ASYMPTOTIC ANALYSIS OF THE ANDERSON PARABOLIC PROBLEM AND THE MOSER'S TYPE OPTIMAL STOPPING PROBLEM

by

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ABSTRACT

HAO ZHANG.Asymptotic analysis of the Anderson parabolic problem and the Moser's type optimal stopping problem. (Under the direction of DR. STANISLAV A. MOLCHANOV)

The central objects of the thesis are the Anderson parabolic problem and the Moser's type optimal stopping problem:

(1) In the lattice parabolic Anderson problem, we study the quenched and annealed asymptotics for the solutions of the lattice parabolic Anderson equation in the situation in which the underlying random walk has long jumps and belongs to the domain of attraction of the stable process.

The i.i.d random potential in our case is unbounded from above with regular Weibull type tails. Similar models but with the local basic Hamiltonian (lattice Laplacian) were analyzed in the very first work on intermittency for the parabolic Anderson problem by J. Gärtner and S. Molchanov.

We will show that the long range model demonstrates the new effect. The annealed (moment) and quenched (almost sure) asymptotics of the solution have the same order in contrast to the case of the local models for which these orders are essentially different.

(2) Concerning Moser's problem, we study two related optimization problems for i.i.d. random variables X_i, $i = 1, 2, ..., n$, referred to as the generalized Moser's problem: a) Find $\max_{\tau \leq n} EX_\tau$ ($\tau \leq n$ are the stopping times). b) Find τ: $P\{X_\tau = M_n\}$=max, here $M_n = \max_{0 \leq i \leq n} X_i$. For the wide class of continuous distribution functions $F_X(x)$ with regular tails, we will present the asymptotic formulas for the optimal thresholds and analyze the relationship between the Moser's type problem and the classical secretary problem with information.

The present paper is structured as follows:

The first two chapters contain preliminary information.

In Chapter 1, we summarize some important properties and results about slowly varying functions.

In Chapter 2, we introduce the Anderson parabolic model, summarize some main results, such as the uniqueness and existence and the asymptotic properties of of the solution $u(t,x)$, for the parabolic Anderson model on Z^d and R^d with homogenous potentials, and discuss some limit theorems for random walks with heavy-tailed long jumps.

In Chapter 3, we prove several results on the annealed and quenched behavior of $u(t,x)$ as $t \to \infty$ with Weibull's potential. We will show that the long range model demonstrates the new effect. The annealed (moment) and quenched (almost sure) asymptotics of the solution have the same order in contrast to the case of the local models for which these orders are essentially different.

In Chapter 4, we study Moser's problem, present the asymptotic formulas for the optimal thresholds of the wide class of continuous distribution functions $F_X(x)$ with regular tails, and analyze the relationship between the Moser's type problem and the classical secretary problem with information.

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CHAPTER 1: SLOWLY VARYING FUNCTIONS AND A TAUBERIAN THEOREM

This chapter contains a review of some technical tools that are important for the applications in chapters 3 and 4.

1.1 Slowly varying functions

Recall that a slow varying function is a positive measurable function f satisfying

$$\frac{f(\lambda x)}{f(x)} \to 1 \quad (x \to \infty), \quad \forall \lambda > 0,$$

and a regularly varying function of index ρ is a measurable function $f > 0$ satisfying

$$\frac{f(\lambda x)}{f(x)} \to \lambda^{\rho} \quad (x \to \infty), \quad \forall \lambda > 0,$$

alternatively written $f \in R_{\rho}$. For details and further references see [Bingham et al., 1989].

In the spirit of the paper [Ben Arous et al., 2005] on REM model, we will introduce some essential technical developments of the pure Weibull case in chapters 3 and 4. Specifically, we will study models when the tail probability of a random variable X has a Weibull type distribution, that is,

$$P\{X > x\} = e^{-\frac{x^{\alpha}}{\alpha} L(x)}, \quad \alpha > 0,$$

where $L(\cdot)$ is a slowly varying function with some additional regularity assumptions. We will start with several definitions and propositions.

The following lemmas and definitions are fundamental to our paper and can be found in [Bingham et al., 1989].

Lemma 1.1. *(Uniform Convergence Theorem by Karamata and Korevaar et al.) If*

$f(x)$ is slowly varying then $\frac{f(\lambda x)}{f(x)} \to 1 (x \to \infty)$ uniformly on each compact λ-set in $(0, \infty)$.

It is known (see [Bingham et al., 1989]) that a function $f(x) \in R_\rho$ iff $f(x)$ admits the Karamata representation

$$f(x) = c(x) \exp\left\{ \int_a^x \frac{\rho + \epsilon(u)}{u} du \right\} \quad (x \geq a) \tag{1.1}$$

for some $a > 0$, where $c(\cdot)$ and $\epsilon(\cdot)$ are measurable functions and $c(x) \to c_0 > 0$, $\epsilon(x) \to 0$ as $x \to \infty$.

Definition 1.1. *(After [Bingham et al., 1989]) The function f is a normalized regularly varying function, or $f \in NR_\rho$, if it can be represented in the form (1.1) with $c(\cdot) =$ constant> 0.*

One of the important properties of a normalized regularly varying function $f(x)$, $f(x) \in NR_\rho$, is provided by the following lemma (see [Bingham et al., 1989, page 24]).

Lemma 1.2. *A positive measurable function f is a normalized regularly varying function, or $f(x) \in NR_\rho$, iff for every $\epsilon > 0$ $\frac{f(x)}{x^{\rho-\epsilon}}$ is ultimately increasing and $\frac{f(x)}{x^{\rho+\epsilon}}$ is ultimately decreasing.*

Another important property of a normalized regularly varying function $f(x) \in NR_\rho$ is given by the following lemma (see [Bingham et al., 1989, page 15]).

