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Objects Invisible from Multiple Directions

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Abstract: The first proposed invisibility cloaks required materials that are highly anisotropic, spatially inhomogeneous and that possess a magnetic response. These properties are still difficult or impractical to achieve in practice, leading to many researchers to explore simplified invisibility schemes that trade perfection for simplicity in design. In this article, we investigate a traditional method by Devaney for constructing multi-angle invisibility devices, i.e. devices that are invisible for a finite number of directions of illumination in the weak scattering limit. We demonstrate that the scattering cross-section of these objects decreases dramatically as the number of invisibility directions is increased.

1. Introduction

In 2006, two papers were published back-to-back in *Science* [1, 2] showing that it is theoretically possible to make an object that is perfectly invisible for all directions of illumination, at least at a single frequency. The "cloaking devices" that were introduced in these articles formed the foundation of a new subfield of invisibility physics that remains an active area of research to this day [3, 4]. An invisible object may be broadly defined as an object that does not scatter or absorb light incident upon it; cloaking devices are a subset of invisibility devices that also exclude any fields from their interior.

The study of invisibility has led to many unusual and even practical possibilities that go beyond simply hiding things. The mathematics of transformation optics, used in developing the original cloaking devices, has been employed to study the possibility of making electromagnetic wormholes [5] and perfect lossless bends in optical fibers [6, 7]. By exploring invisibility for other types of waves, researchers have proposed using invisibility devices as protection against seismic waves [8], water waves [9] and even heat flow [10]. The existence of invisible objects, even in principle, breaks the uniqueness of the inverse scattering problem and allows for the development of devices that create perfect three-dimensional illusions [11].

It is in the context of inverse problems that the original 2006 cloaking papers were the greatest surprise, as it had long been thought that perfect invisibility was impossible even at a single frequency. This broad assumption arose from two papers that appeared by Nachman in 1988 [12] and by Habashy and Wolf in 1993 [13] that studied the uniqueness of the inverse scattering problem. Both showed that the inverse problem should be unique if the scattered field is measured for all directions of incidence and scattering, indicating that invisible objects should not exist. The resolution of this seeming contradiction comes from the recognition that the older uniqueness papers focus on "natural" materials, i.e. non-magnetic, isotropic materials, while perfect cloaks require materials that are magnetic and anisotropic.

The fabrication of materials to make a perfect cloak is extremely difficult, and many demonstrations of cloaking principles have used simplified cloak designs. For example, the first experimental microwave cloak demonstration was two-dimensional and was designed with material parameters that only approximate the ideal case [14]. Furthermore, since 2006 there have been many different cloak designs introduced that generally sacrifice perfection in exchange for ease of construction; these include so-called carpet cloaks [15], interference-based cloaks [16], and geometrical optics-based cloaks [17].

With this in mind, it is worth noting that the possibility of imperfect invisibility was considered long before the seminal cloaking papers. In 1978, Devaney demonstrated [18] that, in the context of weak scattering theory, it is possible to construct an isotropic, non-magnetic scatterer that is invisible for a finite number of directions of illumination. This result is consistent with the Habashy and Wolf result [13] that it is impossible to construct an object out of natural materials that is invisible for a continuous range of illumination directions. Devaney introduced a mathematical technique for designing weak scatterers that are invisible for any finite number of illumination directions.

With the goal of constructing imperfect multi-angle invisibility devices in mind, however, it is of interest to ask how well one can create invisibility using the Devaney approach: If the number of invisibility directions is increased, does an object come closer to perfect invisibility or does it become more strongly scattering for the non-invisibility directions? In this paper, we investigate the Devaney invisibility technique and analyze the properties of such invisible objects when the number of invisibility directions is increased.

2. Multi-Angle Invisibility and the Devaney Operator

To facilitate the construction of multi-angle invisibility devices, we take advantage of the connection between radiation and scattering problems; in particular, we review an important theorem concerning nonradiating sources, i.e. primary radiation sources that ironically produce no radiation. In their 1989 paper [19], Gamliel, Kim, Nachman, and Wolf discussed the conditions a monochromatic field $u(\mathbf{r})$ of wavenumber k would have to meet so that its corresponding source $q(\mathbf{r})$ is nonradiating; we will apply this theorem to the scattering problem and invisible objects.

Theorem 2.1 *Let $q(\mathbf{r})$ be a bounded, nonradiating source distribution of finite support, and let $u(\mathbf{r})$ be the field that $q(\mathbf{r})$ generates. Then $u(\mathbf{r})$ and $q(\mathbf{r})$ are related by the inhomogeneous Helmholtz equation*

$$(\nabla^2 + k^2)u(\mathbf{r}) = -4\pi q(\mathbf{r}), \tag{1}$$

with the boundary conditions

$$u(\mathbf{r}) = \partial_{\mathbf{n}} u(\mathbf{r}) = 0 \textit{ for } \mathbf{r} \in \partial D, \tag{2}$$

where $k > 0$ is a constant, D is the domain of our source, ∂D is the boundary of D, and $\partial_{\mathbf{n}}$ denotes differentiation along the outward normal, where $\mathbf{n}$ is the unit normal vector pointing away from ∂D.

