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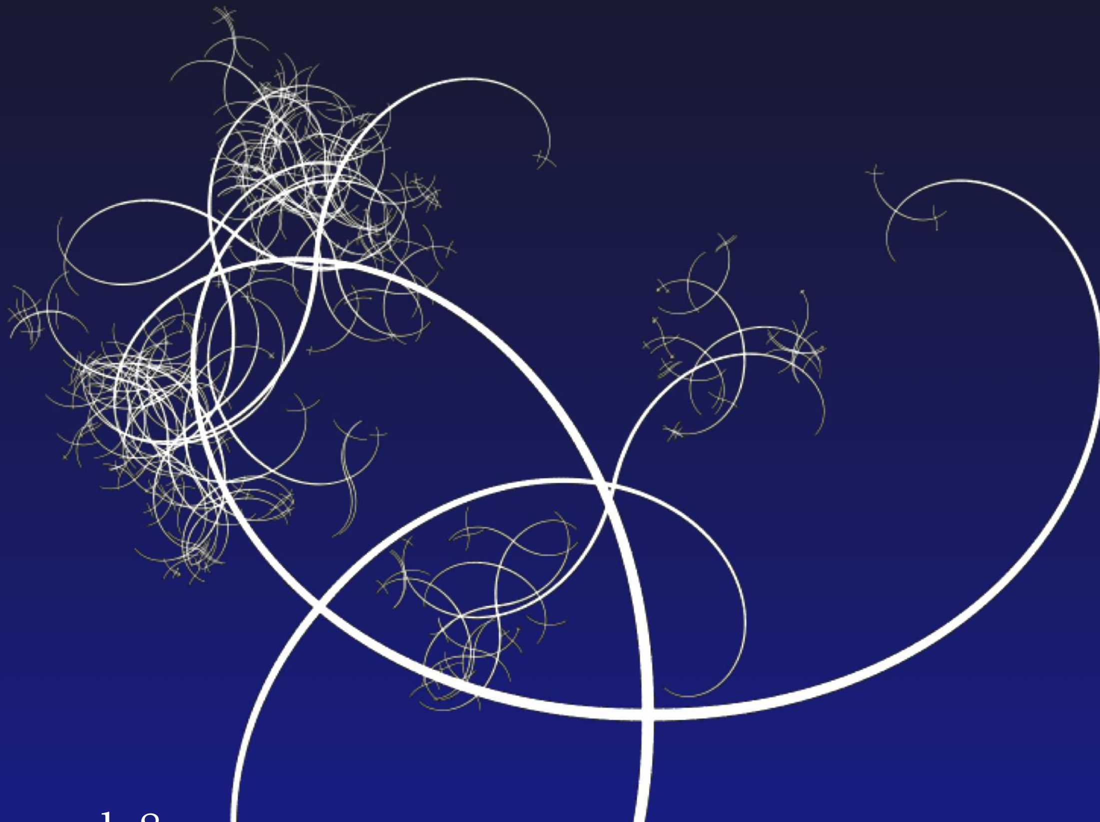

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MARCELO EPSTEIN AND RONALD E. SMELSER

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AN APPRECIATION AND DISCUSSION OF PAUL GERMAIN'S "THE METHOD OF VIRTUAL POWER IN THE MECHANICS OF CONTINUOUS MEDIA I: SECOND-GRADIENT THEORY"

MARCELO EPSTEIN AND RONALD E. SMELSER

Paul Germain's 1973 article on the method of virtual power in continuum mechanics has had an enormous impact on the modern development of the discipline. In this article we examine the historical context of the ideas it contains and discuss their continuing importance. Our English translation of the French original appears elsewhere in this volume (*MEMOCS* **8**:2 (2020), 153–190).

Introduction

Among the many contributions of Paul Germain (1920–2009) to mechanics, this classical 1973 article [1973a] on the method of virtual power in continuum mechanics stands out for its enormous impact on the modern development of the discipline, as evidenced by hundreds of citations and by its direct or indirect influence in establishing a paradigm of thought for succeeding generations. In this article we examine the historical antecedents of the ideas contained in the article and discuss their continuing relevance.

The article was published in French in the *Journal de Mécanique*. It was soon followed by a sequel [Germain 1973b], written in English and dealing with media possessing microstructure. To the best of our knowledge, the original paper had not previously been translated into English, a gap we have sought to remedy by providing our own translation in this issue [Germain 2020].

