - 1\}$$

by putting $n_r(x) = n_r$ if, in the enumeration of Π_r, $Q^{(r)}(x)$ is the $(n_r + 1)^{\text{th}}$ cube of rank r contained in $Q^{(r+1)}(x)$. Then $n_r(x)$ is the $(r + 1)^{\text{th}}$ digit of the base-ν representation of $n(x)$, i.e.,

$$n(x) = \sum_{r=0}^{\infty} n_r(x)\nu^r = \sum_{r=0}^{|x|_h - 1} n_r(x)\nu^r$$

where $|x|_h = d_h(x_0, x)$. Notice that

$$|x|_h = r \quad \text{if and only if} \quad \nu^{r-1} \leq n(x) < \nu^r$$

or equivalently, $|x_n|_h = 1 + \lceil \log_\nu n \rceil$. We also have

$$n_0(x_{i\nu^r}) = n_1(x_{i\nu^r}) = \cdots = n_{r-1}(x_{i\nu^r}) = 0$$

and

$$n_k(x_{i\nu^r}) = n_{k-r}(x_i) \text{ for } k \geq r,$$

hence

$$|x_{i\nu^r}|_h = r + |x_i|_h.$$

Define an additive group (*hierarchical addition*, see [3, 11]) on X by putting

$$n_r(x \dotplus y) = n_r(x) + n_r(y) \mod \nu$$

for each $r \geq 0$. It means that we add the indices for x and y in base-ν except that we "forget to carry the tens" over to the next digit whenever $n_r(x) + n_r(y) \geq \nu$. Proposition 2.1 says that no matter how (X, d_h) has been constructed, we may as well assume $(X, d_h) = (\mathbb{N}, \hat{d}_h)$. Accordingly, we will identify $x_n \in X$ with $n \in \mathbb{N}$ and write $x \dot{+} n$ instead of $x \dot{+} x_n$.

The first cube $Q_0^{(r)}$ of each rank is a subgroup of $(X, \dot{+})$ whose cosets are given by

$$Q^{(r)}(x) = x \dot{+} Q_0^{(r)}.$$

As a result, we have

$$Q^{(r)}(x) \dot{+} Q^{(r)}(y) = Q^{(r)}(x \dot{+} y).$$

Furthermore, since $Q_0^{(m)}$ is a subgroup of $Q_0^{(m+r)}$, we have

$$Q_0^{(m)} \dot{+} Q_0^{(m+r)} = Q_0^{(m+r)}$$

so that

$$Q^{(m)}(x) \dot{+} Q^{(m+r)}(y) = Q^{(m+r)}(x \dot{+} y).$$

Similarly, because $i\nu^r \dot{+} j\nu^r = (i \dot{+} j)\nu^r$ and $Q_i^{(r)} = Q^{(r)}(i\nu^r)$, we have

$$Q_i^{(r)} \dot{+} Q_j^{(r)} = Q_{i \dot{+} j}^{(r)}$$

and it follows from (2.13) that

$$Q_{i\nu^r+k}^{(m)} \dot{+} Q_j^{(m+r)} = Q_{i \dot{+} j}^{(m+r)} \quad \text{for} \quad 0 \leq k < \nu^r.$$

Now (2.13) becomes

$$Q^{(m+r)}(x) = \bigcup_{k=0}^{\nu^r - 1} Q^{(m)}(x \dot{+} k\nu^m) = \bigcup_{k=0}^{\nu^m - 1} Q^{(r)}(x \dot{+} k\nu^r). \tag{2.14}$$

CHAPTER 3: HIERARCHICAL LAPLACIANS

3.1 Averaging Operators and Associated Subspaces of $\mathbb{C}^X$

We define an operator $A_r : \mathbb{C}^X \to \mathbb{C}^X$, the r^{th}*-rank averaging operator*, in the space $\mathbb{C}^X$ of complex-valued functions defined on X by putting

$$A_r f(x) = \frac{1}{\nu^r} \sum_{z \in Q^{(r)}(x)} f(z) \tag{3.1}$$

for $f : X \to \mathbb{C}$. Equivalently,

$$A_r f = \sum_{Q \in \Pi_r} f_Q \mathbf{1}_Q \tag{3.2}$$

where $\mathbf{1}_Q : X \to \{0, 1\}$ is the indicator function of a set $Q \subseteq X$ and

$$f_Q = \frac{1}{|Q|} \sum_{x \in Q} f(x).$$

is the average value of f on the cube Q.

To motivate this definition consider a random walk $\{x_n\}_{n \geq 0}$ beginning at the point $x \in X$ which at each step, jumps with equal probabilities to another point $y \in Q^{(r)}(x) = x \dotplus Q_0^{(r)}$. In other words,

$$x_n = x \dotplus z_1 \dotplus \cdots \dotplus z_n$$

where $\{z_n\}_{n \geq 1}$ is an i.i.d. sequence of uniformly distributed random elements of $Q_0^{(r)}$. It means that, beginning at $x \in X$, the probability, at the n^{th} step, of arriving at

$y \in X$ is given by

$$\mathbf{P}^x(x_n = y) = \frac{\mathbf{1}_r(x,y)}{\nu^r}$$

where

$$\mathbf{1}_r(x,y) = \mathbf{1}_{Q^{(r)}(x)}(y) = \mathbf{1}_{Q^{(r)}(y)}(x) = \mathbf{1}_r(y,x).$$

Then $\mathbf{P}^x$-almost surely, $\{x_n\}$ never leaves the cube $Q^{(r)}(x)$ hence for every $f \in \mathbb{C}^X$ and $n \geq 1$ we have

$$\mathbf{E}^x f(x_n) = \sum_{y \in Q^{(r)}(x)} f(y)\, \mathbf{P}^x(x_n = y) = A_r f(x).$$

We say that A_r *generates a symmetric random walk on cubes of rank* r.

It is clear from (3.2) that $A_r : \mathbb{C}^X \to \mathcal{M}_r$ where $\mathcal{M}_r$ is the subspace of functions which are constant on cubes of rank r. Observe that $f \in \mathcal{M}_r$ if and only if $A_r f = f$. Since every cube of rank $k < r$ is contained in a cube of rank r, we see that

$$k < r \quad \text{implies} \quad \mathcal{M}_r \subseteq \mathcal{M}_k. \tag{3.3}$$

Since every cube of rank r is a disjoint union of exactly ν^{r-k} cubes of rank $k < r$, considering the average of averages, we see that

$$k < r \quad \text{implies} \quad A_k A_r = A_r A_k = A_r. \tag{3.4}$$

Proposition 3.1. *For $1 \leq k \leq r$, if $f \in \mathcal{M}_r$ then for every $g \in \mathbb{C}^X$, we have*

$$A_k f g = f A_k g. \tag{3.5}$$

In other words, A_k treats functions $f \in \mathcal{M}_r$ like constants.

Proof. If $f \in \mathcal{M}_r$ then by (3.3), f is constant on cubes of rank $k \le r$ hence for $g \in \mathbb{C}^X$,

$$(A_k f g)(x) = \frac{1}{\nu^k} \sum_{y \in Q^{(k)}(x)} f(y)g(y) = \frac{1}{\nu^k} \sum_{y \in Q^{(k)}(x)} f(x)g(y) = f(x)(A_k g)(x). \qquad \square$$

Now we must describe two subspaces related to $\mathcal{M}_r$ and corresponding operators related to A_r which are needed in the sequel.

First, the subspace $\mathcal{L}_r$ consists of all $f \in \mathbb{C}^X$ which are constant on cubes of rank $< r$ with $\sum f = 0$ on cubes of rank $\ge r$, i.e.,

$$A_k f = f \;\text{ for }\; 1 \le k < r \qquad \text{and} \qquad A_k f = 0 \;\text{ for all }\; k \ge r. \tag{3.6}$$

We have $f \in \mathcal{L}_r$ if and only if $E_r f = f$ where $E_r : \mathbb{C}^X \to \mathcal{L}_r$ is defined by

$$E_r = A_{r-1} - A_r. \tag{3.7}$$

It follows from (3.4) that

$$k < r \quad \text{implies} \quad E_k E_r = E_r E_k = 0. \tag{3.8}$$

Next, for each $Q \in \Pi_r$ (each cube of rank r), the subspace $\mathcal{L}_Q$ consists of all $f \in \mathcal{L}_r$ which vanish outside of Q. Then $f \in \mathcal{L}_Q$ if and only if $E_Q f = f$ where $E_Q : \mathbb{C}^X \to \mathcal{L}_Q$ is defined by $E_Q = \mathbf{1}_Q E_r$. Since $\mathbf{1}_X = \sum_{Q \in \Pi_r} \mathbf{1}_Q$, it follows that

$$E_r = \sum_{Q \in \Pi_r} E_Q. \tag{3.9}$$

If we write $Q = \bigcup_{i=1}^{\nu} Q_i$ where $Q_1, Q_2, \ldots, Q_\nu$ are the cubes of preceding rank contained in Q then $\mathcal{L}_Q$ consists of all functions which vanish outside of Q and are constant on each subcube Q_i with the sum of these constants being zero, i.e., functions of the form

$$f = E_Q f = \sum_{i=1}^{\nu} c_i \mathbf{1}_{Q_i} \quad \text{with} \quad \sum_{i=1}^{\nu} c_i = 0. \tag{3.10}$$

We obtain the following corollary to Proposition 3.1.

Corollary 3.2. *For $1 \le k \le r$, if $f \in \mathcal{M}_r$ and $g \in \mathcal{L}_k$ then fg is constant on cubes of rank $k-1$ and $\sum fg = 0$ on cubes of rank k hence $fg \in \mathcal{L}_k$. In other words, $\mathcal{L}_k$ absorbs multiplication from functions in $\mathcal{M}_r$.*

Next, we consider the subspace of functions which can be continuously extended to the one-point compactification of X, i.e., functions $f \in \mathbb{C}^X$ having a limit as x approaches the point at infinity. We will prove, in Proposition 3.3, that A_r is invariant on this subspace.

Whenever we write $\lim_{x\to\infty} f(x) = c$ or $f(x) \to c$ as $x \to \infty$, it is equivalent to saying that for every $\varepsilon > 0$, there exists n such that $|x|_h > n$ implies $|f(x) - c| < \varepsilon$.

Proposition 3.3. *If $\lim_{x\to\infty} f(x) = c$ then $\lim_{x\to\infty} A_r f(x) = c$.*

Proof. Because $A_r(f - c) \equiv A_r f - c$ we may assume $c = 0$. Let $\varepsilon > 0$. Then there exists $n > r$ such that $|x|_h > n$ implies $|f(x)| < \varepsilon$. If $|x|_h > n$ then $|y|_h > n$ for every $y \in Q^{(r)}(x)$ hence

$$|A_r f(x)| \le \frac{1}{\nu^r} \sum_{y \in Q^{(r)}(x)} |f(y)| < \varepsilon.$$

Therefore, $\lim_{x\to\infty} A_r f(x) = 0$. □

Notice that whenever $z \in Q_0^{(r)}$, since $Q^{(r)}(x \dot{+} z) = Q^{(r)}(x)$, we have

$$A_r f(x \dot{+} z) = A_r f(x)$$

for all $x \in X$, i.e., each $z \in Q_0^{(r)}$ is a period for $A_r f$. Similarly, we have

$$f(x \dot{+} z) \equiv f(x) \quad \text{for all } f \in \mathcal{M}_r \text{ and } z \in Q_0^{(r)}.$$

It means we may think of $\mathcal{M}_r$ as the space of $Q_0^{(r)}$-periodic functions defined on the group $(X, \dot{+})$. From (2.14) we obtain

$$A_{m+r} f(x) = \frac{1}{\nu^r} \sum_{k=0}^{\nu^r - 1} A_m f(x \dot{+} k\nu^m) = \frac{1}{\nu^m} \sum_{k=0}^{\nu^m - 1} A_r f(x \dot{+} k\nu^r).$$

In particular,

$$A_r f(x) = \frac{1}{\nu} \sum_{k=0}^{\nu-1} A_{r-1} f(x \dot{+} k\nu^{r-1})$$

and therefore

$$-E_r f(x) = A_r f(x) - A_{r-1} f(x) = \frac{1}{\nu} \sum_{k=1}^{\nu-1} A_{r-1} f(x \dot{+} k\nu^{r-1}).$$

LEMMA 3.4. *If* $\lim_{x\to\infty} f(x) = c$ *then* $\lim_{r\to\infty} A_r f(x) = c$ *for every* $x \in X$.

Proof. Again we may assume $c = 0$. First observe that for $r \geq m$ we have

$$A_r f(x) = \frac{1}{\nu^{r-m}} \sum_{k=0}^{\nu^{r-m}-1} A_m f(x \dot{+} k\nu^m).$$

If $m > |x|_h$ and $k > 0$ then we have $x \dot{+} k\nu^m \geq \nu^{m-1}$ so that $|x \dot{+} k\nu^m|_h \geq m$ hence

$$|A_m f(x \dot{+} k\nu^m)| \leq \max_{|y|_h \geq m} |f(y)|.$$

Now let $\varepsilon > 0$ and choose $m > |x|_h$ so large that $\max_{|y|_h \geq m} |f(y)| < \frac{\varepsilon}{2}$. Then we have

$$|A_r f(x)| \leq \frac{|A_m f(x)|}{\nu^{r-m}} + \sum_{k=1}^{\nu^{r-m}-1} \frac{|A_m f(x \dot{+} k\nu^m)|}{\nu^{r-m}} \leq \frac{\|f\|_\infty}{\nu^{r-m}} + \frac{\varepsilon}{2}$$

so that $|A_r f(x)| < \varepsilon$ for all $r > m + \log_\nu(1 + \frac{2}{\varepsilon} \|f\|_\infty)$. $\square$

PROPOSITION 3.5. *If* $\lim_{x\to\infty} f(x) = c$ *then we have*

$$f(x) = c + \sum_{r=1}^{\infty} E_r f(x) = c + \sum_{Q\in\mathcal{V}_1} E_Q f(x)$$

for every $x \in X$.

Proof. Since we have

$$\sum_{r=1}^{n} E_r f(x) = \sum_{r=1}^{n} (A_{r-1} - A_r) f(x) = f(x) - A_n f(x),$$

the result follows from Lemma 3.4. □

3.2 Averaging Operators and Associated Subspaces of $\ell^2(X)$

Let $\ell^2(X)$ be the Hilbert space of square-summable functions on X with inner product and norm

$$\langle \psi, \varphi \rangle = \sum_{x\in X} \psi(x)\overline{\varphi(x)} \qquad \text{and} \qquad \|\psi\|^2 = \sum_{x\in X} |\psi(x)|^2.$$

The matrix element for A_r is given by

$$\langle A_r \delta_x, \delta_y \rangle = \nu^{-r} \mathbf{1}_r(x,y) = \langle \delta_x, A_r \delta_y \rangle \tag{3.11}$$

hence A_r is self-adjoint. Because $A_r^2 = A_r$, it follows that A_r is the orthogonal projection onto the subspace $\mathcal{M}_r$ of $\ell^2(X)$. Similarly, E_r is the orthogonal projection onto $\mathcal{L}_r$ and it follows from (3.3) that

$$\mathcal{L}_r = \mathcal{M}_{r-1} \cap \mathcal{M}_r^{\perp}.$$

For $r < s$, since $\mathcal{M}_{s-1} \subseteq \mathcal{M}_r$,

$$\mathcal{L}_r \cap \mathcal{L}_s \subseteq \mathcal{M}_r \cap \mathcal{M}_r^{\perp} = \{0\} \qquad \text{hence} \qquad \mathcal{L}_r \perp \mathcal{L}_s \;\; \text{for} \;\; r \neq s. \tag{3.12}$$

For each cube $Q \in \Pi_r$, E_Q is the orthogonal projection onto $\mathcal{L}_Q$. Furthermore, (3.12) implies that $\mathcal{L}_Q$ is orthogonal to $\mathcal{L}_{Q'}$ for $Q, Q' \in \mathcal{V}_1$ with $Q \neq Q'$. It follows from (3.9) that

$$\mathcal{L}_r = \bigoplus_{Q \in \Pi_r} \mathcal{L}_Q. \tag{3.13}$$

It follows from (3.10) that $\mathcal{L}_Q$ is finite dimensional with

$$\dim \mathcal{L}_Q = \dim \{(c_1 \ldots, c_\nu) \in \mathbb{C}^\nu : c_1 + \cdots + c_\nu = 0\} = \nu - 1.$$

Together with the second equation in (3.13), this further implies that $\dim \mathcal{L}_r = \infty$. From (3.10), it is immediate that the orthogonal complement of $\mathcal{L}_Q$ consists of all $\psi \in \ell^2(X)$ which are constant on Q. If $\psi \in \left(\bigoplus_{Q \in \mathcal{V}_1} \mathcal{L}_Q\right)^{\perp}$ then ψ is constant on every cube $Q \in \mathcal{V}_1$ so by (2.4), ψ is constant on X which means $\psi \equiv 0$ on X. It follows that

$$\ell^2(X) = \bigoplus_{Q \in \mathcal{V}_1} \mathcal{L}_Q = \bigoplus_{r=1}^{\infty} \mathcal{L}_r \tag{3.14}$$

hence

$$I = \sum_{Q \in \mathcal{V}_1} E_Q = \sum_{r=1}^{\infty} E_r. \tag{3.15}$$

Alternately, for functions in $\ell^2(X)$, (3.15) follows from Proposition 3.5.

