T-ALGEBRAS AND EFIMOV'S PROBLEM

by

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ABSTRACT

ROBERTO PICHARDO MENDOZA. T-Algebras and Efimov's Problem.
(Under the direction of DR. ALAN DOW)

We study the topological properties of minimally generated algebras (as introduced by Koppelberg) and, particularly, the subclass of T-algebras (a notion due to Koszmider) and its connection with Efimov's problem.

We show that the class of T-algebras is a proper subclass of the class of minimally generated Boolean algebras. It is also shown that being the Stone space of a T-algebra is not even finitely productive.

We prove that the existence of an Efimov T-algebra implies the existence of a counterexample for the Stone-Scarborough problem. We also show that the Stone space of an Efimov T-algebra does not map onto the product $(\omega_1 + 1) \times (\omega + 1)$.

We establish the following consistency results. Under CH there exists an Efimov minimally generated Boolean algebra; there are Efimov T-algebra in the forcing extensions obtained by adding ω_2 Cohen or Hechler reals to any model of CH.

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CHAPTER 1: PRELIMINARIES

1.1 Introduction

In this dissertation we focus on the study of the topological properties of the class of Boolean algebras which was introduced and investigated by Koppelberg in [12], the so called minimally generated Boolean algebras. They were present implicitly in [2], [18], and [19] but the notion and the first systematic study are due to Koppelberg. A corollary of her results is that the Stone space of a minimally generated Boolean algebra contains no copy of $\beta\omega$, the Stone-Čech compactification of the integers. This leads naturally to Efimov's problem.

In [8] Efimov raised the following question, does every infinite compact Hausdorff space contain either a nontrivial converging sequence or else a copy of $\beta\omega$? It is known that a negative answer is consistent. For example, in [10] Fedorčuk shows that under $\diamondsuit$, Jensen's combinatorial principle, there is a space which serves as a counterexample to Efimov's problem. Moreover, one can verify that the Boolean algebra of clopen subsets of Fedorčuk's space is minimally generated. Answering a question of Kunen we constructed from CH a space that possesses the properties of Fedorčuk's space listed above. This, in turn, implies that a conjecture of Mercourakis regarding measures on compact spaces [16] is false in any model of CH.

In [14] the concept of T-algebra is introduced and it is shown that every T-algebra is minimally generated. We prove here that they are, in fact, different classes.

Our main results are about Efimov T-algebras, i.e. T-algebras with the property that its Stone space is a counterexample to Efimov's problem. Scarborough-Stone question is We show that the existence of one of this Boolean algebras gives a counterexample to Scarborough-Stone's problem, i.e. a family of sequentially compact spaces whose product is not countably compact. We consider this result interesting because it is the first time (to the best of our knowledge) that a connection between these two longstanding problems is established.

We also show that adding ω_2 Cohen reals to any model of CH produces an Efimov T-algebra and that the same is true if we replace Cohen's poset with Hechler's.

The dissertation is divided as follows. The first chapter serves two purposes: to establish the notation we will use along the text and to introduce the results obtained by Koppelberg and Koszmider. The second chapter contains all the results we obtained without employing any assumptions beyond the usual set of axioms for Set Theory, ZFC. The final chapter is devoted to the theorems involving CH and forcing extensions.

Any topological term not defined explicitly should be understood as in [9]. The corresponding remark applies to set theoretic notions and [15]. Our standard reference for Boolean algebras is the second chapter of [3] (for a more comprehensive book see [17]).

1.2 Set Theory

Given a set X and a cardinal κ, $[X]^\kappa := \{A \subseteq X : |A| = \kappa\}$, where $|A|$ is the cardinality of the set A. Similarly, $[X]^{<\kappa} := \{A \subseteq X : |A| < \kappa\}$ and $[X]^{\leqslant\kappa} := [X]^{<\kappa} \cup [X]^\kappa$.

Given a function $f : X \to Y$ we define $f[A] := \{f(x) : x \in A\}$ and $f^{-1}[B] := \{x \in X : f(x) \in B\}$ for all $A \subseteq X$ and $B \subseteq Y$. We will use $f^{-1}[p]$ instead of $f^{-1}[\{p\}]$ for each $p \in Y$.

As usual, $\mathbb{R}$ and $\mathbb{Q}$ represent the set of real numbers and the collection of all rational points, respectively. $\mathfrak{c}$ will denote the cardinality of $\mathbb{R}$ and CH, Cantor's Continuum Hypothesis, is the statement $\mathfrak{c} = \omega_1$.

A tree is a pair $(T, \leqslant)$, where $\leqslant$ is a partial ordering on T so that $\{s \in T : s < t\}$ is well-ordered by $<$ for each $t \in T$.

Definition 1.1. Given $t \in T$, we denote by $t_T^{\downarrow}$ or simply $t^{\downarrow}$ the set of all predecesors of t in T, i.e. $\{s \in T : s < t\}$.

In addition, let $\mathrm{ht}(t, T)$ (or $\mathrm{ht}(t)$ if the tree is clear from the context) stand for the height of the node $t \in T$. If α is an ordinal, we will denote by T_α or $T(\alpha)$ the αth level of T, i.e. $T_\alpha := \{t \in T : \mathrm{ht}(t) = \alpha\}$. Also $T(< \alpha)$ or $T_{<\alpha}$ denote $\bigcup\{T_\xi : \xi < \alpha\}$. Similarly for $T(\leqslant \alpha)$ and $T_{\leqslant\alpha}$.

A *branch* in T is a subset of T that is linearly ordered and maximal with respect to this property.

An important example of tree is the set $2^{<\varepsilon}$, the collection of all binary functions whose domain is an ordinal $< \varepsilon$, ordered by $s < t$ iff $s \subseteq t$. The following concepts will be used continuously.

Definition 1.2. Let $t \in 2^{<\varepsilon}$ and $\alpha = \operatorname{dom} t$.

1. $t^\frown i := t \cup \{(\alpha, i)\}$ for any $i < 2$. So $t^\frown i : \alpha + 1 \to 2$, $(t^\frown i) \upharpoonright \alpha = t$, and $(t^\frown i)(\alpha) = i$.

2. If $\alpha = \beta + 1$ then $t^* := (t \upharpoonright \beta)^\frown(1 - t(\beta))$. Hence t^* has domain $\beta + 1$, $t \upharpoonright \beta = t^* \upharpoonright \beta$, and $t(\beta) \neq t^*(\beta)$.

1.3 Inverse Systems

An *inverse system* of topological spaces is a double sequence

$$S := \langle X_\alpha, f_{\alpha\beta} : \alpha < \beta < \varepsilon \rangle$$

where ε is an ordinal; X_α is a topological space for each $\alpha < \varepsilon$; if $\alpha < \beta < \varepsilon$ then $f_{\alpha\beta}$ is a continuous map from X_β into X_α, and $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ whenever $\alpha < \beta < \gamma$. The mappings $f_{\alpha\beta}$ are called *bonding maps* of S.

The *inverse limit* of S is the subspace X of the topological product $\prod_{\alpha<\varepsilon} X_\alpha$ defined by $x \in X$ iff $f_{\alpha\beta}(\pi_\beta(x)) = \pi_\alpha(x)$ for all $\alpha < \beta < \varepsilon$, where π_α is the projection from the product onto X_α. It is customary to denote by $\varprojlim S$ the limit of the inverse system.

Definition 1.3. Let X be the limit of $S = \langle X_\alpha, f_{\alpha\beta} : \alpha < \beta < \varepsilon \rangle$. For each $\alpha < \beta \leqslant \varepsilon$ we will adopt the following notation.

1. $f_{\alpha\varepsilon} = \pi_\alpha \upharpoonright X$, the projection onto X_α restricted to X.

2. $f_{\alpha\alpha} : X_\alpha \to X_\alpha$ is the identity map.

3. $S \upharpoonright \beta := \langle X_\gamma, f_{\gamma\delta} : \gamma < \delta < \beta \rangle$.

Observe that $S \upharpoonright \beta$ is itself an inverse system.

The following proposition summarizes the properties we will use later.

Proposition 1.1. Let X be the limit of the inverse system $\langle X_\alpha, f_{\alpha\beta} : \alpha < \beta < \varepsilon \rangle$.

1. If each X_α is compact Hausdorff, then X is compact Hausdorff.

2. If Y is a cofinal subset of ε and $\mathcal{B}_\alpha$ is a base for X_α, for all $\alpha \in Y$, then $\{f_{\alpha\varepsilon}^{-1}[B] : \alpha \in Y \text{ and } B \in \mathcal{B}_\alpha\}$ is a base for X. In particular, X is zero-dimensional if each X_α is zero-dimensional.

3. If $A \subseteq X$ then $\overline{A} = \bigcap_{\alpha<\varepsilon} f_{\alpha\varepsilon}^{-1}[\overline{f_{\alpha\varepsilon}[A]}]$.

1.4 Boolean Algebras

Given a set E, an *algebra of subsets of* E is a collection $B \subseteq \mathcal{P}(E)$ so that

1. $\emptyset, E \in B$.

2. $E \setminus a \in B$ for all $a \in B$.

3. If $a, b \in B$ then $a \cap b \in B$ and $a \cup b \in B$.

Algebras of sets are examples of Boolean algebras. Moreover, Stone's Representation Theorem guarantees that all Boolean algebras are, essentially, algebras of subsets for a suitable set E. For this reason, we will adopt the following convention: The statement B *is a Boolean algebra* means that B is an algebra of subsets of some set that will be denoted by 1.

Definition 1.4. For each $a, b \in B$, $-a$ denotes the complement of a with respect to 1, $a - b := a \cap (-b)$ and $x \triangle y = (x - y) \cup (y - x)$.

A *subalgebra* of B is a set $A \subseteq B$ containing $\emptyset$ and 1 which is closed under complements, unions and intersections. In other words: $-a, a \cup b, a \cap b \in A$ for all $a, b \in A$. In particular, A is itself a Boolean algebra. The symbol $A \leqslant B$ abbreviates the phrase A *is a subalgebra of* B and $A < B$ is equivalent to $A \leqslant B$ and $A \neq B$, i.e. A is a *proper subalgebra* of B.

The statement B *is an extension of* A means that $A \leqslant B$; and we will say that B *is a proper extension of* A when $A < B$.

Given a set $Y \subseteq B$, the smallest subalgebra of B containing Y will be denoted by $[Y]$ and will be called the *Boolean algebra generated by* Y. Note that $[Y]$ is the intersection of all Boolean algebras containing Y and therefore $[Y]$ always exists.

The following result provides us with an alternative way to calculate the Boolean algebra generated by Y. The proof is straightforward and can be found in [3], Lemma 2.4.

Proposition 1.2. $a \in [Y]$ if and only if $a = \bigcup_{i<n}(\bigcap F_i - \bigcup H_i)$ for some integer n and sequences of finite sets $\{F_i : i < n\}, \{H_i : i < n\} \subseteq [Y]^{<\omega}$ (where $\bigcap \emptyset = 1$, by definition).

Let $\varphi(x)$ be a formula whose only free variable is x. We will use $[x \in B : \varphi(x)]$ to denote the Boolean algebra generated by $\{x \in B : \varphi(x)\}$.

The simplest way to extend a Boolean algebra is to add a single element.

Definition 1.5. If $A \leqslant B$ and $x \in B$ then $A(x) := [A \cup \{x\}]$.

As a corollary of Proposition 1.2 one gets $A(x) = \{(a \cap x) \cup (b - x) : a, b \in A\}$.

A *filter* in B is a nonempty set $F \subseteq B \setminus \{\emptyset\}$ so that

1. If $a, b \in F$ the $a \cap b \in F$.

2. $a \in F$, $b \in B$ and $a \subseteq b$ imply $b \in F$.

Let Y be a subset of B with the finite intersection property. The collection $\{b \in B : \exists F \in [Y]^{<\omega}(\bigcap F \subseteq b)\}$ is a filter that will be called *the filter generated by* Y *in* B. The following result is Theorem 2.19 from [3].

Proposition 1.3. If u is a filter in B then the following are equivalent.

1. For each $a \in B$ either $a \in u$ or $-a \in u$.

2. If F is a filter and $u \subseteq F$ then $u = F$.

3. If $a \in B$ satisfies $a \cap b \neq \emptyset$ for all $b \in u$ then $a \in u$.

If u is a filter that satisfies any of the previous statements then u will be called an *ultrafilter* in B.

Proposition 1.4. If Y generates B and u is a filter in B so that $u \cap \{y, -y\} \neq \emptyset$ for all $y \in Y$ then u is an ultrafilter in B.

Proof. We verify Proposition 1.3-(1) for u. Let F and H be finite subsets of Y so that $-(\bigcap F - \bigcup H) \notin u$. Since $-(\bigcap F - \bigcup H) = \bigcup\{-b : b \in F\} \cup \bigcup H$ we have that $-b \notin u$ for all $b \in F$ and $H \cap u = \emptyset$. Our assumption about u implies that $F \subseteq u$ and $\{-b : b \in H\} \subseteq u$. Therefore $\bigcap F - \bigcup H \in u$.

Given $n \in \omega$ and $\{F_i : i < n\}, \{H_i : i < n\} \subseteq [Y]^{<\omega}$ let $b_i := \bigcap F_i - \bigcup H_i$. The previous paragraph shows that $u \cap \{b_i, -b_i\} \neq \emptyset$ for each $i < n$. If $b_k \in u$ for some $k < n$ then $\bigcup_{i<n} b_i \in u$. Otherwise $\{-b_i : i < n\} \subseteq u$ and therefore $-\bigcup_{i<n} b_i = \bigcap_{i<n}(-b_i) \in u$. *Q.E.D.*

St(B) denotes the collection of all ultrafilters in B. For each $a \in B$ we let $a^- := \{u \in \text{St}(B) : a \in u\}$. The proof of the following proposition can be found in [3] (see Theorem 2.21).

Proposition 1.5. For all $a, b \in B$ the following holds.

1. $(a \cap b)^- = a^- \cap b^-$.
2. $(a \cup b)^- = a^- \cup b^-$.
3. $(-a)^- = \text{St}(B) \setminus a^-$.
4. $a \subseteq b$ is equivalent to $a^- \subseteq b^-$.

The *Stone space of* B is the topological space obtained by endowing St(B) with the topology that has $\{a^- : a \in B\}$ as a base. The resulting space is compact Hausdorff zero-dimensional, in fact, a^- is clopen for each $a \in B$.

If $A \leqslant B$ and u is an ultrafilter in B, then $A \cap u$ is an ultrafilter in A. Hence the map $f : \text{St}(B) \to \text{St}(A)$ given by $f(u) = A \cap u$ is well-defined and, moreover, is onto and continuous. f is called the *Stone map.*

Definition 1.6. For any topological space X, $CO(X)$ is the collection of all clopen subsets of X.

$CO(X)$ is an algebra of subsets of X. The following result is known as Stone Duality Theorem (see [3]).

Proposition 1.6. If X is a compact Hausdorff zero-dimensional topological space then the map $h : X \to \mathrm{St}(CO(X))$ given by $h(x) := \{a \in CO(X) : x \in a\}$ is a homemomorphism.

Two Boolean algebras, A and B, are *isomorphic* if there is a bijection $f : A \to B$ so that $f(a \cap b) = f(a) \cap f(b)$, $f(a \cup b) = f(a) \cup f(b)$ and $f(-a) = -f(a)$ for all $a, b \in A$. A standard argument shows that A and B are isomorphic iff $\mathrm{St}(A)$ is homeomorphic to $\mathrm{St}(B)$.

Lemma 1.7. If K is a clopen subset of $\mathrm{St}(B)$ then $K = b^-$ for some $b \in B$.

Proof. For each $p \in K$ there exists $a_p \in p$ so that $a_p^- \subseteq K$. Then $\{a_p^- : p \in K\}$ covers K and therefore there is a finite $F \subseteq K$ so that $K = \bigcup\{a_p^- : p \in F\}$. Set $b = \bigcup\{a_p : p \in F\}$ and use (1) from Proposition 1.5 to obtain $K = b^-$. *Q.E.D.*

Definition 1.7. An *ideal* is a nonempty set $I \subseteq B \setminus \{1\}$ so that

1. $a \cup b \in I$ for all $a, b \in I$.

2. If $a \in I$ and $b \in B$ satisfy $b \subseteq a$ then $b \in I$.

Observe that if I is an ideal then $I^* := \{-a : a \in I\}$ is a filter which will be called *the dual filter of* I. And vice versa: If F is a filter, $F^* := \{-a : a \in F\}$ is an ideal, *the dual ideal of* F.

The ideal I is *maximal* in A if I^* is an ultrafilter in A. In other words (Proposition 1.3), I is maximal iff for all $a \in A$ we have that either $a \in I$ or $-a \in I$. Hence u is an ultrafilter iff u^* is maximal.

A filter F is *principal* if there exists $a \in B \setminus \{\emptyset\}$ so that $F = \{b \in B : a \subseteq b\}$. An ideal is *principal* if its dual filter is principal. Equivalently, the ideal I is principal iff $I = \{b \in B : b \subseteq a\}$ for some $a \in B \setminus \{1\}$.

Let Y be a subset of the Boolean algebra A so that no finite subset of Y covers 1, i.e. for any finite set $F \subseteq Y$ we have $-\bigcup F \neq \emptyset$. Then the collection $\{a \in A : \exists F \in [Y]^{<\omega}(a \subseteq \bigcup F)\}$ is an ideal in A which will be called *the ideal generated by* Y.

1.5 Minimally Generated Boolean Algebras

The concept that gives title to this section was introduced by Koppelberg in [12]. All the results presented in this section were proved originally in [12] or are corollaries of those results. We include the proofs for the convenience of the reader.

Definition 1.8. A Boolean algebra B is a *minimal extension* of A if $A \leqslant B$ and there is no C so that $A < C < B$. In this case we will write $A \leqslant_m B$.

Note that $A = B$ is not excluded from the definition. When $A \leqslant_m B$ and $A \neq B$ se will use the symbol $A <_m B$.

If $A <_m B$ and $x \in B \setminus A$ then $B = A(x)$. Hence any proper minimal extension of A is of the form $A(x)$ for some x.

A sequence $\langle B_\alpha : \alpha < \varepsilon \rangle$ is a *chain* of Boolean algebras if ε is an ordinal and $B_\alpha \leqslant B_\beta$ whenever $\alpha < \beta < \varepsilon$. The chain will be called *continuous* if $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$ for each limit ordinal $\lambda < \varepsilon$.

B is *minimally generated over* A (in symbols, $A \leqslant_{mg} B$) if there is a continuous chain $\langle B_\alpha : \alpha < \varepsilon \rangle$ so that

1. $B_0 = A$,
2. $B_\alpha \leqslant_m B_{\alpha+1}$ whenever $\alpha + 1 < \varepsilon$, and
3. $B = \bigcup_{\alpha<\varepsilon} B_\alpha$.

In the case $A = \{\emptyset, 1\}$ we will say that B is *minimally generated*.

Informally speaking, a Boolean algebra is minimally generated if one can construct it by small, indivisible steps.

A chain satisfying the conditions stated above *witnesses the minimal generation* of B over A.

If $\langle B_\alpha : \alpha < \varepsilon \rangle$ witnesses the minimal generation of B and for each $\alpha < \varepsilon$ we select $a_\alpha \in B_{\alpha+1} \setminus B_\alpha$ then we have the following.

1. $B_\alpha = [a_\beta : \beta < \alpha]$.
2. $B_\alpha <_m B_\alpha(a_\alpha)$.
3. $B = [a_\alpha : \alpha < \varepsilon]$.

If all of the above hold, then $\{a_\alpha : \alpha < \varepsilon\}$ *witnesses the minimal generation* of B.

Definition 1.9. Assume $A \leqslant B$ and $x \in B \setminus A$. Then

1. $I_x^A := \{a \in A : a \subseteq x\}$ and $I_{-x}^A := \{a \in A : a \subseteq -x\}$.

2. J_x^A is the ideal in A generated by $I_x^A \cup I_{-x}^A$, i.e. $a \in J_x^A$ iff there exist $b, c \in A$ so that $a = b \cup c$, $b \subseteq x$, and $c \subseteq -x$..

We will write just I_x, I_{-x}, and J_x when there is no risk of confusion about the Boolean algebra.

Lemma 1.8. If $A < A(x)$ and $a \in A$ then the following are equivalent

1. $a \in J_x$.

2. $a \cap x \in A$.

3. $a - x \in A$.

4. $\{y \in A(x) : y \subseteq a\} \subseteq A$.

Proof. The key observation is that $a \in J_x$ if and only if there exist $b, c \in A$ so that $a = b \cup c$, $b \subseteq x$, and $c \subseteq -x$. *Q.E.D.*

Note that a consequence of condition (4) is that J_x is also an ideal in $A(x)$.

It is worth mentioning that $A(x)$ is not necessarily a minimal extension of A. For example, let A be the algebra of clopen subsets of the topological product $X := (\omega+1) \times 2$, where both factors have the order topology. We claim that $x := \{(\omega, 0), (\omega, 1)\}$ and $y := \{(\omega, 0)\}$ satisfy $A < A(y) < A(x)$. Since $y \notin A$ and $y = x - ((\omega+1) \times \{1\}) \in A(x)$ we only have to show that $x \notin A(y)$. Observe that if $a, b \in A$ are so that $(\omega, 1) \in (a \cap y) \cup (b - y)$ then $(\omega, 1) \in b - y$ and since b is open in X, we get that b must be infinite. In particular, $b - y$ cannot be a subset of x.

If u is an ultrafilter in A and $A \leqslant B$, then u is a subset of B with the finite intersection property and therefore generates a filter F in B. A standard argument involving Zorn's Lemma gives the existence of an ultrafilter v in B containing F. In other words, u can be extended to an ultrafilter in B.

It could be the case that $v = F$, i.e. that F is itself an ultrafilter. When this happens we will say that u *generates an ultrafilter in* B.

Proposition 1.9. If $A < A(x)$ then the following are equivalent.

1. $A <_m A(x)$.
2. J_x is a maximal ideal of A.
3. There is only one ultrafilter in A that can be extended to more than one ultrafilter in $A(x)$.