This note is organized around three of the key ideas found in Germain's article: torsors, the rigidification axiom, and duality. We conclude with a brief discussion of the language in the paper and some of our terminological choices made in its translation.

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Screws or torsors

It may be hard to believe these days, when a standard first-year course in engineering statics counts couples along with forces as fundamental pillars of the discipline, that the very notion of a couple was not introduced formally or otherwise until the year 1803. It was the genius of Louis Poinsot (1777–1859) that conceived, baptized, and formalized this idea at a young age in his *Éleméns de Statique* [Poinsot 1803] and extended it further to kinematical and dynamical theories in later works.[1] Among Poinsot's important statements, we may mention the proof that every system of forces and couples in space can be reduced to a statically equivalent system consisting of a single resultant force through any given point and a single resultant couple. Moreover, a judicious choice of a particular line reduces the system to a force along it and a couple about it (that is, the couple can be represented as a pair of forces on a plane perpendicular to this line). In the English terminology, this contraption is known as a "wrench". Earlier, the Italian mathematician Giulio Giuseppe Mozzi (1730–1813) had proved [1763] that every rigid-body motion can be represented as a "twist", that is, a combination of a translation along a line and a rotation about this line, a result often attributed to Michel Chasles (1793–1880).

Although controversial at first, Poinsot's ideas slowly gained acceptance and were supported, expanded, and promulgated by figures such as Alfred Ferdinand Möbius (1790–1868), Julius Plücker (1801–1868), Felix Klein (1849–1925), Edward John Routh (1831–1907), Robert Stawell Ball (1840–1913), Eduard Study (1862–1930), and Richard von Mises (1883–1953), all of whom helped to generalize the original concept in various physical and mathematical directions. It may have been Plücker who first proposed to consider a single hybrid entity encompassing forces and moments, an entity which he called "dyname", a six-dimensional vector whose first three components represent the force, while the last three represent the couple.[2] With the structure of $\mathbb{R}^3$ in the background, certain additional peculiar operations can be defined, inspired clearly by the original idea in statics. This concept entered the English language as "screw", a term used by Ball in the title of his original treatise [1900] dedicated exclusively to this topic.[3] In his monumental five-volume treatise on rational mechanics, Paul Émile Appell (1855–1930),

[1] A delightful historical account of Poinsot and his times is given in [Grattan-Guinness 2014].

[2] Plücker introduced early on the concept of six coordinates (only four of which are independent) to describe the collection of lines in space. See, e.g., [Plücker 1846]. His mature views on analytic geometry are collected in a later treatise, which has been translated into English [Plücker 1868]. The introductory chapter makes reference to the dyname as an example of his geometric system, although a dyname requires two extra parameters to convey the magnitudes of the force and the moment. An English summary by Plücker himself on his approach can be found in [Plücker 1865; 1866].

[3] Published earlier as [Ball 1876]. In extending the ideas of Plücker, von Mises created his Motorrechnung or "motor calculus". A screw is a motor represented by two collinear vectors.

following Ball, translates it into French as "torseur" [Appell 1902]. The English term "torsor" is reserved for a more abstract concept in algebraic geometry.

Definition. Let $\mathbb{E}^3$ denote the Cartesian affine space of $\mathbb{R}^3$. A *screw* (or *torsor*, or *motor*) is a vector field $\boldsymbol{v}$ over $\mathbb{E}^3$ such that there exists a fixed skew-symmetric matrix $\boldsymbol{W} \in \mathbb{R}^3 \times \mathbb{R}^3$ with the property

$$\boldsymbol{v}_q = \boldsymbol{v}_p + \boldsymbol{W}(q - p) \quad \text{for all } p, q \in \mathbb{E}^3. \tag{1}$$