3.3 Hierarchical Random Walk and Laplacian

The hierarchical Laplacian is defined for each $\psi \in \ell^2(X)$ by

$$\Delta\psi(x) = \sum_{y \in X} p(x,y)\big(\psi(y) - \psi(x)\big) \tag{3.16}$$

where $p(x,y)$ are the transition probabilities for the discrete time hierarchical random walk $\{x_n\}_{n\geq 0}$ whose probability matrix is given by $I + \Delta = [p(x,y)]_{X\times X}$ where I is the identity operator on $\ell^2(X)$, i.e., for each $\psi \in \ell^2(X)$,

$$(I + \Delta)\psi(x) = \sum_{y \in X} p(x,y)\psi(y). \tag{3.17}$$

It means that Δ generates the semigroup $e^{t\Delta} = [p(t,x,y)]_{X\times X}$ for the continuous time random walk, $x_t = x_{N(t)}$, where $N(t)$ is a Poisson process independent of $\{x_n\}_{n\geq 0}$ with intensity equal to one [5, 6]. Our definition of the hierarchical Laplacian follows [16, 17] but sometimes $I + \Delta$ is referred to as the hierarchical Laplacian [14, 10].

To define the discrete time hierarchical random walk, we fix an i.i.d. sequence $\{\rho_n\}_{n\geq 1}$ of random variables supported on the positive integers and we assume there exist constants $p \in (0,1)$ and $\alpha > 0$ such that for every $r \in \mathbb{Z}^+$,

$$(1/p - 1)p^{r+\alpha} \leq \mathbf{P}(\rho = r) \leq (1/p - 1)p^{r-\alpha}. \tag{3.18}$$

In (3.18), we always keep in mind the case where ρ is geometrically distributed, i.e., where $\alpha = 0$. Now, at each time n, the random-walking particle jumps to the site x_n which is uniformly distributed within the cube of rank ρ_n containing x_{n-1}, i.e.,

$$\mathbf{P}\big(x_n = y \,|\, x_{n-1} = x \;\&\; \rho_n = r\big) = \frac{\mathbf{1}_r(x,y)}{\nu^r} = \langle A_r\delta_x, \delta_y\rangle \tag{3.19}$$

Since ρ_n is independent of x_{n-1} the transition probabilities are easily computed:

$$p(x,y) = \mathbf{P}\big(x_n = y \,|\, x_{n-1} = x\big) = \sum_{r=1}^{\infty} \frac{\mathbf{P}(\rho = r)\mathbf{1}_r(x,y)}{\nu^r}. \tag{3.20}$$

In particular, $p(x,y)$ depends only on $d_h(x,y)$, i.e., $p(x,x) = a_1$ and $p(x,y) = a_r$ for $d_h(x,y) = r \geq 1$ where $a_r = \sum_{k=r}^{\infty} \frac{\mathbf{P}(\rho=k)}{\nu^k}$. This allows us to diagonalize Δ. We do this by summing first, for each individual rank r, the terms in (3.17) with $d_h(x,y) = r$, i.e., we first sum over each sphere $Q^{(r)}(x) \diagdown Q^{(r-1)}(x)$ of radius r centered around x. We have

$$\sum_{y:d_h(x,y)=r} \psi(y) = \big\langle \psi, \mathbf{1}_{Q^{(r)}(x) \diagdown Q^{(r-1)}(x)} \big\rangle = \big\langle \psi, \mathbf{1}_{Q^{(r)}(x)} \big\rangle - \big\langle \psi, \mathbf{1}_{Q^{(r-1)}(x)} \big\rangle.$$

Therefore, since $a_r - a_{r+1} = \frac{\mathbf{P}(\rho=r)}{\nu^r}$, summation by parts gives us

$$\sum_{y:y\neq x} p(x,y)\psi(y) = \sum_{r=1}^{\infty} a_r \sum_{y:d_h(x,y)=r} \psi(y) = -a_1\psi(x) + \sum_{r=1}^{\infty} \frac{\mathbf{P}\big(\rho = r\big)\big\langle \psi, \mathbf{1}_{Q^{(r)}(x)} \big\rangle}{\nu^r} \tag{3.21}$$

so that

$$(I + \Delta)\psi(x) = \sum_{r=1}^{\infty} \frac{\mathbf{P}\big(\rho = r\big)\big\langle \psi, \mathbf{1}_{Q^{(r)}(x)} \big\rangle}{\nu^r}. \tag{3.22}$$

Equation (3.22) now becomes

$$I + \Delta = \sum_{r=1}^{\infty} \mathbf{P}(\rho = r) A_r \qquad \text{or} \qquad \Delta = \sum_{r=1}^{\infty} \mathbf{P}(\rho = r)(A_r - I) \tag{3.23}$$

which implies Δ is self-adjoint. If we put $\lambda_r = \mathbf{P}\big(\rho \geq r\big)$, since $E_r = A_{r-1} - A_r$, another summation by parts gives us

$$-\Delta = \sum_{r=1}^{\infty} (\lambda_r - \lambda_{r+1})(I - A_r) = \sum_{r=1}^{\infty} \lambda_r E_r. \tag{3.24}$$

From (3.14), we see that (3.24) diagonalizes Δ. The functional calculus for Δ is given by

$$f(\Delta) = \sum_{r=1}^{\infty} f(-\lambda_r) E_r. \tag{3.25}$$

For any function f which is bounded on $\mathrm{Sp}(\Delta) = \{-\lambda_r : r \geq 1\} \cup \{0\}$, the operator $f(\Delta)$ is bounded with

$$\|f(\Delta)\| = \sup_{\lambda \in \mathrm{Sp}(\Delta)} |f(\lambda)|.$$

Since $\langle A_r \delta_x, \delta_y \rangle = \nu^{-r} \mathbf{1}_r(x, y)$, the matrix element for $f(\Delta)$ is given by

$$f(\Delta)(x, y) = \langle f(\Delta)\delta_x, \delta_y \rangle = \sum_{r=1}^{\infty} f(-\lambda_r) \left(\frac{\mathbf{1}_{r-1}(x, y)}{\nu^{r-1}} - \frac{\mathbf{1}_r(x, y)}{\nu^r} \right) \tag{3.26}$$

The sum in (3.26) can be simplified in two ways depending on whether or not $d_h(x, y) = 0$ [16, 17]. We have

$$f(\Delta)(x, x) = \sum_{k=1}^{\infty} \frac{(\nu - 1) f(-\lambda_k)}{\nu^k} \tag{3.27}$$

and for $d_h(x, y) = r > 0$ we have

$$f(\Delta)(x, y) = -\frac{f(-\lambda_r)}{\nu^r} + \sum_{k=r+1}^{\infty} \frac{(\nu - 1) f(-\lambda_k)}{\nu^k}. \tag{3.28}$$

Alternately, (3.26) can be rewritten in the form

$$f(\Delta)(x, y) = f(-\lambda_1) \mathbf{1}_0(x, y) + \sum_{r=1}^{\infty} \frac{\big(f(-\lambda_{r+1}) - f(-\lambda_r)\big) \mathbf{1}_r(x, y)}{\nu^r}. \tag{3.29}$$

In particular, for $\lambda > 0$, taking $f(x) = \mathbf{1}_{[0,\lambda)}(-x)$ in (3.27), we obtain the expression given in [17] for the integrated density of states for $-\Delta$. We have

$$N[0, \lambda) = \left(1 - \frac{1}{\nu}\right) \sum_{r=0}^{\infty} \frac{\mathbf{1}_{[0,\lambda)}(\lambda_{r+1})}{\nu^r}. \tag{3.30}$$

It follows that the "density" of states for $-\Delta$ is simply a sum

$$n(\lambda) = \sum_{r=1}^{\infty} \frac{(\nu - 1)\delta_{\lambda_r}(\lambda)}{\nu^r}$$

of point masses along $\mathrm{Sp}(-\Delta)$. Our assumption (3.18) allows us to find the asymptotics of $N[0, \lambda)$ as $\lambda \to 0+$. Observe that for every $r \in \mathbb{Z}^+$, by (3.18), we have

$$p^{r+\alpha} \le \lambda_{r+1} \le p^{r-\alpha}. \tag{3.31}$$

Since $\mathbf{1}_{[0,\lambda)}$ is non-increasing on $[0, \infty)$, it follows that for every $r \ge 0$,

$$\frac{\mathbf{1}_{[0,\lambda)}(p^{r-\alpha})}{\nu^r} \le \frac{\mathbf{1}_{[0,\lambda)}(\lambda_{r+1})}{\nu^r} \le \frac{\mathbf{1}_{[0,\lambda)}(p^{r+\alpha})}{\nu^r}. \tag{3.32}$$

The left-hand side of (3.32) is non-zero if and only if $r > \alpha + \log_p \lambda$ and the right-hand side is non-zero if and only if $r > -\alpha + \log_p \lambda$. Summing the geometric series $\left(1 - \frac{1}{\nu}\right) \sum \frac{1}{\nu^r}$ over all $r > \pm\alpha + \log_p \lambda$ (separately), we obtain

$$\frac{\lambda^{s_h/2}}{\nu^{\alpha+1}} \le N[0, \lambda) \le \nu^{\alpha} \lambda^{s_h/2} \tag{3.33}$$

where $s_h = -2\log_p \nu > 0$. The first inequality in (3.33) is valid for $0 \le \lambda \le p^{-\alpha}$ and the second for $0 \le \lambda \le p^{\alpha}$. It immediately implies we have Lifshitz tails in the strong form (see [8, 16, 17])

$$\lim_{\lambda \searrow 0} \frac{\log N[0, \lambda)}{\log \lambda} = \frac{s_h}{2}. \tag{3.34}$$

Applying (3.29) to the semigroup

$$e^{t\Delta} = \sum_{r=1}^{\infty} e^{-\lambda_r t} E_r = [p(t, x, y)]_{X \times X}, \tag{3.35}$$

we obtain transition probabilities for the continuous-time hierarchical random walk

$$\mathbf{P}^x(x_t = y) = p(t,x,y) = e^{-\lambda_1 t}\mathbf{1}_0(x,y) + \sum_{r=1}^{\infty} \frac{(e^{-\lambda_{r+1}t} - e^{-\lambda_r t})\mathbf{1}_r(x,y)}{\nu^r}, \tag{3.36}$$

i.e., the solution to the parabolic problem

$$\tfrac{\partial}{\partial t}p(t,x,y) = \Delta_x p(t,x,y),\ t > 0, \quad p(0,x,y) = \mathbf{1}_0(x,y).$$

In particular, applying (3.27) to $e^{t\Delta}$, we have

$$p(t,x,x) = \sum_{r=1}^{\infty} \frac{(\nu-1)e^{-\lambda_r t}}{\nu^r}. \tag{3.37}$$

Since $p(t,x,y) > 0$ for all $x, y \in X$ and $t > 0$, it follows that either $\int_0^\infty p(t,x,x)dt < \infty$ for all $x \in X$ in which case the process x_t is transient (spends a finite time in each state), or $\int_0^\infty p(t,x,x)dt = \infty$ for all $x \in X$ in which case x_t is recurrent (spends an infinite amount of time in each state).

From (3.36) we obtain the kernel of the resolvent operator $R_\lambda = (\lambda I - \Delta)^{-1}$

$$R_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p(t,x,y)\,dt = \frac{\mathbf{1}_0(x,y)}{\lambda+\lambda_1} + \sum_{r=1}^{\infty} \frac{p_r \mathbf{1}_r(x,y)}{\nu^r(\lambda+\lambda_r)(\lambda+\lambda_{r+1})}, \tag{3.38}$$

and from (3.37),

$$R_\lambda(x,x) = \sum_{r=1}^{\infty} \frac{\nu-1}{\nu^r(\lambda+\lambda_r)}. \tag{3.39}$$

If $\nu p > 1$ (equivalently $s_h > 2$) and $d_h(x,y) = r > 0$, (3.38) shows that

$$\int_0^\infty p(t,x,y)\,dt = R_0(x,y) = \sum_{k=r}^{\infty} \frac{p_k}{\nu^k \lambda_k \lambda_{k+1}}.$$

But (3.18) and (3.31) imply

$$\frac{(1-p)p^{3\alpha}}{(\nu p)^k} \le \frac{p_k}{\nu^k \lambda_k \lambda_{k+1}} \le \frac{(1-p)p^{-3\alpha}}{(\nu p)^k}$$

hence

$$\frac{cp^{3\alpha}}{(\nu p)^r} \le \int_0^\infty p(t,x,y)\,dt \le \frac{cp^{-3\alpha}}{(\nu p)^r} \tag{3.40}$$

where $c = \frac{\nu p(1-p)}{\nu p - 1}$. On the other hand, if $\nu p \le 1$ ($s_h \le 2$), we have

$$\int_0^\infty p(t,x,y)\,dt = \lim_{\lambda \searrow 0} R_\lambda(x,y) = \sum_{k=r}^\infty \frac{p_k}{\nu^k \lambda_k \lambda_{k+1}} \ge (1-p)p^{3\alpha} \sum_{k=r}^\infty \frac{1}{(\nu p)^k} = \infty.$$

It follows that x_t is transient for $s_h > 2$ ($\nu p > 1$) and recurrent for $s_h \le 2$ ($\nu p \le 1$) [16, 17]. We call $s_h = -2\log_p \nu$ the *spectral dimension of* Δ. To further justify this terminology, we will find the asymptotics of $p(t,x,x)$ as $t \to \infty$. Following [17], we will first find the asymptotics of the function

$$\theta(t) = \left(1 - \frac{1}{\nu}\right) \sum_{k=0}^\infty \frac{e^{-p^k t}}{\nu^k}. \tag{3.41}$$

i.e., $p(t,x,x)$ for the case given in [16, 17] where $\mathbf{P}\big(\rho = r\big) = (1/p-1)p^r$. Considering the continuous analogue of $\theta(t)$, i.e., $\tilde{\theta}(t) = \log\nu \int_0^\infty \nu^{-x} e^{-p^x t}\,dx$, we see that $t^{s_h/2}\theta(t)$ is essentially the discrete analogue of an incomplete Gamma function — substituting $y = p^x t$, we have $t^{s_h/2}\tilde{\theta}(t) = \frac{s_h}{2}\int_0^t y^{s_h/2-1} e^{-y}\,dy \to \Gamma\big(1 + \frac{s_h}{2}\big)$ as $t \to \infty$. Replacing $\Gamma(1 + \frac{s_h}{2})$ with a logarithmically periodic function of t, the same thing holds for $\theta(t)$.

Proposition 3.6. [16, 17] *There exists a periodic function* $h(z) = \big(1 - \frac{1}{\nu}\big)\sum_{-\infty}^\infty \frac{e^{-p^{k+z}}}{\nu^{k+z}}$ *such that* $t^{s_h/2}\theta(t) \sim h\left(\log_p t\right)$ *as* $t \to \infty$.

Proof. First, observe that

$$h(\log_p t) = h(z) = \left(1 - \frac{1}{\nu}\right) \sum_{k=-\infty}^\infty \frac{e^{-p^{k+z}}}{\nu^{k+z}} = \left(1 - \frac{1}{\nu}\right) \sum_{k=-\infty}^\infty \frac{e^{-p^{k+\{z\}}}}{\nu^{k+\{z\}}}$$

where $z = z(t) = \log_p t$ and the last equality is from replacing the index k with $k - \lfloor z \rfloor$. Since $\{z+1\} \equiv \{z\}$, this also shows that $h(z) \equiv h(\{z\})$ is periodic with period one. Next, since $t = p^z$ implies $t^{s_h/2} = \nu^{-z}$ and $e^{-p^k t} = e^{-p^{k+z}}$, we have

$$t^{s_h/2}\theta(t) = \left(1 - \frac{1}{\nu}\right)\sum_{k=0}^{\infty}\frac{e^{-p^{k+z}}}{\nu^{k+z}} = \left(1 - \frac{1}{\nu}\right)\sum_{k=\lfloor z\rfloor}^{\infty}\frac{e^{-p^{k+\{z\}}}}{\nu^{k+\{z\}}}.$$

Therefore, since $\lfloor z(t)\rfloor \to -\infty$ as $t \to \infty$,

$$\frac{t^{s_h/2}\theta(t)}{h_1(\log_p t)} = \frac{\sum_{k=\lfloor z\rfloor}^{\infty}\frac{e^{-p^{k+\{z\}}}}{\nu^{k+\{z\}}}}{\sum_{k=-\infty}^{\infty}\frac{e^{-p^{k+\{z\}}}}{\nu^{k+\{z\}}}} \to 1 \quad \text{as} \quad t \to \infty$$

which completes the proof. □

PROPOSITION 3.7. *There exists a periodic function $h(z)$ such that*

$$\frac{1}{\nu^{\alpha+1}} \le \frac{t^{s_h/2}p(t,x,x)}{h(\log_p t)} \le \nu^{\alpha+1} \quad \textit{as} \quad t \to \infty \tag{3.42}$$

hence $p(t,x,x) \asymp t^{-s_h/2}$ as $t \to \infty$.