Lemma 1.3. *Let f be a positive measurable function. Then $f \in NR_\rho$ iff f is differentiable (a.e.) and when $x \to \infty$,*

$$\frac{x f'(x)}{f(x)} \to \rho.$$

1.2 A Tauberian Theorem

Kasahara-de Bruijn's Tauberian theorem (see [Bingham et al., 1989, page 253]) is fundamental to this paper. In the following, $f^{\leftarrow}(y)$ is the generalized inverse function of f, defined $f^{\leftarrow}(y) = \inf\{x : f(x) \geq y\}$.

Theorem 1.1. *(Kasahara-de Bruijn's Tauberian Theorem). Let μ be a measure on $(0, \infty)$ such that*

$$M(\lambda) = \int_0^\infty e^{\lambda x} d\mu(x) < \infty$$

for all $\lambda > 0$. If $0 < \alpha < 1$, $\phi \in R_\alpha$, put $\psi(\lambda) = \lambda/\phi(\lambda) \in R_{1-\alpha}$; then, for $B > 0$,

$$-\ln \mu(x, \infty) \sim B\phi^{\leftarrow}(x) \quad (x \to \infty) \tag{1.2}$$

if and only if

$$\ln M(\lambda) \sim (1-\alpha)(\alpha/B)^{\alpha/(1-\alpha)}\psi^{\leftarrow}(\lambda) \quad (\lambda \to \infty).$$

The following theorem (see [Bingham et al., 1989, page 78]) is critical for the iteration of slowly varying functions.

Theorem 1.2. *(See [Bingham et al., 1989]) If $L(x)$ is a slowly varying function and for some $\lambda_0 > 1$,*

$$\left\{\frac{L(\lambda_0 x)}{L(x)} - 1\right\} \ln L(x) \to 0 \ (x \to \infty) \tag{1.3}$$

then

$$L(xL^\beta(x))/L(x) \to 1 \ (x \to \infty) \ locally \ uniformly \ in \ \beta \in R.$$

Remark 1.1. *Examples of slowly varying functions that satisfy condition (1.3) are (1) $f(x) = \ln^\beta x$ and (2) $f(x) = \ln \ln^\beta x$. Not every slowly varying function satisfies condition (1.3). The function $L(x) = e^{\ln^\beta x}$ $(0 < \beta < 1)$ satisfies the condition (1.3) when $0 < \beta < \frac{1}{2}$ and does not satisfy this condition when $\beta = \frac{1}{2}$. As shown in [Bojanic and Seneta, 1971], the recursive relationship for function $f(x) = e^{\ln^\beta x}(0 < \beta < 1)$ is as follows:*

$$f(xf^{\alpha}(x))/f(x) \to \begin{cases} 1 & if\ 0 < \beta < \frac{1}{2} \\ \exp(\alpha\beta) & \beta = \frac{1}{2} \\ 0 & if\ \alpha < 0\ and\ \frac{1}{2} < \beta < 1 \\ \infty & if\ \alpha > 0\ and\ \frac{1}{2} < \beta < 1. \end{cases} \tag{1.4}$$

The following lemma (see [Bojanic and Seneta, 1971] and [Bingham et al., 1989, page 77, 78]) is also related to the iteration of slowly varying functions.

Lemma 1.4. *(See [Bingham et al., 1989] and [Bojanic and Seneta, 1971]) If $L(x)$ is a non-decreasing slowly varying function, $L(x) > 1$, and $L(x)$ is continuously differentiable, then*

$$\frac{xL'(x)}{L(x)} \ln L(x) \to 0 \tag{1.5}$$

implies

$$L(xL^{\alpha}(x))/L(x) \to 1\ (x \to \infty)\ locally\ uniformly\ in\ \alpha \in R.$$

This lemma is very general and useful. Its main defect is the assumption that $L(x)$ is non-decreasing.

In the following discussion we assume that the slowly varying function $L(x)$ satisfies the condition that $L(xL^{\alpha}(x))/L(x) \to 1$ $(x \to \infty)$ locally uniformly in $\alpha \in R$. Theorem 1.2 and Lemma 1.4 guarantee this assumption.

CHAPTER 2: THE ANDERSON PARABOLIC PROBLEM

The Anderson model originated in solid state physics. In the original Anderson model, the movement of an electron in a disordered environment is described by the non-stationary discrete Schrödinger equation

$$i\hbar\frac{\partial u}{\partial t}(t,x) = (-\hbar^2\Delta + V(x,\omega_m))u(t,x), \tag{2.1}$$

in which the potential $V(x,\omega_m)$ is independent of time.

Let us cite from [Molchanov, 1994] a different situation in which the Anderson parabolic problem arises naturally. Consider the following population problem on the lattice Z^d. Suppose that at time zero the system contains a single particle, which jumps according to the laws of a continuous time random walk. The model has the following four properties:

(1) The infinitesimal transition probabilities of the random walk $X(t), t \geq 0$ are :

$$\begin{cases} P\{X(t+dt) = x+z|X(t)=x\} = \kappa dt, \quad |z|=1. \\ P\{X(t+dt)=x|X(t)=x\} = 1-2d\kappa dt. \end{cases} \tag{2.2}$$

(2) During the time interval dt any particle at $x \in Z^d$ splits into two with probability $\nu(x)dt$ and dies with probability $\mu(x)dt$.

(3) The staying time of each particle at any $x \in Z^d$ is exponentially distributed with parameter $2d\kappa$. We call the rate $2d\kappa$ the diffusion rate.

(4) Each descendent of a particle evolves according to the same law but independently of all other particles.

The moment generating function for the number of particles

$$M_z(t,x,y) = E_x z^{n(t,y)}$$

satisfies the Skorokhod equation:

$$\begin{cases} \frac{\partial M_z}{\partial t} = \mathcal{L}M_z + \nu(x)M_z^2 - (\nu(x)+\mu(x))M_z + \mu(x) \\ M_z(0,x,y) = 1 \ if \ x = y \\ M_z(0,x,y) = 1 \ otherwise, \end{cases} \tag{2.3}$$

where the Laplacian operator $\mathcal{L}M_z(t,x,y) = \kappa \sum_{|x^{'}-x|=1} [M_z(t,x^{'},y) - M_z(t,x,y)]$.