This theorem indicates that a source will be nonradiating, and produce no field outside the source domain, if the field it generates and the normal derivative of the field vanish on the source boundary ∂D. This is consistent with the general observation from diffraction theory that a diffracted field depends on both the field and its normal derivative: If both vanish, then the diffracted field will vanish as well (see, for example, Devaney [20]).

It is to be noted that the second partial derivatives of the field $u(\mathbf{r})$ can be discontinuous, which results in a discontinuous source structure.

In Devaney's 1978 paper [18], he defined an operator that can be used to construct multi-angle invisible objects that we call the *Devaney operator*. His derivation of this operator is done in spatial Fourier space; here we present a more straightforward derivation in real space that makes clear the assumptions used.

Let us assume that we have a field $U_i(\mathbf{r})$ incident upon an object of domain D with inhomogeneous refractive index $n(\mathbf{r})$; the total field $U(\mathbf{r})$ must satisfy the Helmholtz equation with an inhomogeneous wavenumber, i.e.

$$[\nabla^2 + n^2(\mathbf{r})k^2]U(\mathbf{r}) = 0. \tag{3}$$

Following the standard approach in scattering theory, we add the quantity $[k^2 - n^2(\mathbf{r})k^2]U(\mathbf{r})$ to both sides of Eq. (3), to get

$$[\nabla^2 + k^2]U(\mathbf{r}) = -4\pi F(\mathbf{r})U(\mathbf{r}), \tag{4}$$

where we have introduced the scattering potential $F(\mathbf{r})$ as

$$F(\mathbf{r}) = \frac{k^2}{4\pi}[n^2(\mathbf{r}) - 1]. \tag{5}$$

We may simplify Eq. (4) further by noting that the total field must be a combination of the incident field $U_i(\mathbf{r})$ and the scattered field $U_s(\mathbf{r})$,

$$U(\mathbf{r}) = U_i(\mathbf{r}) + U_s(\mathbf{r}), \tag{6}$$

where the incident field is defined to satisfy the free-space Helmholtz equation, $[\nabla^2+k^2]U_i(\mathbf{r}) = 0$. This leaves us with

$$[\nabla^2 + k^2]U_s(\mathbf{r}) = -4\pi F(\mathbf{r})U(\mathbf{r}). \tag{7}$$

This equation should be compared with Eq. (1) for the radiation produced by a primary source. Theorem 2.1 indicates that, for a given incident field $U_i(\mathbf{r})$, the scattered field will be localized to the domain D and will not radiate if the function $U_s(\mathbf{r})$ and its normal derivative vanish on the boundary of D.

This approach was used by Gbur [21] to construct objects that are perfectly invisible for one particular direction of illumination. A similar approach will work for multi-angle invisibility in the case of weak scattering, where it is assumed that $U(\mathbf{r}) \approx U_i(\mathbf{r})$. Then we may write Eq. (7) as

$$[\nabla^2 + k^2]U_s(\mathbf{r}) = -4\pi F(\mathbf{r})U_i(\mathbf{r}). \tag{8}$$

There are, in principle, many ways to design an invisibility object based on Eq. (8) and the constraints of Theorem 2.1. The Devaney approach is equivalent to choosing $U_s(\mathbf{r})$ to be of the form,

$$U_s(\mathbf{r}) \equiv U_i(\mathbf{r})\Phi_1(\mathbf{r}), \tag{9}$$

where $\Phi_1(\mathbf{r})$ is a continuous function with continuous first partial derivatives, and which satisfies the continuity constraints of Theorem 2.1 on the boundary of D. Let us further take $U_i(\mathbf{r}) = U_0 \exp(ik\hat{\mathbf{s}}_1 \cdot \mathbf{r})$, where U_0 is a complex constant amplitude and $\hat{\mathbf{s}}_1$ is a unit vector indicating the direction of incidence of the plane wave. On substitution of this form of $U_s(\mathbf{r})$ into Eq. (8), we can solve for $F(\mathbf{r})$,

$$F(\mathbf{r}) = -\frac{1}{4\pi}(\nabla^2 + 2ik\hat{\mathbf{s}}_1 \cdot \nabla)\Phi_1(\mathbf{r}) \equiv -\frac{1}{4\pi}D[\hat{\mathbf{s}}_1]\Phi_1(\mathbf{r}), \tag{10}$$

where we have defined the Devaney operator as

$$D[\hat{\mathbf{s}}_1] \equiv [\nabla^2 + 2ik\hat{\mathbf{s}}_1 \cdot \nabla]. \tag{11}$$

The scattering potential given by Eq. (10) will not scatter a plane wave incident from direction $\hat{\mathbf{s}}_1$.