Proof. As in (3.32), since the function $\lambda \mapsto e^{-\lambda t}$ is decreasing, by (3.31) we have

$$\theta(p^{-\lceil\alpha\rceil}t) \le \theta(p^{-\alpha}t) \le p(t,x,x) \le \theta(p^{\alpha}t) \le \theta(p^{\lceil\alpha\rceil}t) \tag{3.43}$$

where $\lceil\alpha\rceil = \min\{n \in \mathbb{Z} : n \ge \alpha\}$. Dividing through by $t^{-s_h/2}h(\log_p t)$ and observing that $h\big(\log_p(p^{\pm\lceil\alpha\rceil}t)\big) = h(\log_p t)$, we have

$$\frac{t^{s_h/2}\theta(p^{-\lceil\alpha\rceil}t)}{h\big(\log_p(p^{-\lceil\alpha\rceil}t)\big)} \le \frac{t^{s_h/2}p(t,x,x)}{h(\log_p t)} \le \frac{t^{s_h/2}\theta(p^{\lceil\alpha\rceil}t)}{h\big(\log_p(p^{\lceil\alpha\rceil}t)\big)}. \tag{3.44}$$

As $t \to \infty$, the left-hand side converges to $\nu^{-\lceil\alpha\rceil} \ge \nu^{-\alpha-1}$ while the right-hand side converges to $\nu^{\lceil\alpha\rceil} \le \nu^{\alpha+1}$. □

3.4 Hierarchical Laplacian with Variable Coefficients

The diagonalization (3.24) of the hierarchical Laplacian displays the fact that each eigenvalue $\lambda_r = \mathbf{P}(\rho \geq r)$ is isolated in $\mathrm{Sp}(-\Delta)$ and has multiplicity $\dim \mathcal{L}_r = \infty$. To correct these "defects", we first observe that (3.15) implies we have the further diagonalization

$$-\Delta\psi(x) = \sum_{r=1}^{\infty} \lambda_r \left(\sum_{Q\in\Pi_r} E_Q\psi(x) \right) = \sum_{Q\in\mathcal{V}_1} \lambda_Q E_Q \psi(x) \tag{3.45}$$

where $\lambda_Q = \lambda_r$ for each $Q \in \Pi_r$. In essence, it seems that because the mapping $Q \mapsto \lambda_Q$ from $\mathcal{V}_1$ to $\mathrm{Sp}(-\Delta)$ is constant on each $\Pi_r \subseteq \mathcal{V}_1$, the finite dimensional subspaces, $\mathcal{L}_Q$ for $Q \in \Pi_r$, which should have been the eigenspaces, have instead been collapsed into the infinite dimensional eigenspace $\mathcal{L}_r$.

A *hierarchical Laplacian* $\widetilde{\Delta}$ *with variable coefficients* is a modification of Δ where, in (3.45), we instead require λ_Q to vary for different $Q \in \Pi_r$. We accomplish this by replacing each constant λ_r in (3.24) with a function $\lambda^{(r)} : X \to \mathbb{R}$ which is single-valued on cubes of rank r with different values on different cubes of rank r, i.e.,

$$\lambda^{(r)}(x) = \lambda^{(r)}(y) \quad \text{if and only if} \quad d_h(x,y) \leq r. \tag{3.46}$$

Furthermore, we require that

$$|\lambda^{(r)}(x) - \lambda_r| \leq \sigma\lambda_r \quad \text{for all } \ x \in X \ \text{ and } \ r \geq 1 \tag{3.47}$$

where $\sigma \in (0,1)$ is a coupling constant (measure of disorder) — the condition $\sigma < 1$ ensures that we do not gain any negative spectrum. It means $\widetilde{\Delta}$ is an operator of the form

$$-\widetilde{\Delta}\psi(x) = \sum_{r=1}^{\infty} \lambda^{(r)}(x) E_r \psi(x) = \sum_{Q \in \mathcal{V}_1} \lambda_Q E_Q \psi(x) \tag{3.48}$$

where $\lambda_Q = \lambda_i^{(r)}$ is now the single value of the function $\lambda^{(r)}(x)$ on the cube $Q = Q_i^{(r)}$. If we put $\xi^{(r)}(x) = \lambda^{(r)}(x) - \lambda^{(r+1)}(x)$, then

$$\lambda^{(r)}(x) = \xi^{(r)}(x) + \xi^{(r+1)}(x) + \xi^{(r+2)}(x) + \cdots$$

and $\widetilde{\Delta}$ takes a form similar to (3.23),

$$-\widetilde{\Delta}\psi(x) = \sum_{r=1}^{\infty} \xi^{(r)}(x)(I - A_r)\psi(x). \tag{3.49}$$

The functions $\xi^{(r)}(x)$ are the variable coefficients of $-\widetilde{\Delta}$.

Proposition 3.8. *If $\widetilde{\Delta}$ is defined by* (3.48), *then*

$$\mathrm{Sp}(-\widetilde{\Delta}) \subseteq \{0\} \cup \bigcup_{r=1}^{\infty} \big[(1-\sigma)\lambda_r, (1+\sigma)\lambda_r\big], \tag{3.50}$$

$\widetilde{\Delta}$ is self-adjoint, $\widetilde{\Delta} \le 0$ and $\|\widetilde{\Delta}\| \le 1 + \sigma$.

Proof. The "if" part of (3.46) means $\lambda^{(r)} \in \mathcal{M}_r$. Then by Proposition 3.1, we have

$$E_r(\lambda^{(r)}\psi) = \lambda^{(r)} E_r \psi$$

so that

$$\langle \lambda^{(r)} E_r \varphi, \psi \rangle = \langle \varphi, \overline{\lambda^{(r)}} E_r \psi \rangle.$$

Therefore, since $\lambda^{(r)}(x)$ is real valued, $\widetilde{\Delta}$ is self-adjoint. Since the right-hand side of (3.50) is closed, the condition (3.47) implies (3.50) which further implies $\widetilde{\Delta} \le 0$ and $\|\widetilde{\Delta}\| \le 1 + \sigma$. □

This modification of the Hierarchical Laplacian has the effect of breaking each eigenvalue $\lambda_r \in \mathrm{Sp}(-\Delta)$ with eigenspace $\mathcal{L}_r$ into a collection of eigenvalues

$$\mathrm{range}(\lambda^{(r)}) = \{\lambda_Q : Q \in \Pi_r\} \subseteq \mathrm{Sp}(-\widetilde{\Delta}) \tag{3.51}$$

whose eigenspaces are properly contained in $\mathcal{L}_r$. If the mapping $Q \mapsto \lambda_Q$ turned out to be one-to-one, each subspace $\mathcal{L}_Q$ for $Q \in \mathcal{V}_1$ would itself be an eigenspace of $-\widetilde{\Delta}$ and the multiplicity of each eigenvalue λ_Q would be exactly $\dim \mathcal{L}_Q = \nu - 1$. Note that our definition falls short of requiring the mapping $Q \mapsto \lambda_Q$ to be one-to-one on all of $\mathcal{V}_1$ — it only requires that $\lambda_Q \neq \lambda_{Q'}$ for cubes $Q \neq Q'$ of the same rank. Because the functions $\lambda^{(r)}(x)$ take values in intervals which may overlap, it remains possible for subspaces $\mathcal{L}_{Q_1}, \ldots, \mathcal{L}_{Q_n}$, corresponding to cubes $Q_1, \ldots, Q_n$, with no two of the same rank, to be collapsed into a single eigenspace $\mathcal{L}_{Q_1} \oplus \cdots \oplus \mathcal{L}_{Q_n}$. However, because the right endpoints of the intervals in (3.50) decrease to zero as $r \to \infty$, the multiplicity of an eigenvalue λ is at most $(\nu - 1)|I(\lambda)|$ where

$$I(\lambda) = \big\{r : (1-\sigma)\lambda_r \leq \lambda \leq (1+\sigma)\lambda_r\big\} = \big\{r : \tfrac{\lambda}{1+\sigma} \leq \lambda_r \leq \tfrac{\lambda}{1-\sigma}\big\}$$

hence, all eigenvalues of $-\widetilde{\Delta}$ have finite multiplicity. We will compute a bound on the number $|I(\lambda)|$ which is uniform for all $\lambda \in \mathrm{Sp}(-\widetilde{\Delta})$. If we write $m+1 = \min I(\lambda)$ and $M = \max I(\lambda)$, we have

$$\lambda_{M+1} < \tfrac{\lambda}{1+\sigma} \leq \lambda_M < \cdots < \lambda_{m+2} < \lambda_{m+1} \leq \tfrac{\lambda}{1-\sigma} < \lambda_m$$

so that, letting $\beta = \log_p \frac{1-\sigma}{1+\sigma}$, by (3.31), we have

$$p^{\alpha+1}\tfrac{\lambda}{1+\sigma} \leq p^M < p^{-\alpha}\tfrac{\lambda}{1+\sigma} \qquad \text{and} \qquad p^{\alpha+1}\tfrac{\lambda}{1+\sigma} < p^{m+\beta} \leq p^{-\alpha}\tfrac{\lambda}{1+\sigma}. \tag{3.52}$$

Taking logarithms in (3.52), we have

$$-\alpha < M - \log_p \tfrac{\lambda}{1+\sigma} \le \alpha + 1 \qquad \text{and} \qquad -\alpha \le m + \beta - \log_p \tfrac{\lambda}{1+\sigma} < \alpha + 1.$$

Since $|I(\lambda)| = M - m$, it follows that

$$\beta - (2\alpha + 1) < |I(\lambda)| \le \beta + (2\alpha + 1) \tag{3.53}$$

hence the multiplicity of an eigenvalue for $-\widetilde{\Delta}$ is at most $(\nu - 1)(2\alpha + \beta + 1)$.

In order to rid the spectrum of isolated points, we would like to define the functions $\lambda^{(r)}(x)$ in such a way that for each $r \ge 1$, the eigenvalues $\{\lambda_Q : Q \in \Pi_r\}$ form a dense subset of the interval of length $2\sigma\lambda_r$ centered around λ_r. In this case, because $\mathrm{Sp}(-\widetilde{\Delta})$ is closed, we will have

$$\mathrm{Sp}(-\widetilde{\Delta}) \supseteq \bigcup_{r=1}^{\infty} \big[\lambda_r(1-\sigma), \lambda_r(1+\sigma)\big],$$

and because $\lambda_r \to 0$ as $r \to \infty$, $0 \in \mathrm{Sp}(-\widetilde{\Delta})$, hence we will have equality in (3.50). This way, $\mathrm{Sp}(-\Delta)$ is contained in $\mathrm{Sp}(-\widetilde{\Delta})$ but each isolated eigenvalue $\lambda_r \in \mathrm{Sp}(-\Delta)$ is replaced its corresponding interval in (3.50). Furthermore, as $\sigma \to 0$, $\mathrm{Sp}(-\widetilde{\Delta})$ shrinks to $\mathrm{Sp}(-\Delta)$ and we obtain Δ as a special case of $\widetilde{\Delta}$.

These observations are summarized in Propositions 3.9–3.10.

Proposition 3.9. *Every eigenvalue for $-\widetilde{\Delta}$ has finite multiplicity.*

If the mapping $Q \mapsto \lambda_Q$ from $\mathcal{V}_1$ to $\mathrm{Sp}(-\widetilde{\Delta})$ is one-to-one, the eigenspaces for $-\widetilde{\Delta}$ consist of $\mathcal{L}_Q$ for $Q \in \mathcal{V}_1$ with each eigenvalue having multiplicity $\nu - 1$.

Otherwise, the multiplicity of an eigenvalue for $-\widetilde{\Delta}$ is a multiple of $\nu - 1$ which does not exceed $(\nu - 1)(2\alpha + \beta + 1)$ where $\beta = \log_p \frac{1-\sigma}{1+\sigma}$.

PROPOSITION 3.10. *If for each rank r, the range of $\lambda^{(r)}$ is a dense subset of the r^{th} interval in* (3.50)*, then we have equality in* (3.50) *and* $\|\widetilde{\Delta}\| = 1 + \sigma$.

In the next chapter, we will construct a *random* mapping $Q \mapsto \lambda_Q$ which, almost surely, is one-to-one and satisfies the hypotheses of Proposition 3.10.

3.5 Integrated Density of States

Here we will introduce some notation and give the general framework for computing the density of states. In Proposition 3.11, we will prove that the hierarchical Laplacian with variable coefficients has the same spectral dimension as the Hierarchical Laplacian with constant coefficients.

For a measurable set A, let $\mathcal{N}_L(A)$ be the number of eigenvalues for the problem

$$-\widetilde{\Delta}\psi = \lambda\psi, \quad \psi \equiv 0 \text{ on } X \diagdown Q_0^{(L)}. \tag{3.54}$$

The *density of states measure*, $N(A)$, for $-\widetilde{\Delta}$ (whenever it exists), is defined to be the limit as $L \to \infty$ of the finite volume approximation

$$N_L(A) = \frac{\mathcal{N}_L(A)}{\mathcal{N}_L(\mathbb{R})}. \tag{3.55}$$

Then the *integrated density of states* for $-\widetilde{\Delta}$ is the function $\lambda \mapsto N[0, \lambda)$.

If we let $\mathcal{S}_L$ be the set of all non-degenerate sub-cubes of $Q_0^{(L)}$, i.e.,

$$\mathcal{S}_L = \bigcup_{r=1}^{L} \mathcal{S}_L^{(r)} \quad \text{where} \quad \mathcal{S}_L^{(r)} = \{Q \in \Pi_r : Q \subseteq Q_0^{(L)}\}, \tag{3.56}$$

then the eigenvalues for (3.54) are given by $\{\lambda_Q : Q \in \mathcal{S}_L\}$. Since the multiplicity of each eigenvalue is a multiple of $\nu - 1$ and since we are ultimately concerned with the proportion (3.55), we may as well assume that each time $\lambda_Q \in A$, it contributes a 1 rather than a $\nu - 1$ to the sum $\mathcal{N}_L(A)$, i.e., we compute $\mathcal{N}_L(A)$ by the formula

$$\mathcal{N}_L(A) = \sum_{Q\in\mathcal{S}_L} \mathbf{1}_A(\lambda_Q) = \sum_{r=1}^{L} \mathcal{N}_L^{(r)}(A) \tag{3.57}$$

where

$$\mathcal{N}_L^{(r)}(A) = \sum_{Q\in\mathcal{S}_L^{(r)}} \mathbf{1}_A(\lambda_Q) = \sum_{i<\nu^{L-r}} \mathbf{1}_A(\lambda_i^{(r)}). \tag{3.58}$$

Then for $1 \le r \le L$, since there are ν^{L-r} cubes of rank r contained in $Q_0^{(L)}$,

$$\mathcal{N}_L(\mathbb{R}) = \left|\mathcal{S}_L\right| = \sum_{r=1}^{L} \left|\mathcal{S}_L^{(r)}\right| = \sum_{r=1}^{L} \nu^{L-r} = \frac{\nu^L - 1}{\nu - 1}.$$

Note that if we exponentiate the inequalities (3.52) by $\frac{s_h}{2} = -\log_p \nu$, we obtain

$$\tfrac{1}{\nu^{\alpha+1}}\Big(\tfrac{\lambda}{1+\sigma}\Big)^{s_h/2} \le \tfrac{1}{\nu^M} < \nu^{\alpha}\Big(\tfrac{\lambda}{1+\sigma}\Big)^{s_h/2} \quad \text{and} \quad \tfrac{1}{\nu^{\alpha+1}}\Big(\tfrac{\lambda}{1+\sigma}\Big)^{s_h/2} < \tfrac{1}{\nu^{m+\beta}} \le \nu^{\alpha}\Big(\tfrac{\lambda}{1+\sigma}\Big)^{s_h/2}. \tag{3.59}$$

These inequalities allow us to obtain (3.33–3.34) for the integrated density of states whenever it exists.

Proposition 3.11. *Let* $N_*[0,\lambda) = \liminf_{L\to\infty} N_L[0,\lambda)$ *and* $N^*[0,\lambda) = \limsup_{L\to\infty} N_L[0,\lambda)$. *Then there exist constants* $c_1, c_2 > 0$ *such that*

$$c_1\lambda^{s_h/2} \le N_*[0,\lambda) \le N^*[0,\lambda) \le c_2\lambda^{s_h/2} \tag{3.60}$$

for all $\lambda > 0$ *hence*

$$\lim_{\lambda\searrow 0} \tfrac{\log N_*[0,\lambda)}{\log\lambda} = \lim_{\lambda\searrow 0} \tfrac{\log N^*[0,\lambda)}{\log\lambda} = \tfrac{s_h}{2}. \tag{3.61}$$

Proof. Writing $I(\lambda) = \{m+1, m+2, \ldots, M\}$, we see that for $r \le m$,

$$\lambda < (1-\sigma)\lambda_r \le \lambda_Q$$

for all $Q \in \mathcal{S}_L^{(r)}$ so that $\mathcal{N}_L^{(r)}[0,\lambda) = 0$. On the other hand, for $r > M$, we have

$$\lambda > (1+\sigma)\lambda_r \geq \lambda_Q$$

for all $Q \in \mathcal{S}_L^{(r)}$ so that $\mathcal{N}_L^{(r)}[0,\lambda) = \nu^{L-r}$. It follows that

$$\mathcal{N}_L[0,\lambda) = \sum_{r=m+1}^{M} \mathcal{N}_L^{(r)}[0,\lambda) + \sum_{r=M+1}^{L} \nu^{L-r} = \sum_{r=m+1}^{M} \mathcal{N}_L^{(r)}[0,\lambda) + \frac{\nu^L(\nu^{-M} - \nu^{-L})}{\nu - 1}.$$

But since

$$0 \leq \sum_{r=m+1}^{M} \mathcal{N}_L^{(r)}[0,\lambda) \leq \sum_{r=m+1}^{M} \nu^{L-r} = \frac{\nu^L(\nu^{-m} - \nu^{-M})}{\nu - 1},$$

we have

$$\frac{\nu^L(\nu^{-M} - \nu^{-L})}{\nu - 1} \leq \mathcal{N}_L[0,\lambda) \leq \frac{\nu^L(\nu^{-m} - \nu^{-L})}{\nu - 1} < \frac{\nu^{L-m}}{\nu - 1}.$$

Therefore, since $1 < \frac{\nu^L}{(\nu-1)|\mathcal{S}_L|} < 1 + \nu^{1-L}$, it follows that

$$\nu^{-M} - \nu^{-L} < N_L[0,\lambda) < \nu^{-m} + \nu^{-L}$$

so by (3.59), for $c_1 = \nu^{-\alpha-1}(1+\sigma)^{-s_h/2}$ and $c_2 = \nu^{\alpha}(1+\sigma)^{-s_h/2}$, we obtain

$$c_1\lambda^{s_h/2} - \nu^{-L} < N_L[0,\lambda) < c_2\lambda^{s_h/2} + \nu^{-L} \tag{3.62}$$

which proves (3.60). Finally, it follows from (3.60) that for $\lambda < 1$,

$$\tfrac{\log c_2}{\log \lambda} + \tfrac{s_h}{2} \leq \tfrac{\log N^*[0,\lambda)}{\log \lambda} \leq \tfrac{\log N_*[0,\lambda)}{\log \lambda} \leq \tfrac{\log c_1}{\log \lambda} + \tfrac{s_h}{2}.$$

Letting $\lambda \searrow 0$, we obtain (3.61). $\square$

CHAPTER 4: RANDOM HIERARCHICAL LAPLACIAN

4.1 Definition

To define a *random* hierarchical Laplacian, let $\{\omega_Q : Q \in \mathcal{V}_1\}$ be an independent family of symmetric random variables where for each $r \geq 1$, the random variables $\{\omega_Q : Q \in \Pi_r\}$ corresponding to cubes of rank r, are identically distributed with a continuously differentiable density $f_r(x)$ supported on the interval $[-1, 1]$ with $\sup_{r\geq 1} \|f_r'\|_\infty < \infty$. Then for any two different cubes Q and Q', we have $\omega_Q \overset{\text{law}}{=} \omega_{Q'}$ when Q and Q' have the same rank but we allow for the possibility that ω_Q and $\omega_{Q'}$ are distributed differently whenever Q and Q' have different ranks.