Proof. In order to show that (1) implies (2) assume that $A <_m A(x)$ and let $a \in A$ be arbitrary. Since $A \leqslant A(a \cap x) \leqslant A(x)$ we get $A = A(a \cap x)$ or $A(a \cap x) = A(x)$. In the first case, $a \in J_x$ because $a \cap x \in A$ so assume that $A(x) = A(a \cap x)$. There exist $b, c \in A$ so that

$$x = (b \cap a \cap x) \cup (c - (a \cap x)) = (b \cap a \cap x) \cup (c - a) \cup (c - x).$$

Thus $c - x = \emptyset$ and $x = (b \cap a \cap x) \cup (c - a)$. Clearly, $(-a) \cap x = c - a \in A$ and hence $-a \in J_x$.

Now assume that J_x is maximal. We claim that J_x^*, the dual filter of J_x, is the only ultrafilter in A that does not generate an ultrafilter in $A(x)$. Let u be an ultrafilter in A so that $u \neq J_x^*$. Then there is $a \in u \setminus J_x^*$; therefore $a \in J_x$ and so $a \cap x, a - x \in A$. Let v be an ultrafilter in $A(x)$ satisfying $u \subseteq v$. If $b \in v$ then $b = (c \cap x) \cup (d - x)$ for some $c, d \in A$. Thus

$$b \cap a = (c \cap (a \cap x)) \cup (d \cap (a - x)) \in A.$$

Since u is an ultrafilter in A, we have $A \cap v = u$; therefore $b \cap a \in u$ and $b \cap a \subseteq b$. The previous argument shows that $v \subseteq \{b \in A(x) : \exists c \in u(c \subseteq b)\}$ which implies that they are, in fact, equal and thus u extends to a unique ultrafilter in $A(x)$.

To finish this part of the proof note that if $a \in J_x^*$ then $-a \in J_x$ and therefore $x - a \in A$. Since $x \notin A$ we must have $a \cap x \notin A$ and, in particular, $a \cap x \neq \emptyset$. Hence $J_x^* \cup \{x\}$ can be extended to an ultrafilter in $A(x)$. A similar argument shows that $J_x^* \cup \{-x\}$ shares the same property and then J_x^* can be extended to more than one ultrafilter.

Now assume (3) and let u be the only ultrafilter in A that extends to more than one ultrafilter in $A(x)$.

Suppose that $u \cup \{x\}$ does not have the finite intersection property. Let $a \in u$ be so that $a \cap x = \emptyset$. If v is an ultrafilter in $A(x)$ extending u and $b \in v$ is arbitrary then $b = (c \cap x) \cup (d - x)$ for some $c, d \in A$. Thus $a \cap b = d \cap (a - x) = d \cap a \in A$. Moreover, $a \cap b \in A \cap v = u$ and therefore $v \subseteq \{b \in A(x) : \exists y \in u(y \subseteq b)\}$ so v is the only filter extending u. A contradiction. This implies that $u_0 := \{b \in A(x) : \exists a \in u(a \cap x \subseteq b)\}$ is a filter in $A(x)$. Using Proposition 1.4 we get that u_0 is, in fact, an ultrafilter. The same arguments show that $u_1 := \{b \in A(x) : \exists a \in u(a - x \subseteq b)\}$ is an ultrafilter that extends u. Since any ultrafilter in $A(x)$ must contain either x or $-x$ we have that u_0 and u_1 are the only ultrafilters extending u.

We claim that if $y \in A(x)$ then $y \in A$ or $x \in A(y)$. Before proving this property let us see how we can use it to show that $A <_m A(x)$. Assume that $A < B \leqslant A(x)$ for some Boolean algebra B and let $y \in B \setminus A$. Hence $A(y) \subseteq B \subseteq A(x)$ and since our claim implies that $x \in A(y)$ we conclude that $A(x) = A(y)$ so $B = A(x)$.

The only thing left is to prove the claim. Let $X = \mathrm{St}(A(x))$, $Y = \mathrm{St}(A(y))$ and $Z = \mathrm{St}(A)$. Also let $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$ be the corresponding Stone mappings. Our assumption about u implies that $h \restriction X \setminus \{u_0, u_1\}$ is one-to-one and $h(u_0) = h(u_1) = u$. Since $h = g \circ f$ it must be the case that either f is an injection or g is one-to-one. In the first case, f becomes a homeomorphism. The set $K := \{v \in X : x \in v\}$ is a basic clopen subset of X and therefore $f[K]$ is clopen too. Lemma 1.7 implies that $f[K] = \{w \in Y : b \in w\}$ for some $b \in A(y)$. We claim that $b = x$. Otherwise, $b \triangle x$, the symmetric difference of b and x, is not empty so there exists $v \in X$ with $b \triangle x \in v$. There are two cases: $b - x \in v$ or $x - b \in v$. If $b - x \in v$ then $b \in v$ and thus $b \in v \cap A(y) = f(v)$ so $v \in K$ and hence $x \in v$, a contradiction to $b - x \in v$. Similar reasons show that $x - b \in v$ is impossible too (note that $X \setminus K = \{v \in X : -x \in v\}$). When g is a homeomorphism, the conclusion is $y \in A$. *Q.E.D.*

Corollary 1.10. $A <_m A(x)$ is equivalent to $J_x^* = \{b \in A : b \cap x \notin A\}$.

Proof. Observe that $J_x^* = A \setminus J_x$ iff J_x is a maximal ideal. Lemma 1.8 finishes the proof. *Q.E.D.*

In order to obtain a topological translation of minimality we will introduce the following concept.

Definition 1.10. Let X and Y be compact Hausdorff zero-dimensional topological spaces. We say that X is a *simple extension* of Y if there is a map $f : X \to Y$ such that, for some $p \in Y$, $f^{-1}[y]$ is a singleton for each $y \in Y \setminus \{p\}$ and $|f^{-1}[p]| = 2$.

The sentence *f witnesses that X is a simple extension of Y* means that f is as in the definition.

Proposition 1.11.

1. If f witnesses that X is a simple extension of Y then the Boolean algebra $CO(X)$ is isomorphic to a minimal extension of $CO(Y)$.
2. If $A <_m A(x)$ then $\mathrm{St}(A(x))$ is a simple extension of $\mathrm{St}(A)$ as witnessed by the Stone map. Moreover, the dual filter of J_x is the only point whose fiber is not a singleton.

Proof. Let $p \in Y$ be so that $f^{-1}[p]$ is not a singleton. Fix a clopen set $c \subseteq X$ so that $|c \cap f^{-1}[p]| = 1$ and define $x := f[c]$ and $x' := f[X \setminus c]$. Then $X = c \oplus (X \setminus c)$, where $\oplus$ denotes the sum of topological spaces (as defined in [9]). Since $f \restriction c$ and $f \restriction X \setminus c$ are both injective, we have that c and $X \setminus c$ are homeomorphic to x and x', respectively. Thus X is homeomorphic to $Z := x \oplus x'$. Note that $y \subseteq Z$ is clopen iff there are clopen sets $a, b \subseteq Y$ so that $y \cap x = a \cap x$ and $y - x = b - x$. Hence $CO(X)$ is, essentially, $A(x)$ where $A = CO(Y)$.

It only remains to show that $A <_m A(x)$. Observe that p is not an interior point of x and therefore for all $b \in A$ we have that $b \cap x \in A$ iff $p \notin b$. In other words, $J_x^A := \{b \in A : p \notin b\}$ and hence J_x^A is a maximal ideal so we can invoke Proposition 1.9 to finish the proof of (1).

In order to prove part (2) assume $A <_m A(x)$. Let $f : \mathrm{St}(A(x)) \to \mathrm{St}(A)$ be the Stone map. Observe that the relation $f(u) = v$ is equivalent to $v \subseteq u$.

Proposition 1.9 shows that $J_x^* \cup \{x\}$ and $J_{-x}^* \cup \{-x\}$ extend to ultrafilters u_0 and u_1, respectively, in $A(x)$. Moreover, if u is any other ultrafilter that extends J_x^* then $x \in u$ or $-x \in u$ and therefore $u \in \{u_0, u_1\}$. In other words, $f^{-1}[J_x^*] = \{u_0, u_1\}$. The proof of Proposition 1.9 also shows that if $v \neq J_x^*$ then there is a unique $v' \in \mathrm{St}(A(x))$ so that $v \subseteq v'$, i.e. $f^{-1}[v] = \{v'\}$. *Q.E.D.*

Minimal generation is closely related to inverse systems in topology.

Definition 1.11. Let $S = \langle X_\alpha, f_{\alpha\beta} : \alpha < \beta < \varepsilon \rangle$ be an inverse system.

1. We say that S is *based on simple extensions* if $f_{\alpha,\alpha+1} : X_{\alpha+1} \to X_\alpha$ witnesses that $X_{\alpha+1}$ is a simple extension of X_α.

2. S is *continuous* if for each limit ordinal $\lambda < \varepsilon$ there is a homeomorphism $h : X_\lambda \to \varprojlim S \restriction \lambda$ so that $f_{\alpha\lambda} = \pi_\alpha \circ h$ for all $\alpha < \lambda$, where π_α is the projection from $\prod_{\beta<\alpha} X_\beta$ onto X_α.

3. S is a *simplistic system* if S is continuous, based on simple extensions and X_0 is a singleton.

4. A topological space will be called *simplistic* if it is the limit of a simplistic system.

One can picture simplistic spaces as spaces that were obtained as the result of an iterative process: start with a singleton; to move from stage α to stage $\alpha+1$ select a point and *double* it; and when you reach a limit stage take the inverse limit.

We will show in the next chapter that all compact metrizable zero-dimensional spaces and all compact Hausdorff scattered spaces are simplistic.

Recall the notation established in Definition 1.3.

Proposition 1.12.

1. If B is a minimally generated Boolean algebra then $\mathrm{St}(B)$ is simplistic.

2. If X is simplistic then $CO(X)$ is minimally generated.

Proof. Let $\langle B_\alpha : \alpha < \varepsilon \rangle$ be a chain witnessing the minimal generation of B. For each $\alpha < \varepsilon$ let $X_\alpha := \mathrm{St}(B_\alpha)$ and let $f_{\alpha\beta} : X_\beta \to X_\alpha$ be the Stone map for $\alpha < \beta < \varepsilon$. In view of Proposition 1.11 we only have to show that $S := \langle X_\alpha, f_{\alpha\beta} : \alpha < \beta < \varepsilon \rangle$ is continuous to claim that S is a simplistic system.

Let $\lambda < \varepsilon$ be a limit ordinal and let $Y := \varprojlim S \restriction \lambda$. Define $h : X_\lambda \to Y$ by

$$h(x) = \langle f_{\alpha\lambda}(x) : \alpha < \lambda \rangle.$$

Clearly h is continuous and satisfies the requirement about compositions stated in Definition 1.11. To show that h is one-to-one let $p, q \in X_\lambda$ be so that $p \neq q$. There exist $a, b \in B_\lambda$ so that $a \in p$, $b \in q$ and $a \cap b = \emptyset$. Since λ is limit, $a, b \in B_\alpha$ for some $\alpha < \lambda$ and therefore $a \in f_{\alpha\lambda}(p)$ and $b \in f_{\alpha\lambda}(q)$. Hence $f_{\alpha\lambda}(p) \neq f_{\alpha\lambda}(q)$.

Now we will show that h is onto. Let $x \in Y$ and let $x_\alpha := f_{\alpha lambda}(x)$, i.e. x_α is the αth coordinate of x. Noteh that if $p \in \bigcap\{f_{\alpha\lambda}^{-1}[x_\alpha] : \alpha < \lambda\}$ then $h(p) = x$ so we only have to show that $\{f_{\alpha\lambda}^{-1}[x_\alpha] : \alpha < \lambda\}$ has the finite intersection property because Y is compact. Let $F \in [Y]^{<\omega}$ and define $\beta := \max F$. By definition, $x \in Y$ implies that

$$f_{\alpha\beta}(x_\beta) = f_{\alpha\beta}(f_{\beta\lambda}(x)) = f_{\alpha\lambda}(x) = x_\alpha$$

for all $\alpha \in F$. Now fix $y \in f_{\alpha\lambda}^{-1}[x_\beta]$ (recall that $f_{\beta\lambda}$ is onto) and observe that

$$f_{\alpha\lambda}(y) = f_{\alpha\beta}(f_{\beta\lambda}(y)) = f_{\alpha\beta}(x_\beta) = x_\alpha$$

and therefore $y \in \bigcap\{f_{\alpha\lambda}^{-1}[x_\alpha] : \alpha \in F\}$.

Let $X_\varepsilon := \mathrm{St}(B)$ and for each $\alpha < \varepsilon$ let $f_{\alpha\varepsilon} : X_\varepsilon \to X_\alpha$ be the corresponding Stone map. If Y is the limit of S, then we can define h as before but replacing λ with ε. In the case where ε is limit, we already know that h is a homeomorphism. When $\varepsilon = \gamma + 1$, h is one-to-one because $f_{\gamma\varepsilon}$ is the identity map and therefore h is a homeomorphism. Thus $\mathrm{St}(B)$ is simplistic because is homeomorphic to X_ε.

Let X be the limit of the simplistic system $\langle X_\alpha, f_{\alpha\beta} : \alpha < \beta < \varepsilon\rangle$. For each $\alpha < \varepsilon$ define $B_\alpha := \{f_{\alpha\varepsilon}^{-1}[a] : a \in CO(X_\alpha)\}$. Note that $B_\alpha \leqslant CO(X)$. Since $CO(X_{\alpha+1})$ is isomorphic to a minimal extension of $CO(X_\alpha)$ (Proposition 1.11) we have that there is no proper algebra lying between B_α and $B_{\alpha+1}$, i.e. $B_\alpha <_m B_{\alpha+1}$.

When $\lambda < \varepsilon$ is a limit ordinal, $\{f_{\alpha\lambda}^{-1}[a] : \alpha < \lambda \text{ and } a \in CO(X_\alpha)\}$ is a base of clopen sets of X_λ (Proposition 1.1) which implies $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$ because $f_{\alpha\varepsilon}^{-1}[a] = f_{\lambda\varepsilon}^{-1}[f_{\alpha\lambda}^{-1}[a]]$ for all $\alpha < \lambda$ and all $a \in CO(X_\alpha)$. Hence $\{B_\alpha : \alpha < \varepsilon\}$ witnesses the minimal generation of $CO(X)$. *Q.E.D.*

Lemma 1.13. Assume $A \leqslant_m B \leqslant C$. If $C' \leqslant C$ then $A \cap C' \leqslant_m B \cap C'$.

Proof. Let $A' := A \cap C'$ and $B' := B \cap C'$. If $A' = B'$ then we are done so we can assume that $A' < B'$. It is enough to prove that if $x, y \in B' \setminus A'$ then $y \in A'(x)$.

Since $x, y \in B \setminus A$ we get $y \in A(x)$. Therefore $y = (a \cap x) \cup (b - x)$ for some $a, b \in A$. By letting $c = a - b$, $d = b - a$ and $e = a \cap b$ we get $y = (c \cap x) \cup (d - x) \cup e$ and $\{c, d, e\} \subseteq A$ is pairwise disjoint. We have two cases: If $c, d \in J_x^A$ then $y \in A$ (Lemma 1.8) which together with $y \in C'$ gives $y \in A'$. Otherwise, $J_x^A \cap \{-c, -d\} \neq \emptyset$ (recall that J_x^A is maximal). Assume, for example, that $-c \in J_x^A$. The fact $x \triangle y \subseteq -c$ implies, according to Lemma 1.8, that $x \triangle y \in A$. Also note that $x \triangle y \in C'$ because $x, y \in C'$. So $x \triangle y \in A'$ and thus $y = x \triangle (x \triangle y) \in A'(x)$. *Q.E.D.*

The behavior of the class of simplistic spaces under the traditional topological operations was analyzed by Koppelberg [12] in the context of Boolean algebras. She proved the following.

Proposition 1.14. Let X be a simplistic space.

1. If $f : X \to Y$ is continuous and onto and Y is Hausdorff zero-dimensional then Y is simplistic.
2. If Y is a closed subspace of X then Y is simplistic.

Proof. The hypothesis of (1) implies that $B := \{f^{-1}[c] : c \in CO(Y)\}$ is a subalgebra of $A := CO(X)$ which is isomorphic to $CO(Y)$ and therefore $\mathrm{St}(B)$ is homeomorphic to Y. So it is enough to show that any subalgebra of a minimally generated algebra is minimally generated.

Assume that $\{A_\alpha : \alpha < \varepsilon\}$ witnesses the minimal generation of A. Define by transfinite induction a function h as follows, $h(0) = 0$,

$$h(\alpha + 1) = \min\{\beta < \varepsilon : A_\beta \cap B \setminus A_{h(\alpha)} \neq \emptyset\},$$

and $h(\alpha) = \sup\{h(\beta) : \beta < \alpha\}$ for limit α. Let $\delta := \mathrm{dom}(h)$ and set $B_\alpha := B \cap A_{h(\alpha)}$ for all $\alpha < \delta$. Hence $\langle B_\alpha : \alpha < \delta \rangle$ is a continuous chain whose union is B.

If $\alpha + 1 < \delta$ then $h(\alpha + 1) = \beta + 1$ for some $\beta < \varepsilon$ and therefore $A_\beta \cap B \subseteq A_{h(\alpha)}$ and hence $B_\alpha = A_\beta \cap B$. Since $A_\beta <_m A_{\beta+1}$ and $B \leqslant A$, Lemma 1.13 implies that $B \cap A_\beta <_m B \cap A_{\beta+1}$, i.e. $B_\alpha <_m B_{\alpha+1}$.

To prove (2): Let $\langle X_\alpha, f_{\alpha\beta} : \alpha < \beta < \varepsilon\rangle$ be a simplistic system whose limit is X. For each $\alpha + 1 < \varepsilon$ let $x_\alpha \in X_\alpha$ be the unique point such that $[x_\alpha]_{\alpha+1}$ is not a singleton.

Define a function h by transfinite induction as follows, $h(0) = 0$,

$$h(\alpha + 1) = \min\{\beta < \varepsilon : h(\alpha) < \beta \wedge x_\beta \in Y \restriction \beta\}$$

and if α is limit, $h(\alpha) = \sup\{h(\beta) : \beta < \alpha\}$. Let δ be the domain of h.

For each $\alpha < \beta < \delta$ set $Y_\alpha := Y \restriction h(\alpha)$ and $g_{\alpha\beta} := f_{h(\alpha)h(\beta)} \restriction Y_{h(\beta)}$. Hence Y is the limit of the simplistic system $\langle Y_\alpha, g_{\alpha\beta} : \alpha < \beta < \delta\rangle$. *Q.E.D.*

An *atom* in a Boolean algebra A is an element $a \in A \setminus \{\emptyset\}$ which is $\subseteq$-minimal in $A \setminus \{\emptyset\}$. Equivalently, a is an atom iff for all $b \in A$ either $a \subseteq b$ or $a \subseteq -b$. Therefore, a is an atom iff $\{b \in A : a \subseteq b\}$ is an ultrafilter in A. Using this information and Proposition 1.5 one can prove that $u \in \mathrm{St}(A)$ is an isolated point iff u contains an atom.

Lemma 1.15. Let $A <_m A(x)$. If u is an ultrafilter on $A(x)$ containing $J_x^* \cup \{x\}$ then the following are equivalent.

1. u is an isolated point of $\mathrm{St}(A(x))$.

2. I_x is a principal ideal.

Proof. Assume that I_x is principal. There is $b \in A$ so that $a \in I_x$ iff $a \subseteq b$. In particular, $b \subseteq x$. Since $x \notin A$ we obtain $b \subset x$ and hence $x - b \neq \emptyset$.

We claim that $J_x^* = \{a \in A : x - b \subseteq a\}$. Indeed, if $a \in J_x^*$ then $-a \in J_x$ and hence $x - a \in A$. Since $x - a \subseteq x$ we obtain $x - a \in I_x$ and therefore $x - a \subseteq b$ or equivalently $x - b \subseteq a$. Now assume that $a \in A$ satisfies $x - b \subseteq a$. As before, we get $x - a \subseteq b$. The maximality of J_x gives that $a \in J_x$ or $-a \in J_x$. If $a \in J_x$ then $a \cap x \in A$ and therefore $a \cap x \in I_x$. Hence $a \cap x \subseteq b$ which implies $x \subseteq b$, a contradiction to $b \subset x$. Thus $-a \in J_x$, i.e. $a \in J_x^*$.

To finish the first part of the proof we will show that $x - b$ is an atom in $A(x)$. Let $z \in A(x)$ be so that $z \subseteq x - b$. Then $z = a \cap x$ for some $a \in A$. We have two cases: $a \in J_x^*$ or $-a \in J_x^*$. In the first case we get $x - b \subseteq a$ and thus $x - b \subseteq a \cap x = z$ so $z = x - b$. On the other hand, if $-a \in J_x^*$ we obtain $x - b \subseteq -a$, which implies $a \cap x \subseteq b$, i.e. $z \subseteq b$ and hence $z = \emptyset$.

Now let us assume that I_x is nonprincipal. We must prove that u does not contain an atom. Let $y \in u$ be arbitrary and define $z = y \cap x$. It is enough to show that z is not atom. Since $z \in A(x)$ and $z \subseteq x$ there exists $a \in A$ so that $z = a \cap x$. Note that $J_x^* \cup \{a\}$ has the finite intersection property and therefore $a \in J_x^*$, i.e. $-a \in J_x$. Then $x - a \in A$ and, moreover, $x - a \in I_x$. The nonprincipality of I_x guarantees the existence of $b \in A$ so that $b \subseteq x$ and $b \not\subseteq x - a$. Hence $\emptyset \neq a \cap b \subseteq z$. To show that $z \neq a \cap b$ recall that $x \notin A$, $x - a \in A$ and $x = z \cup (x - a)$, so $z \notin A$ but $a \cap b \in A$. *Q.E.D.*

We will investigate the relation between π-bases and simple extensions. Given a Boolean algebra B, a set $D \subseteq B \setminus \{\emptyset\}$ is *dense* in B if for each $a \in B \setminus \{\emptyset\}$ there is $d \in D$ so that $d \subseteq a$. Note that D is dense in B iff $\{d^- : d \in D\}$ is a π-base for $\mathrm{St}(B)$.

Recall the notation established in Definition 1.4.