It is to be noted, however, that we may also introduce a new function $\Phi_2(\mathbf{r})$ that satisfies the relation,

$$\Phi_1(\mathbf{r}) = D[\hat{\mathbf{s}}_2]\Phi_2(\mathbf{r}), \tag{12}$$

where now $\Phi_2(\mathbf{r})$ must have continuous second and third partial derivatives so that $\Phi_1(\mathbf{r})$ is continuous and has a continuous first partial derivative. Substituting (12) into Eq. (10), the scattering potential of this object will be of the form,

$$F(\mathbf{r}) = -\frac{1}{4\pi} D[\hat{\mathbf{s}}_1] D[\hat{\mathbf{s}}_2] \Phi_2(\mathbf{r}). \tag{13}$$

Because the operators $D[\hat{\mathbf{s}}_1]$ and $D[\hat{\mathbf{s}}_2]$ commute, this object will be invisible for incident plane waves of directions $\hat{\mathbf{s}}_1$ and $\hat{\mathbf{s}}_2$. We can extend this process for an object that is invisible for any finite number of incident plane wave directions,

$$F(\mathbf{r}) = -\frac{1}{4\pi} D[\hat{\mathbf{s}}_1] D[\hat{\mathbf{s}}_2] \cdots D[\hat{\mathbf{s}}_N] \Phi_N(\mathbf{r}), \tag{14}$$

which will be invisible for N directions provided $\Phi_N(\mathbf{r})$ has continuous partial derivatives at least up to $2N-1$ and the function and those derivatives vanish on the boundary.

With the Devaney operator, we can construct multi-angle invisible objects, and we explore their properties in the next section. It is to be noted that Devaney never explicitly constructed objects using this method, so our work represents the first detailed study of them.

3. The Design of Multi-Angle Invisibility Objects

To construct an example of a multi-angle invisibility object, we need to make a specific choice of the function $\Phi_N(\mathbf{r})$. It is taken to satisfy the expression $\Phi_N(\mathbf{r}) = v(r) + iv(r)$, where

$$v(r) = \begin{cases} f(r) & r \in (0, r_0), \\ 0 & \text{otherwise}, \end{cases} \tag{15}$$

where r_0 is the radius of our object. The specific choice of equal real and imaginary parts $v(r)$ for the function $\Phi_N(\mathbf{r})$ was guided by Gbur [21], who found that this balance significantly reduced the imaginary part of the resulting scattering potential. The imaginary part is associated with gain and loss and is in practice much more difficult to create experimentally, so minimizing it is highly desirable.

We consider two sets of invisible objects, one based on polynomials and the other based on the infinitely continuously differential bump function. For the polynomial objects, $f(r)$ is a polynomial of the form

$$f(r) = \sum_{j=0}^{m+1} c_{n+2j} r^{n+2j} \tag{16}$$

with $m+2$ terms, where $n \in \mathbb{N}_0$. To determine the coefficients c_{n+2j}, we set up and solve a system of equations. Let us suppose that we want $\Phi_N(\mathbf{r})$ to be m times continuously differentiable, we need $v(r)$ to also be m times continuously differentiable. Thus, we will need $f(r)$ to satisfy

$$f(r_0) = f'(r_0) = \cdots = f^{(m)}(r_0) = 0. \tag{17}$$

Then, setting up a system of $m+1$ equations,

$$\begin{aligned} f(r_0) &= 0, \\ f'(r_0) &= 0, \\ &\vdots \\ f^{(m)}(r_0) &= 0, \end{aligned} \tag{18}$$

we solve for our coefficients $c_n, \ldots, c_{n+2m}$, where the last coefficient $c_{n+2(m+1)}$ can be freely chosen and we take as $c_{n+2(m+1)} = 1$. For an object with N directions of invisibility, we need $m \geq 2N - 1$.

A second option for $f(r)$ is to take a function that is already infinitely differentiable and has every derivative go to zero at $r = r_0$; we may use the so-called bump function used in the theory of distributions, defined as

$$f(r) = \exp\left[\frac{1}{(r/r_0)^2 - 1}\right]. \tag{19}$$

This function is convenient because we do not need to solve a system of equations and can make an object that is invisible from a large number of directions in a straightforward manner.