For each $r \geq 1$ we define $\omega^{(r)} : X \to [-1, 1]$ by $\omega^{(r)}(x) = \omega_{Q^{(r)}(x)}$ and we define a random coefficient function $\xi^{(r)} : X \to [(1-\sigma)p_r, (1+\sigma)p_r]$, where $p_r = \mathbf{P}(\rho = r)$, by

$$\xi^{(r)}(x) = \big(1 + \sigma\omega^{(r)}(x)\big)p_r \tag{4.1}$$

Then $\xi^{(r)}(x)$ and $\xi^{(r)}(y)$ are independent for $d_h(x, y) > r$ but $\xi^{(r)}(x) = \xi^{(r)}(y)$ whenever $d_h(x, y) \leq r$. For each $\psi \in \ell^2(X)$ we define

$$-\Delta_\omega \psi(x) = \sum_{k=1}^{\infty} \xi^{(k)}(x)(I - A_k)\psi(x) = \sum_{r=1}^{\infty} \lambda^{(r)}(x) E_r \psi(x). \tag{4.2}$$

where

$$\lambda^{(r)}(x) = \sum_{k=r}^{\infty} \xi^{(k)}(x) = \lambda_r + \sigma \sum_{k=r}^{\infty} p_k \omega^{(k)}(x). \tag{4.3}$$

Observe that (3.18) implies

$$p^{\alpha}\sum_{k=r}^{\infty} qp^{k-1}\big(1+\sigma\omega^{(k)}(x)\big) \le \lambda^{(r)}(x) \le p^{-\alpha}\sum_{k=r}^{\infty} qp^{k-1}\big(1+\sigma\omega^{(k)}(x)\big) \tag{4.4}$$

where $q = 1 - p$. Let $\lambda_Q = \lambda_i^{(r)}$ denote the single random value assumed by the function $\lambda^{(r)} : X \to \mathbb{R}$ on the cube $Q = Q_i^{(r)}$ of rank r. It is important to note that $\lambda_Q - \lambda_{Q'}$ is independent of $\lambda_{Q'}$ whenever $Q \subsetneq Q'$. Since $\{\lambda_Q : Q \in \mathcal{V}_1\}$ is a continuous family of random variables, it means, almost surely, the mapping $Q \mapsto \lambda_Q$ is one-to-one. Therefore, each $\mathcal{L}_Q$ is an eigenspace for $-\Delta_\omega$ with eigenvalue λ_Q having finite multiplicity $\dim \mathcal{L}_Q = \nu - 1$.

In Proposition 4.1, we will prove that, almost surely, the functions $\lambda^{(r)}(x)$ satisfy the conditions of Proposition 3.10. Thus, even though the eigenvalues for $-\Delta_\omega$ are random, it follows from Proposition 3.10 that $\mathrm{Sp}(-\Delta_\omega)$ is deterministic.

PROPOSITION 4.1. *For $r \ge 1$, almost surely,* $\overline{\{\lambda_Q : Q \in \Pi_r\}} = \big[(1-\sigma)\lambda_r, (1+\sigma)\lambda_r\big]$.

Proof. For $(a, b) \subseteq \big[(1-\sigma)\lambda_r, (1+\sigma)\lambda_r\big]$, let

$$(z-\varepsilon, z+\varepsilon) = \tfrac{(a,b)-\lambda_r}{\sigma\lambda_r} = \big(\tfrac{a-\lambda_r}{\sigma\lambda_r}, \tfrac{b-\lambda_r}{\sigma\lambda_r}\big).$$

Then $z \in (-1, 1)$ and we have

$$\lambda^{(r)}(x) \in (a,b) \quad \text{if and only if} \quad |\lambda^{(r)}(x) - (1+\sigma z)\lambda_r| < \sigma\varepsilon\lambda_r.$$

In view of (4.3),

$$|\lambda^{(r)}(x) - (1+\sigma z)\lambda_r| \le \sigma\sum_{k=r}^{\infty} p_k|z-\omega^{(k)}(x)| \le 2\sigma\lambda_{r+n} + \sigma\sum_{k=r}^{r+n-1} p_k|z-\omega^{(k)}(x)|. \tag{4.5}$$

Note that by (3.31), $\lambda_{r+n} \le p^{n-2\alpha}\lambda_r$. Choose an integer $n > 2\alpha + 1 + \log_p \frac{\varepsilon}{2}$ so that

$2\lambda_{r+n} < p\varepsilon\lambda_r$. For each cube $Q \in \Pi_{r+n}$ let η_Q be the indicator for the event that $|z - \omega_{Q'}| < (1-p)p^{2\alpha}\varepsilon$ for every subcube $Q' \subseteq Q$ whose rank is between r and $r+n$. Then $\{\eta_Q : Q \in \Pi_{r+n}\}$ is an i.i.d. sequence of Bernoulli random variables with

$$\mathbf{P}\big(\eta_Q = 1\big) = \prod_{k=r}^{r+n} \mathbf{P}\big(|z - \omega^{(k)}| < qp^{2\alpha}\varepsilon\big)^{\nu^{r+n-k}} > 0, \quad \text{where} \quad q = 1-p,$$

hence there almost surely exists a cube $Q \in \Pi_{n+r}$ with $\eta_Q = 1$. Then

$$|z - \omega^{(k)}(x)| < qp^{2\alpha}\varepsilon \quad \text{for all} \quad x \in Q \quad \text{and} \quad r \le k \le r+n.$$

Continuing (4.5), we have

$$\begin{aligned} |\lambda^{(r)}(x) - (1+\sigma z)\lambda_r| &< \sigma p\varepsilon\lambda_r + \sigma qp^{2\alpha}\varepsilon \sum_{k=r}^{r+n-1} p_k \\ &= \sigma p\varepsilon\lambda_r + \sigma qp^{2\alpha}\varepsilon(\lambda_r - \lambda_{r+n}) < \sigma p\varepsilon\lambda_r + \sigma q\varepsilon\lambda_r = \sigma\varepsilon\lambda_r. \end{aligned}$$

Then, almost surely, $\lambda^{(r)}(x) \in (a,b)$ so that $(a,b) \cap \{\lambda_Q : Q \in \Pi_r\} \neq \varnothing$. Now by considering $(a,b) \subseteq \big[(1-\sigma)\lambda_r, (1+\sigma)\lambda_r\big]$ with rational endpoints it follows that, almost surely, $\{\lambda_Q : Q \in \Pi_r\}$ is dense in $\big[(1-\sigma)\lambda_r, (1+\sigma)\lambda_r\big]$. □

4.2 Dependence of Eigenvalues

The eigenvalues are dependent but in a sense we will make precise, λ_Q and $\lambda_{Q'}$ are nearly uncorrelated if the graph distance (2.6) between Q and Q' is large.

Proposition 4.2. *Let $g^{(r)}(\lambda)$ be the density for $\lambda^{(r)}$ and for each $n \ge 1$, let $g_n^{(r)}(\lambda)$ be the density for $\lambda^{(r)} - \lambda^{(r+n)}$. Then*

$$\left|g_n^{(r)}(\lambda) - g_n^{(r)}(\mu)\right| \le \tfrac{c}{(\sigma p_r)^2}|\lambda - \mu| \tag{4.6}$$

for all $\lambda, \mu \in \mathbb{R}$, $r \ge 1$, and $1 \le n \le \infty$ where $c = \frac{8}{\pi}\big(\sup_{r\ge1} \|f_r'\|_\infty\big)^2$.

Proof. Observe that $g_1^{(r)}$ is the density for $\xi^{(r)} = p_r(1+\sigma\omega^{(r)})$ hence

$$g_1^{(r)}(x) = \tfrac{1}{\sigma p_r} f_r\big(\tfrac{x-p_r}{\sigma p_r}\big) \quad \text{so that} \quad \|(g_1^{(r)})'\|_\infty \leq \tfrac{1}{(\sigma p_r)^2}\|f_r'\|_\infty \leq \tfrac{C}{(\sigma p_r)^2}.$$

where $C = \sup_{r\geq 1}\|f_r'\|_\infty$. Then because $\xi^{(r)}, \xi^{(r+1)}, \ldots, \xi^{(r+n-1)}$ are independent,

$$\hat{g}_n^{(r)}(t) = \mathbf{E}e^{it(\lambda^{(r)}-\lambda^{(r+n)})} = \prod_{k=r}^{r+n-1} \hat{g}_1^{(k)}(t).$$

By the Mean Value Theorem, for all λ, μ, we have

$$\big|g_1^{(r)}(\lambda) - g_1^{(r)}(\mu)\big| \leq \tfrac{C}{(\sigma p_r)^2}|\lambda-\mu|.$$

Similarly, since $g_2^{(r)} = g_1^{(r)} * g_1^{(r+1)}$, we have

$$\big|g_2^{(r)}(\lambda) - g_2^{(r)}(\mu)\big| \leq \int g_1^{(r)}(z)|g_1^{(r+1)}(\lambda-z) - g_1^{(r+1)}(\mu-z)|\,dz \leq \tfrac{C}{(\sigma p_r)^2}|\lambda-\mu|.$$

For $n \geq 3$, observe that since $\lambda^{(r)} - \lambda^{(r+n)}$ is symmetric around $\lambda_r - \lambda_{r+n}$,

$$\big|g_n^{(r)}(\lambda) - g_n^{(r)}(\mu)\big| = \frac{1}{2\pi}\bigg|\int_{-\infty}^{\infty}(e^{-it\lambda} - e^{-it\mu})\hat{g}_n^{(r)}(t)\,dt\bigg| \leq \frac{|\lambda-\mu|}{\pi}\int_0^\infty \big|t\hat{g}_n^{(r)}(t)\big|\,dt.$$

But we have

$$\hat{g}_1^{(r)}(t) = \mathbf{E}e^{it\xi^{(r)}} = \int_{(1-\sigma)p_r}^{(1+\sigma)p_r} \tfrac{e^{itx}}{\sigma p_r} f_r\big(\tfrac{x-p_r}{\sigma p_r}\big)dx = \frac{1}{it}\int_{(1-\sigma)p_r}^{(1+\sigma)p_r} \tfrac{e^{itx}}{(\sigma p_r)^2} f_r'\big(\tfrac{x-p_r}{\sigma p_r}\big)dx$$

so that

$$\big|t\hat{g}_1^{(r)}(t)\big| \leq \tfrac{C}{(\sigma p_r)^2}\int_{(1-\sigma)p_r}^{(1+\sigma)p_r} dx = \tfrac{2C}{\sigma p_r}$$

hence

$$\big|t\hat{g}_n^{(r)}(t)\big| = \bigg|t\hat{g}_1^{(r)}(t)\prod_{k=r+1}^{r+n-1}\hat{g}_1^{(k)}(t)\bigg| \leq \tfrac{2C}{\sigma p_r}$$

and

$$\left|t\hat{g}_n^{(r)}(t)\right| = \left|t\hat{g}_1^{(r)}(t)\hat{g}_1^{(r+1)}(t)\hat{g}_1^{(r+2)}(t)\prod_{k=r+3}^{r+n-1}\hat{g}_2^{(k)}(t)\right| \leq \left(\tfrac{2C}{\sigma p_r}\right)^3 t^{-2}$$

so that for $A > 0$,

$$\int_0^\infty \left|t\hat{g}_n^{(r)}(t)\right| dt \leq \tfrac{2C}{\sigma p_r}\int_0^A dt + \left(\tfrac{2C}{\sigma p_r}\right)^3 \int_A^\infty t^{-2}\, dt = \tfrac{2C}{\sigma p_r} A + \left(\tfrac{2C}{\sigma p_r}\right)^3 A^{-1}.$$

Taking $A = \frac{2C}{\sigma p_r}$ minimizes the right-hand side and we obtain

$$\int_0^\infty \left|t\hat{g}_n^{(r)}(t)\right| dt \leq 2\left(\tfrac{2C}{\sigma p_r}\right)^2$$

so that for $n \geq 3$,

$$\left|g_n^{(r)}(\lambda) - g_n^{(r)}(\mu)\right| \leq \tfrac{8C^2}{\pi(\sigma p_r)^2}|\lambda - \mu|$$

hence (4.6) is valid. □

PROPOSITION 4.3. *If* $|h-1| \leq \sigma$ *and* f *is bounded on* $[(1-\sigma)\lambda_r, (1+\sigma)\lambda_r]$ *then*

$$\left|\mathbf{E}f\left(\lambda^{(r)} - \lambda^{(r+n)} + h\lambda_{r+n}\right) - \mathbf{E}f\left(\lambda^{(r)} - \lambda^{(r+n)}\right)\right| \leq cp^n \left\|f\right\|_\infty \tag{4.7}$$

and

$$\left|\mathbf{E}f\left(\lambda^{(r)} - \lambda^{(r+n)} + h\lambda_{r+n}\right) - \mathbf{E}f\left(\lambda^{(r)}\right)\right| \leq 2cp^n \left\|f\right\|_\infty \tag{4.8}$$

where $c = \frac{16(1+\sigma)\left(\sup_{r\geq 1}\|f_r'\|_\infty\right)^2}{\pi\sigma(qp^{2\alpha})^2}$.

Proof. Let $C = \frac{8}{\pi}\left(\sup_{r\geq 1}\|f_r'\|_\infty\right)^2$ as in Proposition 4.2. Since

$$\mathbf{E}f\left(\lambda^{(r)} - \lambda^{(r+n)} + h\lambda_{r+n}\right) = \int f(x) g_n^{(r)}(x - h\lambda_{r+n})\, dx,$$

by (4.6), (3.18), and (3.31), we have

$$\begin{aligned}&\big|\mathbf{E}f\big(\lambda^{(r)}-\lambda^{(r+n)}+h\lambda_{r+n}\big)-\mathbf{E}f\big(\lambda^{(r)}-\lambda^{(r+n)}\big)\big|\\&\qquad\le\int\big|f(x)\big|\big|g_n^{(r)}(x-h\lambda_{r+n})-g_n^{(r)}(x)\big|\,dx\le\tfrac{C|h|\lambda_{r+n}}{(\sigma p_r)^2}\int_{(1-\sigma)\lambda_r}^{(1+\sigma)\lambda_r}\big|f(x)\big|\,dx\\&\qquad\le\tfrac{C|h|\lambda_{r+n}}{(\sigma p_r)^2}2\sigma\lambda_r\,\|f\|_\infty\le\tfrac{2C(1+\sigma)p^{2r+n-2\alpha-2}}{\sigma(qp^{r+\alpha-1})^2}\,\|f\|_\infty=cp^n\,\|f\|_\infty\end{aligned}$$

so that (4.7) is proven. Next, since $\lambda^{(r)}-\lambda^{(r+n)}$ and $\lambda^{(r+n)}$ are independent,

$$\mathbf{E}\big(f\big(\lambda^{(r)}\big)\mid\lambda^{(r+n)}\big)=\varphi\big(\lambda^{(r+n)}\big)$$

where for $\ell\in\operatorname{supp}\lambda^{(r+n)}$, $\varphi(\ell)=\mathbf{E}f\big(\lambda^{(r)}-\lambda^{(r+n)}+\ell\big)$. Then, by (4.7), we have

$$\big|\,\mathbf{E}f\big(\lambda^{(r)}-\lambda^{(r+n)}\big)-\varphi\big(\lambda^{(r+n)}\big)\big|\le cp^n\,\|f\|_\infty\,,$$

almost surely, so that

$$\begin{aligned}&\Big|\,\mathbf{E}f\big(\lambda^{(r)}-\lambda^{(r+n)}+h\lambda_{r+h}\big)-\varphi\big(\lambda^{(r+n)}\big)\Big|\\&\quad\le\Big|\,\mathbf{E}f\big(\lambda^{(r)}-\lambda^{(r+n)}+h\lambda_{r+h}\big)-\mathbf{E}f\big(\lambda^{(r)}-\lambda^{(r+n)}\big)\Big|+cp^n\,\|f\|_\infty\le 2cp^n\,\|f\|_\infty\,.\end{aligned}$$

Taking expectations, we obtain (4.8). □

PROPOSITION 4.4. *For $f(\lambda),g(\lambda)$ bounded, for all $Q\in\Pi_m$ and $Q'\in\Pi_r$, with $Q\subsetneq Q'$,*

$$\big|\mathbf{Cov}\big(f(\lambda_Q),g(\lambda_{Q'})\big)\big|\le cp^{r-m}\,\|f\|_\infty\,\|g\|_\infty \tag{4.9}$$

and for all $Q\in\Pi_m$ and $Q'\in\Pi_n$, with $1\le m\le n<r=d_h(Q,Q')$,

$$\big|\mathbf{Cov}\big(f(\lambda_Q),g(\lambda_{Q'})\big)\big|\le cp^{r-n}\,\|f\|_\infty\,\|g\|_\infty \tag{4.10}$$

where $c=\frac{32(1+\sigma)\left(\sup_{r\ge1}\|f_r'\|_\infty\right)^2}{\pi\sigma(qp^{2\alpha})^2}\left(1+\frac{16(1+\sigma)\left(\sup_{r\ge1}\|f_r'\|_\infty\right)^2}{\pi\sigma(qp^{2\alpha})^2}\right)$.