Differentiating (2.3) at $z = 1$, we obtain the moment equation for the particle field $n(t,y)$. In particular, $u(t,x) = E_x n(t,0)$ satisfies

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}u + V(x)u \\ u(0,x) = \delta_0(x), \end{cases} \tag{2.4}$$

where $V(x) = \nu(x) - \mu(x)$ and $\mathcal{L}u(t,x) = \kappa \sum_{|x^{'}-x|=1} [u(t,x^{'}) - u(t,x)]$.

Next, we replace the term $V(x)$ in the equation (2.4) with $V(x,\omega_m)$. Here, the random variable $V(x,\omega_m)$ belongs to a new probability space (Ω_m, Γ_m, P_m). More precisely, our new model is:

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}u + V(x,\omega_m)u \\ u(0,x) \equiv \rho_0 > 0, \end{cases} \tag{2.5}$$

where $V(x,\omega_m) = \nu(x,\omega_m) - \mu(x,\omega_m)$ and the parameter ρ_0 is the initial density of the population..

For the nonnegative potential $V(x,\omega_m)$, we also can interpret problem (2.5) as a linearized model of chemical reactions. In this case, the solution of the equation describes the evolution of the density of reactants $u(t,x)$ under the influence of a catalyst medium $V(x,\omega_m)$. The interpretation of (2.5) as a linearized model of chemical kinetics is outlined in [Gärtner and Molchanov, 1990], Section 1.2.

Consider the case of the potential $V(x, \omega_m)$ unbounded from above, i.e., $P\{V(\cdot) > a\} > 0$ for any $a > 0$. A typical example is i.i.d. $N(0,1)$ r.v.s. See [Molchanov, 1994] for detailed analysis of the Gaussian case.

General study of the problem (2.5) in the particular case of local diffusion $\mathcal{L} = \kappa\Delta$ was started in a paper by J. Gärtner and S. Molchanov (1990) and later was expanded in many different directions by J. Gärtner and his collaborators. The central idea is the justification of the intermittency phenomenon: the random environment leads to a highly non-uniform population structure (see J. Gärtner and S. Molchanov (1990, 1998), J. Gärtner, W. König and S. Molchanov (2007), S. Molchanov (1994) , S. Molchanov and H. Zhang (2011), J. Gärtner and W. König (2000)). Corresponding effects are known in the physics literature [Molchanov et al., 1988].

2.1 The annealed and quenched asymptotics of $u(t, x)$

The probability measure P that determines the distribution of the random walk in a given environment $\omega \in \Omega$ is referred to as the quenched law, while the probability measure P_m on the random media is referred to as the annealed law. We often use a subindex to indicate the initial position of the walk, so that, e.g., $P_x\{x_0 = x\} = 1$.

Here, and in the future, the symbol $\langle\ \rangle$ means the expectation with respect to the probability measure P_m of the random media. The notation E or E_x will be used for the expectation over the quenched probability measure P for the random walk and fixed ω_m.

2.2 Intermittency

Definition 2.1. *(see [Molchanov, 1994]) Let $u(t, x)$, $t \geq 0$, $x \in Z^d$ be a family of non-negative, homogeneous and ergodic-in-space random fields on a joint probability space $(\Omega_m, \Gamma_m, P_m)$. Suppose that all moment functions of order p, $p = 1, 2, ...$ are* *finite for all $t \geq 0$. In particular, the functions*

$$\Lambda_p(t) = \ln\langle u^p(t,x)\rangle, \; x \in Z^d, \; t \geq 0$$

depend only on t.

If there exists some monotone increasing scale function $A(t)$, the limits

$$\gamma_p = \lim_{t\to\infty} \frac{\Lambda_p(t)}{A(t)} = \lim_{t\to\infty} \frac{\ln\langle u^p(t,x)\rangle}{A(t)}$$

are called the moment Lyapunov exponents with scale $A(\cdot)$.

Lemma 2.1. *For any $p \in N$*

$$\frac{\gamma_p}{p} \leq \frac{\gamma_{p+1}}{p+1}$$

Proof. Applying Hölder's inequality

$$E(fg) \leq (Ef^r)^{\frac{1}{r}}(Eg^s)^{\frac{1}{s}}$$

to $f = u(t,x)^{p+1}$ and $g = u(t,x)^{\frac{p}{p+1}}$, $r = \frac{p+1}{p}$, $s = p+1$, we have $\langle u(t,x)^p\rangle^{\frac{1}{p}} \leq \langle u(t,x)^{p+1}\rangle^{\frac{1}{p+1}}$. □

Definition 2.2. *(see [Molchanov, 1994]) $u(t,x)$ is asymptotically intermittent if*

$$\gamma_1 < \frac{\gamma_2}{2} < \frac{\gamma_3}{3} < \ldots.$$

The concept of intermittency or intermittent random fields is very popular in several branches of modern physics, such as statistical (turbulent) hydrodynamics and magnetohydrodynamics. Let us cite an explanation of intermittency from [Molchanov, 1994]: "... intermittency means that, in contrast with homogenization, the spatial structure of $u(t,\cdot)$ is highly irregular for large t. In one or another sense the essential part of the solution is believed to consist of islands of high peaks which are located far from each other. The sizes of these islands as well as the heights and shapes of the corresponding peaks (both of the potential $V(x,\omega_m)$ and the solution $u(t,\cdot)$) are crucial for different asymptotic questions related to our Anderson problem.... The intermittent random fields are distinguished by the formation of strong pronounced spatial structures: sharp peaks, foliations and others giving the main contribution to the physical processes in such media."