Lemma 1.16. Assume that $A <_m A(x)$ and let u_0 and u_1 be ultrafilters so that $J_x^* \cup \{x^i\} \subseteq u_i$ for each $i < 2$. Let D be a dense subset of A.

1. If u_i is an isolated point of $\mathrm{St}(A(x))$, for some $i < 2$, then $\{d - y : d \in D\} \cup \{y\}$ is dense in $A(x)$, where y is an atom of $A(x)$ so that $y \in u_i \setminus A$.

2. Otherwise, D is dense in $A(x)$.

Proof. If, for example, u_0 is an isolated point then $A(x)$ has an atom $y \in u_0$. Observe that if $y \in A$ then $y \in A \cap u_0 = J_x^*$ (this equality holds because J_x^* is an ultrafilter) and therefore $y \cap x \neq \emptyset$ and $y - x \neq \emptyset$. Since y is an atom this gives $y \subseteq x$ and $y \subseteq -x$, a contradiction. Hence $y \in A(x) \setminus A$ and thus $A(x) = A(y)$. Clearly $\{d - y : d \in D\} \cup \{y\}$ is dense in $A(x)$.

Assume that u_0 and u_1 are not isolated points. We claim that A is dense in $A(x)$. To show this let $z \in A(x)$ be arbitrary. There exist $a, b \in A$ so that $z = (a \cap x) \cup (b - x)$. If $a \notin J_x^*$ then $a \in J_x$ and therefore $a \cap x \in A$ and $a \cap x \subseteq z$. So assume that $a \in J_x^*$. Then $-a \in J_x$ or equivalently $x - a \in A$. Moreover, $x - a \in I_x$. Since I_x is nonprincipal there is $c \in A$ so that $c \subseteq x$ and $c \not\subseteq x - a$, i.e. $c \cap a \neq \emptyset$. Therefore $c \cap a \subseteq a \cap x \subseteq z$. *Q.E.D.*

We turn now our attention to tree π-bases in simplistic spaces. Recall that a tree π-base for a topological space is a π-base which forms a tree when ordered by reverse inclusion.

It is worth mentioning that $\beta\omega \setminus \omega$, the remainder of the Stone-Čech compactification of the integers, has a tree π-base as proved in [1]. Compare this fact with Corollary 1.18 and Proposition 1.19.

Definition 1.12. Let B be a Boolean algebra. A set $T \subseteq B \setminus \{\emptyset\}$ will be called a *tree in B* if the following conditions hold.

1. For each $x, y \in T$ if $x \neq y$ then $x \cap y \in \{\emptyset, x, y\}$.

2. $\supseteq$ is a tree ordering for T (in the set-theoretic sense).

A *tree algebra* is a Boolean algebra generated by a tree.

Proposition 1.17. If B is a minimally generated Boolean algebra then there is a tree algebra $A \leqslant_{mg} B$ which is dense in B.

Proof. Fix $X \subseteq B$ and a well-ordering $\prec$ on X so that $\langle B_x : x \in X\rangle$ witnesses the minimal generation of B, where $B_x := [y \in X : y \prec x]$ for all $x \in X$.

Define $S := \{x \in X : B_x \text{ is dense in } B_x(x)\}$ and $T := X \setminus S$. From the previous lemma we know that, without loss of generality, we can assume that each $x \in T$ is an atom in $B_x(x)$. This assumption implies immediately that T satisfies (1) from Definition 1.12.

To show that $(T, \supseteq)$ is a tree in B observe that if $x, y \in T$ satisfy $x \supset y$ then $y \notin B_x(x)$ which implies $x \prec y$.

Using induction on $(X, \prec)$ and the previous lemma one can show that $[y \in X : y \prec x]$ is dense in B_x for all $x \in X$ and therefore $A := [T]$ is dense in B. It remains to show that $A \leqslant_{mg} B$. Observe that if $X = T$ then $A = B$ so we will assume $S \neq \emptyset$.

Let $\lhd$ be the binary relation on X given by $x \lhd y$ iff $(x, y \in T \wedge x \prec y) \vee (x, y \in S \wedge x \prec y) \vee (x \in T \wedge y \in S)$. In other words, $\lhd$ is a well-ordering in X that puts T before S. If we show that the sequence $C_x := [y \in X : y \lhd x]$, $x \in X$, witnesses the minimal generation of B then the proof will be done. Indeed, if z is the $\lhd$-least element of S then $A = C_z$ and the sequence $\langle C_x : x \in S\rangle$ witnesses $A \leqslant_{mg} B$.

We only have to show that $C_x \leqslant_m C_x(x)$ for each $x \in X$. In order to do this let $y \in X$ be so that $y \lhd x$. We aim to prove that $x \cap y \in C_x$ or $y - x \in C_x$ (propositions 1.8-(2) and 1.2). We will break the argument in cases.

When $x \in T$ we have $y \prec x$ and since x is an atom in $B_x(x)$ we obtain $x \subset y$ or $y \subset -x$. Moreover, $B_x <_m B_x(x)$ implies $x \cap y \in B_x$ or $x - y \in B_x$. Therefore $x \cap y = \emptyset$ or $x - y = \emptyset$.

Assume $x \in S$. If $y \prec x$ we use the fact $B_x <_m B_x(x)$ to get $x \cap y \in B_x$ or $x - y \in B_x$. Now note that $x \in S$ implies $B_x \leqslant C_x$. Finally, if $x \prec y$ then $y \in T$ which implies that y is an atom in $B_y(y)$ and $x \in B_y(y)$ so $x \cap y \in \{\emptyset, y\} \subseteq C_x$. *Q.E.D.*

Corollary 1.18. Any simplistic space has a tree π-base. In particular, every closed subset of a simplistic space has a tree π-base.

Proof. We know that any minimally generated Boolean algebra contains a dense tree algebra so it is enough to prove that every tree algebra has a dense tree.

Let B be a tree algebra and denote by $\mathcal{T}$ the collection of all trees in B that generate B. We define a partial ordering $\leqslant$ on $\mathcal{T}$ by $T_0 \leqslant T_1$ iff $T_0 \subseteq T_1$ and no element of $T_1 \setminus T_0$ contains an element from T_0. In order to show that $\mathcal{T}$ has a $\leqslant$-maximal element let $\mathcal{C}$ be a chain in $\mathcal{T}$ and define $T := \bigcup \mathcal{C}$. Note that if $t \in T_0 \in \mathcal{C}$ and $s \in T_1 \in \mathcal{C}$ satisfy $t \subset s$ then our definition of $\leqslant$ implies that $s \in T_0$. This remark implies that $T \in \mathcal{T}$ and, moreover, that T is an upper bound for $\mathcal{C}$.

Let T be a $\leqslant$-maximal element of $\mathcal{T}$. We claim that T is dense. Observe that if $F \subseteq T$ is finite and nonempty then $\bigcap F \in T \cup \{\emptyset\}$. Since T generates B one can use Proposition 1.2 and the previous remark to obtain that any element of B is of the form $\bigcup_{i<n}(t_i - \bigcup F_i)$ for some $\{F_i : i < n\} \subseteq [T]^{<\omega}$ and $\{t_i : i < n\} \subseteq T$. Hence we only have to show that if $t \in T$ and $F \in [T]^{<\omega}$ then $a := t - \bigcup F$ contains an element of T whenever $a \neq \emptyset$.

Assume otherwise. Property (1) from the definition of tree implies that, without loss of generality, we can assume that $\bigcup F \subseteq t$ and F is pairwise disjoint.

We claim that $T' := T \cup \{a\} \in \mathcal{T}$. The only non-trivial part is to show that $W := \{s \in T' : s \supset a\}$ is well-ordered by $\supseteq$. Observe that if $s \in T$ satisfies $a \subset s$ then $s \cap t \neq \emptyset$. Therefore (part (1) of Definition 1.12) $W = \{s \in T : s \supseteq t\} \cup Y$, where $Y := \{s \in T : a \subset s \subseteq t\}$ so we only have to show that $(Y, \supseteq)$ is well-ordered.

If $s \in Y$ and $r \in F$ satisfy $r \cap s \neq \emptyset$ then we get $r \subseteq s$ or $s \subseteq r$. Since $a \subset s$ the second case is impossible. Thus $r \subseteq s$. This simple remark shows that if $s \in Y$ then $s = a \cup \bigcup F'$, where $F' := \{q \in F : s \cap q \neq \emptyset\}$. Hence Y is finite. Moreover, Y is a chain because $s \cap s' \neq \emptyset$ for all $s, s' \in Y$.

To conclude the proof note that $T \neq T_0$ and $T \leqslant T_0$, a contradiction to the maximality of T. *Q.E.D.*

It is proved in [12] that $\mathrm{Fr}(\omega_1)$, the free Boolean algebra on ω_1 generators, is not minimally generated. Since the topological product 2^{ω_1} is homeomorphic to $\mathrm{St}(\mathrm{Fr}(\omega_1))$ we have that no simplistic space maps onto 2^{ω_1} (Proposition 1.14). There is a topological proof of this fact in [4].

In particular, no simplistic space maps onto $2^{\mathfrak{c}}$ or equivalently:

Proposition 1.19. No simplistic space contains a copy of $\beta\omega$, the Stone-Čech compactification of the integers.

This remark leads naturally to Efimov's problem [8].

Definition 1.13. Let X be an infinite compact Hausdorff space. X is an *Efimov space* if X contains neither a copy of $\beta\omega$ nor a copy of the ordinal $\omega + 1$ (i.e. no infinite converging sequence).

One of the well-known examples of an Efimov space was constructed by Fedorčuk [10] assuming $\diamondsuit$. His space is simplistic. We will present later an improvement in the sense that we will obtain an Efimov simplistic space assuming only CH.

1.6 T-Algebras

All the notions and results provided in this section were introduced by Koszmider in [14].

Definition 1.14. An *acceptable tree* T is a subset of $2^{<\varepsilon}$, for some ordinal ε, so that

1. If $t \in T$ then $\operatorname{dom} t$ is a successor ordinal.

2. For all $t \in 2^{<\varepsilon}$ we have that $t^\frown 0 \in T$ is equivalent to $t^\frown 1 \in T$.

3. If $t \in T$ and $\alpha < \varepsilon$ then $t \restriction \alpha + 1 \in T$.

In order to simplify notation we will adopt the following terminology, *x is minimal for (A, u)* means that $A <_m A(x)$ and $u = J_x^*$. According to Corollary 1.10, x is minimal for (A, u) iff $u = \{b \in A : b \cap x \notin A\}$ is an ultrafilter in A.

Recall that the tree-ordering on $2^{<\varepsilon}$ is given by $s < t$ iff $s \subset t$.

From now on we will start using the notation introduced in Definition 1.2.

Definition 1.15. Let T be an acceptable tree and let A be a Boolean algebra. A is a *T-algebra* if

1. There is a function $a : T \to A$ whose range, $\{a_t : t \in T\}$, generates A (it will be a common practice to write a_t instead of $a(t)$).

2. For each $t \in T$, a_t is minimal for (A_t, u_t), where $A_t := [a_s : s < t]$ and u_t is the filter generated by $\{a_s : s < t\}$ in A_t.

3. For any $t \in T$ we have $a(t^*) = -a_t$. In other words, if $s^\frown 0 \in T$ then $a_{s^\frown 0} = -a_{s^\frown 1}$.

Naturally, a collection $\{a_t : t \in T\}$ as described in the definition *witnesses that A is a T-algebra.*

Remark. Let us observe that condition (2) above holds iff (i) $\{a_s : s < t\}$ has the finite intersection property and (ii) $a_t - a_s \in A_t$ for all $s < t$. Clearly (2) implies (i) and for (ii) recall that if a_t is minimal for (A_t, u_t) then $u_t = \left(J_{a_t}^{A_t}\right)^*$, i.e. u_t is the dual filter of the ideal $\{c \in A_t : a_t \cap c \in A_t\}$; thus $c \in u_t$ iff $a_t - c = a_t \cap (-c) \in A_t$ and, in particular, (ii) is true. On the other hand, if we assume (i) and (ii) and we let A_t and u_t be as defined in (2) then u_t is an ultrafilter in A_t (Proposition 1.4) and (ii) implies that $u_t \subseteq \left(J_{a_t}^{A_t}\right)^*$ so they are equal and thus a_t is minimal for (A_t, u_t).

In general the function $a : T \to A$ does not preserve order, i.e. $s < t$ does not imply $a_s \subset a_t$. Condition (3) is symmetric since $t^\frown 0 \in T$ if and only if $t^\frown 1 \in T$, thus if $t^\frown 1 \in T$ then $a(t^\frown 1) = -a(t^\frown 0)$.

A remarkable property about T-algebras is that the branches of T determine the points of their Stone space and vice versa.

Proposition 1.20. If A is a T-algebra as witnessed by $\{a_t : t \in T\}$, then

$$\mathrm{St}(A) = \{u_b : b \text{ is a branch in } T\},$$

where u_b is the filter generated by $\{a_t : t \in b\}$ in A.

Proof. Assume that $b \subseteq T$ is a branch and let $t \in T \setminus b$ be arbitrary. There exists $s \in b$ so that $s^* \leqslant t$. Therefore $\mathrm{ht}(t) = \mathrm{ht}(s) + \alpha$, for some ordinal α (note that ht refers here to the height with respect to T). We will use induction on α to show that $Y := u_s \cup \{a_s\}$ generates an ultrafilter in $A_t(a_t)$. In particular, $u_b \cap \{a_t, -a_t\} \neq \emptyset$ so we can use Proposition 1.4 to conclude that u_b is an ultrafilter in A.

When $\alpha = 0$, $t = s^*$ and since a_s is minimal for (A_s, u_s) we have that Y generates an ultrafilter in $A_s(a_s) = A_t(a_t)$. Now assume that for all $r \in T$ and $\beta < \alpha$ satisfying $s^* \leqslant r$ and $\mathrm{ht}(r) = \mathrm{ht}(s) + \beta$ we get that Y generates an ultrafilter in $A_r(a_r)$. Let $t \in T$ be so that $s^* \leqslant t$ and $\mathrm{ht}(t) = \mathrm{ht}(s) + \alpha$.

If $\alpha = \beta + 1$ then there is $r \in T$ so that $s^* \leqslant r < t$ and $\mathrm{ht}(r) = \mathrm{ht}(s) + \beta$. In particular, $A_t = A_r(a_r)$. The minimality of a_t for (A_t, u_t) implies that u_t is the only ultrafilter in A_t that can be extended to more than one ultrafilter in $A_t(a_t)$ (Proposition 1.9). Our inductive hypothesis guarantees that F, the filter generated by Y in A_t, is an ultrafilter. Since $a_s \in Y \subseteq F$ and $-a_s = a_{s^*} \in u_t$ we get $F \neq u_t$ so F (and therefore Y) generates an ultrafilter in $A_t(a_t)$. Finally, assume that α is a limit ordinal. Since a_t is minimal for (A_t, u_t) and $a_s \notin u_t$ we get $a_t \cap a_s \in A_t$ and therefore there exist $\{F_i : i < n\}, \{H_i : i < n\} \subseteq [t^{\downarrow}]^{<\omega}$ (recall Definition 1.1) so that $a_t \cap a_s = \bigcup_{i<n}(\bigcap F_i - \bigcup H_i)$. There is $r \in t^{\downarrow}$ so that $F_i, H_i \subseteq r^{\downarrow}$ for all $i < n$ because α is limit. Then $a_t \cap a_s \in A_r$ and according to our inductive hypothesis there is a finite set $F \subseteq Y$ such that $\bigcap F \subseteq a_t \cap a_s$ or $\bigcap F \subseteq -(a_t \cap a_s)$. Thus $\bigcap F \subseteq a_t$ or $a_s \cap \bigcap F \subseteq -a_t$. Showing that Y generates an ultrafilter in $A_t(a_t)$ (Proposition 1.4).

We claim that if $u \in \mathrm{St}(A)$ then there is a branch $b \subseteq T$ so that $u_b \subseteq u$. Since u_b is an ultrafilter, $u = u_b$ and this finishes the proof of the proposition.

The proof of the previous claim is by induction. First, $T_0 = \{\emptyset^\frown 0, \emptyset^\frown 1\}$ and hence there is $b_0 \in T_0$ so that $a(b_0) \in u$. Now assume that for some ordinal α we have defined $\{b_\beta : \beta < \alpha\} \subseteq T$ satisfying

1. $b_\beta \in T_\beta$ and $a(b_\beta) \in u$ for all $\beta < \alpha$.

2. $\beta < \gamma < \alpha$ implies $b_\beta < b_\gamma$.

Let $f := \bigcup\{b_\beta : \beta < \alpha\}$. If $f^\frown 0 \in T$ then $a(f^\frown k) \in u$ for some $k < 2$ and we set $b_\alpha := f^\frown k$. Otherwise, $\{b_\beta : \beta < \alpha\}$ is already a branch in T. *Q.E.D.*

Corollary 1.21. Every T-algebra is minimally generated.

Proof. Assume that A and $\{a_t : t \in T\}$ are as in the previous argument. Let $\prec$ be a well-ordering on T so that $s \prec t$ whenever $\mathrm{ht}(s) < \mathrm{ht}(t)$.

Define $Y \subseteq T$ by $t \in Y$ iff $t = r^\frown 0$ for some $r \in 2^{<\varepsilon}$. For each $x \in Y$ let $Y_x := \{y \in Y : y \prec x\}$ and $B_x := [a_t : t \in Y_x]$. Then $\langle B_x : x \in Y\rangle$ is a continuous chain whose union is A.

Let $x \in Y$ be arbitrary. The set $S := Y_x \cup \{t^* : t \in Y_x\}$ is an acceptable tree. In fact, if $\alpha := \mathrm{ht}(x, T)$ then $S_\beta = T_\beta$ for all $\beta < \alpha$ and $S_\alpha = (T_\alpha \cap Y_x) \cup \{t^* : t \in T_\alpha \cap Y_x\}$. Thus $x_S^\downarrow = x_T^\downarrow$ and $x^\downarrow := x_S^\downarrow$ is a branch in S. Clearly $\{a_t : t \in S\}$ witnesses that B_x is an S-algebra. Let u be the ultrafilter generated by $\{a_t : t \in x^\downarrow\}$ in B_x. Since a_x is minimal for (A_x, u_x) we have that $u \cup \{x\}$ and $u \cup \{-x\}$ have the finite intersection property and therefore u can be extended to more than one ultrafilter in $B_x(x)$. On the other hand, if v is an ultrafilter in B_x and $v \neq u$ then there is a branch $b \subseteq S$ and $t \in b$ so that v is generated by $\{a_s : s \in b\}$ and $t^* \in x^\downarrow$. Hence $a_x \cap a_t \in A_x \leqslant B_x$ because a_x is minimal for (A_x, u_x) and $a_t \notin u_x$. Therefore $a_x \cap a_t \in v$ or $-(a_x \cap a_t) \in v$ which implies that there exists $c \in v$ so that $c \subseteq x$ or $c \subseteq -x$ (recall that $a_t \in v$), i.e. v generates an ultrafilter in $B_x(x)$. Proposition 1.9 implies that $B_x <_m B_x(x)$. *Q.E.D.*

CHAPTER 2: IN ZFC

As the title suggests all the results contained in this chapter are consequences of the usual axioms of set theory, ZFC, in the sense that no forcing techniques and no extra axioms are used.

The chapter is divided into two sections. The first one is dedicated to general properties of minimally generated Boolean algebras and, specifically, of T-algebras. The second section focuses on properties of those spaces which are Efimov and can be obtained as Stone spaces of T-algebras.

2.1 General Results

As promised in the previous chapter we will show that some important classes of topological spaces are simplistic. Let us start with the Cantor space: 2^ω.

For each $t \in 2^{<\omega}$ let $[t] := \{f \in 2^\omega : t \subseteq f\}$. Then $\{[t] : t \in 2^{<\omega}\}$ is a base of clopen sets for the topological product 2^ω and therefore it generates the Boolean algebra $CO(2^\omega)$.

Lemma 2.1. Let $T := 2^{<\omega} \setminus \{\emptyset\}$. Then $CO(2^\omega)$ is a T-algebra.

Proof. Set $A := CO(2^\omega)$. We will define $a : T \to A$ by letting $a(t^\frown 0) := [t^\frown 0]$ and $a(t^\frown 1) := 2^\omega \setminus a(t^\frown 0)$, for all $t \in 2^{<\omega}$.

Since, for all $t \in 2^{<\omega}$, $[t^\frown 1] = [t] \setminus a(t^\frown 0)$ one gets $A = [a_s : s \in T]$.

Let $s, t \in T$ be so that $s \leqslant t$. We claim that $a(t^\frown 0) \subseteq a_s$. To prove this note that $s = r^\frown i$ for some $r \in 2^{<\omega}$ and $i < 2$. If $i = 0$ then, by definition, $a(t^\frown 0) \subseteq a(r^\frown 0) = a_s$. When $i = 1$ we have that $a(r^\frown 0) \cap a(t^\frown 0) = \emptyset$ and hence $a(t^\frown 0) \subseteq -a(r^\frown 0) = a(r^\frown 1) = a_s$.

An immediate corollary of the claim is that, for all $t \in T$, the collection $\{a_s : s < t\}$ has the finite intersection property and therefore the filter generated by it in $A_t := [a_s : s < t]$ is an ultrafilter (Proposition 1.4).

Another consequence of the claim is that $(-a_s) \cap a(t^\frown 0) = \emptyset \in A_{t^\frown 0}$ for all $s \leqslant t$. Hence (propositions 1.8-(2) and 1.2) we obtain that $a(t^\frown 0)$ is minimal for $(A_{t^\frown 0}, u_{t^\frown 0})$.

Which implies that $a(t^\frown 1)$ is minimal for $(A_{t^\frown 1}, u_{t^\frown 1})$ and therefore condition (2) from Definition 1.15 holds. *Q.E.D.*

If X is metric compact and zero-dimensional, then X has countable weight and therefore is homeomorphic to a closed subspace of 2^ω. Hence propositions 1.21 and 1.14 imply that X is simplistic. A more elaborated example is the following.

Example 2.1. If X is compact Hausdorff and scattered then $CO(X)$ is a T-algebra for some acceptable tree.

Proof. Since X is compact and scattered there is an ordinal δ so that $X = \bigcup\{X_\alpha : \alpha \leqslant \delta\}$ where X_α is the set of isolated points of $X \setminus \bigcup\{X_\xi : \xi < \alpha\}$ and $X_\delta \neq \emptyset$.