Given a screw, its *core matrix* $\boldsymbol{W}$ is unique, as can be easily proved by assuming the contrary and using the definition. In $\mathbb{R}^3$ with the standard orientation every skew-symmetric matrix $\boldsymbol{W}$ can be equivalently represented by a vector, say $\boldsymbol{w}$, such that for all vectors $\boldsymbol{u}$ the identity

$$\boldsymbol{W}\boldsymbol{u} = \boldsymbol{w} \wedge \boldsymbol{u} \tag{2}$$

is satisfied, where, following the French engineering tradition, a wedge denotes the usual cross product of vectors.[4] Thus, Equation (1) can be replaced by

$$\boldsymbol{v}_q = \boldsymbol{v}_p + \boldsymbol{w} \wedge (q - p). \tag{3}$$

The relation between the components W_{ij} of $\boldsymbol{W}$ and the components w_k of the *core vector* $\boldsymbol{w}$ is

$$w_k = -\tfrac{1}{2}\varepsilon_{kij} W_{ij}, \tag{4}$$

where ε_{kij} are the components of the Levi-Civita alternating symbol.

Equiprojectivity. As a vector field, a screw satisfies the property of *equiprojectivity*. Its name derives from the fact that, due to the skew-symmetry of $\boldsymbol{W}$, (1) implies that

$$(\boldsymbol{v}_q - \boldsymbol{v}_p) \cdot (q - p) = 0, \tag{5}$$

where a dot is used for the ordinary Cartesian inner product. It follows that any two vectors $\boldsymbol{v}_p$ and $\boldsymbol{v}_q$ have the same projection on the line determined by their points of application, namely, $\boldsymbol{v}_q \cdot (q - p) = \boldsymbol{v}_p \cdot (q - p)$. The converse of this statement is known as the *theorem of Delassus*.[5] It states that every equiprojective vector field is a screw. From the viewpoint of equiprojectivity, it is a straightforward matter to formulate a theory of screws in infinite-dimensional Hilbert spaces, a generalization that so far does not seem to have a direct bearing on continuum mechanics.

[4] In $\mathbb{R}^3$ there is a definite relation between the cross product of vectors and the Grassmann or wedge product of multivectors.

[5] Named after Étienne Delassus (1868–1926), a French mathematician who made important contributions to the theory of partial differential equations. The "theorem" appears in the first few pages of his book [Delassus 1913]. Delassus uses the terminology of *fields of moments* for screws and *special fields* for equiprojective fields, and then proves their equivalence. The book contains interesting contributions to the theory and applications of nonholonomic constraints.

The elements of reduction of a screw at a point. It follows from the definition that to completely describe a screw it is sufficient to specify its core vector $\boldsymbol{w}$ and its value $\boldsymbol{v}_p$ at an arbitrary point p. Put differently, a screw is completely defined by the pair $(\boldsymbol{w}, \boldsymbol{v}_p)$, denoted also by $\{\mathcal{W}\}_p$, whose entries are *the elements of reduction of the screw* $\boldsymbol{v}$ *at the point* p. This observation should suffice to convince ourselves that the collection $\mathcal{W}$ of all possible screws constitutes a six-dimensional real vector space, where vector addition and multiplication by a scalar are defined in the obvious way. As such, we can define its *dual space* $\mathcal{W}^*$ consisting of all the scalar-valued linear operators (or *forms*) on $\mathcal{W}$. An element of reduction in the dual space will be denoted with square (rather than curly) brackets, such as $[\mathcal{T}]_p$.

The two fundamental examples. The first fundamental example of a screw space (or torsor space) in mechanics is the space of *twists* (or *kinematic torsors*, or *distributors*), namely, the space $\mathcal{C}$ of rigid-body velocity fields. Indeed, choosing any point $p \in \mathbb{E}^3$ such a field is represented by

$$\boldsymbol{v}_x = \boldsymbol{v}_p + \boldsymbol{\omega} \wedge (x - p) \quad \text{for all } x \in \mathbb{E}^3, \tag{6}$$

where $\boldsymbol{\omega}$ is the *angular velocity vector*, which, incidentally, is the core vector of the twist.

The second fundamental example of relevance to mechanics is the space $\mathcal{T}$ of *wrenches* (or *static torsors*), each of whose elements is a field of moments of any system of forces and couples. We know, since Poinsot's pioneering work, that this field can be represented as

$$\boldsymbol{m}_x = \boldsymbol{m}_p + \boldsymbol{f} \wedge (x - p), \tag{7}$$

where $\boldsymbol{f}$ is the force *resultant* of the system, which turns out to be the core of the wrench.