Intermittency corresponds to very irregular behavior of the solution u. In the case of a stationary random field $V(x, \omega_m)$, intermittency corresponds to the fact that there are some small but more and more widely spaced peaks absorbing the total mass of the solution u. See Gärtner and Molchanov (1990), Section 1.1, for a detailed interpretation of intermittency in this case. A detailed understanding of the geometric structure of intermittent solutions, therefore, would be extremely useful.

The Anderson parabolic model admits three essentially different variants with respect to the media.

(a) Homogenous models. In this case, we mainly restrict ourselves to i.i.d. random variables with restrictions on the tails $log\langle e^{tV(0)}\rangle < \infty$, $\forall\ t > 0$. Here, the potential $V(x, \cdot)$ is independent of the time t and the position x . We mainly restrict ourselves to the Gaussian case and more general Weibull type potentials.

(b) Stationary models. The characteristics of this situation are (1) The potentials $V(x, \cdot)$ $V(x, \cdot)$ are dependent with covariance $\gamma(x)$; (2) The potentials $V(x, \cdot)$ $V(x, \cdot)$ do not depend on time. The Poissonian type shot noise potential was considered in [Carmona and Molchanov, 1995].

(c) Non stationary models. The potential $V(t, x, \omega_m)$ depends on t explicitly, and time correlations decrease rapidly. See [Carmona and Molchanov, 1994] for details.

Definition 2.3. *(see [Molchanov, 1994]) If there exists some monotone increasing scale function $a(t)$, the limits*

$$\tilde{\gamma} = \lim_{t\to\infty} \frac{\ln u(t, x)}{a(t)} \quad (P_m - a.s.)$$

are called a.s. Lyapunov exponents or quenched Lyapunov exponents.

We shall henceforth assume that the cumulant generating function of the $V(x, \omega_m)$ is finite on the positive half axis:

$$H(t) = log\langle e^{tV(0)}\rangle < \infty, \quad \forall\, t > 0.$$

2.3 The Anderson parabolic problem on Z^d with homogeneous random medium

2.3.1 The existence and uniqueness of the solution $u(t, x)$

We will assume throughout the section that that the potential $V(\cdot)$ consists of independent, identically distributed random variables with continuous distribution function F satisfying $F(x) < 1$ for all x (i.e. $V(\cdot)$ is unbounded from above a.s.). In this paper we will discuss only a homogeneous environment.

Under the condition $\langle e^{\lambda V(\cdot)}\rangle = \Psi(\lambda) < \infty$, $\forall(\lambda \in R^1)$, the particle field $u(t, x)$ has all statistical moments and

$$m_k(t) = \langle u^k(t, 0)\rangle = \langle u^k(t, x)\rangle, \quad k = 1, 2, ...$$

Proposition 2.1. *(Feynman-Kac formula) The integral over trajectories*

$$u(t, x) = E_x e^{\int_0^t V(x_s)ds}$$

is the solution of

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}u + V(x, \omega_m)u \\ u(0, x) = 1, \end{cases}$$

where X_s, $s \geq 0$, is a random walk on Z^d with generator $\mathcal{L}$ and corresponding expectation E_x.

Proof. By definition,

$$
\begin{aligned}
&u(t+\Delta t, x) = E_x e^{\int_0^{t+\Delta t} V(x_s)ds} \\
&= E_x e^{\int_0^{\Delta t} V(x_s)ds} E_{x_{\Delta t}} e^{\int_{\Delta t}^{t+\Delta t} V(x_s)ds} \\
&= E_x e^{\int_0^{\Delta t} V(x_s)ds} u(t, x_{\Delta t}) \\
&= E_x (1 + \textstyle\int_0^{\Delta t} V(x_s)ds + \Delta t) u(t, x_{\Delta t}) \\
&= (1 - 2d\Delta\kappa t)(1 + V(x)\Delta t + o(\Delta t))u(t,x) + (1 + O(\Delta t)) \sum_{|x'-x|=1} u(t,x')\kappa\Delta t \\
&= u(t,x) + V(x)\Delta t u(t,x) + \kappa\Delta t \sum_{|x'-x|=1} (u(t,x') - u(t,x)) + o(\Delta t).
\end{aligned}
$$

So, we have that

$$\tfrac{u(t+\Delta t,x)-u(t,x)}{\Delta t} = \kappa \sum_{|x'-x|=1} (u(t,x') - u(t,x)) + V(x)u(t,x) + \tfrac{o(\Delta t)}{\Delta t}.$$

As $\Delta t \to 0$, the above equation is $\frac{\partial u}{\partial t}(t,x) = \kappa \sum_{|x'-x|=1} (u(t,x') - u(t,x)) + V(x,\omega_m)u(t,x)$.

$\square$

Due to the proposition (2.1), the quenched representation of the first moment is

$$u(t, \omega_m, x) = E_x e^{\int_0^t V(x_s,\omega_m)ds}.$$

At the same time,

$$m_1(t,0) = \langle E_0 e^{\int_0^t V(x_s,\omega_m)ds} \rangle = E_0 \langle e^{\int_0^t V(x_s,\omega_m)ds} \rangle$$

is the corresponding annealed first moment.

The existence and uniqueness of the solution $u(t,x)$ for the Anderson parabolic problem on Z^d with homogeneous random medium are solved in [Gärtner and Molchanov, 1990]. To facilitate our discussion, we introduce the non-decreasing function

$$\phi(r) := \log \frac{1}{1 - F(r)}, \quad r \in R$$

and its left-continuous inverse

$$\psi(s) := \{\min r : \phi(r) \geq s\}, \quad s > 0.$$

Note that ψ is strictly increasing and $\phi(\psi(s)) = s$ for all $s > 0$.