Let $\prec$ be a well-ordering for X so that if $\alpha < \beta$ then $x \prec y$ for all $x \in X_\alpha$ and $y \in X_\beta$. Let z be the $\prec$-maximum element of X_δ (observe that X_δ is finite because X is compact so z is well defined). Note that z is actually the maximum element of X.

For all $\alpha \leqslant \delta$ and for each $x \in X_\alpha \setminus \{z\}$ let W_x be a clopen subset of X satisfying $W_x \setminus \bigcup\{X_\xi : \xi < \alpha\} = \{x\}$. Note that $W_z := X \setminus \bigcup\{W_x : x \in X_\delta \setminus \{z\}\}$ is a clopen set so that $W_z \cap X_\delta = \{z\}$.

Let ε be the order type of $(X \setminus \{z\}, \prec)$. Then there is a bijection $h : \varepsilon \to X \setminus \{z\}$ so that $h(\alpha) \prec h(\beta)$ whenever $\alpha < \beta$.

Let $f : \varepsilon \to 2$ be the constant zero function, i.e. $f(\alpha) = 0$ for all $\alpha < \varepsilon$. Then $T := \{(f \upharpoonright \alpha)^\frown i : \alpha < \varepsilon \text{ and } i < 2\}$ is an acceptable tree and $T \subseteq 2^{<\varepsilon}$. For all $\alpha < \varepsilon$ define

$$a((f \upharpoonright \alpha)^\frown 0) := X \setminus W_{h(\alpha)} \text{ and } a((f \upharpoonright \alpha)^\frown 1) := W_{h(\alpha)}.$$

Given $\alpha < \varepsilon$ we have that $\{W_{h(\alpha)} \setminus \bigcup\{W_{h(\xi)} : \xi \in H\} : H \in [\alpha]^{<\omega}\}$ is a local base for X at $h(\alpha)$ and therefore $\{a_t : t \in T\}$ generates $CO(X)$.

Fix $\alpha < \varepsilon$ and let $t := (f \upharpoonright \alpha)^\frown 0$. We will show that (i) $\{a_s : s < t\}$ has the finite intersection property and (ii) $\forall s < t(a_t - a_s \in A_t)$ (see the Remark following Definition 1.15). Let $s < t$ be arbitrary. Then $s = (f \upharpoonright \beta)^\frown 0$ for some $\beta < \alpha$. Therefore $h(\alpha) \in a_s$ and hence (i) holds. Since $a_t - a_s$ is compact open and $a_t - a_s = W_{h(\beta)} \setminus W_{h(\alpha)}$, there is a finite set $F \subseteq \beta + 1$ so that $a_t - a_s = \bigcup\{W_{h(\xi)} \setminus \bigcup\{W_{h(\eta)} : \eta \in H_\xi\} : \xi \in F\}$, where $H_\xi \in [\xi]^{<\omega}$

for each $\xi \in F$. This proves (ii). The argument for t^* is similar so we omit it.

Q.E.D.

Recall that a subset A of a topological space is *crowded* if every point of A is an accumulation point of A.

We will use the following result in the proof of Lemma 2.2. If $S = \langle Y_\alpha, g_{\alpha\beta} : \alpha < \beta < \varepsilon \rangle$ is an inverse system such that all bonding maps are onto and $H := \{\alpha < \varepsilon : Y_\alpha \text{ is crowded}\}$ is cofinal with ε, then $Y := \varprojlim S$ is crowded. To prove this let $y \in Y$ be arbitrary and let U be an open subset of Y which contains y. Since H is cofinal, there is $\alpha \in H$ so that $y \in g_{\alpha\varepsilon}^{-1}[W] \subseteq U$ for some open set $W \subseteq Y_\alpha$ (Proposition 1.1-(2)). Thus $W \setminus \{g_{\alpha\varepsilon}(y)\} \neq \emptyset$ and hence $U \setminus \{y\} \neq \emptyset$.

Lemma 2.2. Let X be the limit of the simplistic system $S = \langle X_\alpha, f_{\alpha\beta} : \alpha < \beta < \varepsilon \rangle$. If $A, B \subseteq X$ and $\alpha < \varepsilon$ are so that $\overline{f_{\alpha\varepsilon}(A)} \cap \overline{f_{\alpha\varepsilon}(B)}$ contains a crowded set Y_α then there exists a crowded set Y_ε contained in $\overline{A} \cap \overline{B}$ so that $f_{\alpha\varepsilon}(Y_\varepsilon) = Y_\alpha$.

Proof. For each $\alpha < \gamma$ let $Y_\alpha := f_{\alpha\gamma}[Y_\gamma]$. To prove the lemma it suffices to show that there is a sequence $\langle Y_\tau : \gamma \leqslant \tau < \varepsilon \rangle$ so that each Y_τ is a crowded subset of $\overline{f_{\tau\varepsilon}[A]} \cap \overline{f_{\tau\varepsilon}[B]}$ and $f_{\alpha\tau}[Y_\tau] = Y_\alpha$ for all $\gamma \leqslant \alpha < \tau$. Indeed, given a sequence as the one described above, we can let Y_ε be the limit of the inverse system $\langle Y_\alpha, f_{\alpha\beta} \restriction Y_\beta : \alpha < \beta < \varepsilon \rangle$ to obtain a crowded subset of X (see the remark previous to Lemma 2.2) which satisfies $f_{\gamma\varepsilon}[Y_\varepsilon] = Y_\gamma$ and

$$Y_\varepsilon \subseteq \bigcap_{\gamma < \alpha < \varepsilon} f_{\alpha\varepsilon}^{-1}(\overline{f_{\alpha\varepsilon}(A)} \cap \overline{f_{\alpha\varepsilon}(B)}) = \overline{A} \cap \overline{B}$$

(Proposition 1.1). We will use induction on τ. Assume that Y_ξ has been defined for all $\xi < \tau$.

If $\tau = \alpha + 1$ then $f_{\alpha\tau}$ witnesses that X_τ is a simple extension of X_α and therefore there is a point $p \in X_\tau$ so that $Y_\tau := f_{\alpha\tau}^{-1}[Y_\alpha] \setminus \{p\}$ is a crowded subset of X_τ. Since $f_{\beta\tau} = f_{\beta\alpha} \circ f_{\alpha\tau}$ for all $\beta \leqslant \alpha$, a straightforward argument shows that Y_τ is as required. When τ is a limit ordinal it is enough to let Y_τ be the limit of the inverse system $\langle Y_\alpha, f_{\alpha\beta} \restriction Y_\beta : \alpha < \beta < \tau \rangle$.

Q.E.D.

A situation we will face several times is the following: X is a compact Hausdorff zero-dimensional topological space for which $CO(X)$ is a T-algebra as witnessed by $\{a_t : t \in T\}$. Given $x \in X$, a straightforward application of propositions 1.6 and 1.20 gives the existence of a branch b in T so that

1. the ultrafilter $\{c \in CO(X) : x \in c\}$ is generated by $\{a_t : t \in b\}$.

We claim that (1) is equivalent to the following statements.

2. $\{\bigcap\{a_t : t \in F\} : F \in [b]^{<\omega}\}$ is a local base for X at x.

3. $\bigcap\{a_t : t \in b\} = \{x\}$.

Indeed, (2) is a consequence of (1) because X is zero-dimensional and (2) implies (3) because X is Hausdorff. To finish the argument let us assume (3) and let $x \in c \in CO(X)$. Then $\bigcap\{a_t : t \in b\} \subseteq c$ and since X is compact, there is a finite set $F \subseteq b$ so that $\{a_t : t \in F\} \subseteq c$. Therefore (1) holds.

We will use the equivalence of these properties frequently.

Lemma 2.3. Let X be a compact Hausdorff zero-dimensional topological space for which $CO(X)$ is a T-algebra as witnessed by $\{a_t : t \in T\}$.

1. If $s, t \in T$ are comparable and $W := \bigcap\{a_r : r < s\} - a_s$ then $a_t \cap W \in \{\emptyset, W\}$. In other words, a_t does not split W.

2. If $\{t_n : n \in \omega\}$ is a chain in T and $\{x_n, y_n\} \subseteq \bigcap\{a_r : r < t_n\} - a(t_n)$ for all $n \in \omega$ then $\overline{\{x_n : n \in \omega\}} \cap \overline{\{y_n : n \in \omega\}} \neq \emptyset$.

Proof. To prove (1) let us start by fixing a branch $b \subseteq T$ so that $s, t \in b$. Then $T' := b \cup \{t^* : t \in b\}$ is an acceptable tree and $B := [a_r : r \in T']$ is a T'-algebra. Since $s^{\downarrow} \cup \{s^*\}$ (Definition 1.1) is a branch in T' we have that $\{a_r : r < s\} \cup \{-a_s\}$ generates an ultrafilter u in B. Now note that if $a_t \cap W \neq \emptyset$ then $\{a_t, -a_s\} \cup \{a_r : r < s\}$ has the finite intersection property and therefore we can extend it to an ultrafilter in B. Clearly that ultrafilter must be u and hence $a_t \in u$. This implies that $\{-a_t, -a_s\} \cup \{a_r : r < s\}$ cannot be extended to an ultrafilter; i.e. $W - a_t = \emptyset$.

In order to prove the second part of the Lemma we can assume, without loss of generality, that $t_n < t_{n+1}$ for all $n \in \omega$. Therefore, $\{x_k : k > n\} \subseteq \bigcap\{a_r : r < t_n\}$ for each $n \in \omega$. Thus there exists $z \in \overline{\{x_n : n \in \omega\}} \cap \bigcap_{n<\omega} \bigcap\{a_r : r \leqslant t_n\}$. The remarks preceeding the Lemma show that there is a branch $b \subseteq T$ so that $\bigcap\{a_t : t \in b\} = \{z\}$. Hence $\{t_n : n \in \omega\} \subseteq b$. We will show that $z \in \overline{\{y_n : n \in \omega\}}$. Let F be a finite subset of b. There is an integer m so that $x_m \in \bigcap\{a_t : t \in F\}$ so we can invoke part (1) to obtain that, for each $t \in F$, $\bigcap\{a_r : r < t_m\} - a(t_m) \subseteq a_t$. In particular, $y_m \in \bigcap\{a_t : t \in F\}$. *Q.E.D.*

The following result shows that the converse of Corollary 1.21 is false.

Theorem 2.4. There is a minimimally generated Boolean algebra which is not a T-algebra for any acceptable tree T.

Proof. Our strategy is to construct a simplistic system so that the clopen algebra of its limit is as required in the statement.

Let us start by enumerating all rational numbers in the Cantor set: $2^\omega \cap \mathbb{Q} = \{q_n : n \in \omega\}$. Define, by induction, a sequence $\langle Y_0^m, g_m^{m+1} : m \in \omega\rangle$ of topological spaces and continuous maps so that

1. $Y_0^0 := 2^\omega$,
2. $Y_0^{m+1} = Y_0^m \oplus \{(q_0, m+1)\}$, and
3. $g_m^{m+1} : Y_0^{m+1} \to Y_0^m$ satisfies $g_m^{m+1} \restriction Y_0^m$ is the identity map and $g_m^{m+1}(q_0, m+1) = q_0$.

In other words, Y_0^{m+1} is obtained from Y_0^m by splitting q_0 into two points and making one of them isolated.

We define $g_m^{m+k+1} := g_{m+k}^{m+k+1} \circ g_m^{m+k}$ by induction on k to obtain $\langle Y_0^m, g_n^m : n < m < \omega\rangle$, an inverse system based on simple extensions. Let Y_1 be its limit and let $h_0^m : Y_1 \to Y_0^m$ be the corresponding projection map. Observe that Y_1 contains two kinds of points: If $\langle x_n : n \in \omega\rangle \in Y_1$ then either $x_0 = x_n$ for all $n \in \omega$ or there exists $k \in \omega$ so that $x_i = q_0$ for $i \leqslant k$ and $x_i = (q_0, k)$ when $i > k$. Moreover, the points belonging to the second kind form a sequence converging to $\langle q_0, q_0, \ldots\rangle$. From here one can show that Y_1 is homeomorphic to the subspace $(2^\omega \times \{\omega\}) \cup (\{q_0\} \times \omega)$ of the topological product $2^\omega \times (\omega + 1)$, i.e. Y_1 is the result of adding a converging sequence to q_0 to the space $Y_0 := 2^\omega$.

Applying the process described above to Y_1 and q_1 one gets Y_2 and, in general, an inverse system $\langle Y_m, h_n^m : n < m < \omega\rangle$, where each Y_m is homeomorphic to the subspace $(2^\omega \times \{\omega\}) \cup (\{q_i : i \leqslant m\} \times \omega) \subseteq 2^\omega \times (\omega + 1)$ and h_m^{m+1} collapses the new converging sequence to a point: $h_m^{m+1}(q_m, i) = (q_m, \omega)$ for all $i < \omega$. Let X_0 be the limit of this inverse system.

X_0 is homeomorphic to the space obtained by endowing the set $2^\omega \cup ((\mathbb{Q} \cap 2^\omega) \times \omega)$ with the following topology: Each (q, m) is isolated and a local base for $r \in 2^\omega$ is given by all sets of the form

$$W \cup ((W \cap \mathbb{Q}) \times \omega) \setminus F),$$

where W is an arbitrary clopen subset of 2^ω which contains r and F is a finite set. Moreover, when $r \notin \mathbb{Q}$ one can take $F = \emptyset$.

Let $\{(r_\alpha, m_\alpha) : \alpha < \mathfrak{c}\}$ be an enumeration of all pairs (r, m) so that

(a) $r : 2^{<\omega} \to \mathbb{Q} \cap 2^\omega$ and $m : 2^{<\omega} \to \omega$.

(b) For all $g : \omega \to 2$ the sequence $\langle r(g \restriction n) : n \in \omega\rangle$ converges.

(c) If $f, g \in {}^\omega 2$ satisfy $f \neq g$ then

$$\lim_{n\to\infty} r(f \restriction n) \neq \lim_{n\to\infty} r(g \restriction n).$$

For each $\alpha < \mathfrak{c}$ we will obtain, by transfinite induction, a function $g_\alpha \in {}^\omega 2$ so that $x_\alpha := \lim_{n\to\infty} r_\alpha(g_\alpha \restriction n)$ satisfies $x_\alpha \notin \{x_\xi : xi < \alpha\} \cup \mathbb{Q}$ and, at the same time, we will construct a topology $\mathcal{T}_\alpha$ for $X_\alpha := X_0 \cup \{(x_\xi, 0) : \xi < \alpha\}$ in such a way that $S := \langle X_\alpha, f_{\alpha\beta} : \alpha < \beta < \mathfrak{c}\}$ is a continuous inverse system and

1. For all $\beta < \alpha$, $f_{\beta\alpha}$ is given by $f_{\beta\alpha}(x_\xi, 0) = x_\xi$ whenever $\beta \leqslant \xi < \alpha$ and $f_{\beta\alpha} \restriction X_\beta$ is the identity map.
2. The sequence $e_\alpha := \{(r_\alpha(g_\alpha \restriction n), m_\alpha(n)) : n \in \omega\}$ converges to x_α in $\mathcal{T}_\alpha$.
3. $\mathcal{T}_\alpha \cup \{e_\alpha \cup \{(x_\alpha, 0\}, X_\alpha \setminus e_\alpha\}$ is a subbase for $\mathcal{T}_{\alpha+1}$.

Observe that according to this prescription the inverse system is based on simple extensions. More precisely, for each α the point x_α is doubled and e_α becomes a converging sequence to the twin of x_α, namely $(x_\alpha, 0)$.

We only have to explain how to get $\mathcal{T}_{\alpha+1}$ from $\mathcal{T}_\alpha$. Condition (c) above implies that

$$|\{x_\xi : \xi < \alpha\}| < \mathfrak{c} = |\{\lim r_\alpha(g \upharpoonright n) : g \in {}^\omega 2\}|$$

and therefore we can find $g_\alpha \in {}^\omega 2$ for which $x_\alpha := \lim r_\alpha(g_\alpha \upharpoonright n)$ works.

As one can verify, a local base at x_α in $\mathcal{T}_\alpha$ is given by all sets of the form

$$(W \setminus \bigcup\{e_\xi : \xi \in F\}) \cup ((W \cap \{x_\xi : \xi \in \alpha \setminus F\}) \times \{0\})$$

where W is a clopen set in X_0 containing x_α and F is an arbitrary finite subset of α. In particular, property (2) holds and this completes the induction.

Let X be the limit of S. By gluing all the inverse systems involved in the construction of X one can verify that X is simplistic and hence $A := CO(X)$ is minimally generated.

X will be identified with $X_0 \cup \{(x_\alpha, 0) : \alpha < \mathfrak{c}\}$ in such a way that the subspace topology for $2^\omega \subseteq X$ is the Cantor set topology and $\{\{(x_\alpha, 0)\} \cup e_\alpha \setminus F : F \in [e_\alpha]^{<\omega}\}$ is a local base of clopen sets at $(x_\alpha, 0)$ for all $\alpha < \mathfrak{c}$.

Seeking a contradiction let us assume that $\{a_t : t \in T\}$ witnesses that A is a T-algebra for some acceptable tree T. For each $t \in 2^{<\omega} \setminus \{\emptyset\}$ we will define inductively $f(t) \in T$, $q(t) \in \mathbb{Q}$, $\ell(t) \in \omega$, and $W(t) \in CO(2^\omega)$ in such a way that the following is true for all t and all $i < 2$.

(1t) If $s < t$ then $f(s) < f(t)$.

(2t) $f(t^*) = f(t)^*$.

(3t) $W(t)$ has diameter $< 1/2^{|t|}$.

(4t) $q(t) \in W(t) \subseteq a(f(t^*))$.

(5t) $W(t^\frown i) \subseteq W(t) \subseteq a_s$ for all $s < f(t^\frown i)$.

(5t) $a(f(t^\frown i)) \cap W(t) \neq \emptyset$.

(6t) $(q(t), \ell(t)) \in a(f(t^*))$.

Let $t_0 \in T$ be so that $a(t_0) \cap 2^\omega \neq \emptyset \neq 2^\omega - a(t_0)$ but $2^\omega \subseteq a_s$ for all $s < t_0$. Define $f(\emptyset^\frown 0) = t_0$ and $f(\emptyset^\frown 1) = t_0^*$. Let $i < 2$ be arbitrary. Since $a(f(\emptyset^\frown i))$ is open, there exist a rational number $q(\emptyset^\frown i) \in a(f(\emptyset^\frown (1-i)))$ and an integer $\ell(\emptyset^\frown i)$ such that $(q(\emptyset^\frown i), \ell(\emptyset^\frown i)) \in a(f(\emptyset^\frown (1-i)))$. Let $W(\emptyset^\frown i)$ be a clopen subset of the Cantor set with diameter $< 1/2$ and satisfying $q(\emptyset^\frown i) \in W(\emptyset^\frown i) \subseteq a(f(\emptyset^\frown (1-i)))$. This is the base of the induction.

Assume that for some $n \in \omega$ and for all $t \in 2^{\leqslant n}$ we have defined $f(t)$, $q(t)$, $\ell(t)$, and $W(t)$ as required. Fix $t \in 2^{\leqslant n}$ and let $\tilde{t} \in T$ be so that $f(t) < \tilde{t}$, $W(t) \cap a(\tilde{t}) \neq \emptyset \neq W(t) - a(\tilde{t})$, and $W(t) \subseteq a_s$ for all $s < \tilde{t}$. Set $f(t^\frown 0) = \tilde{t}$ and $f(t^\frown 1) = \tilde{t}^*$. As before, for each $i < 2$ we can find $q(t^\frown i)$, $\ell(t^\frown i)$, and $W(t^\frown i)$ satisfying all the requirements and this completes the induction.

The rules $t \mapsto q(t)$ and $t \mapsto \ell(t)$ define maps $q : 2^{<\omega} \to 2^\omega \cap \mathbb{Q}$ and $\ell : 2^{<\omega} \to \omega$. Conditions (1t), (2t), and (4t) imply that there exists $\alpha < \mathfrak{c}$ so that $(q, \ell) = (r_\alpha, m_\alpha)$. Let $t_n := g_\alpha \restriction n$ for all $n \in \omega$. The sets $H := \{q(t_n) : n \in \omega\}$ and e_α have disjoint closures in X. On the other hand, conditions (4t) and (6t) give

$$\{q(t_n), (q(t_n), \ell(t_n))\} \subseteq \bigcap \{a_s : s < f(t_n)\} - a(f(t_n))$$

and since $\{f(t_n) : n \in \omega\}$ is a chain in T, Lemma 2.3 guarantees that $\overline{H} \cap \overline{e_\alpha} \neq \emptyset$. A contradiction. *Q.E.D.*

Lemma 2.5. Let X be a compact Hausdorff zero-dimensional space. If $c \subseteq X$ and $p \in c$ satisfy

1. c is closed and
2. $\{p\} = c \cap \overline{X \setminus c}$

then c is minimal for $(CO(X), u)$, where $u := \{a \in CO(X) : p \in a\}$.

Proof. Let $A := CO(X)$. Clearly u is an ultrafilter in A and condition (2) implies that $u \cup \{c\}$ and $u \cup \{-c\}$ have the finite intersection property. Therefore u extends to more than one ultrafilter in $A(c)$.

To show that u is the only one with this property let v be an ultrafilter in A so that $u \neq v$. Since X is compact Hausdorff, there is $q \in X$ so that $\{q\} = \bigcap v$. The assumption $u \neq v$ implies that $q \neq p$ and therefore $q \notin c$ or $q \notin \overline{X \setminus c}$. Thus there exists $a \in v$ satisfying $a \cap c = \emptyset$ or $a - c = \emptyset$. In other words, $a \subseteq -c$ or $a \subseteq c$. This shows that the filter generated by v in $A(c)$ is an ultrafilter. *Q.E.D.*

The proof of the following result follows from Definition 1.15 and it provides a method to create extensions of T-algebras.

Lemma 2.6. Let A be a T-algebra as witnessed by $\{a_t : t \in T\}$ and let b be a branch in T. If $f := \bigcup b$ then

1. $T' := T \cup \{f^\frown 0, f^\frown 1\}$ is an acceptable tree.