One significant concern with our approach is that we need to avoid an "apples vs. oranges" comparison of objects with an increasing number of invisibility directions. The Devaney operator will change the refractive index profile of an object and may reduce the overall amplitude of the scattering potential, so an overall downward trend in the scattered power may simply be the result of the scattering potential itself going to zero—the object itself disappearing. Conversely, an upward trend in the scattered power may represent an overall increase in the refractive index. To avoid these cases, we normalize our scattering potentials after applying the Devaney operators, i.e. we introduce a normalized potential $F_{\text{norm}}(\mathbf{r})$ such that

$$F_{\text{norm}}(\mathbf{r}) = \frac{F(\mathbf{r})}{\|F(\mathbf{r})\|_{L^1(R)}}, \tag{20}$$

where we use the L^1 norm

$$\|F(\mathbf{r})\|_{L^1(R)} = \int_R |F(\mathbf{r})|\, d^2\mathbf{r}. \tag{21}$$

All of our objects are therefore taken to have $\|F_{\text{norm}}(\mathbf{r})\|_{L^1(R)} = 1$. Because we are working in the weak scattering regime, there is a linear relationship between the scattering potential and the scattered power and the overall norm of the scattering potential does not affect the physics of the scattering process.

When the object is illuminated from one of the invisibility directions $\hat{\mathbf{s}}_i$, with $i \in \{1, \ldots, N\}$, the scattered field can readily be found to be of the form,

$$U_s(\mathbf{r}) = D[\hat{\mathbf{s}}_1]D[\hat{\mathbf{s}}_2]\cdots\cancel{D[\hat{\mathbf{s}}_i]}\cdots D[\hat{\mathbf{s}}_N]\Phi_N(\mathbf{r}), \tag{22}$$

where the ith Devaney operator is removed. For any other directions of incidence, however, the scattered field must be calculated numerically. We consider a two-dimensional scattering geometry for simplicity; then the scattered field can be found by the Green's function formula,

$$U_s(\mathbf{r}) = \int_R i\pi H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|)F_{\text{norm}}(\mathbf{r}')U_i(\mathbf{r}')\, d^2\mathbf{r}', \tag{23}$$

where $H_0^{(1)}$ is the Hankel function of the first kind.

Of particular interest will be the total power P scattered by the object as a function of the direction of incidence; for a two-dimensional geometry, this direction can be characterized by an angle θ_i with respect to the x-axis in the xy-plane, where $\theta_i \in \{0, 2\pi\}$. For each direction of incidence, we calculate the scattered field $U_s(\mathbf{r}) = U_s(r, \theta)$ in polar coordinates, and then calculate the total power radiated by our scattered field for a given incident angle θ_i according to the formula

$$P = \int_0^{2\pi} r|U_s(r, \theta')|^2\, d\theta' \tag{24}$$

for a fixed value of r. We expect that the total scattered power will be $P = 0$ for incidence along an invisibility direction and nonzero along any other direction.

4. Simulations of Multi-Angle Invisibility Objects

For our simulations, we consider two-dimensional objects ranging from one to six directions of invisibility, both of the polynomial and bump function types. In each case, "Object 1" is invisible from one direction, "Object 2" is invisible from two directions, and so forth. For all cases, we take the incident wave amplitude $U_0 = 1$, the radius of the scatterer to be $r_0 = 1$, and the wavelength to be $\lambda = 1$ (arbitrary units). We study the scattered field within a region $R = [-2, 2] \times [-2, 2]$. For the scattering objects constructed from polynomials, we take $n = 0$ in Eq. (16) and design the function $f(r)$ to be 11 times continuously differentiable, i.e. $m = 11$. Each of the invisible objects comes from applying the Devaney operator to the same function $\Phi_N(\mathbf{r})$, i.e. if we want an object invisible from three directions we apply the Devaney operator 3 times.

For Object 1, the direction of invisibility is taken to be $\theta = 0°$. For objects with more directions of invisibility, the directions are evenly spaced so that the angle between them is $360°/N$, where N denotes the number of directions from which a particular object is invisible.

To construct the scattering potentials for the polynomial objects, we construct our system of equations (18) and solve the system to find the polynomial coefficients for $f(r)$ and consequently $v(r)$ and $\Phi_N(\mathbf{r})$. We then apply the Devaney operator for each desired direction of invisibility to construct $F(\mathbf{r})$ following Eq. (14). For our examples, the gradient and Laplacian operations were numerically evaluated using MATLAB's `grad` and `del2` functions, respectively. Finally, the scattering potential is normalized according to Eq. (20).

The process for bump function objects is even more straightforward: the Devaney operator is applied to the bump function for each desired direction of invisibility, and then the resulting scattering potential is normalized.