As is stated in [Gärtner and Molchanov, 1990], the problem (2.5) admits at most one nonnegative solution. Existence of a nonnegative solution is equivalent to

$$u(t,x) = E_x e^{\int_0^t V(x_s)ds} < \infty. \tag{2.6}$$

It is easily seen from the representation in (2.6) that assumption (2.6) is equivalent to the fact that all moments and correlations of the $u(t,x)$ are finite for all time. To decide whether (2.6) is fulfilled, we need to compare the large deviation of the position of the random walk $(x(s), P)$, or the speed of decay of the probability that the random walk $(x(s), P)$ hits a point y in the time interval $[0,t]$, with the speed of growth of $\max\limits_{x(s), s<t} V(x)$.

We know that if the particle's waiting time $\tau \sim exp(\kappa)$, then the number of jumps during time t has a Poisson distribution with parameter $2d\kappa t$, that is,

$$P_x(N(t) = n) = \frac{(2d\kappa t)^n}{n!} e^{-2d\kappa t}.$$

We will use the following notation: x^+ and x^- denote the positive and negative parts of $x \in R$, respectively. We let $\log_+ x = \log x$ if $x > e$ and $\log_+ x = 1$ otherwise.

The class ϱ_0 of function $\theta : Z^d \to R_+$ is defined as

$$\varrho_0 = \{\theta : \limsup_{|x|\to\infty} \frac{\log_+ \theta(x)}{|x| \log |x|} < 1\}. \tag{2.7}$$

The class u_0 of function $\theta : Z^d \to R_+$ is defined as

$$u_0 = \{\theta : \limsup_{|x|\to\infty} \frac{\log \log_+ \theta(x)}{\log |x|} < 1,\ a.s.\}. \tag{2.8}$$

Condition (2.8) is slightly stronger than condition (2.7).

Theorem 2.1. *(See [Gärtner and Molchanov, 1990]) Assume that the initial datum* ρ_0 *belongs to class* ϱ_0 *in (2.7) a.s.*

a) If

$$\langle(\frac{V_+(0)}{\log_+ V(0)})^d\rangle < \infty \quad \textit{with } V_+(0) \;=\; \textit{max } \{V(0), e\},$$

then a.s. the Anderson parabolic problem (2.5) has a unique nonnegative solution u:

$$u(t,x) = E_x e^{\int_0^t V(x_s)ds}\rho_0(x).$$

b) If

$$\langle(\frac{\nu^+}{\log_+ V})^d\rangle = \infty$$

and either $d \geq 2$ *or* $d = 1$ *and* $\langle \log(1+\mu^-)\rangle < \infty$

then a.s. there is no nonnegative solution to (2.5).

2.3.2 The annealed and quenched asymptotic properties of $u(t,x)$

The annealed and quenched asymptotic results can be summarized in the following theorems.

Theorem 2.2. *(See [Gärtner and Molchanov, 1990]) For every* $p \in N$ *and every* $t \geq 0$,

$$\exp\{H(pt) - 2\kappa dpt\}\langle u_0^p\rangle \leq \langle u^p(t,0)\rangle \leq \exp\{H(pt)\}\langle u_0^p\rangle.$$

In particular, $\langle u^p(t,0)\rangle < \infty$ *iff* $H(pt) < \infty$*. If* $H(t) < \infty$ *for all* $t > 0$ *and either the random potential is unbounded from above (i.e.* $\operatorname{ess\,sup} V = \infty$*) or the random variables* $V(x)$, $x \in Z^d$*, are independent and* $\operatorname{ess\,sup} V \neq 0$*, then*

$$\lim_{t\to\infty} \frac{\ln\langle u^p(t,0)\rangle}{H(pt)} = 1, \quad p \in N.$$

A more precise version of the above result is

$$\ln\langle u^p(t,0)\rangle = H(pt) - 2dpkt + o(t).$$

Theorem 2.3. *(See [Gärtner and Molchanov, 1990]) Under the above assumptions,* *with probability one for each $x \in Z^d$ the nonnegative solution u(t,x) to the random Cauchy problem has the following asymptotic behavior as $t \to \infty$:*

a) If

$$\lim_{s\to\infty} \frac{\log \psi(s)}{s} = \frac{1}{\gamma}$$

for some $\gamma > d$, then

$$\phi(\frac{\log u(t,x)}{t}) \sim \frac{\gamma}{\gamma - d} d \log t.$$

b) If

$$\lim_{s\to\infty} \frac{\log \psi(s)}{s} = 0$$

and

$$\lim_{s\to\infty} [\psi(\theta s) - \psi(s)] = \infty \quad for \quad each\ \theta > 1,$$

then

$$\phi(\frac{\log u(t,x)}{t}) \sim d \log t.$$

c) If

$$\lim_{\theta \downarrow 1} \limsup_{s\to\infty} [\psi(\theta s) - \psi(s)] = 0,$$

then

$$\frac{\log u(t,x)}{t} = \psi(d \log t) + O(1).$$

For problem (2.5) with a nonnegative homogeneous initial condition, the second order asymptotics of the statistical moments $\langle u(t,0)^p \rangle$ and the almost sure growth of $u(t,0)$ as $t \to \infty$ were studied in [Gärtner and Molchanov, 1998].

In the following, without loss of generality we take $\rho_0 = 1$.

The particular case when the $V(x_s, \omega_m)$ are i.i.d. $N(0, \sigma^2)$ r.v.s is also essential:

$$P\{V(\cdot) > a\} = \frac{1}{\sqrt{2\pi}\sigma} \int_a^\infty e^{-\frac{x^2}{2\sigma^2}} dx \underset{a\to\infty}{\sim} \frac{e^{-\frac{a^2}{2\sigma^2}}}{a\sqrt{2\pi}\sigma}.$$

It is close to the Weibull situation with $\alpha = 2$.