2. If x is minimal for (A, u_b), then $A(x)$ is a T'-algebra as witnessed by $\{a_t : t \in T'\}$, where $a(f^\frown 0) := x$ and $a(f^\frown 1) := -x$.

In [13] Koppelberg showed that the topological product of simplistic spaces may not be simplistic. Her example is the product of the Alexandroff double arrow on the Cantor set and 2^ω. In view of Theorem 2.4 one can ask if the situation is different for the class of Stone spaces of T-algebras.

Recall that one can identify 2^ω with the Cantor Middle Third Set using ternary expansions. Hence we will consider 2^ω as a subspace of the interval $[0, 1]$.

The Alexandroff double arrow on 2^ω is the subspace $2^\omega \times \{0, 1\}$ of the square $[0, 1] \times [0, 1]$ with the lexicographic order topology. Another way to obtain this space is by splitting each point x of the Cantor set into two points x^- and x^+ and defining an order by declaring $x^- < x^+$ and using the induced order of $[0, 1]$ otherwise.

Example 2.2. Let X be the Alexandroff's double arrow on 2^ω. Then $CO(X)$ is a T-algebra.

Proof. We know that $CO(2^\omega)$ is a T'-algebra where $T' := 2^{<\omega}$. Let $\{a_t : t \in T'\}$ be as described in the proof of Lemma 2.1. For each $t \in T'$ let $c_t := a_t \times \{0, 1\}$. The Boolean algebra generated by $\{c_t : t \in T'\}$ is isomorphic to $CO(2^\omega)$ and if, for each $f \in 2^\omega$, we let $c(f^\frown 0) := (2^\omega \cap [0,f] \times \{0\}) \cup (2^\omega \cap [0,f) \times \{1\})$ and $c(f^\frown 1) := (2^\omega \cap (f,1] \times \{0\}) \cup (2^\omega \cap [f,1] \times \{1\})$ then $\{c_t : t \in 2^{\leqslant\omega}\}$ generates $CO(X)$ and $c(f^\frown 0)$ is minimal for $([c(f \restriction n) : 0 < n < \omega], u_f)$, where u_f is the ultrafilter generated by $\{c(f \restriction n) : 0 < n < \omega\}$. Therefore (Lemma 2.6) $CO(X)$ is a $2^{\leqslant\omega}$-algebra as witnessed by $\{c_t : t \in 2^{\leqslant\omega}\}$. *Q.E.D.*

2.2 Efimov T-algebras

Definition 2.1. A Boolean algebra A will be called Efimov if $\mathrm{St}(A)$ is an Efimov space. In particular, an Efimov T-algebra is a T-algebra for which its Stone space is Efimov.

If A is an Efimov T-algebra then (Proposition 1.6) $\mathrm{St}(A)$ is homeomorphic to $\mathrm{St}(CO(X))$, where $X := \mathrm{St}(A)$, and therefore the Boolean algebras A and $CO(X)$ are isomorphic. Thus $CO(X)$ is an Efimov T-algebra. In other words, the existence of an Efimov T-algebra is equivalent to the existence of a zero-dimensional Efimov space X for which $CO(X)$ is a T-algebra.

We will denote by ω^* the collection of all nonprincipal ultrafilters in ω.

Given a space X, a sequence $\{x_n : n \in \omega\} \subseteq X$ and $r \in \omega^*$ we will say that the point $x \in X$ is an r-limit of $\{x_n : n \in \omega\}$ (in symbols, $x = r\text{-}\lim x_n$) if $\{n \in \omega : x_n \in U\} \in r$ for any neighborhood U of x. X will be called r-compact if every sequence in X has an r-limit.

If X is compact and $r \in \omega^*$ then for any sequence $\{x_n : n \in \omega\} \subseteq X$ we have that $\bigcap\{\overline{\{x_n : n \in a\}} : a \in r\} \neq \emptyset$ (finite intersection property) and therefore $\{x_n : n \in \omega\}$ has an r-limit.

A straightforward argument shows that r-limits are preserved by continuous functions, i.e. if $f : X \to Y$ is continuous and $\{x_n : n \in \omega\}$ is a sequence in X whose r-limit is x then $f(x)$ is the r-limit of $\{f(x_n) : n \in \omega\}$.

Recall that the Scarborough-Stone question is: *Must every product of sequentially compact spaces be countably compact?*

We will use the following property in the proof of Theorem 2.7: If $\{X_r : r \in \omega^*\}$ is a family of topological spaces so that X_r is not r-compact for each r then $X := \prod\{X_r : r \in \omega^*\}$ is not countably compact. To show that this is the case fix, for each r, a sequence $\{x_n^r : n \in \omega\}$ in X_r which has no r-limit and define $x_n := \langle x_n^r : r \in \omega^*\rangle$ for all $n \in \omega$. We claim that $S := \{x_n : n \in \omega\}$ has no cluster point. Indeed, if $y \in X$ is a cluster point for S then, by definition, for each neighborhood U of y the set $U^\dagger := \{n \in \omega : x_n \in U\}$ is infinite and therefore $\{U^\dagger : U$ is a neighborhood of $y\}$ generates a nonprincipal ultrafilter $u \in \omega^*$ and since the projection map $\pi_u : X \to X_u$ is continuous, it must be the case that $\pi_u(y)$ is a u-limit of $\{x_n^u : n \in \omega\}$. A contradiction.

Theorem 2.7. If there is an Efimov T-algebra then the Scarborough-Stone question has a negative answer.

Proof. Let X be a zero-dimensional Efimov space and assume that $CO(X)$ is a T-algebra as witnessed by $\{a_t : t \in T\}$ (see the remark following Definition 2.1). We will construct a family $\{X_r : r \in \omega^*\}$ of sequentially compact spaces so that X_r is not r-compact for each r.

As we mentioned after Definition 2.1 for any point $x \in X$ there is a branch b in T so that $\bigcap\{a_t : t \in b\} = \{x\}$. Note that if b is finite then x is an isolated point of X. Since X is Efimov, X is infinite and compact so X possesses an accumulation point and hence T has an infinite branch. This branch will hit, for each integer n, the nth level of the tree, $T(n)$. Let $\{t_n : n \in \omega\} \subseteq T$ be an incressing sequence satisfying $t_n \in T(n)$, for all $n \in \omega$.

For each integer n fix a branch $b_n \subseteq T$ satisfying $\{t_k : k < n\} \cup \{t_n^*\} \subseteq b_n$. Then $\bigcap\{a_s : s \in b_n\} = \{w_n\}$ for some $w_n \in X$. Set $W := \{w_k : k \in \omega\}$ and note that our construction gives $W \cap \bigcap\{a_s : s < t_n\} - a(t_n) = \{w_n\}$.

Let $r \in \omega^*$ be arbitrary. The fact that X is compact implies the existence of $x_r \in X$ so that $x_r = r\text{-}\lim w_n$. Thus there is a branch b_r satisfying $\bigcap\{a_s : s \in b_r\} = \{x_r\}$. Moreover, $x_r \notin W$ because W is discrete and infinite. Define $B_r := [a_t : t \in b_r]$ and let u_r be the only ultrafilter in B_r such that $\{a_t : t \in b_r\} \subseteq u_r$. Observe that the map $f_r : X \to \mathrm{St}(B_r)$ given by $f_r(x) := \{c \in B_r : x \in c\}$ is continuous, onto and satisfies $f_r^{-1}[u_r] = \{x_r\}$. Therefore the subspace $X_r := \mathrm{St}(B_r) \setminus \{u_r\}$ is not r-compact because $\{f_r(w_n) : n \in \omega\}$ has no r-limit.

The only thing left is to show that X_r is sequentially compact. To do this we will prove that X_r is scattered (i.e. X_r does not contain a crowded subspace) and countably compact.

Let $\{x_n : n \in \omega\}$ be an infinite subset of X_r. For each $n \in \omega$ there is $y_n \in X$ so that $f_r(y_n) = x_n$. Since X is Efimov, $\{y_n : n \in \omega\}$ possesses more than one accumulation point. In particular, $\{y_n : n \in \omega\}$ accumulates to some $y \in X \setminus \{x_r\}$ and thus $f_r(y)$ is an accumulation point of $\{x_n : n \in \omega\}$ in X_r. Hence X_r is countably compact. It is worth mentioning that this is the only part of the proof where being Efimov is used.

Note that $T' := b_r \cup \{t^* : t \in b_r\}$ is an acceptable tree and that B_r is a T'-algebra as witnessed by $\{a_t : t \in T'\}$. If b is a branch in T' then $b = b_r$ or $b = t^{\downarrow} \cup \{t^*\}$ for some $t \in b_r$. Therefore, for each $y \in X_r$ there exists $t_y \in b_r$ so that y is the ultrafilter generated by $\{a_s : s < t_y\} \cup \{-a(t_y)\}$. We are ready to show that X_r is scattered: let E be a nonempty subset of X_r; since b_r is well-ordered, there exists $z \in E$ so that $t_z = \min\{t_y : y \in E\}$. By definition, $U := \{p \in \mathrm{St}(B_r) : -a(t_z) \in p\}$ is a clopen subset of $\mathrm{St}(B_r)$ and our choice of t_z guarantees that $U \cap E = \{z\}$ so E has an isolated point.

To finish the argument let us show that if Y is scattered and countably compact then Y is sequentially compact. Let $H \in [Y]^{\omega}$. Since Y is scattered there is an ordinal δ for which $Y = \bigcup\{Y_\alpha : \alpha < \delta\}$, where Y_α is the set of isolated points of $Y \setminus \bigcup\{Y_\xi : \xi < \alpha\}$. Let $\alpha < \delta$ be the least ordinal for which $Y_\alpha \cap H' \neq \emptyset$, where H' is the set of accumulation points of H. If $z \in H' \cap Y_\alpha$ then, by definition, there is an open set U so that $U \setminus \bigcup\{Y_\xi : \xi < \alpha\} = \{z\}$. Clearly $S := H \cap U$ is infinite. We claim that S converges to z, i.e. any neighborhood of z contains all but finitely many elements of S. Let V be a neighborhood of z. There is an open set V_0 so that $z \in V_0 \subseteq \overline{V_0} \subseteq U \cap V$. Then $U \setminus \overline{V_0}$ is an open set contained in $\bigcup\{Y_\xi : \xi < \alpha\}$ and therefore it does not contain an accumulation point of H so $H \cap U \setminus \overline{V_0}$ is finite. Hence $|S \setminus V| < \omega$ as claimed. *Q.E.D.*

We mentioned in the previous chapter that no simplistic space maps onto the product 2^{ω_1}. As the following result guarantees, there is a simpler product which is not the continuous image of any Stone space of an Efimov T-algebra.

A continuous map $f : X \to Y$ is *irreducible* if f is onto and whenever F is a proper closed subset of X we get $f[F] \neq Y$. Observe that if U is an open subset of X then $F := X\setminus(U\cap f^{-1}[\mathrm{int} f[X\setminus U]]$ is closed and $f[F] = Y$ which implies that $f[U]\cap \mathrm{int} f[X\setminus U] = \emptyset$ and therefore

$$Y \setminus f[X \setminus U] \subseteq f[U] \subseteq \overline{Y \setminus f[X \setminus U]}.$$

Hence, when f is closed and irreducible, $f[\overline{U}]$ is a regular closed subset of Y. We will use this property several times during the proof of Theorem 2.8.

Assume that $f : X \to Y$ is continuous and onto and X is compact. We claim that there exists a closed set $K \subseteq X$ so that $f[K] = Y$ and $f \restriction K$ is irreducible. This is a consequence of Zorn's Lemma. Let $\mathcal{F}$ be the collection of all closed sets $F \subseteq X$ so that $f[F] = Y$. Then $X \in \mathcal{F}$ and if $\mathcal{C}$ is a $\subseteq$-chain in $\mathcal{F}$ then $\{F \cap f^{-1}[y] : F \in \mathcal{C}\}$ has the finite intersection property for all $y \in Y$. Indeed, if $\{F_k : k < n\} \subseteq \mathcal{C}$ then there is $m < n$ so that $\bigcap\{F_k : k < n\} = F_m$ and hence $F_m \cap f^{-1}[y] \neq \emptyset$. Thus $\bigcap \mathcal{C} \in \mathcal{F}$ and therefore $\mathcal{F}$ has a $\subseteq$-minimal element K which is the closed set whose existence was claimed.

Theorem 2.8. If X is the Stone space of an Efimov T-algebra, then X does not map continuously onto $Y := (\omega_1 + 1) \times (\omega + 1)$.

Proof. Seeking a contradiction assume that $f : X \to Y$ is continuous and onto.

Let K be a compact subset of X for which $f[K] = Y$ and $f \restriction K$ is irreducible. Since $f \restriction K$ is closed and $(\omega_1, \omega) \in \overline{\{\omega_1\} \times \omega}$, there exists $q \in K \cap \overline{f^{-1}[\{\omega\} \times \omega]}$. Observe that for any neighborhood U of q the set $\{n \in \omega : (\omega_1, n) \in f[U \cap K]\}$ is infinite.

Let $\{a_t : t \in T\}$ be a family witnessing that $CO(X)$ is a T-algebra (see the remark following Definition 2.1) and fix a brach $b \subseteq T$ so that $\bigcap\{a_s : s \in b\} = \{q\}$. We will use the following notation: for each $t \in T$ let $\Delta(t) := \bigcap\{a_s : s \leqslant t\}$.

We claim that there are two sequences $\{t_{n_i} : i \in \omega\}$ and $\{n_i : i < \omega\}$ so that $t_{n_{i+1}}$ is the least node in b for which there is an integer $n_{i+1} > n_i$ so that

$$(\omega_1, n_{i+1}) \in f\left[a(t^*_{n_{i+1}}) \cap \bigcap\{a(t_{n_j}) : j < i\} \cap K\right]$$

and n_{i+1} is the smallest integer having this property.

The construction uses finite induction. For each $x \in K \cap f^{-1}[\{\omega\} \times \omega]$ there exists a branch b_x in T so that $\bigcap\{a_s : s \in b_x\} = \{x\}$; hence $b_x \neq b$ so there is $s_x \in b$ such that $s^*_x \in b_x$. Let $t_{n_0} := \min\{t_x : x \in K \cap f^{-1}[\{\omega\} \times \omega]\}$. Clearly $t_{n_0} = t_x$ for some $x \in K \cap f^{-1}[\{\omega\} \times \omega]$ and thus $f(x) = (\omega_1, m)$ for some integer m. This gives $(\omega_1, m) \in f[a(t^*_{n_0}) \cap K]$ so we can let n_0 be the least integer satisfying this property.

Now assume that $\{t_{n_j} : j \leqslant i\}$ and $\{n_j : j \leqslant i\}$ have been constructed. Since $U := \bigcap\{a(t_{n_j}) : j \leqslant i\}$ is a neighborhood of q, we have that $(\omega_1, \ell) \in f[U \cap K]$ for infinitely many $\ell \in \omega$. Using the method described in the previous paragraph one can guarantee the existence of $t_{n_{i+1}}$ and n_{i+1} as required.

Define $I := \{n_i : i < \omega\}$ and for each $n \in I$ set $c_n := a(t_n^*) \cap \bigcap\{a(t_k) : k < n\}$. Let

$$W := \operatorname{int}\left(\bigcup_{n\in I} f[c_n \cap K] \cap (\omega_1 \times \{n\})\right)$$

and for each $n \in I$ let $V_n := K \cap f^{-1}[W \cap (\omega_1 \times \{n\})]$. Hence V_n is open in K and $f[\bigcup\{V_n : n \in I\}] = W$ so we will restrict our attention to the compact set $K_0 := K \cap \overline{\bigcup}\{V_n : n \in I\}$.

Observe that the minimality of each t_n, $n \in I$, gives $(\omega_1, n) \in f[K \cap \Delta(t_n^*)]$.

We claim that $K_0 \cap \Delta(t_n^*) \cap f^{-1}[(\omega_1, n)] \neq \emptyset$ for all $n \in I$. To show this we only have to prove that $\mathcal{F} = \{K_0 \cap a_s \cap f^{-1}[(\omega_1, n)] : s \leqslant t_n^*\}$ has the finite intersection property. So let $F \subseteq (t_n^*)^{\downarrow} \cup \{t_n^*\}$ be finite and define $c := c_n \cap \bigcap\{a_s : s \in F\} \cap K$. Since $f \restriction K$ is irreducible and c is a clopen subset of K, the set $f[c]$ is regular closed and contains (ω_1, n) (because $K \cap \Delta(t_n^*) \subseteq c$). Hence (ω_1, n) is in the closure of $\operatorname{int}(f[c] \cap (\omega_1 \times \{n\})) \subseteq W \cap (\omega_1 \times \{n\})$ so we can use the fact that f is closed to get

$$\emptyset \neq c \cap \overline{V_n} \cap f^{-1}[(\omega_1, n)] \subseteq \bigcap\{a_s : s \in F\} \cap K_0 \cap f^{-1}[(\omega_1, n)],$$

and therefore $\mathcal{F}$ has the finite intersection property.

Fix, for each $n \in I$, a point $x_n \in K_0 \cap \Delta(t_n^*)$ so that $f(x_n) = (\omega_1, n)$. Then $\{x_k : n < k < \omega\} \subseteq \bigcap\{a_s : s < t_n\}$.

There are only finitely many $n \in I$ so that $f[\overline{V_n} \cap K \cap \Delta(t_n^*)] \not\subseteq \{(\omega_1, n)\}$. In effect, assume that $E \in [I]^\omega$ and $\{y_n : n \in E\}$ satisfy $y_n \in \overline{V_n} \cap K \cap \Delta(t_n^*)$ and $f(y_n) \neq (\omega_1, n)$ for all $n \in E$. Since, for each $n \in E$, $f[\overline{V_n}] \subseteq \overline{f[V_n]} \subseteq (\omega_1 + 1) \times \{n\}$, there exists $\alpha < \omega_1$ for which $\{f(y_n) : n \in E\} \subseteq (\alpha + 1) \times (\omega + 1)$ and hence $\overline{\{f(y_n) : n \in E\}} \cap \overline{\{f(x_n) : n \in E\}} = \emptyset$. Clearly $\{x_n : n \in E\}$ and $\{y_n : n \in E\}$ must have disjoint closures but this contradicts Lemma 2.3-(2).

We will construct two functions $e : 2^{<\omega} \to T$ and $H : 2^{<\omega} \to [I]^\omega$ so that for all $i < 2$ the following holds.

1. $e(s) < e(t)$ whenever $s < t$.

2. $e(t^*) = e(t)^*$.

3. $H(t^\frown i) = \{n \in H(t) : x_n \in a(e(t^\frown i))\}$.

4. If $e(t) < s < e(t^\frown i)$ then $\{n \in H(t) : x_n \in a(s^*)\}$ is finite.

If $\tilde{b} := \{s \in T : \exists n \in I(s < t_n)\}$ is a branch in T and $\{x\} = \bigcap\{a_s : s \in \tilde{b}\}$ then any neighborhood U of x contains a set of the form $\bigcap\{a_s : s \in F\}$ where F is a finite subset of $\tilde{b}$; thus there exists $n \in I$ so that $s < t_n$ for any $s \in F$ and therefore $\{x_k : n < k < \omega\} \subseteq U$. In other words, if $\tilde{b}$ is a branch, X is not an Efimov space.

Start the induction by letting $e(\emptyset)$ be the least node $s \in T$ so that $\bigcup\{t_n : n \in I\} \subseteq s$ (the previous paragraph guarantees that there is a node in T that extends it) and $\{n \in I : x_n \in a(s)\}$ is infinite (since $\{x_n : n \in I\}$ is infinite and $\{a_s, a(s^*)\}$ is a partition of X such a node must exist). And set $H(\emptyset) := \{n \in I : x_n \in a(e(\emptyset))\}$.

Now assume that $e(t)$ and $H(t)$ have been defined for some $t \in 2^{<\omega}$. We claim that $e(t)$ has an extension $s \in T$ that *splits* $H(t)$, i.e. both sets $\{n \in H(t) : x_n \in a_s\}$ and $\{n \in H(t) : x_n \in a(s^*)\}$ are infinte. Otherwise we can build by transfinite induction a branch $\tilde{b} \subseteq T$ so that $e(t) \in \tilde{b}$ and for all $r \in \tilde{b}$ the set $\{n \in H(t) : x_n \in a(r^*)\}$ is finite (when $e(\emptyset) \leqslant r < e(t)$, condition (4) takes care and if $r < e(\emptyset)$ then $t_n \leqslant r < t_{n+1}$ for some $n \in H$ and therefore $\{x_k : n{+}1 < k < \omega\} \subseteq a_r$). Hence the infinite sequence $\{x_n : n \in H(t)\}$ converges to $p \in \bigcap\{a_r : r \in \tilde{b}\}$; contradicting that X is an Efimov space. Since there is a node that splits $H(t)$ there is a minimal node $e(t^\frown 0)$ which does the splitting. The rest is to define $e(t^\frown 1) := e(t^\frown 0)^*$ and $H(t^\frown i)$ as described in (3) to complete the induction.

For each $g \in 2^\omega$ define $\overline{g} := \bigcup\{e(g \restriction n) : n \in \omega\}$ and $\Delta_g := \bigcap\{a_s : s < \overline{g}\}$. We will show that $\overline{\{x_n : n \in I\}} \cap \Delta_g \cap K_0 \neq \emptyset$ by proving that $\{\overline{\{x_n : n \in I\}} \cap a_s \cap K_0 : s < \overline{g}\}$ has the finite intersection property. Let F be a finite set of predecessors of $\overline{g}$ in T. There exists an integer m so that $s < e(g \restriction m)$ for each $s \in F$. Thus $\{x_n : n \in H(g \restriction m)\} \setminus a_s$ is finite for all $s \in F$ and hence $\overline{\{x_n : n \in H(g \restriction m)\}} \cap K_0 \cap \bigcap\{a_s : s \in F\} \neq \emptyset$.