Let us first consider the properties of the polynomial objects. Figure 1 shows the real and imaginary parts of the normalized scattering potential for objects with an odd number of invisibility directions. For one direction of invisibility, the object is highly asymmetric, but as the number of directions is increased, the scattering potential quickly starts to appear rotationally symmetric. The objects possess a significant imaginary part, which means that these objects inherently provide gain and loss for the illuminating field. Several years ago, Hurwitz and Gbur [22] noted that objects designed to be invisible for a single direction typically possess parity-time (PT) symmetry in their scattering potentials, and our results agree with that observation. More recently, Krešić et al. [23] derived similar non-Hermitian invisibility objects using a nonconformal coordinate transformation.

For each polynomial object, we then numerically calculated the scattered field using Eq. (23) and the total scattered power by Eq. (24) as a function of incident angle. Figure 2 shows the real part of the total field $U(\mathbf{r}) = U_i(\mathbf{r}) + U_s(\mathbf{r})$ for polynomial Object 3 for an invisibility direction and a non-invisibility direction. As expected, the field outside the object in the former case is simply the incident field, while in the latter case one can see significant distortion of the total field.

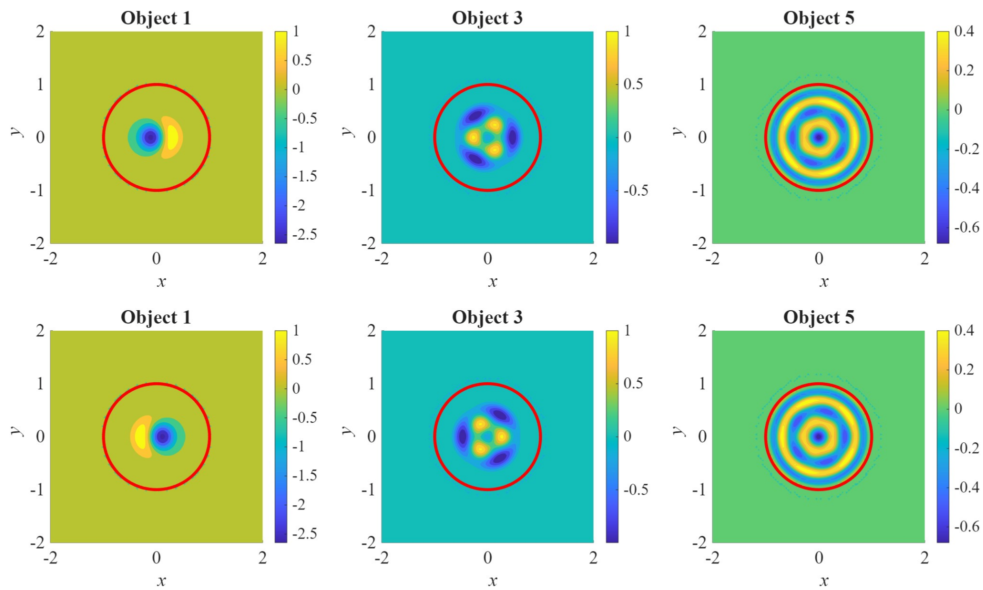

Fig. 1. The graphs of $\Re[F_{\text{norm}}(\mathbf{r})]$ (top) and $\Im[F_{\text{norm}}(\mathbf{r})]$ (bottom) of the scattering potentials of our odd numbered polynomial objects. The red circle indicates the bounds of the invisible object.

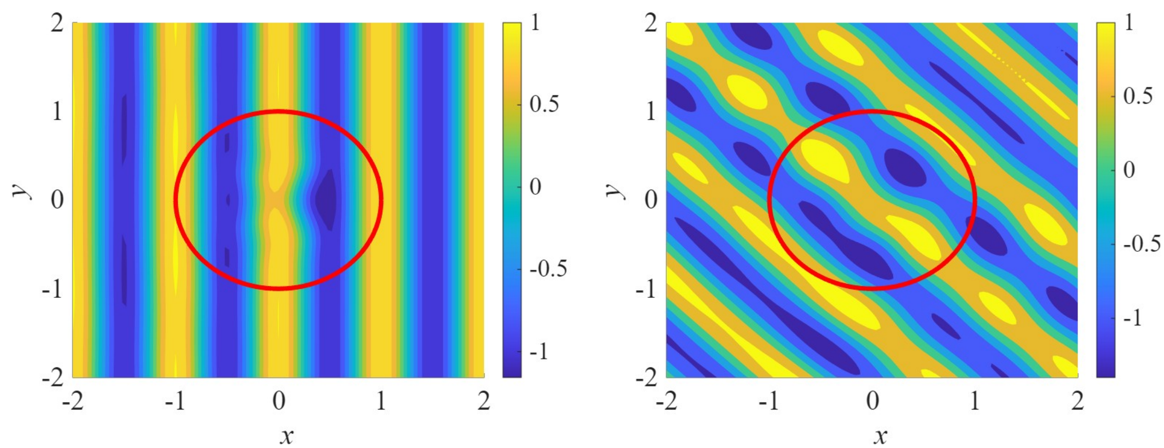

Fig. 2. The real part of the total field $U(\mathbf{r}) = U_i(\mathbf{r}) + U_s(\mathbf{r})$ for Object 3 of the polynomial objects, at angles $\theta = 0°$ and $\theta = 45°$.