Duality and inner product. In a Lagrangian mechanics framework, generalized forces at a configuration are elements of the dual of the tangent space of the configuration manifold at the point representing the configuration. The evaluation of a force (a *covector*) on a tangent vector ("virtual velocity") is interpreted as a *virtual power*. According to this mental paradigm, therefore, the space of wrenches should be regarded as the dual space of the space of twists. If the space of twists were to be endowed with an *inner product*, each covector (a wrench) would be naturally identified with a vector (a twist), and the action of the former on the latter would consist of their inner product.

In the case of a rigid body, however, we know exactly what the natural inner product should be. Indeed, the virtual power $\mathcal{P}$ of a system of forces (defined, say, by $[\mathcal{T}]_p$) on a field of rigid-body virtual velocities (defined by $\{\mathcal{C}\}_p$) is given by

$$\mathcal{P} = [\mathcal{T}]_p \cdot \{\mathcal{C}\}_p = \boldsymbol{f} \cdot \boldsymbol{v}_p + \boldsymbol{m}_p \cdot \boldsymbol{\omega}. \tag{8}$$

On the right-hand side of this equation, we are using the ordinary dot product of $\mathbb{R}^3$, while the middle part of the equation introduces the desired inner product in the six-dimensional space of twists. We remark that this definition is consistent in the sense that, as can be easily verified, the result is independent of the point of reduction p chosen. Note that the core of one screw acts on the field element of the other, and vice versa. This operation was formalized by von Mises, who call it a *scalar product*. He also introduced a generalized cross product, called the *motor product*, and discussed the physical meaning of both operations [von Mises 1924a].[6]

The rigidification axiom

Another important concept found in Germain's work is the *axiom of rigidification*, which, just as the concept of a wrench, echoes back to older traditions. Its earliest manifestation is the *principle of solidification* usually attributed to the Flemish scientist Simon Stevin (circa 1548–1620). In general terms, this principle establishes that the state of equilibrium of a (deformable) body is not altered if any part of it is replaced by a rigid body of the same geometry. Without providing an explicit statement of the principle, Stevin used it in his work on the equilibrium of fluids at rest [1586].[7] Clairaut, Euler, Poinsot, and other important scientists made use of this principle in their treatment of equilibrium of continuous media [Truesdell 1968].[8]

Within the framework of the principle of virtual power, a crucial role is played by the axiom of rigidification in the following form. The virtual power of the *internal forces* vanishes for all rigid-body virtual velocity fields. When this axiom is attached to the principle of virtual power, the equations of equilibrium (or motion) of a continuum, including the balance of angular momentum, are obtained directly. An interesting parallel can be drawn between this axiom and Walter Noll's axiom of *objectivity* [1963] or the equivalent derivation of the laws of continuum mechanics from the invariance of an energy equation under superposed rigid-body motions [Green and Rivlin 1964].

Since the principle of virtual power postulates an *identity*, valid for *all* possible virtual velocity fields,[9] all conclusions obtained from the application of this

[6] A follow-up paper appeared in the same volume, and has been translated into English [von Mises 1924b].

[7] For a thorough historical treatment, see [Casey 1992].

[8] A historical account in connection with virtual power can be found in [Capecchi 2012].

[9] The admissible virtual velocity fields in a particular theory are, naturally, restricted to a given functional space. Nevertheless, it is always assumed that this space contains the space of twists $\mathcal{C}$.

principle are legitimate for the system under consideration.[10] Consider, for example, the wrench $[\mathcal{T}_{\text{int}}]$ of the internal forces integrated over any subbody. Since, by the axiom of rigidification, the virtual power of this wrench must vanish identically for all twists, we conclude that $[\mathcal{T}_{\text{int}}]$ must vanish identically, whether the body be deformable or not. This fact is usually interpreted as a manifestation of Newton's third law (of action and reaction). The condition of equilibrium is obtained, therefore, as the vanishing of the wrench $[\mathcal{T}_{\text{ext}}]$ of the external forces.

Duality

The mystical, religious, and philosophical appeal of the notion of *duality* is as old as recorded history and need not be considered here. It is, however, interesting to note that Isaac Newton (1642–1727) himself devoted much of his creative energies to hermetic writings, including a translation of the so-called Emerald Tablet in which Newton finds that, "That which is below is like that which is above and that which is above is like that which is below to do the miracles of one only thing."[11] In short, the idea that there is some kind of automatic correspondence of concepts at two complementary levels of discourse entered more or less explicitly into a scientific (and prescientific) description of the universe. Already Archimedes (circa 287–212 BCE) envisioned the law of the lever as some kind of compensating effect between forces and virtual displacements to produce the vanishing of the "one only thing", which is virtual work.