Let us formulate the annealed (moment) asymptotics and quenched (P_m-a.s.) asymptotics of $u(t,0)$ in the Gaussian case and the random walk with Laplacian operator

$$\mathcal{L}f(x) = \tfrac{\kappa}{2d} \sum_{|z-x|=1} [f(x+z) - f(x)].$$

Theorem 2.4. *(See [Gärtner and Molchanov, 1990] and [Molchanov, 1994])*

(a) For every $p \geq 1$ and every $t \geq 0$,

$$\exp\{\frac{p^2t^2}{2}\sigma^2 - pkt\} \leq \langle u^p(t,0)\rangle \leq \exp\{\frac{p^2t^2}{2}\sigma^2\},$$

or

$$\lim_{t\to\infty} \frac{\ln\langle u^p(t,0)\rangle}{t^2} = \frac{p^2}{2}\sigma^2.$$

A more precise version of (a) is

$$(a')\ \ln\langle u^p(t,0)\rangle = \frac{p^2t^2}{2}\sigma^2 - pkt + o(t)$$

(b) P_m-a.s.: $\lim_{t\to\infty} \frac{\ln u(t,0)}{t\sqrt{\ln t}} = \sqrt{2d}\sigma.$

Theorem 2.5. *(See [Gärtner and Molchanov, 1990] and [Molchanov, 1994])*

P_m-a.s.: and $t \to \infty$,

$$\ln u(t,0) = t\sqrt{2d\ln t} - 2dkt + o(t).$$

Let us stress that the annealed and quenched asymptotics have completely different orders. It is a general feature of all models with the local basic Hamiltonian (see [Gärtner and Molchanov, 1990], [Molchanov, 1994], etc.). We will see that for the long range Hamiltonian the opposite situation obtains, that is, the annealed and quenched asymptotics have the same order.

2.4 The random walk on Z^d with heavy-tailed long jumps

The asymptotic analysis of the Anderson parabolic problem, the surrounding bifurcations (depending on the tail behavior of the random potential), the phenomenon of the intermittency, etc., are the central topics of the remaining part of this chapter and Chapter 3.

We will assume that $a(z) = a(-z) \geq 0$, $a(0) = 0$ and $\sum_{z \neq 0} a(z) = 1$, i.e., κ is the rate of the exponentially distributed time that the underlying random walk spends in each site $x \in Z^d$. The random walk $X(s), s \geq 0$ has the following infinitesimal transition probabilities:

$$P\{X(s+ds) = x + z | X(s) = x\} = \kappa a(z) ds, z \neq 0.$$
$$(2.9)$$
$$P\{X(s+ds) = x | X(s) = x\} = 1 - \kappa a(z) ds.$$

We call the rate κ the "diffusivity."

Let us introduce the operator $\mathcal{L}$ of this random walk and call it the basic Hamiltonian:

$$\mathcal{L} f(x) = \kappa \sum_{|z|=1} [f(x+z) - f(x)] a(z), \tag{2.10}$$

where $f \in l^\infty(Z^d)$.

If $\mathcal{L} f(x) = \kappa \Delta f(x) = \frac{\kappa}{2d} \sum_{|z|=1} [f(x+z) - f(x)]$ is the lattice Laplacian then we can call the underlying random walk the lattice diffusion with diffusivity $\kappa > 0$.

We will consider here the non-local random walk with long jumps:

$$\mathcal{L} f(x) = \kappa \sum_{z \neq 0} [f(x+z) - f(x)] a(z). \tag{2.11}$$

Regularity conditions on $a(z)$ will be presented later in lemma 2.3.

Clearly, the transition probabilities $p(t, x, y)$ depend on the difference $z = y - x$ (i.e., the process X_s has independent increments); due to the symmetry of $a(z)$, they are symmetric. Furthermore, it follows easily from (2.9) that

$$\frac{\partial p}{\partial t} = \kappa (\mathcal{L} p)(t, 0, y), \;\; p(0, 0, y) = \delta_0(y). \tag{2.12}$$

This equation is most easily analyzed using the Fourier transform (characteristic function). To do so we use the following lemma.

Lemma 2.2. *Define* $\hat{\mathcal{L}}(\varphi) = \kappa \sum_{z\neq 0}(1 - cos(\varphi, z))a(z)$.
Then, $\widehat{\mathcal{L}f}(\varphi) = -\hat{f}(\varphi)\hat{\mathcal{L}}(\varphi)$.

Proof. By definition,

$$\begin{aligned}
\widehat{\mathcal{L}f}(\varphi) &= \kappa \sum_x e^{i(\varphi,x)} \sum_{z\neq 0} a(z)(f(x+z) - f(x)) \\
&= \kappa \sum_{z\neq 0} a(z)[e^{-i(\varphi,z)} \sum_x e^{i(\varphi,x+z)} f(x+z) - \sum_x e^{i(\varphi,x)} f(x)] \\
&= \kappa \sum_{z\neq 0} a(z)(e^{-i(\varphi,z)} - 1)\hat{f}(\varphi) = -\kappa \hat{f}(\varphi) \sum_{z\neq 0}(1 - cos(\varphi, z))a(z) \\
&= -\hat{f}(\varphi)\hat{\mathcal{L}}(\varphi).
\end{aligned}$$

□

In the subsequent discussion we use the following notation:

$$\widehat{\mathcal{L}f}(\varphi) = \hat{f}(\varphi)\hat{\mathcal{L}}(\varphi),\ \ \hat{f}(\varphi) = \sum_{x\in Z^d} e^{i(\varphi,x)} f(x),\ \ \hat{\mathcal{L}}(\varphi) = \kappa \sum_{z\neq 0}(1 - cos(\varphi, z))a(z). \tag{2.13}$$

Operator $\mathcal{L}$ is bounded and self adjoint in $L^2(Z^d)$. In the dual Fourier space $L^2(T^d, d\varphi)$ it acts as the operator of multiplication by $-\hat{\mathcal{L}}(\varphi)$.