Figure 3 shows $P(\theta)$ for each of our six polynomial objects as a function of incident angle θ; it can be seen that the scattered power is effectively zero at each of the designed invisibility directions. It is to be noted that the power does not go completely to zero, which we attribute to the effects of our discrete computation. It can be seen that the scattered power increases away from the invisibility direction, though it will remain low for a finite angular range around that direction.

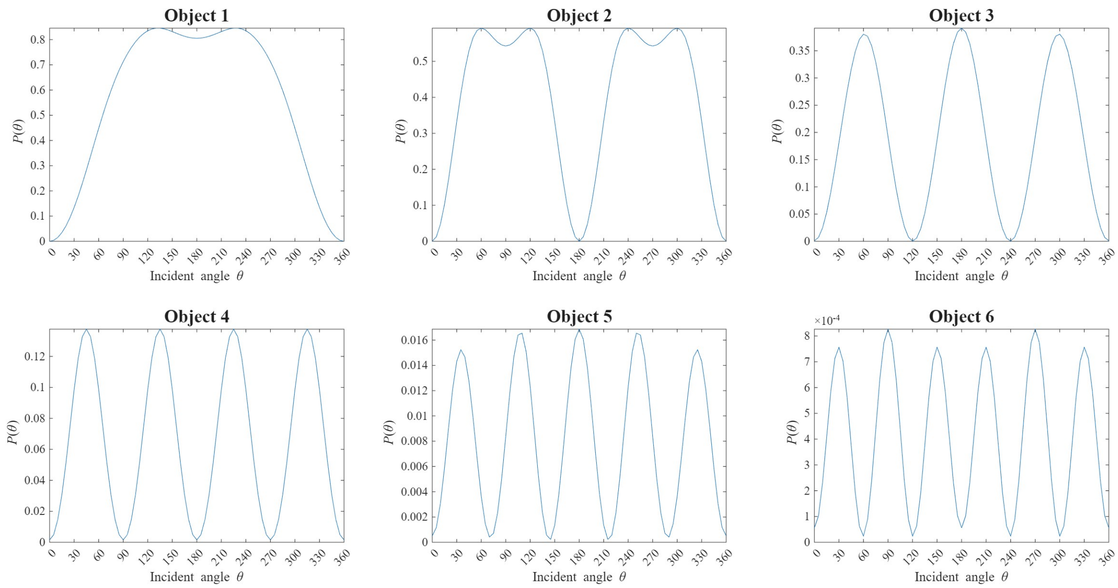

Fig. 3. The total scattered power $P(\theta)$ as a function of incident angle θ for each of our six polynomial objects.

Looking at the vertical axis scales in Fig. 3, it seems clear that the total scattered power drops dramatically as we increase the number of invisibility directions; the object becomes more globally invisible.

We may perform similar calculations for the bump function objects. Figure 4 shows the real and imaginary parts of the normalized scattering potential for objects with an odd number of invisibility directions. As in the polynomial case, the objects show balanced gain and loss. It is to be noted that, with an increase in the number of invisibility directions, the center of the scatterer appears to become effectively hollow; we will comment further on this momentarily.

The total scattered power as a function of incident angle for the first six bump function objects is shown in Fig. 5. Again, we can see that the object is, as predicted, invisible for the designed incident directions. As in the polynomial object case, we can see that the trend is for the overall scattered power to decrease as the number of invisibility directions is increased.

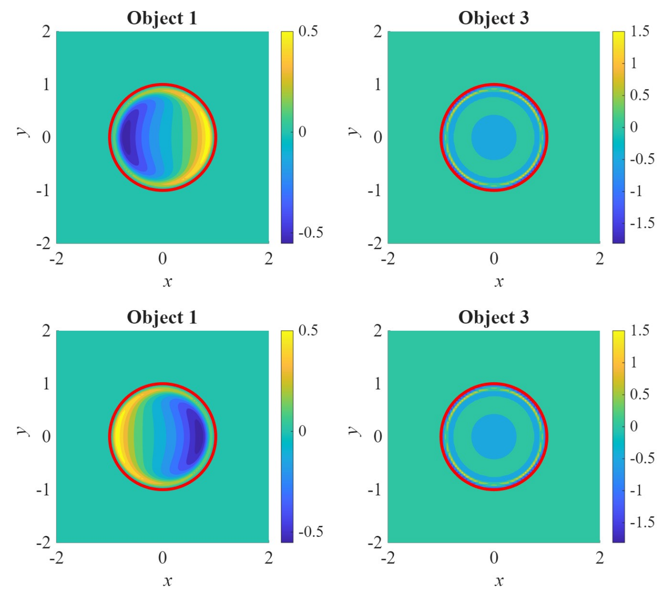

Fig. 4. The graphs of $\Re[F_{\text{norm}}(\mathbf{r})]$ (top) and $\Im[F_{\text{norm}}(\mathbf{r})]$ (bottom) of the scattering potentials of our odd numbered bump function objects. The red circle indicates the bounds of the invisible object.