We will prove by contradiction that the set

$$\{g \in 2^\omega : \exists \alpha < \omega_1(f^{-1}[(\alpha+1) \times (\omega+1)] \cap \Delta_g \cap K_0 \neq \emptyset)\}$$

is finite. So let us assume that it is infinite. In this case the set contains an infinite sequence

S converging to some $\rho \in 2^\omega$. For each $g \in S$ there exists $y_g \in \Delta_g \cap K_0$ and $\alpha_g < \omega_1$ such that $f(y_g) \in (\alpha_g + 1) \times (\omega + 1)$. Since S is countable, $\{f(y_g) : g \in S\} \subseteq (\alpha + 1) \times (\omega + 1)$ for some $\alpha < \omega_1$. Therefore $\{x_n : n \in I\}$ and $\{y_g : g \in S\}$ have disjoint closures. On the other hand, if $g \in S \setminus \{\rho\}$ then there is an integer m so that $(\rho \restriction m + 1)^* = g \restriction m + 1$ (i.e. $\rho \restriction m = g \restriction m$ and $\rho(m) \neq g(m)$). Let $\{g_k : k \in \omega\} \subseteq S$ and $\{m_k : k \in \omega\} \subseteq \omega$ be so that $(\rho \restriction m_k + 1)^* = g_k \restriction m_k + 1$ and $m_k \neq m_\ell$ whenever $k \neq \ell$ (recall that S is infinite and converges to ρ in 2^ω). For each $k \in \omega$ define $r_k := e(\rho \restriction m_k + 1)$ and fix a point $z_k \in \overline{\{x_n : n \in I\}} \cap K_0 \cap \Delta_{g_k}$. Then

$$\{z_k, y_{g_k}\} \subseteq \Delta_{g_k} \subseteq \Delta(e(g_k \restriction m_k + 1)) = \Delta(r_k^*) = \bigcap_{s < r_k} a_s - a(r_k).$$

So we can apply Lemma 2.3-(2) to obtain $\overline{\{z_k : k \in \omega\}} \cap \overline{\{y_{g_k} : k \in \omega\}} \neq \emptyset$ and therefore the closures of $\{x_n : n \in I\}$ and $\{y_g : g \in S\}$ are not disjoint. A contradiction.

For the rest of the proof we will fix $g \in 2^\omega$ in the complement of the finite set discussed in the previous paragraph. Then, for each $\alpha < \omega_1$, we have that

$$\bigcap_{n<\omega} f^{-1}[(\alpha + 1) \times (\omega + 1)] \cap \Delta(e(g \restriction n)) \cap K_0 = \emptyset,$$

and since $\Delta(e(g \restriction n + 1)) \subseteq \Delta(e(g \restriction n))$ there must be $m_\alpha \in \omega$ for which $f^{-1}[(\alpha + 1) \times (\omega + 1)] \cap \Delta(e(g \restriction m_\alpha)) \cap K_0 = \emptyset$. Fix $n \in \omega$ so that $\{\alpha < \omega_1 : m_\alpha = n\}$ is uncountable and observe that $f[K_0 \cap \Delta(e(g \restriction n))] \subseteq \{\omega_1\} \times (\omega + 1)$. Set $r := g \restriction n$.

If $\{s_k : k \leqslant \ell\} \subseteq 2^{<\omega}$ satisfies $r < s_k < s_{k+1}$ for all $k < \ell$ then for each $m \in H(s_\ell)$ we have (recall condition (3) above) $x_m \in \overline{V_m} \cap \bigcap\{a(e(s_k)) : k \leqslant \ell\} = \overline{V_m \cap \bigcap\{a(e(s_k)) : k \leqslant \ell\}}$ and hence we can use the fact that $f \restriction K$ is irreducible to get that $(\omega_1, m) = f(x_m)$ is in the closure of $\mathrm{int} f[V_m \cap \bigcap\{a(e(s_k)) : k \leqslant \ell\}]$. In particular, $f[V_m \cap \bigcap\{a(e(s_k)) : k \leqslant \ell\}] \cap (\omega_1 \times \{m\})$ is uncountable. The fact that $(2^{<\omega})^{<\omega}$ is countable implies the existence of an ordinal $\gamma < \omega_1$ so that for any integer ℓ, for any increasing sequence $\{s_k : k \leqslant \ell\} \subseteq 2^{<\omega}$ with $r < s_0$, and for each $m \in H(s_\ell)$ the set $f[V_m \cap \bigcap\{a(e(s_k)) : k \leqslant \ell\}] \cap ((\gamma + 1) \times \{m\})$ is infinite.

Let $B := [a_s : s < e(r)]$ and let $h : X \to \mathrm{St}(B)$ be the corresponding Stone map:

$h(x) := \{c \in B : x \in c\}$. Denote by u the ultrafilter generated by $\{a_s : s < e(r)\}$ in B. We claim that $u \notin h[K_1]$, where $K_1 := \overline{\bigcup\{V_m : m \in H(r)\}} \cap f^{-1}[(\gamma+1) \times (\omega+1)]$. Indeed, if $z \in K_1$ satisfies $h(z) = u$ then $z \in \Delta(e(r))$ and our choice for n guarantees that $f(z) \in \{\omega_1\} \times (\omega+1)$, contradicting that $z \in K_1$.

For any $x \in K_1$ the fact that $u \not\subseteq h(x)$ implies that there exists $t_x \in T$ so that $t_x < e(r)$ and $x \in -a(t_x) = a(t_x^*)$. A compactness argument shows that for any closed nonempty set $C \subseteq K_1$ the set $C^\dagger := \{t_x : x \in C\}$ has a maximum element. In particular, there is $y_1 \in K_1$ so that $t_{y_1} = \max K_1^\dagger$.

Let s_0 be an immediate successor of r (i.e. $s_0 = r^\frown i$ for some $i < 2$) so that $y_0 \notin a(e(s_0))$. If $K_1 \cap a(e(s_0)) \neq \emptyset$ let $y_1 \in K_1 \cap a(e(s_0))$ be so that $t_{y_1} = \max(K_1 \cap a(e(s_0)))^\dagger$. And repeat: let s_1 be an immediate successor of s_0 so that $y_1 \notin a(e(s_1))$ and if $K_1 \cap a(e(s_0)) \cap a(e(s_1)) \neq \emptyset$ then let y_2 be an element of this set so that $t_{y_2} = \max(K_1 \cap a(e(s_0)) \cap a(e(s_1)))^\dagger$. Since $t_{y_0} > t_{y_1} > t_{y_2} > \dots$ the process must stop after finitely many steps, in other words, there is $\ell < \omega$ such that $K_1 \cap \bigcap\{a(e(s_k)) : k \leqslant \ell\} = \emptyset$ and $y_k \in K_1 \cap \bigcap\{a(e(s_i)) : i < \ell\} - a(e(s_i))$ for all $k \leqslant \ell$.

Recall that γ was selected to make $f[V_m \cap \bigcap\{a(e(s_k)) : k \leqslant \ell\}] \cap ((\gamma+1) \times \{m\})$ infinite for all $m \in H(s_\ell)$ which implies that $K_1 \cap \bigcap\{a(e(s_k)) : k \leqslant \ell\} \neq \emptyset$. This contradiction finishes the proof. *Q.E.D.*

CHAPTER 3: CONSISTENCY RESULTS

3.1 Simplistic Efimov Spaces and CH

We mentioned in Chapter 1 that Fedorčuk's constructed a simplistic Efimov space assuming $\diamondsuit$. In this section we will show that $\diamondsuit$ can be replaced with CH.

Džamonja and Plebanek [7] show that any Efimov space constructed from the the Cantor space in an inverse limit of length ω_1 using simple extensions refutes a conjecture of Mercourakis concerning measures on compact spaces. Fedorčuk's example proves that Mercourakis' conjecture is false in any model of $\diamondsuit$. It is asked in [7] if CH suffices to refute Mercourakis' conjecture, and our construction answers this affirmatively.

Efimov spaces have been constructed from CH before (see, for example, [7] and [11]) but ours is the first example which is simplistic.

Theorem 3.1. Under CH, there exists an Efimov space that can be obtained as the limit of a simplistic system of length ω_1.

Proof. We will prove the theorem by recursively defining a simplistic system $\langle f_{\alpha\beta}, X_\alpha : \alpha < \beta < \omega_1 \rangle$ whose limit is an Efimov space and so that $|X_\alpha| = \omega_1$ for all $\alpha < \omega_1$. This recursion uses a special notation that will be introduced in the following two paragraphs.

First of all, *sequence* means, from now on, *infinite sequence.* Recall that a countable set E converges to a point p in a topological space if any neighborhood U of p contains all but finitely many elements of E, i.e. $|E \setminus U| < \omega$. Let us fix a partition $\{P_\alpha : \alpha < \omega_1\}$ of ω_1 so that P_α is an uncountable subset of $\omega_1 \setminus \alpha$ for all $\alpha < \omega_1$. Given $\alpha < \omega_1$, we will denote by α' the unique ordinal satisfying $\alpha \in P_{\alpha'}$. Therefore, $\alpha' \in P_{\alpha''}$.

Assume that we have constructed the simplistic system up to and including γ, i.e. $\langle f_{\alpha\beta}, X_\alpha : \alpha < \beta \leqslant \gamma \rangle$ has been defined. Hence we have X_γ. We say that a converging sequence D in X_γ is *special* if for every $\xi < \gamma$, $f_{\xi\gamma}[D]$ is a finite subset of X_ξ. Using CH and $|X_\gamma| = \omega_1$ we can fix an enumeration $\{D_\xi : \xi \in P_\gamma\}$ of all special sequences in X_γ.

Note that once the recursion is complete we will have an ω_1-sequence $\{D_\xi : \xi < \omega_1\}$ which lists all special converging sequences in X_η for all $\eta < \omega_1$. Of course this list is constructed during the recursion. Let $\beta < \gamma$. Since $\beta \in P_{\beta'}$, we get $\beta' \leqslant \beta$ and hence X_β, $X_{\beta'}$, and $f_{\beta'\beta}$ have been constructed. Under these circumstances there are countable sets $E \subseteq X_\beta$ such that $E \cap f^{-1}_{\beta'\beta}[d]$ is a singleton for all $d \in D_\beta$. Such a set E is called a *selector for* D_β. Again by CH we may enumerate all such selectors as $\{E_\xi : \xi \in P_\beta\}$. Doing this for each $\beta < \omega_1$ would result in an ω_1-sequence $\{E_\xi : \xi < \omega_1\}$ listing all selectors for D_η for all $\eta < \omega_1$. Again we mention that this list is constructed during the recursion. Now, given $\beta < \alpha < \gamma$ and $x \in X_\alpha$, define $D_\beta(x) := \{d \in D_\beta : f_{\beta'\alpha}(x) \neq d\}$.

Assume that for some $\varepsilon < \omega_1$ we have defined a continuous inverse system of simple extensions $\langle f_{\gamma\beta}, X_\gamma : \gamma < \beta < \varepsilon\rangle$ so that the following holds for each $\beta < \varepsilon$.

1. $X_0 = 2^\omega$, the Cantor set.
2. If $\beta + 1 < \varepsilon$ and $i < 2$, then there exist A^i_β, a closed subset of X_β, and H^i_β, an infinite subset of E_β, so that
   (a) $A^0_\beta \cap A^1_\beta = \{x_\beta\}$,
   (b) $X_\beta = A^0_\beta \cup A^1_\beta$,
   (c) $X_{\beta+1} = A^0_\beta \oplus A^1_\beta$ (the topological sum), and
   (d) $f_{\beta,\beta+1}$ is the projection map.
   (e) $f^{-1}_{\beta'\beta}[e] \subseteq A^i_\beta$ for all $e \in H^i_\beta$.
3. If $x \in U \in \mathcal{T}_\beta$, where $\mathcal{T}_\beta$ is the topology of X_β, then for each finite set $F \subseteq \beta$ there exists W, a clopen subset of X_β, such that $x \in W \subseteq U$ and W *takes care of* (x, F), i.e.
$$f^{-1}_{\gamma'\gamma}[d] \cap f_{\gamma\beta}[W] \cap f_{\gamma\beta}[X_\beta \setminus W] = \emptyset,$$
   for all $\gamma \in F$ and $d \in D_\gamma(x)$.

Condition (3) is equivalent to saying that $W \cap f^{-1}_{\gamma'\beta}[d]$ is a preimage of a clopen subset of $f^{-1}_{\gamma'\gamma}[d]$.

Observe that each X_α will be compact nonempty and, moreover, that the limit of this inverse system will be a simplistic space and therefore (Proposition 1.19) it will not contain a copy of $\beta\omega$. Hence our main concern is to get rid of all converging sequences.

If ε is a limit ordinal, then X_ε is the limit of this inverse system. To verify (3) let $x \in U_0 \in \mathcal{T}_\varepsilon$ be arbitrary. Fix a finite set $F \subseteq \varepsilon$. Since $\{f_{\xi\varepsilon}^{-1}[V] : \xi < \varepsilon,\ V \in \mathcal{T}_\xi\}$ is a base for $\mathcal{T}_\varepsilon$, there exists $\alpha < \varepsilon$ and $U \in \mathcal{T}_\alpha$ so that $F \subseteq \alpha$ and $x \in f_{\alpha\varepsilon}^{-1}[U] \subseteq U_0$. Apply the inductive hypothesis to α, $f_{\alpha\varepsilon}(x)$, U, and F to get a clopen set W in X_α which takes care of $(f_{\alpha\varepsilon}(x), F)$ and satisfies $f_{\alpha\varepsilon}(x) \in W \subseteq U$. For each $\gamma \in F$ we have $D_\gamma(x) = D_\gamma(f_{\alpha\varepsilon}(x))$, so $f_{\alpha\varepsilon}^{-1}[W]$ is a clopen subset of X_ε which takes care of (x, F).

Now assume that $\varepsilon = \alpha + 1$. Observe that $\alpha'' \leqslant \alpha' \leqslant \alpha$ and therefore E_α and $D_{\alpha'}$ have been defined at this stage. Since X_0 is compact metrizable and $\alpha < \omega_1$, X_α is compact metrizable too. In particular the family $\{f_{\alpha'\alpha}^{-1}[e] : e \in E_\alpha\}$ must have an accumulation point, i.e. there exists a point x_α in X_α so that the set $\{e \in E_\alpha : f_{\alpha'\alpha}^{-1}[e] \cap V \neq \emptyset\}$ is infinite for each neighborhood V of x_α.

The next step is to find an infinite set $H_\alpha \subseteq E_\alpha$ so that $\langle f_{\alpha'\alpha}^{-1}[e] : e \in H_\alpha\rangle$ converges to x_α, i.e. for each neighborhood U of x_α all but finitely many $e \in H_\alpha$ satisfy $f_{\alpha'\alpha}^{-1}[e] \subseteq U$. Let's start by fixing a local decreasing base $\{B_n : n \in \omega\}$ for x_α.

We face two cases. When $\alpha' < \alpha$ apply (3) to construct a sequence $\langle W_n : n \in \omega\rangle$ of clopen subsets of X_α which satisfies

(i) $x_\alpha \in W_0 \subseteq B_0$,

(ii) $x_\alpha \in W_{n+1} \subseteq W_n \cap B_{n+1}$, and

(iii) W_n takes care of $(x_\alpha, \{\beta\})$,

for each $n \in \omega$.

Define $E_\alpha^n := \{e \in E_\alpha : f_{\alpha'\alpha}^{-1}[e] \cap W_n \neq \emptyset\}$. We claim that for all but possibly one $e \in E_\alpha^n$ we get $f_{\alpha'\alpha}^{-1}[e] \subseteq W_n$. To prove this assertion let $e \in E_\alpha^n$ and $d \in D_{\alpha'}(x_\alpha)$ be so that $f_{\alpha''\alpha'}(e) = d$. Now note that if $f_{\alpha'\alpha}^{-1}[e] \setminus W_n \neq \emptyset$, then $e \in f_{\alpha'\alpha}[W_n] \cap f_{\alpha'\alpha}[X_\alpha \setminus W_n] \cap f_{\alpha''\alpha'}^{-1}[d]$, a clear contradiction to (iii). Find an infinite set $H_\alpha \subseteq E_\alpha$ so that $H_\alpha \setminus E_\alpha^n$ is finite for all $n \in \omega$ and observe that this H_α works.

For the case $\alpha = \alpha'$ we have $E_\alpha \subseteq X_\alpha$ and $f^{-1}_{\alpha'\alpha}[e] = \{e\}$, for each $e \in E_\alpha$. Clearly any subsequence of E_α which converges to x_α will work as H_α.

Let $\alpha = \{\beta_k : k < \omega\}$ be an enumeration of α. Use (3) to construct $\{e_n : n \in \omega\} \subseteq H_\alpha$, $g : \omega \to \omega$, and $\{U_n : n \in \omega\}$ so that for each $n \in \omega$

(I) $U_0 = X_\alpha$,

(II) U_n is clopen in X_α and takes care of $(x_\alpha, \{\beta_k : k \leqslant n\})$,

(III) g is an increasing function,

(IV) $x_\alpha \in U_{n+1} \subseteq B_{g(n)} \subseteq U_n \setminus f^{-1}_{\alpha'\alpha}[e_n]$, and

(V) $f^{-1}_{\alpha'\alpha}[e_n] \subseteq U_n$.

We are going to partition $X_\alpha \setminus \{x_\alpha\}$. Let n be an arbitrary integer. Observe that the set $V_n := U_n \setminus U_{n+1}$ is clopen and takes care of $(x_\alpha, \{\beta_k : k \leqslant n\})$. Now, given $i < 2$, define

$$b^i_n := \{x_\alpha\} \cup \bigcup_{k=n}^{\infty} V_{2k+i}.$$

The following holds for each $i < 2$.

(a) b^i_n is closed for all $n < \omega$.

(b) If U a neighborhood of x_α, then $b^i_m \subseteq U$, for some $m < \omega$.

We claim that b^i_n takes care of $(x_\alpha, \{\beta_k : k \leqslant n\})$. Let $k \leqslant n$ and $d \in D_{\beta_k}(x_\alpha)$ be arbitrary. If $y \in f_{\beta_k\alpha}[b^i_n] \cap f_{\beta_k\alpha}[X_\alpha \setminus b^i_n]$, then we have two possibilities: $y \in f_{\beta_k\alpha}[V_{2\ell+i}]$, for some $\ell \geqslant n$, or $y = f_{\beta_k\alpha}(x_\alpha)$. In the first case we get $y \in f_{\beta_k\alpha}[V_{2\ell+i}] \cap f_{\beta_k\alpha}[X_\alpha \setminus V_{2\ell+i}]$ and $k \leqslant n \leqslant \ell \leqslant 2\ell + i$, so $f_{\beta'_k\beta_k}(y) \neq d$.

Now assume that $y = f_{\beta_k\alpha}(x_\alpha)$. Since $d \in D_{\beta_k}(x_\alpha)$, we get $d \neq f_{\beta'_k\alpha}(x_\alpha) = f_{\beta'_k\beta_k}(f_{\beta_k\alpha}(x_\alpha)) = f_{\beta'_k\beta_k}(y)$.

For each $i < 2$, set $A^i_\alpha := b^i_0$ and $H^i_\alpha := \{e_{2n+i} : n \in \omega\}$. Let $X_{\alpha+1} := (A^0_\alpha \times \{0\}) \cup (A^1_\alpha \times \{1\})$ and declare open all the sets of the form $(U^0 \times \{0\}) \cup (U^1 \times \{1\})$, where U^i is open in the subspace topology of $A^i_\alpha \subseteq X_\alpha$. The map $f_{\alpha,\alpha+1} : X_{\alpha+1} \to X_\alpha$ is defined by $f_{\alpha,\alpha+1}(x, i) = x$.

To complete the induction we check that property (3) holds. Assume that $(x, i) \in U \in \mathcal{T}_{\alpha+1}$ are arbitrary and fix a finite set $F \subseteq \alpha + 1$. If $x \neq x_\alpha$, find $U_0 \in \mathcal{T}_\alpha$ so that $(x, i) \in U_0 \times \{i\} \subseteq U$ and $x_\alpha \notin U_0$. Let W_0 be a clopen subset of X_α which takes care of $(x, F \setminus \{\alpha\})$ and satisfies $x \in W_0 \subseteq U_0$. Set $W := W_0 \times \{i\}$ and note that for all $\beta \in F$ we have $f_{\beta,\alpha+1}[W] = f_{\beta\alpha}[W_0]$ and $f_{\beta,\alpha+1}[X_{\alpha+1} \setminus W] = f_{\beta\alpha}[X_\alpha \setminus W_0]$ (recall that $f_{\alpha\alpha}$ is the identity map). Since $D_\beta((x, i)) = D_\beta(x)$ we conclude that W takes care of $((x, i), F)$.

Assume now that $x = x_\alpha$. Find $n \in \omega$, so that $b^i_n \times \{i\} \subseteq U$ and $F \setminus \{\alpha\} \subseteq \{\beta_k : k \leqslant n\}$. Define $W := b^i_n \times \{i\}$. For each $\beta \in F$ and $d \in D_\beta((x_\alpha, i))$ we have that $f_{\beta\alpha}(x_\alpha) \notin f^{-1}_{\beta'\beta}[d]$ and

$$f_{\beta,\alpha+1}[W] \cap f_{\beta,\alpha+1}[X_{\alpha+1} \setminus W] = \{f_{\beta\alpha}(x_\alpha)\} \cup \left(f_{\beta\alpha}[b^i_n] \cap f_{\beta\alpha}[X_\alpha \setminus b^i_n]\right).$$

Therefore W takes care of $((x_\alpha, i), F)$.

Let X be the limit of our inverse system and let $\pi_\alpha : X \to X_\alpha$ be the bonding map for each $\alpha < \omega_1$. In order to check that X is an Efimov space, assume that S is a converging sequence in X. Let $\gamma < \omega_1$ be the least ordinal so that $\pi_\gamma[S]$ is infinite. Since $\pi_\gamma[S]$ is a convergent sequence in X_γ, there exists $\beta \in P_\gamma$ so that $\pi_\gamma[S] = D_\beta$. Since $f_{\gamma\beta} \circ \pi_\beta = \pi_\gamma$, we can find an infinite set $S_0 \subseteq S$ so that π_β is one-to-one on S_0 and $\pi_\beta[S_0]$ is a selector from D_β, i.e. there exists $\alpha \in P_\beta$ so that $\pi_\beta[S_0] = E_\alpha$. Property (2) provides two infinite subsets of S_0, namely S^0_0 and S^1_0, so that $\pi_\alpha[S^i_0] \subseteq A^i_\alpha$ for each $i < 2$. Therefore $\pi_{\alpha+1}[S^0_0]$ and $\pi_{\alpha+1}[S^1_0]$ cannot converge to the same point. This contradiction ends the proof. *Q.E.D.*

3.2 Forcing Extensions

In this section we turn our attention to the existence of Efimov T-algebras in the models obtained by adding Cohen and Hechler reals.