The characteristic function of the random walk is

$$E_0 e^{i(\varphi,X_t)} = \sum_{y\in Z^d} p(t, 0, y)e^{i(\varphi,y)} = \hat{p}(t, 0, \varphi),$$

where $\varphi \in [-\pi, \pi)^2 = T^2$. From (2.11) we have

$$\frac{\partial \hat{p}}{\partial t} = -\kappa\hat{\mathcal{L}}(\varphi), \quad \hat{p}(0, 0, \varphi) = 1, \tag{2.14}$$

which gives the explicit expression

$$\hat{p}(t, 0, \varphi) = e^{-\kappa t\hat{\mathcal{L}}(\varphi)}.$$

If $\sum_{z\neq 0} a(z)|z|^2 < \infty$, then $\hat{\mathcal{L}}(\varphi) \approx \frac{(B\varphi,\varphi)}{2}$, $|\varphi| \to 0$. For fixed $\lambda \in R^2$:

$$Ee^{i(\lambda\frac{X(t)}{\sqrt{t}})} \underset{t\longrightarrow\infty}{\longrightarrow} Ee^{i\frac{\kappa(B\lambda,\lambda)}{2}}$$

i.e., $\frac{Xts}{\sqrt{t}}$, $s \in [0,1]$, is asymptotically two-dimensional Brownian motion with correlation matrix κB (calculation not shown). Here $det\ B > 0$ if $Span(z : a(z) > 0) = Z^d$.

We are interested in the situation $\sum\limits_{z\neq 0} a(z)|z|^2 = \infty$. As usual in the theory of stable distributions we impose the regularity condition on $a(z)$ given by (2.15).

Lemma 2.3. *(See [Feng et al., 2010]) Suppose*

$$a(z) = \frac{h(\theta)}{|z|^{2+\alpha}}\left(1 + O\left(\frac{1}{|z|^2}\right)\right),\ \ z \neq 0 \tag{2.15}$$

with $0 < \alpha < 2$, $\theta = \arg\frac{z}{|z|} \in [-\pi,\pi) = T^1$, $h \in C^2(T^1)$, $h > 0$, *and so satisfies the heavy tails assumption. Then*

$$\begin{aligned} &a)\hat{\mathcal{L}}(\varphi) = c_\alpha|\varphi|^\alpha\mathcal{H}(\gamma) + O(|\varphi|^2)\ as\ \varphi \to 0,\\ &\ where\ \mathcal{H}(\gamma) = \int\limits_{-\pi}^{\pi} h(\theta)|\cos(\theta-\gamma)|^\alpha d\theta\ and\ c_\alpha = \int\limits_{0}^{\infty} \tfrac{1-\cos t}{t^{1+\alpha}}dt, \\ &b)P\{x(t) = x\} \underset{t\to\infty}{\longrightarrow} \tfrac{1}{t^{d/2}}St_{\beta,\mathcal{H}}(\tfrac{x}{t^{1/\alpha}})(1+o(1))\ uniformly\ in\ x \in Z^d. \end{aligned} \tag{2.16}$$

This lemma is the local form of the usual statement that

$$\frac{x(t)}{t^{\frac{1}{\alpha}}} \xrightarrow[t\to\infty]{law} St_{\alpha,\mathcal{H}}(\cdot) \Leftrightarrow P\{\frac{x(t)}{t^{\frac{1}{\alpha}}} \in \Gamma\} = \int_\Gamma St_{\alpha,\mathcal{H}}(z)dz. \tag{2.17}$$

In addition, this local form indicates the absence of “large deviations.” A similar “global” theorem was published recently in [Molchanov et al., 2007]. We will give a sketch of the proof following the idea of [Molchanov et al., 2007]. See [Feng et al., 2010] for the detailed proof.

Proof. We have $\hat{\mathcal{L}}(\varphi) = \sum\limits_{z\neq 0} a(z)(1-\cos(\varphi\cdot z))$.

Let us consider the following integral $I(\varphi)$, which will give a good approximation of $\hat{\mathcal{L}}(\varphi)$, $\varphi \in [-\pi,\pi)^2 = T^2$:

$I(\varphi) = \sum_{\vec{n}\neq 0} I_{\vec{n}}(\varphi)$, where $I_{\vec{n}}(\varphi) = \int\limits_{A(\vec{n})} \frac{h(\frac{\vec{x}}{|\vec{x}|})}{|\vec{x}|^{2+\alpha}}(1-\cos(\varphi\cdot\vec{x}))d\vec{x},\ A(\vec{n}) = \{\vec{x} : |\vec{x}-\vec{n}|_\infty \leq$

$\frac{1}{2}\}$.

For large $|\vec{n}|$, the leading term in the expansion of $I_{\vec{n}}(\varphi)$ is equal to

$$\psi_{\vec{n}}(\varphi) := \frac{1}{|\vec{n}|^{2+\alpha}} h\left(\frac{\vec{n}}{|\vec{n}|}\right)(1-\cos(\vec{n}\cdot\varphi)),$$

while the remaining terms in the expansion of $I_{\vec{n}}(\varphi)$ will give a contribution of order $O(|\varphi|^2)$, $|\varphi| \to 0$. With $\sum\limits_{\vec{n}\neq 0} \psi_{\vec{n}}(\varphi) = \hat{\mathcal{L}}(\varphi)$, we have

$$\hat{\mathcal{L}}(\varphi) = I(\varphi) + O(|\varphi|^2), \ |\varphi| \to 0.$$

Some calculations give

$$I(\varphi) = \int\limits_{R^2 - A(0)} \frac{h(\frac{\vec{x}}{|\vec{x}|})}{|\vec{x}|^{2+\alpha}}(1-\cos(\varphi\cdot\vec{x}))dx = c_\alpha|\varphi|^\alpha \mathcal{H}(\gamma),$$

where $\mathcal{H}(\gamma) = \int\limits_{-\pi}^{\pi} h(\theta)|\cos(\theta-\gamma)|^\alpha d\theta$, $\gamma = \arg\varphi$, $\mathcal{H}(\gamma) \in C(T^1)$, $\mathcal{H}(\gamma) > 0$ and $c_\alpha = \int\limits_0^\infty \frac{1-\cos t}{t^{1+\alpha}} dt$. Then: $\hat{\mathcal{L}}(\varphi) = c_\alpha|\varphi|^\alpha\mathcal{H}(\gamma) + O(|\varphi|^2)$, $|\varphi| \to 0$.