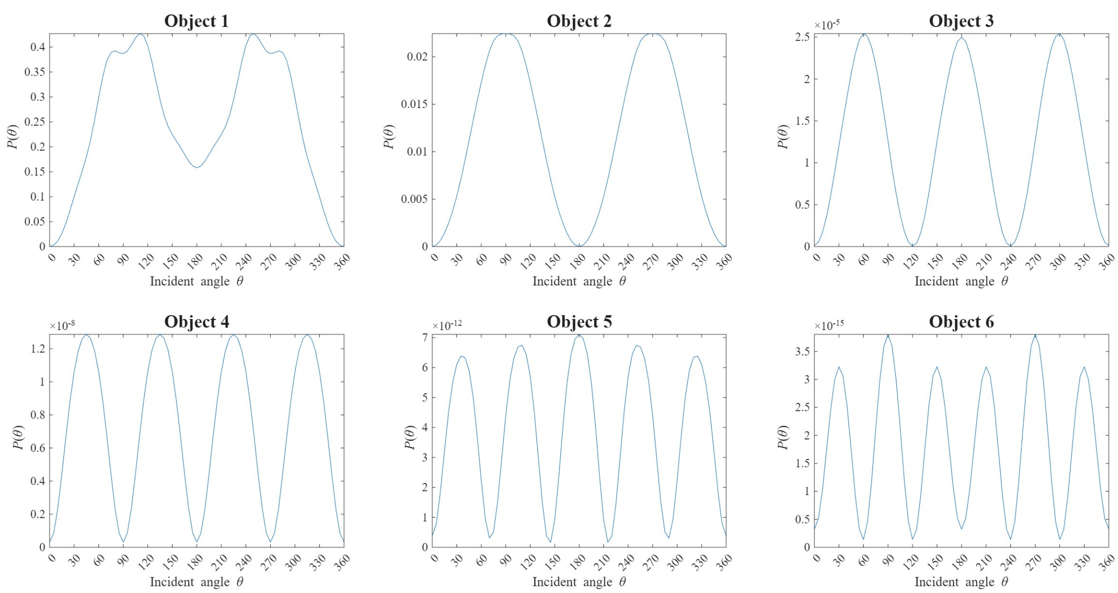

Fig. 5. The total scattered power $P(\theta)$ as a function of incident angle θ for each of our six bump function objects.

This trend of decreasing power with increasing number of invisibility directions can be illustrated by defining $P_{\max}(N)$ as the maximum scattered power for an object with N invisibility directions. In Fig. 6, we plot $P_{\max}(N)$ as a function of N for all six of the polynomial and bump function objects. It is clear that the maximum power scattered decreases rapidly as the number of invisibility directions increases. The effect is more pronounced for the bump function objects than it is for the polynomial objects. It appears that the Devaney approach, though it never produces an object perfectly invisible from all directions, will produce objects that are quite invisible in practice.