The following poset was introduced by Koszmider in [14].

Definition 3.1. Let A be a Boolean algebra and let u be an ultrafilter in A.

1. $\mathbb{P}(A, u) := \{(p_0, p_1) : p_0, p_1 \in A \setminus u \text{ and } p_0 \cap p_1 = \emptyset\}$. We will adopt the following convention: if $p \in \mathbb{P}(A, u)$ then p_0 and p_1 will represent the first and second coordinate of p, respectively.

2. Given $p, q \in \mathbb{P}(A, u)$ we define $p \leqslant q$ iff $q_i \subseteq p_i$ for each $i < 2$.

One can picture Koszmider's poset as a way to force a subset of 1 (we are always assuming that all Boolean algebras are algebras of subsets of a fixed set 1). Indeed, if G is a generic filter, then one can let $g := \bigcup\{p_0 : p \in G\}$. We will refer to g as the generic set added by $\mathbb{P}(A, u)$.

Observe that two conditions p and q are compatible iff $(p_0 \cup q_0) \cap (p_1 \cup q_1) = \emptyset$. As a corollary of this remark one gets that $p_0 \subseteq g \subseteq -p_1$ for all $p \in G$. Indeed, if $x \in g$ then $x \in q_0$ for some $q \in G$ and since p and q are compatible it must be the case that $x \notin p_1$.

Under certain circumstances g becomes minimal for (A, u).

Proposition 3.2. If u does not contain atoms and $\dot{g}$ is the name for the generic set added by $\mathbb{P}(A, u)$, then $\dot{g}$ is forced to be minimal for (A, u).

Proof. Let $P := \mathbb{P}(A, u)$ and let G be a P-generic filter. According to Proposition 1.10 we only have to prove that $u = \{a \in A : a \cap g \notin A\}$.

Let $a \in A \setminus u$ be arbitrary. We claim that $D_a := \{p \in P : a \subseteq p_0 \cup p_1\}$ is dense in P. Indeed, if $p \in P$ then $q := (p_0 \cup (a - p_1), p_1) \in D_a$ and $q \leqslant p$. In particular, if $p \in G \cap D_a$ then $a \cap g = p_0 \in A$.

For each $a \in u$ define $E_a := \{p \in P : \forall i < 2(a \cap p_i \neq \emptyset)\}$. To show that E_a is dense let $p \in P$ be arbitrary. Then $a - (p_0 \cup p_1) \in u$ and since u contains no atoms, there exist $c_0 \in A$ so that $\emptyset \neq c_0 \subset a - (p_0 \cup p_1)$ and $c_1 := (a - (p_0 \cup p_1)) - c_0 \neq \emptyset$. Hence $q := (p_0 \cup c_0, p_1 \cup c_1) \in E_a$ and $q \leqslant p$.

Seeking a contradiction, assume that $a \cap g \in A$ for some $a \in u$. Then $a \cap g \in u$ or $-(a \cap g) \in u$. In the first case there exists $p \in G$ so that $p \in G \cap E_{a \cap g}$ and hence $(a \cap g) \cap p_1 \neq \emptyset$ so $g \cap p_1 \neq \emptyset$ which is impossible. when $-(a \cap g) \in u$ we get $a - g \in u$ so there is $q \in G \cap E_{g-a}$ and therefore $\emptyset \neq q_0 \cap (a - g) \subseteq q_0 \cap (-g)$. A contradiction. *Q.E.D.*

One is tempted to write $-g = \bigcup\{p_1 : p \in G\}$ when u is nonprincipal but this may not be the case. For example, if A is the Boolean algebra generated by $\{(\omega+1) \setminus F : F \in [\omega]^{<\omega}\}$ and $u = \{a \in A : \omega \in a\}$ then u is nonprincipal (if $a \in u$ then $a = (\omega + 1) \setminus F$ for some finite $F \subseteq \omega$ and hence a is not atom in A). If $p \in \mathbb{P}(A, u)$ then $\omega \notin p_0 \cup p_1$ and therefore $\omega \in -g$ but $\omega \notin \bigcup\{p_1 : p \in G\}$ for any $\mathbb{P}(A, u)$-generic filter G.

In the construction of an Efimov simplistic space under CH the main problem was to select carefully the point to be doubled in order to destroy the convergence of the sequence in turn. When we force with Koszmider's poset the minimal generic set added destroys the convergence of any ground model sequence in $\mathrm{St}(A)$ which converges to u.

Lemma 3.3. Let X be the Stone space of the Boolean algebra A. Assume that $\{x_n : n \in \omega\} \subseteq X$ is an infinite sequence converging to $u \in X$. For each $n \in \omega$ let $\dot{y}_n$ be a $\mathbb{P}(A,u)$-name so that $\Vdash x_n \subseteq \dot{y}_n \in \mathrm{St}(A(\dot{g}))$, where $\dot{g}$ is a name for the generic set added by $\mathbb{P}(A,u)$. Then

$$\Vdash |\{n \in \omega : \dot{g} \in \dot{y}_n\}| = |\{n \in \omega : -\dot{g} \in \dot{y}_n\}| = \omega.$$

Proof. It suffices to show that $D^i_n := \{p : \exists m \geqslant n(p_i \in x_m)\}$ is dense in $P := \mathbb{P}(A,u)$ for all $n \in \omega$ and $i < 2$. Indeed, observe that $q \in D^0_n$ implies that, for some $m \geqslant n$, we get $q \Vdash$ "$q_0 \in x_m \subseteq \dot{y}_m$ and $q_0 \subseteq \dot{g}$," i.e. $q \Vdash \dot{g} \in \dot{y}_m$. Similarly, $q \in D^i_n$ gives $q \Vdash -\dot{g} \in \dot{y}_\ell$, for some $\ell \geqslant n$.

Note that the assumption $x_n \to u$ is equivalent to $\forall a \in u \exists k \in \omega \forall n \geqslant k(a \in x_n)$. Now let $p \in P$ be arbitrary. Then $-(p_0 \cup p_1) \in u$ and therefore there is $m \geqslant n$ such that $-(p_0 \cup p_1) \in x_m$ and $x_m \neq u$. Let $a \in u \setminus x_m$ satisfying $a \subseteq -(p_0 \cup p_1)$. Hence $q := (p_0 \cup a, p_1) \in D^0_n$ and $q \leqslant p$. A similar argument shows that D^1_n is dense. *Q.E.D.*

Theorem 3.4. In the model obtained by adding ω_2 Cohen reals to a model of CH there is an Efimov T-algebra, where $T = \bigcup\{2^{\alpha+1} : \alpha < \omega_1\}$

Proof. Let $E := 2^\omega$. Define by transfinite induction a finite support iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2\rangle$ and, for each $\alpha < \omega_2$, two P_α-names, $\dot{T}_\alpha$ and $\{\dot{a}_t : t \in \dot{T}_\alpha\}$, such that P_α forces the following

1. $\dot{T}_\alpha$ is an acceptable tree and $\{\dot{a}_t : t \in \dot{T}_\alpha\}$ is a family of subsets of E which witnesses that $\dot{A}_\alpha := [\dot{a}_t : t \in \dot{T}_\alpha]$ is a $\dot{T}_\alpha$-algebra.
2. $\dot{Q}_\alpha$ is the finite support product $\prod\{\mathbb{P}((\dot{A}_\alpha)_b, u_b) : b$ is a countable branch of $\dot{T}_\alpha\}$, where $(\dot{A}_\alpha)_b := [\dot{a}_t : t \in b]$, u_b is the ultrafilter generated by $\{\dot{a}_t : t \in b\}$ in $(\dot{A}_\alpha)_b$ and u_b does not contain atoms.

3. When α is a limit ordinal, $\dot{T}_\alpha = \bigcup\{\dot{T}_\xi : \xi < \alpha\}$.

4. If $\alpha = \beta + 1$ then $\dot{T}_\alpha = \dot{T}_\beta \cup \{(\bigcup b)^\frown i : b$ is a countable branch of $\dot{T}_\beta$ and $i < 2\}$, $\dot{a}_{(\bigcup b)^\frown 0}$ is the generic set added by $\mathbb{P}((\dot{A}_\beta)_b, u_b)$ and $\dot{a}_{(\bigcup b)^\frown 1} = -\dot{a}_{(\bigcup b)^\frown 0}$ for each countable branch b in $\dot{T}_\beta$.

Let $T_0 := 2^{<\omega} \setminus \{\emptyset\}$ and $\{a_t : t \in T_0\}$ be as described in the proof of Proposition 2.1. At stage α, (3) explains what to do when α is limit. If $\alpha = \beta+1$, we note that $P_\alpha = P_\beta * \dot{Q}_\beta$ and therefore Q_β introduces all the generic sets needed to fulfill (1) and (4) (recall propositions 2.6 and 3.2).

Working in the forcing extension given by P_α we have that if b is a countable branch in T_α, then $(A_\alpha)_b$ is a countable Boolean algebra and therefore $\mathbb{P}((A_\alpha)_b, u_b)$ is a countable non-atomic poset. Thus (see [15], Exercise (C4) from Chapter VII) $\mathbb{P}((A_\alpha)_b, u_b)$ is forcing isomorphic to $\mathrm{Fn}(\omega, 2)$, i.e. both posets yield the same generic extension. Hence Q_α can be viewed as the finite support product of ω_1 copies (recall that we are assuming CH in the ground model) of $\mathrm{Fn}(\omega, 2)$. In other words, Q_α is forcing isomorphic to $\mathrm{Fn}(\omega_1, 2)$. This argument shows that P, the limit of the iteration, produces the same forcing extension as $\mathrm{Fn}(\omega_2, 2)$.

The argument contained in the next paragraph takes place in the generic extension obtained by forcing with P.

From (3) and (4) we obtain that $T := \bigcup\{T_\xi : \xi < \omega_2\}$ is equal to $\bigcup\{2^{\alpha+1} : \alpha < \omega_1\}$. Hence T is an acceptable tree and, moreover, $A := [a_t : t \in T]$ is a T-algebra. We only need to show that X, the Stone space of A, has no infinite converging sequences. Seeking a contradiction assume that $\{x_n : n \in \omega\} \subseteq X$ is infinite and converges to $x_\omega \in X$. Then for each $n \leqslant \omega$ there is a branch b_n in T so that $\{x_n\}$ is the ultrafilter generated by $\{a_t : t \in b_n\}$ in A. By passing to a subsequence we can assume that $x_\omega \notin \{x_n : n \in \omega\}$ and therefore for each integer n there is a node $s_n \in b_\omega$ so that $s_n^* \in b_n$. Note that if $m \in \omega$ and $W := \{y \in X : -a(s_n) \in y\}$ then W is a clopen subset of X for which $x_\omega \in X \setminus W$ and hence the set $\{n \in \omega : x_n \in W\}$ is finite. Thus $\{n \in \omega : s_n = s_m\}$ is finite too. Once again we can pass to a subsequence and assume that $s_n \neq s_m$ whenever $m < n < \omega$. There is a permutation $\pi : \omega \to \omega$ so that $m < n < \omega$ implies $s_{\pi(m)} < s_{\pi(n)}$ and since $x_{\pi(n)} \to x_\omega$, there is no loss of generality in assuming that $\{s_n : n \in \omega\}$ is increasing. $\bigcup b_\omega$ is a binary function whose domain is ω_1 and therefore there exists $s_\omega \in b_\omega$ so that $s_n < s_\omega$ for all $n \in \omega$. Observe that this defines an increasing function $s : \omega + 1 \to b_\omega$ given by $s(n) = s_n$. Recall that $x_n \to x_\omega$ is equivalent to $\forall t \in b_\omega \exists m \in \omega \forall n \geqslant m(a_t \in x_n)$. Let $t \in b_\omega$ be so that $t \geqslant s_\omega$. Then the previous remark and Lemma 2.3 imply that for some $m \in \omega$ and for all $n \geqslant m$ we get $\bigcap\{a_r : r < s_n\} - a(s_n) \subseteq a_t$ (the set on the left is an element of the ultrafilter x_n).

Q.E.D.

In the case of the Hechler poset the argument relies on the following property that we isolated during our analysis of the problem.

Definition 3.2. Let $X = \{x_\alpha : \alpha < \omega_1\}$ be a topological space. We say that X has the *stationary set property* (X has the SP, for short) if it possesses a familiy of countable compact subsets $\{c_\alpha : \alpha < \omega_1\}$ so that

1. $x_\alpha \in c_\alpha$ for all $\alpha < \omega_1$, and
2. for any stationary set $S \subseteq \omega_1$ the set $X \setminus \bigcup\{c_\alpha : \alpha \in S\}$ is compact.

Lemma 3.5. If X has the stationary set property then X is countably compact.

Proof. Let E be an infinite countable subset of X. If $E \cap c_\alpha$ is infinite for some $\alpha < \omega_1$ then we use the fact that c_α is compact to obtain an accumulation point for E.

Without loss of generality let us assume that $E \cap c_\alpha$ is finite for all α. An argument involving the Pressing Down Lemma gives the existence of a stationary set $S \subseteq \omega_1$ and a finite set $F \subseteq E$ so that $\alpha \in S$ implies $E \cap c_\alpha = F$. Hence $K := X \setminus \bigcup\{c_\alpha : \alpha \in S\}$ is compact and contains the infinite set $E \setminus F$ so E has an accumulation point. *Q.E.D.*

Recall that Hechler forcing, P, is the set of all pairs $(s, f) \in \omega^{<\omega} \times \omega^\omega$ ordered by $(s, f) \leqslant (t, g)$ iff $t \subseteq s$, $g(n) \leqslant f(n)$ for each $n \in \omega$, and $g(n) \leqslant s(n)$ for all $n \in \mathrm{dom}(s) \setminus \mathrm{dom}(t)$.

Lemma 3.6. Hechler's poset preserves the stationary set property.

Proof. Let us assume that X has the SP and let $\dot{S}$ be a P-name for a stationary subset of ω_1.

For each $\alpha < \omega_1$ fix, if possible, a condition $(s_\alpha, f_\alpha) \in P$ so that $(s_\alpha, f_\alpha) \Vdash \alpha \in \dot{S}$. Let $H := \{\alpha < \omega_1 : \exists p \in P(p \Vdash \alpha \in \dot{S})\}$. For each $\alpha \in H$ fix a condition (s_α, f_α) so that $(s_\alpha, f_\alpha) \Vdash \alpha \in \dot{S}$. Since P is ccc and $\dot{S}$ is forced to be stationary, H contains a stationary set S_1. Applying the Pressing Down Lemma we obtain a a stationary set $S_0 \subseteq S_1$ in the ground model and a function $s \in \omega^{<\omega}$ so that $s = s_\alpha$ for all $\alpha \in S_0$.

Hence $X \setminus \bigcup\{c_\alpha : \alpha \in S_0\}$ is compact. The fact that X is locally compact and locally countable (hence zero-dimensional) implies that there is a compact clopen set $K \subseteq X$ so that $X \setminus \bigcup\{c_\alpha : \alpha \in S_0\} \subseteq K$. From Lemma 3.5 we obtain that $X \setminus K$ is countably compact. Observe that the proof will be complete if we show that there is $h \in \omega^\omega$ satisfying $(s, h) \Vdash X \setminus \bigcup\{c_\alpha : \alpha \in \dot{S}\} \subseteq K$.

For each $t \in \omega^{<\omega}$ define $S_t := \{\alpha \in S_0 : \forall n \in \operatorname{dom}(t)(f_\alpha(n) \leqslant t(n))\}$. Let us show that if $X \setminus K \subseteq \bigcup\{c_\alpha : \alpha \in S_t\}$ then $X \setminus K \subseteq \bigcup\{c_\alpha : \alpha \in S_{t^\frown m)}\}$ for some integer m. Let $n = \operatorname{dom} t$ and for each $k \in \omega$ set $U_k := \bigcup c_\alpha : \alpha \in S_t$ and $f_\alpha(n) < k\}$. Then $\{U_k : k \in \omega\}$ is an increasing sequence of open sets which covers $X \setminus K$ so $X \setminus K \subseteq U_m$ for some m. Hence m is as required.

Using the property given in the previous paragraph we can construct by induction a function $h : \omega \to \omega$ so that $X \setminus K \subseteq \bigcup\{c_\alpha : \alpha \in S_{h\restriction m}\}$ for all $m \in \omega$ (note that $(S_\emptyset = S_0)$.

We claim that $(s, h) \Vdash X \setminus \bigcup\{c_\alpha : \alpha \in \dot{S}\}$. We will prove the statement by showing that for any $y \in X \setminus K$ the set $D_y := \{(t, g) : \exists \alpha < \omega_1 (y \in c_\alpha \text{ and } (t, g) \Vdash y \in c_\alpha)\}$ is dense below (s, h). Let $y \in X \setminus K$ and $(t, g) \leqslant (s, h)$ be arbitrary. By definition of h there is $\alpha \in S_{h\restriction m}$ so that $y \in c_\alpha$, where $m = \operatorname{dom} t$. Thus $p := (t, f_\alpha + g)$ satisfies $p \in D_y$ (because $p \leqslant (s, f_\alpha))$ and $p \leqslant (t, g)$. *Q.E.D.*

We will say that a poset $\mathbb{P}$ preserves SP if any space which has the stationary set property still has the property after forcing with $\mathbb{P}$.

Lemma 3.7. Let $\langle P, \dot{Q}_\alpha : \alpha < \varepsilon \rangle$ be a finite support iteration of ccc posets. If for any $\alpha < \varepsilon$ we have that $\Vdash$ "$\dot{Q}_\alpha$ preserves SP" then P_ε preserves SP.

Proof. Assume that X is a space which has the SP and let $\dot{S}$ be a P_ε-name for a stationary subset of ω_1 forced by $p \in P_\varepsilon$.

Set $H := \{\alpha < \omega_1 : \exists q \leqslant p(q \Vdash \alpha \in \dot{S})\}$. For each $\alpha \in H$ fix $p_\alpha \leqslant p$ satisfying $p_\alpha \Vdash \alpha \in \dot{S}$. We face two cases.

Fist assume that some $\{\alpha_n : n \in \omega\} \subseteq \varepsilon$ has supremum ε. Then $H = \bigcup_n \{\beta \in H : p_\beta \in P_{\alpha_n})\}$. Since P_ε is ccc, H contains a stationary set so there exist $S_1 \subseteq H$ and $\lambda \in \{\alpha_n : n \in \omega\}$ so that S_1 is stationary and $\alpha \in S_1$ implies $p_\alpha \in P_\lambda$.

Let $G_\lambda := G \cap P_\lambda$, where G is a P_λ-generic filter. Then G_λ is P_λ-generic and we can use the fact that P_λ is ccc to obtain a condition $q \leqslant p$ such that $q \Vdash$ "$\{\alpha \in S_1 : p_\alpha \in G_\lambda\}$ is stationary."} Let $\dot{S}_0$ be a P_λ-name for this stationary set. Our inductive hypothesis implies that P_λ preserves SP and hence $q \Vdash_\lambda$ "$X \setminus \bigcup\{c_\alpha : \alpha \in \dot{S}_0\}$ is compact," which together with $q \Vdash_\varepsilon \dot{S}_0 \subseteq \dot{S}$ gives $q \Vdash_\varepsilon$ "$X \setminus \bigcup\{c_\alpha : \alpha \in \dot{S}\}$ is compact."

Now assume that ε has cofinality ω_1. Then the argument can be reduced to the proof that $\mathrm{Fn}(\omega_1, 2)$ preserves SP. For the Cohen poset it is enough to show that if $S \subseteq \omega_1$ is a stationary set from the ground model and G is the generic filter then $\mathrm{Fn}(\omega_1, 2) \Vdash$ "$X \setminus \bigcup\{c_\alpha : \alpha \in \dot{S}_0\}$ is compact," where $\dot{S}_0$ is a name for the $\{\alpha \in S : (\alpha, 0) \in G\}$. Let K be a compact clopen set so that $X \setminus K \subseteq \bigcup\{c_\alpha : \alpha \in S\}$. Denote by E the set of $\xi \in S$ so that for some $y_\xi \in c_\xi \setminus K$ and $\beta_\xi > \xi$ we have $y_\xi \notin \bigcup\{c_\alpha : \alpha \in S \setminus (\beta_\xi + 1)\}$. Seeking a contradiction let us assume that E is uncountable.

Let M be a countable elementary submodel of some $H(\theta)$ so that S, K, $\{(y_\xi, \beta_\xi) : \xi \in E\} \in M$ (of course, X and $\{c_\alpha : \alpha < \omega_1\}$ are also elements of M). Define $\delta := M \cap \omega_1$. Since $\{y_\xi : \xi \in E \cap \delta\}$ is infinite, $E \cap \delta$ has a cofinal set E_0 such that $\{y_\xi : \xi \in E_0\}$ converges to some $z \in X$ (i.e. any neighborhood of z contains all but finitely many y_ξ's, $\xi \in E_0$). Since K is open and $\{y_\xi : \xi \in E_0\} \subseteq X \setminus K$, we get $z \in X \setminus K$. Also note that $z \in c_\alpha$ for some $\alpha \in S$ and therefore $\{\xi \in E_0 : y_\xi \notin c_\alpha\}$ is finite. If $\xi \in E_0$ is so that $y_\xi \in c_\alpha$ then $\xi \in M$. By definition, $y_\xi \notin \bigcup\{c_\gamma : \gamma \in S \setminus (\beta_\xi + 1)\}$ and, in particular, $\alpha \leqslant \beta_\xi$; thus $\alpha \in M$. Since we are assuming that E is uncountable, M thinks that E_0 is uncountable and therefore $\{\xi \in E : y_\xi \in c_\alpha\}$ is, indeed, uncountable. But $|c_\alpha| \leqslant \omega$ and hence there exists $\gamma \in E$ so that $|\{\xi \in E : y_\xi = y_\gamma\}| = \omega_1$. Note that this implies the existence of a $\xi \in E$ satisfying $y_\xi = y_\gamma$ and $\beta_\gamma < \xi$. A contradiction to $y_\gamma \in c_\xi$.