Proof. Let $C = \frac{16(1+\sigma)\left(\sup_{r\geq 1}\|f_r'\|_\infty\right)^2}{\pi\sigma(qp^{2\alpha})^2}$. Because $\lambda_Q - \lambda_{Q'}$ and $\lambda_{Q'}$ are independent,

$$\mathbf{E}\big(f(\lambda_Q)g(\lambda_{Q'}) \;\big|\; \lambda_{Q'}\big) = \mathbf{E}\big(f(\lambda_Q) \;\big|\; \lambda_{Q'}\big)g(\lambda_{Q'}) = \varphi(\lambda_{Q'})g(\lambda_{Q'})$$

where $\varphi(\ell) = \mathbf{E}f(\lambda_Q - \lambda_{Q'} + \ell)$ for $\ell \in \operatorname{supp}\lambda^{(r)}$. By (4.8) we have

$$\big|\varphi(\ell) - \mathbf{E}f(\lambda_Q)\big| \leq 2Cp^{r-m}\left\|f\right\|_\infty \leq cp^{r-m}\left\|f\right\|_\infty \tag{4.11}$$

so that, almost surely,

$$\Big|\,\mathbf{E}\big(f(\lambda_Q)g(\lambda_{Q'}) \;\big|\; \lambda_{Q'}\big) - g(\lambda_{Q'})\,\mathbf{E}f(\lambda_Q)\Big| \leq cp^{r-m}\left\|f\right\|_\infty |g(\lambda_{Q'})|\,. \tag{4.12}$$

Taking expectations in (4.12) and using $\mathbf{E}|g(\lambda_{Q'})| \leq \|g\|_\infty$, we obtain (4.9).

Now to prove (4.10), let $x \in Q$, $y \in Q'$, and write $\lambda^{(r)} = \lambda^{(r)}(x) = \lambda^{(r)}(y)$. By independence of $\lambda_Q - \lambda^{(r)}$, $\lambda_{Q'} - \lambda^{(r)}$, and $\lambda^{(r)}$, we have

$$\mathbf{E}\big(f(\lambda_Q)g(\lambda_{Q'}) \;\big|\; \lambda^{(r)}\big) = \varphi(\lambda^{(r)})\psi(\lambda^{(r)})$$

where $\varphi(\ell) = \mathbf{E}f(\lambda_Q - \lambda^{(r)} + \ell)$ and $\psi(\ell) = \mathbf{E}g(\lambda_{Q'} - \lambda^{(r)} + \ell)$ for all $\ell \in \operatorname{supp}\lambda^{(r)}$. Let $s = \mathbf{E}f(\lambda_Q)$ and $t = \mathbf{E}g(\lambda_{Q'})$. Then just like in (4.11) we have

$$|\varphi(\ell) - s| \leq 2Cp^{r-m}\left\|f\right\|_\infty \qquad \text{and} \qquad |\psi(\ell) - t| \leq 2Cp^{r-n}\left\|g\right\|_\infty$$

so that

$$\begin{aligned}|\varphi(\lambda^{(r)})\psi(\lambda^{(r)}) - st| &\leq \left(2Cp^{r-m} + 2Cp^{r-n} + 2Cp^{r-m}2Cp^{r-n}\right)\left\|f\right\|_\infty\left\|g\right\|_\infty \\ &\leq 4C(1+C)p^{r-n}\left\|f\right\|_\infty\left\|g\right\|_\infty \leq cp^{r-n}\left\|f\right\|_\infty\left\|g\right\|_\infty.\end{aligned} \tag{4.13}$$

Since $\mathbf{Cov}\big(f(\lambda_Q), g(\lambda_{Q'})\big) = \mathbf{E}\big(\varphi(\lambda^{(r)})\psi(\lambda^{(r)}) - st\big)$, taking expectations, in (4.13), we obtain (4.10). □

Corollary 4.5. *For any two measurable sets A and B we have*

$$\left|\mathbf{Cov}\big(\mathbf{1}_A(\lambda_Q), \mathbf{1}_B(\lambda_{Q'})\big)\right| \le cp^{r-m} \tag{4.14}$$

for all $Q \in \Pi_m$, $Q' \in \Pi_r$, with $Q \subsetneq Q'$, and

$$\left|\mathbf{Cov}\big(\mathbf{1}_A(\lambda_Q), \mathbf{1}_B(\lambda_{Q'})\big)\right| \le cp^{r-n} \tag{4.15}$$

for all $Q \in \Pi_m$, $Q' \in \Pi_n$, with $1 \le m \le n < r = d_h(Q, Q')$, where

$$c = \tfrac{32(1+\sigma)\left(\sup_{r\ge 1}\|f_r'\|_\infty\right)^2}{\pi\sigma(qp^{2\alpha})^2}\left(1 + \tfrac{16(1+\sigma)\left(\sup_{r\ge 1}\|f_r'\|_\infty\right)^2}{\pi\sigma(qp^{2\alpha})^2}\right).$$

Proof. Apply Proposition 4.4 to $f(\lambda) = \mathbf{1}_A(\lambda)$ and $g(\lambda) = \mathbf{1}_B(\lambda)$. □

4.3 Density of States Measure

As in (3.56), let $\mathcal{S}_L$ be the set of all non-degenerate sub-cubes of $Q_0^{(L)}$ so that the eigenvalues of the spectral problem

$$-\Delta_\omega \psi = \lambda\psi, \quad \psi \equiv 0 \text{ on } X \setminus Q_0^{(L)} \tag{4.16}$$

are given by $\{\lambda_Q : Q \in \mathcal{S}_L\}$. Then as in (3.57–3.58), $\mathcal{N}_L(A)$ is the random number of eigenvalues for (4.16) which belong to a measurable set $A \subseteq \mathrm{Sp}(-\Delta_\omega)$, and $\mathcal{N}_L^{(r)}(A)$ is the random number of eigenvalues of rank r which belong to A. We see that

$$\mathbf{E}\mathcal{N}_L^{(r)}(A) = \sum_{Q\in\mathcal{S}_L^{(r)}} \mathbf{P}(\lambda_Q \in A) = \nu^{L-r}\,\mathbf{P}(\lambda^{(r)} \in A) \tag{4.17}$$

and

$$\mathbf{E}\mathcal{N}_L(A) = \sum_{r=1}^{L} \nu^{L-r}\,\mathbf{P}(\lambda^{(r)} \in A). \tag{4.18}$$

Similarly, if we put

$$\mathcal{N}_L f = \int f(\lambda)\mathcal{N}_L(d\lambda) = \sum_{Q\in\mathcal{S}_L} f(\lambda_Q) \tag{4.19}$$

and

$$\mathcal{N}_L^{(r)} f = \int f(\lambda)\mathcal{N}_L^{(r)}(d\lambda) = \sum_{Q\in\mathcal{S}_L^{(r)}} f(\lambda_Q), \tag{4.20}$$

we have

$$\mathbf{E}\mathcal{N}_L^{(r)} f = \nu^{L-r}\,\mathbf{E} f(\lambda^{(r)}) = \nu^{L-r}\int f(\lambda) g^{(r)}(\lambda)\,d\lambda \tag{4.21}$$

and

$$\mathbf{E}\mathcal{N}_L f = \sum_{r=1}^{L} \nu^{L-r}\,\mathbf{E} f(\lambda^{(r)}) = \sum_{r=1}^{L} \nu^{L-r}\int f(\lambda) g^{(r)}(\lambda)\,d\lambda. \tag{4.22}$$

THEOREM 4.6. *For bounded functions $f(\lambda)$ and $g(\lambda)$ on $\mathrm{Sp}(-\Delta_\omega)$ we have*

$$\bigl|\,\mathbf{Cov}\bigl(\mathcal{N}_L f, \mathcal{N}_L g\bigr)\bigr| \lesssim \nu^L\bigl(L^2 + (\nu p)^L\bigr)\,\|f\|_\infty\,\|g\|_\infty \tag{4.23}$$

and for $1 \le k \le r \le L$ we have

$$\bigl|\,\mathbf{Cov}\bigl(\mathcal{N}_L^{(k)} f, \mathcal{N}_L^{(r)} g\bigr)\bigr| \lesssim \nu^L\bigl(L + (\nu p)^L\bigr)\,\|f\|_\infty\,\|g\|_\infty \tag{4.24}$$

where $x \lesssim y$ means $x = O(y)$ as $L \to \infty$.

Proof. Replacing f and g by $f/\|f\|_\infty$ and $g/\|g\|_\infty$ if necessary, we may, without loss of generality, assume $\|f\|_\infty \le 1$ and $\|g\|_\infty \le 1$.

For $i, j < \nu^{L-k}$, considering the isometry $\varphi : X \to X$ which swaps the cubes $Q_i^{(k)}$ and $Q_j^{(k)}$, we see that $\bigl(\lambda_i^{(k)}, \mathcal{N}_L^{(r)} g\bigr) \overset{\text{law}}{=} \bigl(\lambda_j^{(k)}, \mathcal{N}_L^{(r)} g\bigr)$. This means we have

$$\mathbf{Cov}\bigl(\mathcal{N}_L^{(k)} f, \mathcal{N}_L^{(r)} g\bigr) = \sum_{i<\nu^{L-k}} \mathbf{Cov}\bigl(f(\lambda_i^{(k)}), \mathcal{N}_L^{(r)} g\bigr) = \nu^{L-k}\,\mathbf{Cov}\bigl(f(\lambda_0^{(k)}), \mathcal{N}_L^{(r)} g\bigr).$$

But

$$\mathbf{Cov}\big(f(\lambda_0^{(k)}), \mathcal{N}_L^{(r)} g\big) = \mathbf{Cov}\big(f(\lambda_0^{(k)}), g(\lambda_0^{(r)})\big) + \sum_{n=1}^{L-r} \sum_{i=\nu^{n-1}}^{\nu^n-1} \mathbf{Cov}\big(f(\lambda_0^{(k)}), g(\lambda_i^{(r)})\big).$$

Since $Q_0^{(k)} \subseteq Q_0^{(r)}$, by (4.9), we have

$$\big| \mathbf{Cov}\big(f(\lambda_0^{(k)}), g(\lambda_0^{(r)})\big)\big| \lesssim p^{r-k}.$$

For $\nu^{n-1} \leq i < \nu^n$ we have $d_h\big(Q_0^{(k)}, Q_i^{(r)}\big) = r + n$ so by (4.10),

$$\big| \mathbf{Cov}\big(f(\lambda_0^{(k)}), g(\lambda_i^{(r)})\big)\big| \lesssim p^{(r+n)-r} = p^n$$

hence

$$\left| \sum_{i=\nu^{n-1}}^{\nu^n-1} \mathbf{Cov}\big(f(\lambda_0^{(k)}), g(\lambda_i^{(r)})\big)\right| \lesssim (\nu^n - \nu^{n-1}) p^n < (\nu p)^n.$$

It means that

$$\big| \mathbf{Cov}\big(\mathcal{N}_L^{(k)} f, \mathcal{N}_L^{(r)} g\big)\big| \lesssim \nu^{L-k} \left(p^{r-k} + \sum_{n=1}^{L-r} (\nu p)^n \right) = \nu^L \left(\frac{p^r}{(\nu p)^k} + \frac{1}{\nu^k} \sum_{n=1}^{L-r} (\nu p)^n \right). \quad (4.25)$$

Now, because we have

$$\mathbf{Cov}\big(\mathcal{N}_L f, \mathcal{N}_L g\big) = \sum_{r=1}^{L} \sum_{k=1}^{r} 2\, \mathbf{Cov}\big(\mathcal{N}_L^{(k)} f, \mathcal{N}_L^{(r)} g\big)$$

it follows from (4.25) that

$$\begin{aligned} \big| \mathbf{Cov}\big(\mathcal{N}_L f, \mathcal{N}_L g\big)\big| &\lesssim \nu^L \sum_{r=1}^{L} \left[\sum_{k=1}^{r} \frac{p^r}{(\nu p)^k} + \left(\sum_{k=1}^{r} \frac{1}{\nu^k} \right) \left(\sum_{n=1}^{L-r} (\nu p)^n \right) \right] \\ &\leq \nu^L \sum_{r=1}^{L} \left[\sum_{k=1}^{r} \frac{p^r}{(\nu p)^k} + \sum_{n=1}^{L-r} (\nu p)^n \right] \end{aligned}$$

For $\nu p \neq 1$ we have

$$\sum_{k=1}^{r} \frac{p^r}{(\nu p)^k} + \sum_{n=1}^{L-r} (\nu p)^n = \tfrac{\nu p}{|\nu p - 1|} \left(\tfrac{1}{\nu p} |p^r - \nu^{-r}| + \big| (\nu p)^{L-r} - 1 \big| \right) \lesssim L + (\nu p)^{L-r}$$

and for $\nu p = 1$ we have

$$\sum_{k=1}^{r} \frac{p^r}{(\nu p)^k} + \sum_{n=1}^{L-r} (\nu p)^n = r p^r + L - r \leq L + (\nu p)^{L-r}$$

so that

$$\big| \mathbf{Cov}\big(\mathcal{N}_L f, \mathcal{N}_L g\big) \big| \lesssim \nu^L \sum_{r=1}^{L} \big(L + (\nu p)^{L-r} \big) = \nu^L \left(L^2 + \sum_{r=0}^{L-1} (\nu p)^r \right) \lesssim \nu^L \big(L^2 + (\nu p)^L \big).$$

From (4.25) we also find that

$$\big| \mathbf{Cov}\big(\mathcal{N}_L^{(k)} f, \mathcal{N}_L^{(r)} g\big) \big| \lesssim \nu^L \left(1 + \sum_{n=1}^{L-r} (\nu p)^n \right) \lesssim \nu^L \big(L + (\nu p)^L \big)$$

which completes the proof. □

Corollary 4.7. *For any two measurable sets $A, B \subseteq \mathrm{Sp}(-\Delta_\omega)$ we have*

$$\big| \mathbf{Cov}\big(\mathcal{N}_L(A), \mathcal{N}_L(B)\big) \big| \lesssim \nu^L \big(L^2 + (\nu p)^L \big) \tag{4.26}$$

and for $1 \leq k \leq r \leq L$ we have

$$\big| \mathbf{Cov}\big(\mathcal{N}_L^{(k)}(A), \mathcal{N}_L^{(r)}(B)\big) \big| \lesssim \nu^L \big(L + (\nu p)^L \big) \tag{4.27}$$

where $x \lesssim y$ means $x = O(y)$ as $L \to \infty$.

Proof. Apply Theorem 4.6 to $f(\lambda) = \mathbf{1}_A(\lambda)$ and $g(\lambda) = \mathbf{1}_B(\lambda)$. □

The empirical measures for $\{\lambda_Q : Q \in \mathcal{S}_L\}$ and $\{\lambda_Q : Q \in \mathcal{S}_L^{(r)}\}$ are given by

$$N_L(A) = \frac{\mathcal{N}_L(A)}{|\mathcal{S}_L|} = \frac{(\nu-1)\mathcal{N}_L(A)}{(1-\nu^{-L})\nu^L} \qquad \text{and} \qquad N_L^{(r)}(A) = \frac{\mathcal{N}_L^{(r)}(A)}{\left|\mathcal{S}_L^{(r)}\right|} = \frac{\mathcal{N}_L^{(r)}(A)}{\nu^{L-r}},$$

hence

$$N_L f = \int f(\lambda) N_L(d\lambda) = \frac{1}{|\mathcal{S}_L|} \sum_{Q \in \mathcal{S}_L} f(\lambda_Q) \tag{4.28}$$

and

$$N_L^{(r)} f = \int f(\lambda) N_L^{(r)}(d\lambda) = \frac{1}{\left|\mathcal{S}_L^{(r)}\right|} \sum_{Q \in \mathcal{S}_L^{(r)}} f(\lambda_Q). \tag{4.29}$$

Observe that by (4.17) and (4.21),

$$\mathbf{E} N_L^{(r)}(A) = \frac{\mathbf{E}\mathcal{N}_L^{(r)}(A)}{\left|\mathcal{S}_L^{(r)}\right|} = \mathbf{P}(\lambda^{(r)} \in A) \qquad \text{and} \qquad \mathbf{E} N_L^{(r)} f = \frac{\mathbf{E}\mathcal{N}_L^{(r)} f}{\left|\mathcal{S}_L^{(r)}\right|} = \mathbf{E} f(\lambda^{(r)}).$$

Therefore, since

$$\mathbf{Var} N_L^{(r)}(A) \precsim \frac{\nu^L\big(L + (\nu p)^L\big)}{\left|\mathcal{S}_L^{(r)}\right|^2} \precsim \frac{L}{\nu^L} + p^L \to 0 \quad \text{as} \quad L \to \infty,$$

and similarly, $\mathbf{Var} N_L^{(r)} f \precsim (L\nu^{-L} + p^L)\, \|f\|_\infty^2$, we see that, almost surely, as $L \to \infty$,

$$N_L^{(r)}(A) \to \mathbf{P}(\lambda^{(r)} \in A) \quad \text{and} \quad N_L^{(r)} f \to \mathbf{E} f(\lambda^{(r)}) = \int f(\lambda) g^{(r)}(\lambda)\, d\lambda. \tag{4.30}$$

Also, by (4.18), we have

$$\mathbf{E} N_L(A) = \frac{\mathbf{E}\mathcal{N}_L(A)}{|\mathcal{S}_L|} = \frac{1}{1-\nu^{-L}} \sum_{r=1}^{L} \frac{(\nu-1)\, \mathbf{P}(\lambda^{(r)} \in A)}{\nu^r}$$

and by (4.22), we have

$$\mathbf{E} N_L f = \frac{\mathbf{E}\mathcal{N}_L f}{|\mathcal{S}_L|} = \frac{1}{1-\nu^{-L}} \sum_{r=1}^{L} \frac{(\nu-1)\, \mathbf{E} f(\lambda^{(r)})}{\nu^r}.$$

Let

$$N(A) := \lim_{L\to\infty} \mathbf{E} N_L(A) = \sum_{r=1}^{\infty} \frac{(\nu-1)\,\mathbf{P}(\lambda^{(r)} \in A)}{\nu^r} \tag{4.31}$$

and

$$Nf := \lim_{L\to\infty} \mathbf{E} N_L f = \sum_{r=1}^{\infty} \frac{(\nu-1)\,\mathbf{E} f(\lambda^{(r)})}{\nu^r} \tag{4.32}$$

Then because

$$\mathbf{Var} N_L(A) \lesssim \frac{\nu^L\big(L^2 + (\nu p)^L\big)}{|\mathcal{S}_L|^2} \lesssim \frac{L^2}{\nu^L} + p^L \to 0 \quad \text{as} \quad L \to \infty,$$

we see that almost surely, $N_L(A) \to N(A)$ as $L \to \infty$.