In mathematical terms, a finite-dimensional vector space U automatically implies the existence of a dual space U^* of the same dimension, consisting of all the scalar-valued linear functions on U. Moreover, there exists a canonical isomorphism between the dual $(U^*)^*$ of U^* and the original U, as can be shown without difficulty. We have, in fact, already considered above the example of the duality between the six-dimensional vector spaces of wrenches and twists mutually involved in the production of virtual power.

In the Lagrangian conception of classical mechanics, the configuration space of a system with a finite number of degrees of freedom is a finite-dimensional differentiable manifold $\mathcal{Q}$. At each point $q \in \mathcal{Q}$, namely at every configuration of the system, the tangent space $T_q\mathcal{Q}$ is a vector space, interpreted physically as the carrier of all possible virtual velocities $\boldsymbol{v}_q$ away from this configuration. A force at q is, therefore, an element $\boldsymbol{f}_q$ of the dual space $T_q^*\mathcal{Q}$, that is, a real-valued

[10] See, for example, [Salençon 2016].

[11] Newton's manuscript, in his always refreshingly legible handwriting, is housed in King's College Library of Cambridge University and cataloged under the identifier Keynes MS. 28. It is also digitally available [Newton 2010]. Considering that the date of publication of the first edition of the *Principia* is 1687, it is rather interesting to remark that this and other alchemy-related manuscripts by Newton have been dated approximately to the decade of the 1680s.

linear function on $T_q\mathcal{Q}$. The evaluation $\boldsymbol{f}_q(\boldsymbol{v}_q)$ is, therefore, interpreted as the virtual power produced by the force on the velocity. The crux of Germain's article consists of the extension of these ideas to the infinite dimensional realm.

Following Segev [1986], if a continuous medium $\mathcal{B}$ is regarded as a differentiable manifold with boundary, its configuration space $\mathcal{Q}$ can be regarded as the set of all C^p embeddings in the physical space (for some $p \geq 1$). This set is known to sustain the structure of an infinite-dimensional Banach manifold. Its tangent space at a configuration $q \in \mathcal{Q}$ is a Banach space $T_q\mathcal{Q}$ representing the collection of all virtual velocity fields at q. Its dual space can, therefore, be interpreted as a generalized force in continuum mechanics, a concept that embraces both external forces and stresses in their full generality. To make his point crystal clear, however, Germain restricts the configuration manifold $\mathcal{Q}$ by effectively identifying it with a Hilbert space $\mathcal{V}$, a Banach space whose norm is induced by an inner product. As explained by Germain, this assumption affords a description of forces and stresses completely analogous to the finite-dimensional counterpart, avoiding the important technicalities of measures and distributions.

Some terminological remarks

Quite apart from the ordinary stylistic difficulties involved in literary or scientific translation, Germain's paper offers an additional challenge even for those familiar with the French language. It arises from the theoretical framework within which the discipline of mechanics is taught in France and other European countries, a framework that involves not only terminological but also conceptual differences with the prevailing tradition in English-speaking countries. These and other matters of general interest pertinent to the background and its history were the motivation for this short article.

We turn now to some specific vocabulary choices:

- The mechanics literature in French (and other romance languages) uses the term *effort* to designate quite generally any mechanical interaction. Thus, an effort may be an external force or an internal stress. There is no such simple equivalent in English. In Germain's English abstract of the paper under consideration, the term "effort" was rendered as "strength". In his second paper, however, written this time in English, Germain occasionally uses the term "force", placing it between quotation marks. He never uses the term "strength". Our policy has been to use the word "force" for all occurrences of effort, since both the context and the presence of an adjective are sufficient to make the meaning clear.

- We have rendered "produit scalaire" as "inner product", to avoid any possible ambiguity.

- The French term “déformation” is translated as “strain” and must not be confused with the English “deformation”.
- The expression “lois de comportement” is rendered as “constitutive laws”.
- Germain often uses the terms “energy” and “power” interchangeably. We have respected this slight inaccuracy.

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