Now we know that $|E| \leqslant \omega$ and therefore exists $\gamma < \omega_1$ so that such that $\alpha \in S \setminus \gamma$ implies $c_\alpha \setminus K \subseteq \bigcup\{c_\xi : \xi \in S \setminus \beta\}$ for any $\beta < \omega_1$. By enlarging K, if necessary, and removing all ordinals $\alpha \leqslant \gamma$ from S we may assume that $X \setminus \bigcup\{c_\alpha : \alpha \in S\}$. Thus for any $\alpha \in S$ we have that $\mathrm{Fn}(\omega_1, 2) \Vdash c_\alpha \setminus K \subseteq \bigcup\{a_\beta : \beta \in \dot{S}\}$. *Q.E.D.*

Now we have all the ingredients needed to show that in the model obtained by adding ω_2 Hechler reals there is an Efimov T-algebra.

Theorem 3.8. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2\rangle$ be a finite support iteration where $\Vdash_\alpha$ "$\dot{Q}_\alpha$ is the Hechler poset" for each $\alpha < \omega_2$. Then there is an Efimov T-algebra in the forcing extension given by P_ε.

Proof. Let us start by noting that for each $\alpha < \omega_2$ of cofinality ω_1 we can replace the condition given in the statement of the theorem by $\Vdash_\alpha \dot{Q}_\alpha = \mathrm{Fn}(\omega_1, 2)$ because the iteration has finite support. This modification makes clear that the iteration adds a T-algebra A just like we did in the proof of Theorem 3.4. We will show that A is Efimov. The rest of the argument follows closely the one given for Theorem 3.4. *Q.E.D.*

3.3 A Final Note

As part of the research done for this dissertation we found the following result which, in principle, does not relate to the field of T-algebras or Efimov's problem but we consider that it is relevant specially when contrasted with the results about separation by open sets from [6].

For the next two results we will use the poset $\mathbb{P} = 2^{<\omega}$ ordered by $s \leqslant t$ if s extends t. Observe that this is the exact opposite of the partial tree ordering for $2^{<\omega}$

Lemma 3.9. There is a family $\{U(t, i) : t \in \mathbb{P} \text{ and } i < 2\}$ of subsets of ω so that

1. If $s < t$ then $U(t, i) \subseteq U(s, i)$ for all $i < 2$.
2. $U(t, 0) \cap U(t, 1) = \emptyset$ for all t.
3. If F is a finite antichain in $\mathbb{P}$ and $f : F \to 2$ then $\bigcap\{U(t, f(t)) : t \in F\}$ is infinite.
4. $n \in U(t, 0) \cup U(t, 1)$ for all $n \in \omega$ and each $t \in 2^n$.

Proof. We will use induction on the levels of $\mathbb{P}$ to construct the family. For level 1 we only have two nodes: $\mathbf{0}$ and $\mathbf{1}$. Partition ω into five infinite parts, a_0, a_1, b_0, b_1 and c. Now define $U(\mathbf{0},0) = a_0 \cup a_1$, $U(\mathbf{0},1) = b_0 \cup b_1$, $U(\mathbf{1},0) = a_0 \cup b_0$ and $U(\mathbf{1},1) = a_1 \cup b_1$.

Assume that for $n \in \omega$ we have constructed $\{U(t,i) : t \in 2^{\leqslant n} \text{ and } i < 2\}$ in such a way that conditions (1), (2) and (4) from the lemma hold and the following are also true.

(i) If f is a binary function whose domain is an antichain contained in $2^{\leqslant n}$ then $\bigcap\{U(t, f(t)) : t \in \operatorname{dom} f\}$ is infinite.

(ii) $\omega \setminus \bigcup\{U(t,i) : t \in 2^{\leqslant n} \text{ and } i < 2\}$ is infinite.

Let $\{t_k : k < 2^n\}$ be an enumeration of 2^n. For each $k < 2^n$ let $\{f^k_\ell : \ell < m\}$ be an enumeration of all binary functions whose domain is an antichain in $2^{\leqslant n}$ and no element of its domain is compatible with t_k. Using induction on $\ell < m$ we obtain four pairwise disjoint infinite sets a^0_k, a^1_k, b^0_k and b^1_k so that each one of them intersects $\bigcap\{U(t, f^k_\ell(t)) : t \in \operatorname{dom}(f^k_\ell)\}$ in an infinite set for all $\ell < m$.

To finish the construction fix a partition $\{c^i_k : k < 2^n \text{ and } i < 2\} \cup \{d\} \subseteq [\omega]^\omega$ of $\omega \setminus \bigcup\{U(t,i) : t \in 2^{\leqslant n} \wedge i < 2\}$ and define $U(t_k{}^\frown i, 0) := U(t_k, 0) \cup a^i_k \cup c^0_k$ and $U(t_k{}^\frown i, 1) := U(t_k, 1) \cup b^i_k \cup c^1_k$ (and if the integer $n+1$ is not an element of $U(t_k{}^\frown i, 0) \cup U(t_k{}^\frown i, 1)$ then add it to exactly one of them). *Q.E.D.*

Given two sets A and B we say that $A \subseteq^* B$ if $B \setminus A$ is finite. If S is an infinite set and for each $A \in \mathcal{A}$ we have $S \subseteq^* A$ then S is a *pseudointersection* of $\mathcal{A}$.

For an infinite set $S \subseteq \omega$ and a function $h : \omega \to [0,1]$ we will say that $h[S]$ *converges to* p (in symbols, $h[S] \to p$) if the sequence $\langle h(x_n) : n \in \omega\rangle$ converges to p, where $S = \{x_n : n \in \omega\}$ and $x_n < x_{n+1}$ for each $n < \omega$.

Recall that two sets A and B in a topological space X are completely separated if there is a continuous real-valued map f so that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$. In other words, if there is a continuous function that separates them.

If τ is a topology for X and P is any forcing notion then it could be the case that, in the generic extension obtained by forcing with P, τ is no longer a topology for X due to the presence of new subsets of τ but τ will always be a base for some topology for X. Hence, whenever we refer to the topological space (X, τ) (or simply X) we will be referring to the topology on X that has τ as a base.

It is proved in [6] that if two sets from the ground model are separated by open sets in the generic extension obtained by adding any number of Cohen reals then they are separated by open sets in the ground model.

Theorem 3.10. CH implies that there exist a first countable Tychonoff zero-dimensional space X and two sets $A_0, A_1 \subseteq X$ which are not completely separated but after adding a Cohen real they are completely separated.

Proof. Let $\mathbb{P}$ and $\{U(t,i) : t \in \mathbb{P} \wedge i < 2\}$ be as in the previous lemma. Use CH to fix $\{h_\alpha : \omega \leqslant \alpha < \omega_1\}$, an enumeration of all functions from ω into the interval $[0,1]$.

The strategy is to define a topology on ω_1 in such a way that ω is open discrete and each $\omega \leqslant \alpha < \omega_1$ will have a neighborhood base of the form $\{\{\alpha\} \cup S_\alpha \setminus n : n \in \omega\}$ for some $S_\alpha \in [\omega]^\omega$.

We will obtain this topology by induction. To be accurate, at stage α we will get three sets: D_α, a maximal antichain in $\mathbb{P}$; x_α, a function from D_α into 2; and $S_\alpha \in [\omega]^\omega$ satisfying the following.

(1α) $S_\alpha \subseteq^* U(t, x_\alpha(t))$ for each $t \in D_\alpha$.

(2α) $\beta < \alpha$ implies $|S_\beta \cap S_\alpha| < \omega$.

(3α) One of the following conditions holds.

- (a) $h_\alpha[S_\xi]$ does not converge for some $\xi \leqslant \alpha$.
- (b) There exist $k < 2$ and $\xi \leqslant \alpha$ so that $x_\xi \equiv k$ (i.e. $x_\xi(t) = k$ for all $t \in D_\alpha$) and $h_\alpha[S_\xi]$ does not converge to k.

(4α) For each binary function x whose domain is a finite antichain in $\mathbb{P}$ the set $\bigcap\{U(t, x(t)) : t \in \operatorname{dom} x\} \setminus \bigcup\{S_\xi : \xi \in a\}$ is infinite for each $a \in [\alpha \setminus \omega]^{<\omega}$.

Before going over the details of the induction let us show that a sequence satisfying all the given requirements provides us with the required space. Indeed, for each $k < 2$ define $A_k := \{\alpha < \omega_1 : x_\alpha \equiv k\}$ and assume that $h : X \to [0,1]$ is continuous. Let $\alpha < \omega_1$ be so that $h \restriction \omega = h_\alpha$. h's continuity implies that condition (3α-a) fails and therefore (3α-b) must hold. Hence there exists $\xi \leqslant \alpha$ so that $x_\xi \equiv k$, for some $k < 2$, and $h_\alpha[S_\xi]$ does not converge to k. Clearly $\xi \in A_k$ and $h(\xi) \neq k$. Therefore A_0 and A_1 cannot be separated by a continuous function.

On the other hand, if G is a $\mathbb{P}$-generic filter, let $g := \bigcup G$ and define $U_k := \bigcup\{U(g \restriction n, k) : n \in \omega\}$ for each $k < 2$. Observe that if $\alpha \in X \setminus \omega$ then $g \restriction m \in D_\alpha$ for some integer m and therefore

$$S_\alpha \subseteq^* U(g \restriction m, x_\alpha(g \restriction m)).$$

This property and the fact that $\alpha \in \overline{U_0}$ if and only if $S_\alpha \cap U_0$ is infinite imply that $\overline{U_0} \cap \overline{U_1} = \emptyset$ (recall item (2) from Lemma 3.9). The same property implies that if $\alpha \in A_k$ then $\alpha \in \overline{U(g \restriction m, k)} \subseteq \overline{U_k}$; in other words, $A_k \subseteq \overline{U_k}$. Therefore A_0 and A_1 are forced to be completely separated.

The only thing left is to construct the sequence. To do this let us assume that we are at stage α and we have defined $\{(D_\beta, x_\beta, S_\beta) : \omega \leqslant \beta < \alpha\}$ so that conditions (1β)-(4β) are satisfied for all β. For each binary function x for which $\operatorname{dom} x$ is a maximal antichain let

$$\mathcal{F}(x) := \{U(t, x(t)) \setminus I : t \in \operatorname{dom} x \wedge I \in \mathcal{I}\},$$

where $\mathcal{I} := \{\bigcup\{S_\xi : \xi \in a\} : a \in [\alpha \setminus \omega]^{<\omega}\}$. Observe that $\mathcal{F}(x)$ is countable and hence it has pseudointersections.

Seeking a contradiction, which comes at the end of the proof, we assume that for every maximal antichain D, for all functions $x : D \to 2$ and for every pseudointersection S of $\mathcal{F}(x)$ the set $h_\alpha[S]$ converges and if $x \equiv k$ for some $k < 2$ then $h_\alpha[S] \to k$.

Note that if S and S' are pseudointersections of $\mathcal{F}(x)$ then $S \cup S'$ is also a pseudointersection of $\mathcal{F}(x)$ and therefore $h_\alpha[S \cup S']$ converges to some real number r. Thus $h_\alpha[S] \to r$ and $h_\alpha[S'] \to r$. Hence, if x is a binary function for which $\operatorname{dom} x$ is a maximal antichain in $\mathbb{P}$ then there is a real number $\varphi(x)$ so that $h_\alpha[S] \to \varphi(x)$ for any pseudointersection S of $\mathcal{F}(x)$.

We claim that if D is a maximal antichain and $x \in 2^D$, the map $x \mapsto \varphi(x)$ is continuous, where 2^D is equipped with the product topology. Since D is countable, we only have to prove that if $\{x_n : n \in \omega\} \subseteq 2^D$ converges to $x \in 2^D$ then $\varphi(x_n) \to \varphi(x)$. If this were not the case then we would be able to find $\varepsilon > 0$ so that infinitely many n's satisfy $|\varphi(x_n) - \varphi(x)| > \varepsilon$. Without loss of generality let us assume that this happens for all $n \in \omega$. Now let H_n be a pseudointersection of $\mathcal{F}(x_n)$. By removing finitely many elements from H_n we can assume that $|h_\alpha(k) - \varphi(x)| > \varepsilon$ for all $k \in H_n$.

Write D as an increasing union of finite sets, $D = \bigcup_n F_n$, and enumerate $\mathcal{I} = \{I_n : n \in \omega\}$. Let $S = \{k_n : n \in \omega\}$ be a sequence of integers satisfying $k_n \in H_n \cap \bigcap\{U(t, x_n(t)) : t \in F_n\} \setminus (k_{n-1} \cup I_n)$. Observe that for each $t \in D$ and $n \in \omega$ there exists $m > n$ so that $t \in F_m$ and $x_i \restriction F_m = x \restriction F_m$ for all $i \geqslant m$. Hence $\{k_i : i \geqslant m\} \subseteq U(t, x(t)) \setminus I_n$ (recall that $F_m \subseteq F_i$). This proves that S is a pseudointersection of $\mathcal{F}(x)$ and therefore $h_\alpha[S] \to \varphi(x)$. In particular, there is an $n \in \omega$ so that $|h_\alpha(k_n) - \varphi(x)| < \varepsilon$, but $k_n \in H_n$. This contradiction shows that $\varphi \restriction 2^D$ is continuous.

For any set $Y \subseteq \mathbb{P}$ define $Y^\downarrow := \{t \in \mathbb{P} : \exists s \in Y (t < s)\}$.

Claim: Let E_0 and E_1 be maximal antichains. If $y_0 : E_0 \to 2$ and $y_1 : E_1 \to 2$ *agree on cones* (i.e. $y_0(s) = y_1(t)$ whenever $s \in E_0$ and $t \in E_1$ are comparable) then $\varphi(y_0) = \varphi(y_1)$.

Proof of the Claim: To show that $\varphi(y_0) = \varphi(y_1)$ we only have to prove that $\mathcal{F}(y_0)$ and $\mathcal{F}(y_1)$ have a common pseudointersection.

Let us start by proving that $E := (E_0 \setminus E_1^\downarrow) \cup (E_1 \setminus E_0^\downarrow)$ is a maximal antichain. Given $s, t \in E$ there are two cases: Both belong to the same E_i (so they are incompatible) or (without loss of generality) $s \in E_0 \setminus E_1^\downarrow$ and $t \in E_1 \setminus E_0^\downarrow$. In the second case we obtain $s \nleqslant t$ and $t \nleqslant s$ and therefore s and t are incompatible. To prove maximality let $t \in \mathbb{P}$ be arbitrary. Since E_i is maximal, there exists $t_i \in E_i$ which is incompatible with t for each $i < 2$. If, for example, $t_0 \in E_1^\downarrow$ then $t_0 < s$ for some $s \in E_1$ and thus s and t are compatible, which gives $s = t_1$. Clearly, $t_1 \notin E_0^\downarrow$ so $t_1 \in E$.

The function $y := y_0 \restriction (E_0 \setminus E_1^\downarrow) \cup y_1 \restriction (E_1 \setminus E_0^\downarrow)$ has domain E and agrees on cones with y_0 and y_1.

Let S be a pseudointersection of $\mathcal{F}(y)$ and let $i < 2$ be arbitrary. In order to prove that S is a pseudointersection of $\mathcal{F}(y_i)$ let F be a finite subset of E_i. For each $t \in F$ there exists

$t' \in E$ so that $t \leqslant t'$. Therefore

$$S \subseteq^* \bigcap \{U(t', y(t')) : t \in F\} \setminus I \subseteq \bigcap \{U(t, y_i(t)) : t \in F\} \setminus I,$$

for all $I \in \mathcal{I}$. Which finishes the proof of the Claim.

Fix a sequence of positive real numbers $\{\varepsilon_n : n \in \omega\}$ so that $\sum_{n<\omega} \varepsilon_n < 1/3$.

The fact that $2^{<\omega}$ embeds densely in $\omega^{<\omega}$ implies that everything we have done so far can be coded for $\omega^{<\omega}$ via the embedding. To simplify the next arguments we will switch to $\mathbb{P} = \omega^{<\omega}$ and keep the notation we developed for $2^{<\omega}$.

For each $n \in \omega$ the set D_n of all functions from n into ω is a maximal antichain in $\mathbb{P}$ and therefore $\varphi \restriction 2^{D_n}$ is continuous. Moreover, 2^{D_n} is compact and therefore φ is uniformly continuous so there exists a finite set $F_n \subseteq D_n$ so that for all $x, y \in 2^{D_n}$ if $x \restriction F_n = y \restriction F_n$ then $|\varphi(x) - \varphi(y)| < \varepsilon_n$. By enlarging F_n we can assume that there is an integer m_n so that F_n is the set of all functions from n into m_n and $m_n < m_{n+1}$.

The set $D^0 := \{t^\frown i : \exists n \in \omega(t \in D_n) \wedge \forall k < n(t(k) < m_k) \wedge m_n \leqslant i < \omega\}$ is a maximal antichain in $\mathbb{P}$. Let F^0 be a finite subset of D^0 so that $x \restriction F^0 = y \restriction F^0$ implies $|\varphi(x) - \varphi(y)| < 1/3$ for all $x, y : D^0 \to 2$. Let $\ell < \omega$ be large enough so that $F^0 \subseteq \omega^{<\ell}$.

For all $1 \leqslant k \leqslant \ell$ define $x_k : D_k \to 2$ by $x_k(t) = 1$ iff $x \restriction i \in F^0$ for some $i \leqslant k$. Also let $y_k : D_k \to 2$ be given by $y_k(t) = x_k(t \restriction (k-1)^\frown 0)$. If $t \in F_k$ then t and $t \restriction (k-1)$ have the same predecessors and since $F_k \cap F^0 = \emptyset$ we obtain $x_k \restriction F_k = y_k \restriction F_k$ which implies $|\varphi(x_k) - \varphi(y_k)| < \varepsilon_k$.

On the other hand, if $s \in D_{k-1}$ and $t \in D_k$ satisfy $t < s$ then $y_k(t) = x_{k-1}(s^\frown 0)$ and therefore x_{k-1} and y_k agree on cones. Hence $\varphi(y_k) = \varphi(x_{k-1})$.

The two previous paragraphs show that $|\varphi(x_\ell) - \varphi(x_1)| < \sum_{k=1}^{\ell} \varepsilon_k < 1/3$. Note that $x_1 \equiv 0$ and thus $\varphi(x_1) = 0$ which gives $\varphi(x_\ell) < 1/3$.

For each $t \in D^0$ fix a $t' \in D_\ell$ which is compatible with t. The function $z : D^0 \to 2$ defined by $z(t) = x_\ell(t')$ agrees on cones with x_ℓ and hence $\varphi(z) = \varphi(x_\ell)$. If $y : D^0 \to 2$ satisfies $y \equiv 1$ then $z \restriction F^0 = y \restriction F^0$ so $|\varphi(z) - \varphi(y)| < 1/3$ and since $\varphi(y) = 1$ we conclude that $\varphi(x_\ell) = \varphi(z) > 1/3$, a contradiction. This ends the proof of the theorem. *Q.E.D.*

REFERENCES

[1] B. Balcar, J. Pelant, and P. Simon, *The space of ultrafilters on N^* covered by nowhere dense sets*. Fundamenta Mathematicae, CX:11-24, 1980.

[2] J. Baumgartner, P. Komjath, *Boolean algebras in which every chain and every antichain is countable*, Fund. Math. 111, 1981, pp. 125-133.

[3] W.W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Berlin; New York : Springer-Verlag, 1974

[4] A. Dow, *Efimov spaces and the splitting number*. Spring Topology and Dynamical Systems Conference. Topology Proc. **29** (2005), no. 1, 105–113.

[5] A. Dow, D.H. Fremlin, *Compact sets without converging sequences in the random real model*, preprint (2005).

[6] Alan Dow, Franklin D. Tall and William A.R. Weiss, *New Proofs of the Consistency of the Normal Moore Space Conjecture*, Topology and its Applications **37** (1990) 33-51.

[7] M. Džamonja, G. Plebanek, *On Efimov spaces and Radon measures*, Topology Appl. **154** (2007), no. 10, 2063-2072.

[8] B. Efimov, *The imbedding of the Stone-Čech compactifications of discrete spaces into bicompacta*, Dokl. Akad. Nauk SSSR **189** (1969), 244-246, Translation: Soviet Math. Dokl. 10 (1969) 1391-1394.

[9] R. Engelking, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from the Polish by the author. MR **91**c:54001

[10] V. V. Fedorčuk, *Completely closed mappings, and the consistency of certain general topology theorems with the axioms of set theory*, Matematicheskiĭ Sbornik. Novaya Seriya **99** (**141**) (1976), 1-26 (Russian); English transl. Mathematics of the USSR. Sbornik **28** (1976), 3-33, 135. MR 53 #14379, Zbl. 0371.54045

[11] K.P. Hart, *Efimov's problem*, Open Problems in Topology II, (E. Pearl, ed.), Elsevier Science, 2007, pp. 171-177.

[12] S. Koppelberg, *Minimally generated Boolean algebras*, Order **5** (1989), no. 4, 393-406. MR **90g**:06022

[13] S. Koppelberg, *Counterexamples in minimally generated Boolean algebras*, Proceedings of the 16th Winter School on Abstract Analysis. Charles University, Praha, 1988. pp. 27-36.

[14] P. Koszmider, *Forcing minimal extensions of Boolean algebras*, Trans. Amer. Math. Soc. **351** (8) (1999), pp. 3073-3117.

[15] K. Kunen, *Set theory. An introduction to independence proofs*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam, 1980.

[16] S. Mercourakis, *Some remarks on countably determined measures and uniform distribution of sequences*, Monats. Math. **121** (1996), 79–101.

[17] R. Sikorski, *Boolean Algebras*, Springer-Verlag, 1964. MR **31:2178**.

[18] S. Shelah, *On uncountable Boolean algebras with no uncountable pairwise comparable or incomparable sets of elements*, Notre Dame Journal of Formal Logic 22, 1981, pp. 301-308.

[19] S. Shelah, *Constructions of many complicated uncountable structures and Boolean algebras*, Israel J. Math. Vol. 45, No. 2-3, 1983, pp. 100-146.

[20] Vaughan, J.E., *Countably Compact and Sequentially Compact Spaces*, Handbook of Set-Theoretic Topology, (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 569-602.