Proposition 4.8. *For each measurable set* $A \subseteq \mathrm{Sp}(-\Delta_\omega)$*,* $\lim_{L\to\infty} \mathbf{Var}\big[N_L(A)\big] = 0$*. Therefore, with probability one,* $\lim_{L\to\infty} N_L(A) = N(A)$*.*

It is clear from (4.31) that the measure $N(d\lambda)$, which depends on the parameter $0 < \sigma < 1$, is supported on $\mathrm{Sp}(-\Delta_\omega)$ and has a continuous distribution function and density given by

$$N(0,\lambda] = \sum_{r=1}^{\infty} \frac{\nu-1}{\nu^r}\, \mathbf{P}\big(\lambda^{(r)} \le \lambda\big) \tag{4.33}$$

and

$$n(\lambda) = \tfrac{d}{d\lambda} N(0,\lambda] = \sum_{r=1}^{\infty} \frac{\nu-1}{\nu^r} g^{(r)}(\lambda). \tag{4.34}$$

Since almost surely, $\lambda^{(r)}$ lies between $(1 \pm \sigma)\lambda_r$, we may write

$$\operatorname{supp} \lambda^{(r)} = (1+\sigma)[p^\beta \lambda_r, \lambda_r]$$

where $\beta = \log_p \frac{1-\sigma}{1+\sigma}$, and we have

$$\mathrm{Sp}(-\Delta_\omega) = (1+\sigma) \bigcup_{r=1}^{\infty} [p^\beta \lambda_r, \lambda_r]. \tag{4.35}$$

Observe that (3.31) implies that for all $r \geq 1$,

$$p^{1+2\alpha} \leq \tfrac{\lambda_{r+1}}{\lambda_r} \leq p^{1-2\alpha}. \tag{4.36}$$

The expression (4.35) shows that $\mathrm{Sp}(-\Delta_\omega)$ is connected, i.e., $\mathrm{Sp}(-\Delta_\omega) = [0, 1+\sigma]$, if and only if $p^\beta \leq \inf_{r\geq 1} \frac{\lambda_{r+1}}{\lambda_r}$ which by (4.36) is the case whenever

$$\tfrac{1-p^{1+2\alpha}}{1+p^{1+2\alpha}} \leq \sigma < 1.$$

On the other hand, whenever

$$0 < \sigma < \tfrac{1-p^{1-2\alpha}}{1+p^{1-2\alpha}},$$

the union in (4.35) is disjoint and it is impossible for two eigenvalues of different rank assume the same value. Notice that the set

$$I(\lambda) = \{r \geq 1 : (1-\sigma)\lambda_r \leq \lambda \leq (1+\sigma)\lambda_r\}$$

contains every rank for which it is possible that some eigenvalue $\lambda_i^{(r)}$ assumes the value $\lambda \in \mathrm{Sp}(-\Delta_\omega)$, i.e., $\{r : g^{(r)}(\lambda) > 0\} \subseteq I(\lambda)$. In particular, (3.53) implies there are approximately $\beta \pm (2\alpha+1)$ values of r where $g^{(r)}(\lambda) > 0$ and for each of these,

$$-\alpha - \beta \leq r - \log_p \tfrac{\lambda}{1+\sigma} \leq \alpha + 1. \tag{4.37}$$

It follows that the sum for $n(\lambda)$ is actually finite — if we write

$$I(\lambda) = \{m+1, m+2, \ldots, M\}$$

then (4.34) becomes

$$n(\lambda) = \sum_{r=m+1}^{M} \frac{\nu - 1}{\nu^r} g^{(r)}(\lambda)$$

(see also Lemma 5.7 below).

The inequalities (3.59) allow us to obtain Lifshitz tails for the integrated density of states in the strong form (see [8]).

PROPOSITION 4.9. *There exist constants* $c_1, c_2 > 0$ *such that*

$$c_1 \lambda^{s_h/2} \le N(0,\lambda] \le c_2 \lambda^{s_h/2}$$

for all $\lambda \in \mathrm{Sp}(-\Delta_\omega)$ *hence* $\lim\limits_{\lambda \searrow 0} \frac{\log N(0,\lambda]}{\log \lambda} = \frac{s_h}{2}$.

Proof. Proposition 3.11. □

4.4 Hierarchical Random Walk in a Random Environment

The semigroup $e^{t\Delta_\omega}$ generates a continuous time Markov process x_t on (X, d_h), i.e.,

$$\mathbf{E}^x f(x_t) = e^{t\Delta_\omega} f(x) = \sum_{Q \in \mathcal{V}_1} e^{-\lambda_Q t} E_Q f(x) = \sum_{r=1}^{\infty} e^{-\lambda^{(r)}(x)t} E_r f(x).$$

Starting at the point x, the process waits for an exponentially distributed time τ with $\mathbf{P}(\tau \ge t) = e^{-(1+\sigma)t}$ and then jumps uniformly into the cube of rank $\rho_x \in \{0, 1, 2, \ldots\}$ containing x, where ρ_x is independent of τ and has the random distribution

$$\mathbf{P}(\rho_x = 0) = 1 - \frac{\lambda^{(1)}(x)}{1+\sigma} \qquad \text{and} \qquad \mathbf{P}(\rho_x = r) = \frac{\xi^{(r)}(x)}{1+\sigma} \quad \text{for} \quad r > 0.$$

The transition probabilities $p(t, x, y) = \mathbf{P}^x(x_t = y)$ are found to be given by

$$p(t,x,x) = \sum_{r=1}^{\infty} \frac{(\nu-1)e^{-\lambda^{(r)}(x)t}}{\nu^r} \quad \text{and} \quad p(t,x,y) = -\frac{e^{-\lambda^{(r)}(x)t}}{\nu^r} + \sum_{k=r+1}^{\infty} \frac{(\nu-1)e^{-\lambda^{(k)}(x)t}}{\nu^k}$$

for $d_h(x, y) = r > 0$. Computations similar to those in Section 3.3 and in [16, 17] show that this process is transient if $s_h > 2$ ($\nu p > 1$) and recurrent if $s_h \le 2$ ($\nu p \le 1$).

CHAPTER 5: EIGENVALUE STATISTICS

5.1 Preliminaries

In this section we will study the distribution of eigenvalues for $-\Delta_\omega$ in the near vicinity of a given point $\lambda \in \mathrm{Sp}(-\Delta_\omega)$ (see [12, 10]). Ignoring multiplicities (since they are all the same), there are $|\mathcal{S}_L| = 1 + \nu + \nu^2 + \cdots + \nu^{L-1}$ eigenvalues for the spectral problem (4.16). The set of these eigenvalues is a point process (see [7, 2]) in the interval $[0, 1+\sigma]$. We will apply the transformation $x \mapsto |\mathcal{S}_L|(x - \lambda)$ to center and scale this process so that the length of the smallest interval almost surely containing it is approximately $|\mathcal{S}_L|$ and this interval is situated about the origin in the same way $\mathrm{Sp}(-\Delta_\omega)$ is situated about λ. Another way to look at it is, we are really looking at the spectrum

$$\mathrm{Sp}(H_L^\lambda) = \{|\mathcal{S}_L|(\lambda_Q - \lambda) : Q \in \mathcal{S}_L\} \tag{5.1}$$

of the operator

$$H_L^\lambda = -|\mathcal{S}_L|(\lambda + \Delta_\omega)\mathbf{1}_{Q_0^L} .$$

For a bounded measurable set $A \subseteq \mathbb{R}$, let

$$A_L^\lambda = \lambda + \tfrac{1}{|\mathcal{S}_L|}A = \left\{\lambda + \tfrac{x}{|\mathcal{S}_L|} : x \in A\right\}$$

and observe that

$$\mu_L^\lambda(A) := \mathcal{N}_L(A_L^\lambda) = \sum_{Q\in\mathcal{S}_L} \mathbf{1}_{A_L^\lambda}(\lambda_Q) = \sum_{Q\in\mathcal{S}_L} \mathbf{1}_A\big(|\mathcal{S}_L|(\lambda_Q - \lambda)\big) = \big|A \cap \mathrm{Sp}(H_L^\lambda)\big| \quad (5.2)$$

gives the number $\mu_L^\lambda(A)$ of eigenvalues for H_L^λ belonging to A. We will prove that the expected number $\mathbf{E}\mu_L^\lambda(A) = \mathbf{E}\mathcal{N}_L(A_L^\lambda)$ of eigenvalues for H_L^λ belonging to A becomes proportional to $|A|$ (the Lebesgue measure of A) and for pairwise disjoint measurable sets, $A_1, \ldots, A_n$, the random numbers $\mu_L^\lambda(A_1), \ldots, \mu_L^\lambda(A_n)$ of eigenvalues for H_L^λ converge in distribution to independent integer-valued random variables as $L \to \infty$. In other words, the set $\mathrm{Sp}(H_L^\lambda)$ converges to a Poisson point process. In view of Proposition 4.8, we should expect that as $L \to \infty$,

$$\mathbf{E}\mathcal{N}_L(A_L^\lambda) = |\mathcal{S}_L|\,\mathbf{E}N_L(A_L^\lambda) \approx |\mathcal{S}_L| \int_{A_L^\lambda} n(x)\,dx = \int_A n(\lambda + \tfrac{x}{|\mathcal{S}_L|})\,dx \approx n(\lambda)|A|,$$

i.e., the intensity measure of this limiting Poisson process is simply $n(\lambda)$ times Lebesgue measure. It means that μ_L^λ converges weakly as $L \to \infty$ to an integer-valued random measure μ^λ which possesses the property that for any collection $A_1, \ldots, A_n$ of pairwise disjoint measurable sets, $\mu^\lambda(A_1), \ldots, \mu^\lambda(A_n)$ is a collection of independent Poissonian distributed random variables with

$$\mathbf{E}\mu^\lambda(A) = n(\lambda)|A|.$$

It is sufficient to prove that for every continuous function $f \geq 0$ with compact support,

$$\lim_{L\to\infty} \mathbf{E}e^{-\mu_L^\lambda f} = \lim_{L\to\infty} \mathbf{E}e^{-\mathcal{N}_L f_L^\lambda} = e^{-n(\lambda)\int(1-e^{-f(x)})dx}$$

where

$$f_L^\lambda(x) = f\big(|\mathcal{S}_L|(x-\lambda)\big)$$

and

$$\mu_L^\lambda f = \int f(x)\mu_L^\lambda(dx) = \mathcal{N}_L f_L^\lambda = \sum_{Q\in\mathcal{S}_L} f_L^\lambda(\lambda_Q) = \sum_{Q\in\mathcal{S}_L} f\big(|\mathcal{S}_L|(\lambda_Q-\lambda)\big).$$

We will further prove that the set of eigenvalues for H_L^λ of individual rank also converges to a Poisson process with intensity equal to Lebesgue measure times the term in the series for $n(\lambda)$ contributed by eigenvalues of that rank.

We first need to collect some lemmas for the proof.

LEMMA 5.1. *Let $z_{n,k}$ and $w_{n,k}$ be two triangular arrays of complex numbers. If there exists a constant $c > 0$ such that $|z_{n,k}| \leq \frac{c}{n}$, $|w_{n,k}| \leq \frac{c}{n}$, and $|z_{n,k} - w_{n,k}| \leq \frac{c}{n^2}$ for all n, k with $1 \leq k \leq n$, then*

$$\left|\prod_{k=1}^{n}(1+z_{n,k}) - \exp\left(\sum_{k=1}^{n} w_{n,k}\right)\right| \leq \frac{C}{n} \tag{5.3}$$

for every $n \geq 1$ where $C = c(1+ce^c)e^{c(2+ce^c)}$.

Proof. First observe that for all $z, w \in \mathbb{C}$, we have

$$\big|e^w - (1+z)\big| \leq |w-z| + \sum_{n=2}^{\infty}\frac{|w|^n}{n!} \leq |w-z| + |w|^2 e^{|w|}$$

so that

$$\big|e^w_{n,k} - (1+z_{n,k})\big| \leq \frac{c}{n^2} + \left(\frac{c}{n}\right)^2 e^{c/n} \leq \frac{c+c^2e^c}{n^2}.$$

Therefore, using the formula

$$\prod_{k\in S} x_k - \prod_{k\in S} y_k = \sum_{\varnothing\neq T\subseteq S}\left[\left(\prod_{k\notin T} y_k\right)\left(\prod_{k\in T}(x_k-y_k)\right)\right] \tag{5.4}$$

for a difference of products with $S = \{1, 2, \dots, n\}$, we have

$$\left|\prod_{k=1}^{n}(1+z_{n,k}) - \prod_{k=1}^{n} e^{w_{n,k}}\right| \le \sum_{\varnothing \ne T \subseteq S} \left(\prod_{k \notin T} |1+z_{n,k}| \cdot \prod_{k \in T} \left|e^{w_{n,k}} - (1+z_{n,k})\right| \right).$$

But using our hypotheses and the the inequality $\left(1+\frac{c}{n}\right)^n \le e^c$, we have

$$\begin{aligned}
&\prod_{k \notin T} |1+z_{n,k}| \cdot \prod_{k \in T} \left|e^{w_{n,k}} - (1+z_{n,k})\right| \\
&\qquad \le \prod_{k \notin T} \left(1+\frac{c}{n}\right) \cdot \prod_{k \in T} \frac{c(1+ce^c)}{n^2} = \left(1+\frac{c}{n}\right)^n \left(\frac{c(1+ce^c)}{n^2+nc}\right)^{|T|} \le e^c \left(\frac{c(1+ce^c)}{n^2}\right)^{|T|}
\end{aligned}$$

hence

$$\left|\prod_{k=1}^{n}(1+z_{n,k}) - \prod_{k=1}^{n} e^{w_{n,k}}\right| \le e^c \sum_{\varnothing \ne T \subseteq S} \left(\frac{c(1+ce^c)}{n^2}\right)^{|T|} = e^c \left[\left(1 + \frac{c(1+ce^c)}{n^2}\right)^n - 1 \right].$$

Finally, applying the inequality

$$\left|(1+z)^n - 1\right| \le n|z|(1+|z|)^{n-1}, \tag{5.5}$$

we have

$$\begin{aligned}
\left|\prod_{k=1}^{n}(1+z_{n,k}) - \prod_{k=1}^{n} e^{w_{n,k}}\right| &\le e^c \cdot n \cdot \frac{c(1+ce^c)}{n^2} \cdot \left(1 + \frac{c(1+ce^c)}{n^2}\right)^{n-1} \\
&\le \frac{c(1+ce^c)e^c}{n} \cdot e^{c(1+ce^c)/n} \le \frac{C}{n}
\end{aligned}$$

which establishes (5.3). □

LEMMA 5.2. *Let* $\{z_Q, w_Q : Q \in \mathcal{S}_L^{(r)}\} \subseteq \mathbb{C}$ *and assume there exists a constant* $c > 0$ *such that* $|z_Q| \le 1 + \frac{c}{\nu^L}$, $|w_Q| \le \frac{c}{\nu^L}$, *and* $|z_Q - (1+w_Q)| \le \frac{c}{\nu^{2L}}$ *for all* $Q \in \mathcal{S}_L^{(r)}$*. Then*

$$\left| \prod_{Q \in \mathcal{S}_L^{(r)}} (z_Q)^\nu - \prod_{Q \in \mathcal{S}_L^{(r)}} (1+\nu w_Q) \right| \le \frac{e^{c/\nu^{r+1}}}{\nu^{r-1}} \nu^{2-L}.$$

Proof. Using the formula (5.4) for a difference of products, we have

$$\left| \prod_{Q\in\mathcal{S}_L^{(r)}} (z_Q)^\nu - \prod_{Q\in\mathcal{S}_L^{(r)}} (1+\nu w_Q) \right| \le \sum_{\varnothing\neq\mathcal{T}\subseteq\mathcal{S}_L^{(r)}} \left(\prod_{Q\notin\mathcal{T}} |z_Q|^\nu \cdot \prod_{Q\in\mathcal{T}} \left|(z_Q)^\nu - (1+\nu w_Q)\right| \right).$$