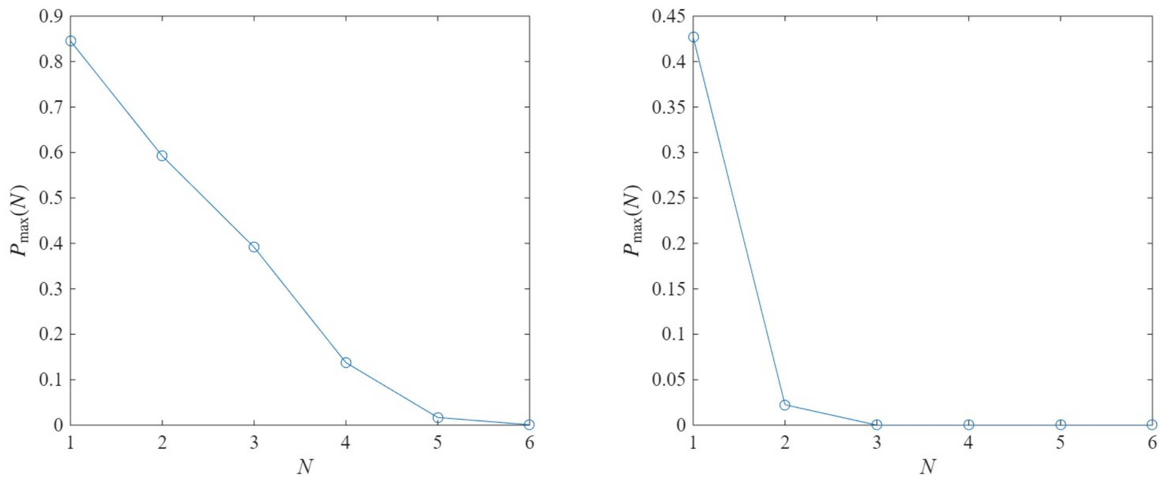

Fig. 6. The graph of $P_{max}(N)$ against N for all six of our objects. The graph on the left is for the polynomial objects, and the graph on the right is for the bump function objects. Note how for both the polynomial objects and the bump function objects $P_{max}(N)$ decreases as N increases.

We noted that Fig. 4 suggests that the bump function objects become effectively hollow as the number of invisibility directions increases. It was natural to investigate whether this means that the objects start to behave like true invisibility cloaks, with the total field being excluded from a central region and zero within. Figure 7 shows $|U_s(\mathbf{r})|^2$ for all six of our bump function objects for an incident angle of $\theta = 0°$, and it is to be noted that the intensity of the scattered field becomes much less than $|U_0|^2 = 1$. The only way to get a zero field in the central region is for the incident and the scattered fields to cancel within, and it is clear that the scattered field does not behave this way.

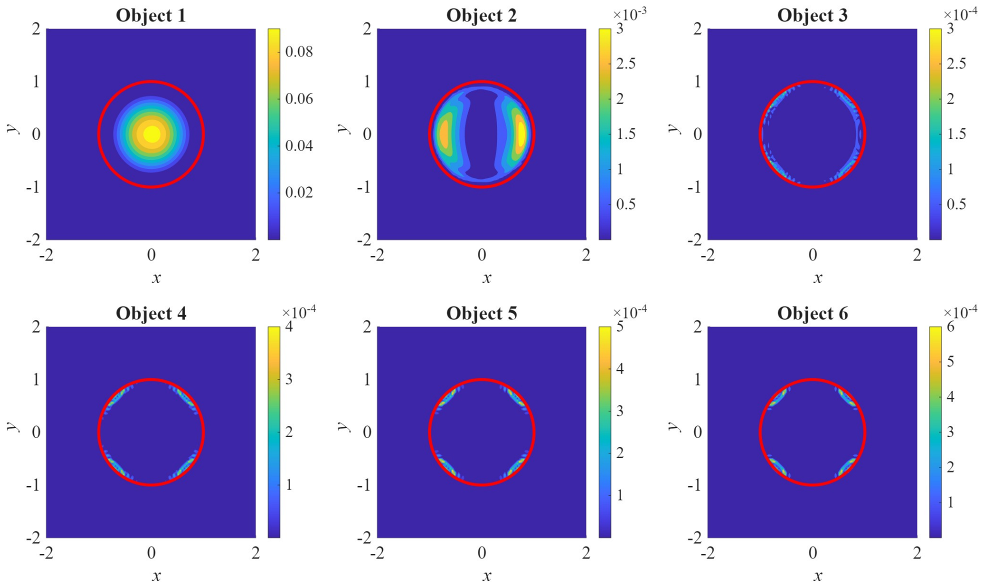

Fig. 7. The graphs of $|U_s(\mathbf{r})|^2$ for all six of our bump function objects, with the incident wave coming in at the invisibility angle $\theta = 0°$.

5. Conclusions

Traditional "perfect" invisibility cloaks rely on the use of anisotropic materials with both an electric and magnetic response, materials that are to this day difficult to artificially fabricate. The objective of our paper was to see if is it possible to construct objects out of simpler materials that are highly invisible even if they are not theoretically perfect. Using a construction first introduced by Devaney to demonstrate the possibility of multi-angle invisible objects under weak scattering conditions, we have shown that the Devaney objects become increasingly invisible as the number of invisibility directions are increased.

These results leave open a number of potential follow-up investigations. First, it is reasonable to look at whether Devaney's multi-angle invisibility approach can be extended beyond the weak scattering approximation. Furthermore, we have noted that our objects possess significant gain and loss, which is also a feature that would be difficult to fabricate in practice; it is worthwhile to explore whether Devaney's approach can be modified to create objects for which $F(\mathbf{r})$ is completely real-valued. Finally, it is worth exploring whether the Devaney approach, beyond weak scattering, can be used to design multi-angle invisibility cloaks that completely exclude fields from a central region.

We have noted that there has been a broad trend in invisibility research to trade "perfection" for "simplicity," and our work with the Devaney operator shows that there are still more avenues to be explored.

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