But since

$$\left|(z_Q)^\nu - (1+w_Q)^\nu\right| \le \left|(z_Q) - (1+w_Q)\right| \sum_{k=0}^{\nu-1} |z_Q|^{\nu-k}|1+w_Q|^k \le \frac{\nu c}{\nu^{2L}}\left(1+\frac{c}{\nu^L}\right)^\nu$$

and

$$\left|(1+w_Q)^\nu - (1+\nu w_Q)\right| \le \nu^2|w_Q|^2(1+|w_Q|)^{\nu-2} \le \frac{\nu^2c^2}{\nu^{2L}}\left(1+\frac{c}{\nu^L}\right)^\nu$$

we have

$$|(z_Q)^\nu - (1+\nu w_Q)| \le \frac{\nu c}{\nu^{2L}}\left(1+\frac{c}{\nu^L}\right)^\nu + \frac{\nu^2c^2}{\nu^{2L}}\left(1+\frac{c}{\nu^L}\right)^\nu \le \frac{\nu^2(c+1)^2}{\nu^{2L}}\left(1+\frac{c}{\nu^L}\right)^\nu$$

so that

$$\prod_{Q\notin\mathcal{T}} |z_Q|^\nu \cdot \prod_{Q\in\mathcal{T}} \left|(z_Q)^\nu - (1+\nu w_Q)\right|$$
$$\le \prod_{Q\notin\mathcal{T}} \left(1+\frac{c}{\nu^L}\right)^\nu \prod_{Q\in\mathcal{T}} \left(\frac{\nu^2(c+1)^2}{\nu^{2L}}\left(1+\frac{c}{\nu^L}\right)^\nu\right) = \left(1+\frac{c}{\nu^L}\right)^{\nu^{L-r+1}} \left(\frac{\nu^2(c+1)^2}{\nu^{2L}}\right)^{|\mathcal{T}|}$$

hence

$$\left| \prod_{Q\in\mathcal{S}_L^{(r)}} (z_Q)^\nu - \prod_{Q\in\mathcal{S}_L^{(r)}} (1+\nu w_Q) \right| \le \left(1+\frac{c}{\nu^L}\right)^{\nu^{L-r+1}} \sum_{\varnothing\neq\mathcal{T}\subseteq\mathcal{S}_L^{(r)}} \left(\frac{\nu^2(c+1)^2}{\nu^{2L}}\right)^{|\mathcal{T}|}$$
$$= \left(1+\frac{c}{\nu^L}\right)^{\nu^{L-r+1}} \left[\left(1+\frac{\nu^2(c+1)^2}{\nu^{2L}}\right)^{\nu^{L-r}} - 1\right].$$

Finally, applying the inequalities $\left(1+\frac{c}{n}\right)^n \le e^c$ and (5.5), we obtain

$$\left| \prod_{Q\in\mathcal{S}_L^{(r)}} (z_Q)^\nu - \prod_{Q\in\mathcal{S}_L^{(r)}} (1+\nu w_Q) \right| \le e^{c/\nu^{r+1}} \nu^{L-r} \frac{1}{\nu^{2L-2}} \left(1+\frac{\nu^2(c+1)^2}{\nu^{2L}}\right)^{\nu^{L-r}-1}$$
$$\le \frac{e^{c/\nu^{r+1}}}{\nu^r} e^{\nu^2(c+1)^2/\nu^{L+r}} \nu^{2-L} \le \frac{e^{c/\nu^{r+1}}}{\nu^{r-1}} \nu^{2-L}. \qquad \square$$

Observe that if $f(x)$ and $g(x)$ are continuous and f has compact support then

$$\int f_L^\lambda(x) g(x)\,dx = \frac{1}{|\mathcal{S}_L|}\int f(x) g(\lambda + \tfrac{x}{|\mathcal{S}_L|})\,dx \approx \frac{(\nu-1)g(\lambda)}{\nu^L}\int f(x)\,dx. \tag{5.6}$$

It will be necessary to keep the error in the approximation (5.6) of order $O(\nu^{-2L})$.

LEMMA 5.3. *Assume $f(x)$, $g(x)$, and $g'(x)$ are continuous and compactly supported. Define an operator T_f by*

$$T_f g(\lambda) = \int f_L^\lambda(x) g(x)\,dx - \frac{(\nu-1)g(\lambda)}{\nu^L}\int f(x)\,dx.$$

Then

$$|T_f g(\lambda)| \;\le \frac{\|g\|_\infty + \|g'\|_\infty}{\nu^{2L-1}}\int (1+\nu|x|)|f(x)|dx. \tag{5.7}$$

Proof. We have

$$\begin{aligned}|T_f g(\lambda)| \;&\le \frac{1}{|\mathcal{S}_L|}\int \big|f(x)\big|\big|g(\lambda + \tfrac{x}{|\mathcal{S}_L|}) - (1-\nu^{-L})g(\lambda)\big|dx \\ &\le \frac{|g(\lambda)|}{\nu^L|\mathcal{S}_L|}\int |f(x)|dx + \frac{1}{|\mathcal{S}_L|}\int \big|f(x)\big|\big|g(\lambda+\tfrac{x}{|\mathcal{S}_L|}) - g(\lambda)\big|dx \\ &\le \frac{\|g\|_\infty \int |f(x)|dx}{\nu^L|\mathcal{S}_L|} + \frac{\|g'\|_\infty \int |x||f(x)|dx}{|\mathcal{S}_L|^2}.\end{aligned}$$

Because $|\mathcal{S}_L| \ge \nu^{L-1}$, this implies (5.7). □

LEMMA 5.4. *Let $f \ge 0$ be a continuous function with compact support and let $g(x)$ be a continuously differentiable density for a random variable X. Then*

$$\mathbf{E}e^{-f_L^\lambda(X)} = 1 - \frac{(\nu-1)g(\lambda)}{\nu^L}\int (1-e^{-f(x)})dx + \varepsilon$$

where $\nu^{2L}|\varepsilon| \le \nu\big(\|g\|_\infty + \|g'\|_\infty\big)\int (1+\nu|x|)(1-e^{-f(x)})dx$.

Proof. We observe that

$$\mathbf{E}e^{-f_L^\lambda(X)} = \int g(x)e^{-f_L^\lambda(x)}dx = 1 - \int g(x)(1 - e^{-f_L^\lambda(x)})dx$$
$$= 1 - \frac{(\nu-1)g(\lambda)}{\nu^L}\int(1 - e^{-f(x)})dx - T_{\tilde{f}}g(\lambda)$$

where $\tilde{f}(x) = 1 - e^{-f(x)}$, and then we apply Lemma 5.3. □

LEMMA 5.5. *Let $f, g, h \geq 0$ be continuous functions with compact support with h and g continuously differentiable and g the density for some random variable X. Then*

$$\mathbf{E}\big[h(\lambda - X)e^{-f_L^\lambda(X)}\big] = (h * g)(\lambda) - \frac{(\nu-1)g(\lambda)h(0)}{\nu^L}\int(1 - e^{-f(x)})\,dx + \varepsilon$$

where $\nu^{2L}|\varepsilon| \leq \nu\big(\|g\|_\infty + \|g'\|_\infty\big)\big(\|h\|_\infty + \|h'\|_\infty\big)\int(1+\nu|x|)(1 - e^{-f(x)})dx.$

Proof. We have

$$\mathbf{E}\big[h(\lambda - X)e^{-f_L^\lambda(X)}\big] = (h * g)(\lambda) - \int g(x)h(\lambda - x)(1 - e^{-f_L^\lambda(x)})dx$$
$$= (h * g)(\lambda) - \frac{(\nu-1)g(\lambda)h(0)}{\nu^L}\int\tilde{f}(x)dx - T_{\tilde{f}}\varphi(\lambda)$$

where $\tilde{f}(x) = 1 - e^{-f(x)}$ and $\varphi(x) = g(x)h(\lambda - x)$. But

$$\|\varphi\|_\infty + \|\varphi'\|_\infty \leq \|g\|_\infty\|h\|_\infty + \|g'\|_\infty\|h\|_\infty + \|g\|_\infty\|h'\|_\infty$$
$$= \big(\|g\|_\infty + \|g'\|_\infty\big)\big(\|h\|_\infty + \|h'\|_\infty\big) - \|g'\|_\infty\|h'\|_\infty,$$

hence by Lemma 5.3,

$$\big|T_{\tilde{f}}\varphi(\lambda)\big| \leq \frac{\big(\|g\|_\infty + \|g'\|_\infty\big)\big(\|h\|_\infty + \|h'\|_\infty\big)}{\nu^{2L-1}}\int(1+\nu|x|)\tilde{f}(x)dx$$

which is our claim. □

Lemma 5.6. *Let f, g, h, and X be as in Lemma 5.5 with $h(0) = 0$. If*

$$\nu^{\gamma/5} \geq \max\left\{\nu,\ \|g\| + \|g'\|,\ \|h\| + \|h'\|,\ \int(1+\nu|x|)(1-e^{-f(x)})dx\right\},$$

and $z = \int(1-e^{-f(x)})dx$ then

$$\mathbf{E}\left[e^{-f_L^\lambda(X)}\left(1 - \frac{(\nu-1)h(\lambda-X)z}{\nu^{L-1}}\right)\right] = 1 - \frac{(\nu-1)\big[g(\lambda)+\nu(h*g)(\lambda)\big]z}{\nu^L} + \varepsilon$$

where $|\varepsilon| \leq \nu^{\gamma-2L}$.

Proof. According to Lemma 5.4, $\mathbf{E}e^{-f_L^\lambda(X)} = 1 - \nu^{-L}(\nu-1)g(\lambda)z + \varepsilon_1$ where

$$|\varepsilon_1| \leq \frac{\nu\big(\|g\|_\infty + \|g'\|_\infty\big)\int(1+\nu|x|)(1-e^{-f(x)})dx}{\nu^{2L}} \leq \frac{\nu^{3\gamma/5}}{\nu^{2L}}$$

and since $h(0) = 0$, according to Lemma 5.5,

$$\frac{(\nu-1)z}{\nu^{L-1}}\,\mathbf{E}\big[e^{-f_L^\lambda(X)}h(\lambda-X)\big] = \nu^{1-L}(\nu-1)(h*g)(\lambda)z + \varepsilon_2$$

where

$$\frac{\nu^{L-1}|\varepsilon_2|}{(\nu-1)z} \leq \frac{\nu(\|g\|_\infty + \|g'\|_\infty)(\|h\|_\infty + \|h'\|_\infty)\int(1+\nu|x|)(1-e^{-f(x)})dx}{\nu^{2L}} \leq \frac{\nu^{4\gamma/5}}{\nu^{2L}}$$

so that

$$|\varepsilon_2| \leq (\nu-1)z\nu^{1+4\gamma/5-3L} \leq (\nu-1)\nu^{1+\gamma-3L}.$$

Then

$$\mathbf{E}\left[e^{-f_L^\lambda(X)}\left(1 - \frac{(\nu-1)h(\lambda-X)z}{\nu^{L-1}}\right)\right] = 1 - \frac{(\nu-1)\big[g(\lambda)+\nu(h*g)(\lambda)\big]z}{\nu^{L-1}} + \varepsilon_1 - \varepsilon_2$$

and we have $|\varepsilon_1 - \varepsilon_2| \leq \nu^{3\gamma/5-2L} + (\nu-1)\nu^{1+\gamma-3L} \leq \nu^{\gamma-2L}$. □

Lemma 5.7. *Let f be continuous with compact support and let $A = \{x : f(x) \neq 0\}$. If M is an integer which exceeds* $\max I(\lambda)$ *and L is taken so large that $|x| \leq p_M|\mathcal{S}_L|$ for every $x \in A$, then*

$$\bigcup_{x\in A} I(\lambda + \tfrac{x}{|\mathcal{S}_L|}) \subseteq \{1, 2, \ldots, M\} \tag{5.8}$$

hence $n(\lambda + \frac{x}{|\mathcal{S}_L|}) = \sum_{r=1}^{M} \nu^{-r} g^{(r)}(\lambda + \frac{x}{|\mathcal{S}_L|})$, $\mathcal{N}_L(A_L^\lambda) = \sum_{r=1}^{M} \mathcal{N}_L^{(r)}(A_L^\lambda)$, *and*

$$\mathcal{N}_L f_L^\lambda = \sum_{r=1}^{M} \mathcal{N}_L^{(r)} f_L^\lambda \quad \textit{almost surely.} \tag{5.9}$$

Proof. Since M exceeds $\max I(\lambda)$, we have $\lambda > (1+\sigma)\lambda_M$. It means for each $r > M$,

$$\lambda + \tfrac{x}{|\mathcal{S}_L|} > (1+\sigma)(\lambda_M - p_M) = (1+\sigma)\lambda_{M+1} \geq (1+\sigma)\lambda_r$$

which establishes (5.8). It also implies $g^{(r)}(\lambda + \frac{x}{|\mathcal{S}_L|}) = 0$ for each $x \in A$ hence

$$\mathbf{P}\big(\lambda^{(r)} \in A_L^\lambda\big) = 0 \;\text{ for each } r > M$$

which establishes (5.9). □

Let $g_n^{(r)}$ be the density for

$$\lambda^{(r)} - \lambda^{(r+n)} = \xi^{(r)} + \xi^{(r+1)} + \cdots + \xi^{(r+n-1)}$$

Then $g_1^{(r)}$ is the density for $\xi^{(r)}$ so we have

$$g_{n+1}^{(r)} = g_n^{(r)} * g_1^{(r+n)} \tag{5.10}$$

and

$$g^{(r)} = g_n^{(r)} * g^{(r+n)} \tag{5.11}$$

where $g^{(r)}$ is the density for $\lambda^{(r)}$.

Now put

$$h_n = \sum_{r=1}^{n} \nu^{n-r} g^{(r)}_{n-r+1} = \nu^{n-1} g^{(1)}_n + \nu^{n-2} g^{(2)}_{n-1} + \cdots + \nu g^{(n-1)}_2 + g^{(n)}_1. \tag{5.12}$$

Notice that $h_1 = g^{(1)}_1$ and for $n > 1$, by (5.10), we obtain the recursive formula

$$h_n = \nu h_{n-1} * g^{(n)}_1 + g^{(n)}_1 = (\nu h_{n-1} + \delta) * g^{(n)}_1. \tag{5.13}$$

By (5.11), we obtain $n(\lambda)$ by convolution of $\nu^{-M} h_M(\lambda)$ with $(\nu-1)g^{(M+1)}(\lambda)$

$$n(\lambda) = \sum_{r=1}^{M} \frac{(\nu-1)g^{(r)}(\lambda)}{\nu^r} = \frac{(\nu-1)(h_M * g^{(M+1)})(\lambda)}{\nu^M} \tag{5.14}$$

where M exceeds $\max I(\lambda)$.

Let $\mathcal{F}_{\geq r}$ be the σ-algebra generated by all eigenvalues of rank at least r, i.e.,

$$\mathcal{F}_{\geq r} = \sigma\big(\lambda_Q : Q \in \mathcal{V}_r\big) \quad \text{and similarly,} \quad \mathcal{F}_{>r} = \sigma\big(\lambda_Q : Q \in \mathcal{V}_{r+1}\big).$$

Proposition 5.8. *Let $f \geq 0$ be a continuous function with compact support and let M and L be as in Lemma 5.7. Then for $2 \leq n \leq M+1$,*

$$\mathbf{E}\big(e^{-\mathcal{N}_L f^\lambda_L} \,\big|\, \mathcal{F}_{\geq n}\big) = e^{-\sum_{r=n}^{M} \mathcal{N}^{(r)}_L f^\lambda_L} \prod_{Q \in \mathcal{S}^{(n)}_L} \left(1 - \frac{z(\nu-1)h_{n-1}(\lambda - \lambda_Q)}{\nu^{L-1}}\right) + \varepsilon_n \tag{5.15}$$

where $z = \int (1 - e^{-f(x)})dx$,

$$\|\varepsilon_n\|_\infty \leq \nu^{3-L} \sum_{k=1}^{n-1} \frac{e^{\nu^{\gamma-k}}}{\nu^k} \tag{5.16}$$

and $\gamma > 6$ is chosen so large that

$$\nu^{\gamma/6} > \max\big\{\textstyle\int (1+\nu|x|)(1-e^{-f(x)})dx,\ \|g^{(n)}_1\|_\infty + \|(g^{(n)}_1)'\|_\infty,\ \|h_n\|_\infty + \|h'_n\|_\infty\big\}$$

for all $n \leq M+1$.

Proof. The proof is by induction. We will first establish (5.15) for $n = 2$. Because $\mathcal{N}_L^{(r)} f_L^\lambda$ is $\mathcal{F}_{\geq 2}$-measurable for $r \geq 2$, it follows from (5.9) that

$$\mathbf{E}\big(e^{-\mathcal{N}_L f_L^\lambda} \mid \mathcal{F}_{\geq 2}\big) = e^{-\sum_{r=2}^{M} \mathcal{N}_L^{(r)} f_L^\lambda}\, \mathbf{E}\big(e^{-\mathcal{N}_L^{(1)} f_L^\lambda} \mid \mathcal{F}_{\geq 2}\big). \tag{5.17}$$

Note also that

$$e^{-\mathcal{N}_L^{(1)} f_L^\lambda} = \prod_{Q \in \mathcal{S}_L^{(1)}} e^{-f_L^\lambda(\lambda_Q)} = \prod_{Q \in \mathcal{S}_L^{(2)}} \prod_{Q^{(1)} \subseteq Q} e^{-f_L^\lambda(\xi^{(1)} + \lambda_Q)}.$$

Since $\mathbf{E}\big(e^{-\mathcal{N}_L^{(1)} f_L^\lambda} \mid \mathcal{F}_{\geq 2}\big)$ depends only on λ_Q for $Q \in \mathcal{S}_L^{(2)}$, since the $\xi^{(1)}$'s are i.i.d. and independent of $\mathcal{F}_{\geq 2}$, and since each cube contains ν cubes of preceding rank,

$$\mathbf{E}\big(e^{-\mathcal{N}_L^{(1)} f_L^\lambda} \mid \mathcal{F}_{\geq 2}\big) = \psi_2\big(\lambda_Q : Q \in \mathcal{S}_L^{(2)}\big)$$

where for constants $\big\{\ell_Q : Q \in \mathcal{S}_L^{(2)}\big\} \subseteq \operatorname{supp}(\lambda^{(2)})$,

$$\psi_2\big(\ell_Q : Q \in \mathcal{S}_L^{(2)}\big) = \mathbf{E} \prod_{Q \in \mathcal{S}_L^{(2)}} \prod_{Q^{(1)} \subseteq Q} e^{-f_L^\lambda(\xi^{(1)} + \ell_Q)} = \prod_{Q \in \mathcal{S}_L^{(2)}} \Big(\mathbf{E} e^{-f_L^\lambda(\xi^{(1)} + \ell_Q)}\Big)^{\nu}.$$

Since $x \mapsto g_1^{(1)}(x - \ell) = h_1(x - \ell)$ is the density for $\xi^{(1)} + \ell$ and because

$$\nu\big(\|g_1^{(1)}\|_\infty + \|(g_1^{(1)})'\|_\infty\big) \int (1 + \nu|x|)(1 - e^{-f(x)})dx \leq \nu^{\gamma/2},$$

by Lemma 5.4, we see that

$$\left|\mathbf{E} e^{-f_L^\lambda(\xi^{(1)} + \ell)} - \left(1 - \frac{z(\nu - 1)h_1(\lambda - \ell)}{\nu^L}\right)\right| \leq \frac{\nu^{\gamma/2}}{\nu^{2L}}.$$

But since $\nu^{\gamma/6} \geq \max\{z, \|h_1\|_\infty\}$,

$$\left|\frac{z(\nu - 1)h_1(\lambda - \ell)}{\nu^L}\right| \leq \frac{(\nu - 1)\nu^{\gamma/3}}{\nu^L} \leq \frac{\nu^{\gamma/2}}{\nu^L}.$$

It follows by Lemma 5.2 with $c = \nu^{\gamma/2}$ that

$$\left| \psi_2\big(\lambda_Q : Q \in \mathcal{S}_L^{(2)}\big) - \prod_{Q \in \mathcal{S}_L^{(2)}} \Big(1 - \tfrac{z(\nu-1)}{\nu^L} \nu h_1(\lambda - \lambda_Q)\Big) \right| \leq \frac{e^{\nu^{\gamma-1}}}{\nu} \nu^{2-L}$$

hence taking

$$\begin{aligned} \varepsilon_2 &= \mathbf{E}\big(e^{-\mathcal{N}_L f_L^\lambda} \mid \mathcal{F}_{\geq 2}\big) - e^{-\sum\limits_{r=2}^{M} \mathcal{N}_L^{(r)} f_L^\lambda} \prod_{Q \in \mathcal{S}_L^{(2)}} \Big(1 - \tfrac{z(\nu-1)h_1(\lambda-\lambda_Q)}{\nu^{L-1}}\Big) \\ &= \left(\psi_2\big(\lambda_Q : Q \in \mathcal{S}_L^{(2)}\big) - \prod_{Q \in \mathcal{S}_L^{(2)}} \Big(1 - \tfrac{z(\nu-1)h_1(\lambda-\lambda_Q)}{\nu^{L-1}}\Big)\right) e^{-\sum\limits_{r=2}^{M} \mathcal{N}_L^{(r)} f_L^\lambda} \end{aligned}$$

and keeping in mind that $f \geq 0$, we obtain (5.15) for $n = 2$.

Now assume (5.15) has been proven for n. Observe that

$$e^{-\mathcal{N}_L^{(n)} f_L^\lambda} = \prod_{Q \in \mathcal{S}_L^{(n)}} e^{-f_L^\lambda(\lambda_Q)} = \prod_{Q \in \mathcal{S}_L^{(n+1)}} \prod_{Q^{(n)} \subseteq Q} e^{-f_L^\lambda(\xi^{(n)} + \lambda_Q)}.$$

Subtracting ε_n and then dividing (5.15) by $e^{-\mathcal{N}_L^{(r)} f_L^\lambda}$ for $r \geq n+1$, we obtain

$$\begin{aligned} \Big(\mathbf{E}\big(e^{-\mathcal{N}_L f_L^\lambda} \,\big|\, \mathcal{F}_{\geq n}\big) - \varepsilon_n\Big) e^{\sum\limits_{r=n+1}^{M} \mathcal{N}_L^{(r)} f_L^\lambda} &= e^{-\mathcal{N}_L^{(n)} f_L^\lambda} \prod_{Q \in \mathcal{S}_L^{(n)}} \Big(1 - \tfrac{z(\nu-1)h_{n-1}(\lambda-\lambda_Q)}{\nu^{L-1}}\Big) \\ &= \prod_{Q \in \mathcal{S}_L^{(n+1)}} \prod_{Q^{(n)} \subseteq Q} \Big[e^{-f_L^\lambda(\xi^{(n)} + \lambda_Q)} \Big(1 - \tfrac{z(\nu-1)h_{n-1}(\lambda-\xi^{(n)}-\lambda_Q)}{\nu^{L-1}}\Big)\Big]. \end{aligned} \tag{5.18}$$

Since $\mathcal{F}_{>n} \subseteq \mathcal{F}_{\geq n}$, we have

$$\mathbf{E}\big(e^{-\mathcal{N}_L f_L^\lambda} \mid \mathcal{F}_{>n}\big) = \mathbf{E}\big(\, \mathbf{E}\big(e^{-\mathcal{N}_L f_L^\lambda} \mid \mathcal{F}_{\geq n}\big) \mid \mathcal{F}_{>n}\big).$$

Re-conditioning the right-hand side of (5.18) on $\mathcal{F}_{>n}$, keeping in mind that each $e^{-\mathcal{N}_L^{(r)} f_L^\lambda}$ is $\mathcal{F}_{>n}$-measurable for $r \geq n+1$, we obtain

$$\mathbf{E}\big(e^{-\mathcal{N}_L f_L^\lambda} \mid \mathcal{F}_{>n}\big) = \psi_{n+1}\big(\lambda_Q : Q \in \mathcal{S}_L^{(n+1)}\big) e^{-\sum\limits_{r=n+1}^{M} \mathcal{N}_L^{(r)} f_L^\lambda} + \mathbf{E}\big(\varepsilon_n \mid \mathcal{F}_{>n}\big)$$

where

$$\psi_{n+1}\big(\ell_Q : Q \in \mathcal{S}_L^{(n+1)}\big) = \mathbf{E}\prod_{Q\in\mathcal{S}_L^{(n+1)}}\prod_{Q^{(n)}\subseteq Q}\Big[e^{-f_L^\lambda(\xi^{(n)}+\ell_Q)}\Big(1-\tfrac{z(\nu-1)h_{n-1}(\lambda-\xi^{(n)}-\ell_Q)}{\nu^{L-1}}\Big)\Big]$$
$$= \prod_{Q\in\mathcal{S}_L^{(n+1)}}\bigg(\mathbf{E}\Big[e^{-f_L^\lambda(\xi^{(n)}+\ell_Q)}\Big(1-\tfrac{z(\nu-1)h_{n-1}(\lambda-\xi^{(n)}-\ell_Q)}{\nu^{L-1}}\Big)\Big]\bigg)^{\nu}.$$

Since $x \mapsto g_1^{(n)}(x-\ell)$ is the density for $\xi^{(n)}+\ell$ and because $h_{n-1}(0)=0$, by (5.13) and Lemma 5.6,

$$\Big|\mathbf{E}\Big[e^{-f_L^\lambda(\xi^{(n)}+\ell)}\Big(1-\tfrac{z(\nu-1)h_{n-1}(\lambda-\xi^{(n)}-\ell)}{\nu^{L-1}}\Big)\Big] - \Big(1-\tfrac{z(\nu-1)h_n(\lambda-\ell)}{\nu^L}\Big)\Big| \le \nu^{\gamma-2L}$$

Furthermore, since

$$\Big|\mathbf{E}\Big[e^{-f_L^\lambda(\xi^{(n)}+\ell)}\Big(1-\tfrac{z(\nu-1)h_{n-1}(\lambda-\xi^{(n)}-\ell)}{\nu^{L-1}}\Big)\Big]\Big| \le 1+\tfrac{\nu^\gamma}{\nu^L} \quad\text{and}\quad \Big|1-\tfrac{z(\nu-1)h_n(\lambda-\ell)}{\nu^L}\Big| \le 1+\tfrac{\nu^\gamma}{\nu^L},$$

it follows by Lemma 5.2 with $c=\nu^\gamma$ that

$$\bigg|\psi_{n+1}\big(\lambda_Q : Q\in\mathcal{S}_L^{(n+1)}\big) - \prod_{Q\in\mathcal{S}_L^{(n+1)}}\Big(1-\tfrac{z(\nu-1)h_n(\lambda-\lambda_Q)}{\nu^{L-1}}\Big)\bigg| \le \frac{e^{\nu^{\gamma-n-1}}}{\nu^n}\nu^{2-L}.$$

Finally, taking

$$\varepsilon_{n+1} = \mathbf{E}\big(e^{-\mathcal{N}_L f_L^\lambda} \,\big|\, \mathcal{F}_{>n}\big) - e^{-\sum_{r=n+1}^{M}\mathcal{N}_L^{(r)}f_L^\lambda}\prod_{Q\in\mathcal{S}_L^{(n+1)}}\Big(1-\tfrac{z(\nu-1)h_n(\lambda-\lambda_Q)}{\nu^{L-1}}\Big)$$
$$= \bigg[\psi_{n+1}\big(\lambda_Q : Q\in\mathcal{S}_L^{(n+1)}\big) - \prod_{Q\in\mathcal{S}_L^{(n+1)}}\Big(1-\tfrac{z(\nu-1)h_n(\lambda-\lambda_Q)}{\nu^{L-1}}\Big)\bigg]e^{-\sum_{r=n+1}^{M}\mathcal{N}_L^{(r)}f_L^\lambda} + \mathbf{E}\big(\varepsilon_n \,\big|\, \mathcal{F}_{>n}\big)$$

and keeping in mind that $f \ge 0$, we obtain (5.15) for $n+1$. □

COROLLARY 5.9. *Let M, L and γ be as in Lemma 5.8. Then*

$$\mathbf{E}\big(e^{-\mathcal{N}_L f_L^\lambda} \,\big|\, \mathcal{F}_{>M}\big) = \prod_{Q\in\mathcal{S}_L^{(M+1)}}\Big(1-\tfrac{(\nu-1)h_M(\lambda-\lambda_Q)\int(1-e^{-f(x)})dx}{\nu^{L-1}}\Big) + O(\nu^{-L}) \quad \text{as } L\to\infty. \tag{5.19}$$

5.2 Proof of Poisson Statistics

THEOREM 5.10. *For every continuous compactly supported function $f \geq 0$,*

$$\lim_{L\to\infty} \mathbf{E} e^{-\mathcal{N}_L f_L^\lambda} = e^{-n(\lambda)\int(1-e^{-f(x)})dx}. \tag{5.20}$$

Proof. Apply Lemma 5.1 to (5.19). Writing $z = \int(1-e^{-f(x)})dx$, we see that

$$\mathbf{E}\big(e^{-\mathcal{N}_L f_L^\lambda} \,\big|\, \mathcal{F}_{>M}\big) = \exp\Bigg(-\sum_{Q\in\mathcal{S}_L^{(M+1)}} \frac{z(\nu-1)h_M(\lambda-\lambda_Q)}{\nu^{L-1}}\Bigg) + O(\nu^{-L}).$$

as $L\to\infty$. We may rewrite the sum inside the exponent as an integral with respect to the empirical measure $N_L^{(M+1)}(dx)$ of eigenvalues of rank $M+1$. We have

$$\sum_{Q\in\mathcal{S}_L^{(M+1)}} \frac{h_M(\lambda-\lambda_Q)}{\nu^{L-1}} = \frac{1}{\big|\mathcal{S}_L^{(M+1)}\big|}\sum_{Q\in\mathcal{S}_L^{(M+1)}} \frac{h_M(\lambda-\lambda_Q)}{\nu^M} = \int \frac{h_M(\lambda-x)}{\nu^M} N_L^{(M+1)}(dx)$$

so that, as $L\to\infty$,

$$\mathbf{E}\big(e^{-\mathcal{N}_L f_L^\lambda} \,\big|\, \mathcal{F}_{>M}\big) = \exp\Bigg(-\frac{z(\nu-1)}{\nu^M}\int h_M(\lambda-x)N_L^{(M+1)}(dx)\Bigg) + O(\nu^{-L}).$$

By (4.30), $N_L^{(M+1)}(dx)$ converges weakly as $L\to\infty$ to $N^{(M+1)}(dx) = g^{(M+1)}(x)dx$. Therefore, applying (5.14)

$$n(\lambda) = \frac{(\nu-1)(h_M * g^{(M+1)})(\lambda)}{\nu^M} = \frac{\nu-1}{\nu^M}\int h_M(\lambda-x)g^{(M+1)}(x)\,dx,$$

we see that

$$\lim_{L\to\infty} \mathbf{E}\big(e^{-\mathcal{N}_L f_L^\lambda} \,\big|\, \mathcal{F}_{>M}\big) = e^{-n(\lambda)z} = e^{-n(\lambda)\int(1-e^{-f(x)})dx}.$$

Taking expectations, we obtain our result. □

5.3 Statistics for Eigenvalues of Rank r

THEOREM 5.11. *$\lim_{L\to\infty} \mathbf{E}e^{-\mathcal{N}_L^{(r)} f_L^\lambda} = e^{-z(\nu-1)g^{(r)}(\lambda)/\nu^r}$ where $z = \int(1-e^{-f(x)})dx$.*

Proof. We may directly condition on $\lambda^{(L)}$. Observe that the random variables

$$\lambda_Q - \lambda^{(L)} = \lambda^{(r)} + \lambda^{(r+1)} + \cdots + \lambda^{(L-1)}, \text{ for } Q \in \mathcal{S}_L^{(r)},$$

are i.i.d. with density $g_{L-r}^{(r)}$ and independent of $\lambda^{(L)}$. Then

$$\mathbf{E}\big(e^{-\mathcal{N}_L^{(r)} f_L^\lambda} \,\big|\, \lambda^{(L)}\big) = \psi(\lambda^{(L)})$$

where for $\ell \in \operatorname{supp}(\lambda^{(L)})$,

$$\psi(\ell) = \mathbf{E}\prod_{Q\in\mathcal{S}_L^{(r)}} e^{-f_L^\lambda(\lambda_Q - \lambda^{(L)} + \ell)} = \left(\mathbf{E}e^{-f_L^\lambda(\lambda^{(r)} - \lambda^{(L)} + \ell)}\right)^{\nu^{L-r}}.$$

By Proposition 4.2, there exists a constant c such that for all x and y,

$$\big|g_n^{(r)}(x) - g_n^{(r)}(y)\big| \le \frac{c|x-y|}{(\sigma\lambda_r)^2} \tag{5.21}$$

uniformly for all $r \ge 1$ and $1 \le n \le \infty$. From this it follows that $\|g_n^{(r)}\|_\infty \le \frac{2c}{\sigma\lambda_r}$ and $\|(g_n^{(r)})'\|_\infty \le \frac{c}{(\sigma\lambda_r)^2}$ for all n. Therefore, by Lemma 5.4

$$\mathbf{E}e^{-f_L^\lambda(\lambda^{(r)} - \lambda^{(L)} + \ell)} = 1 - \frac{z(\nu-1)g_{L-r}^{(r)}(\lambda-\ell)}{\nu^L} + \varepsilon$$

where

$$|\varepsilon| \le \nu^{1-2L}\big(\tfrac{2c}{\sigma\lambda_r} + \tfrac{c}{(\sigma\lambda_r)^2}\big)\int(1+\nu|x|)(1-e^{-f(x)})dx$$

hence, applying Lemma 5.1, we see that

$$\mathbf{E}\big(e^{-\mathcal{N}_L^{(r)} f_L^\lambda} \,\big|\, \lambda^{(L)}\big) = \exp\left(-\frac{z(\nu-1)g_{L-r}^{(r)}(\lambda - \lambda^{(L)})}{\nu^r}\right) + O(\nu^{-L}) \text{ as } L \to \infty.$$

Now observe that since $g^{(r)} = g^{(r)}_{L-r} * g^{(L)}$, (5.21) implies that

$$\begin{aligned}\left|g^{(r)}_{L-r}(\lambda) - g^{(r)}(\lambda)\right| &\leq \int g^{(L)}(x)\left|g^{(r)}_{L-r}(\lambda) - g^{(r)}_{L-r}(\lambda - x)\right| dx \\ &\leq \frac{c \int x g^{(L)}(x)\, dx}{(\sigma\lambda_r)^2} = \frac{c\,\mathbf{E}\lambda^{(L)}}{(\sigma\lambda_r)^2} \leq \frac{c(1+\sigma)\lambda_L}{(\sigma\lambda_r)^2}.\end{aligned}$$

Therefore, since $\lambda_L = O(p^L)$ as $L \to \infty$, we have

$$\left|g^{(r)}_{L-r}(\lambda - \lambda^{(L)}) - g^{(r)}(\lambda)\right| \leq \frac{c\lambda^{(L)}}{(\sigma\lambda_r)^2} + \frac{c(1+\sigma)\lambda_L}{(\sigma\lambda_r)^2} \leq \frac{2c(1+\sigma)\lambda_L}{(\sigma\lambda_r)^2} \to 0 \ \text{ as } \ L \to \infty$$

hence $g^{(r)}_{L-r}(\lambda - \lambda^{(L)}) \to g^{(r)}(\lambda)$, almost surely, and we obtain our result. □

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