$\square$

Corollary 2.1. *Under our condition on $a(z)$,*

$$E_0 e^{i(\varphi\frac{X(t)}{t^{1/\alpha}})} \underset{t\to\infty}{\longrightarrow} e^{-\kappa c_\alpha|\varphi|^\alpha \mathcal{H}(Arg\varphi)}$$

(a center symmetric distribution with parameter $0 < \alpha < 2$ and angular measure $\mathcal{H}(\gamma)$, $\gamma = \arg\varphi$).

Corollary 2.2. *Under our condition on $a(z)$, the random walk $X(S)$ on Z^2 is transient for any $0 < \alpha < 2$.*

In fact:

$$p(t,0,0) = \frac{1}{(2\pi)^2}\int_{T^2} e^{-\kappa t\hat{\mathcal{L}}(\varphi)}d\varphi \Rightarrow \int_0^\infty p(t,0,0)dt = \frac{1}{(2\pi)^2\kappa}\int_{T^2}\frac{d\varphi}{\hat{\mathcal{L}}(\varphi)} < \infty.$$

Let us return to the problem (2.5) and impose some technical conditions on the function $L(x)$ in the spirit of the paper [Ben Arous et al., 2005].

CHAPTER 3: THE PARABOLIC ANDERSON MODEL WITH LONG RANGE BASIC HAMILTONIAN AND WEIBULL TYPE RANDOM POTENTIAL

In this chapter, we are concerned with the potential $V(x, \omega_m)$ such that $\mathbf{E}e^{tV(\cdot)} < \infty$ for all $t > 0$ in the following two Weibull type forms:

$$P\{V(\cdot) > x\} = \exp\{-h(x)\} = \exp\{-\frac{x^\alpha}{\alpha}\}, \tag{3.1}$$

and

$$P\{V(\cdot) > x\} = \exp\{-h(x)\} = \exp\{-\frac{x^\alpha}{\alpha}L(x)\} \quad \textit{with } \alpha > 1, \tag{3.2}$$

where $L(x)$ is a slowly varying function with some restrictions (see below). The main results in this chapter are published in [Molchanov and Zhang, 2012].

For a tail probability of the form $P\{V(\cdot) > x\} = \exp\{-\frac{x^\alpha}{\alpha}\}$, we have the annealed asymptotic result of $u(t, x)$ as follows.

3.1 The annealed asymptotic property of $u(t, 0)$ with Weibull potential $V(x, \omega_m)$:$P\{V(\cdot) > x\} = \exp\{-\frac{x^\alpha}{\alpha}\}$.

For a tail probability of the form $P\{V(\cdot) > x\} = \exp\{-\frac{x^\alpha}{\alpha}\}$, we have the following annealed asymptotic result for $u(t, x)$.

Theorem 3.1. *For every $p \in N$ and every $t \geq 0$,*

$$\exp\{\frac{p^{\alpha'} t^{\alpha'}}{\alpha'} - p\kappa t + O(\ln t)\} \leq \langle u^p(t, x)\rangle \leq \exp\{\frac{p^{\alpha'} t^{\alpha'}}{\alpha'} + O(\ln t)\},$$

i.e., for the scale $A(t) = t^{\alpha'}$,

$$\gamma_p = \lim_{t\to\infty} \frac{\ln\langle u^p(t, x)\rangle}{t^{\alpha'}} = \frac{p^{\alpha'}}{\alpha'}, \quad \textit{where } \frac{1}{\alpha} + \frac{1}{\alpha'} = 1.$$

Remark 3.1. *Except for the specific calculation of the Laplace transformation, this theorem is the direct corollary of the corresponding general result from [Gärtner and*

Molchanov, 1990].

Proof. (a) Lower estimate of the annealed asymptotics of $m_p(t)$. The first moment of the solution $u(t,0)$ is

$$m_1(t) = \langle u(t,0)\rangle \geq \langle e^{tV(0)}\rangle e^{-\kappa t}.$$

For the Weibull tail we calculate the term $\langle e^{tV(0)}\rangle$ as follows:

$$\langle e^{tV(0)}\rangle = \int_0^\infty e^{tx-\frac{x^\alpha}{\alpha}} x^{\alpha-1}dx.$$

Changing variables by setting $x = t^\beta y$ and selecting $\beta : 1+\beta = \alpha\beta$ gives

$$\langle e^{tV(0)}\rangle = \int_0^\infty e^{t^{\alpha'}(y-\frac{y^\alpha}{\alpha})} t^{\alpha\beta} y^{\alpha-1}dy,$$

where $\beta = \frac{1}{\alpha-1}$, $\alpha' = \frac{\alpha}{\alpha-1} = 1+\beta$.

The term $y - \frac{y^\alpha}{\alpha}$ is maximal when $y = 1$. Then, using Laplace's method, we obtain:

$$\int_0^\infty e^{t^{\alpha'}(y-\frac{y^\alpha}{\alpha})} t^{\alpha\beta} y^{\alpha-1}dy = e^{\frac{t^{\alpha'}}{\alpha'}+\frac{\alpha'}{2}\ln t+\frac{1}{2}\ln(\frac{2\pi}{\alpha-1})+o(1)}.$$

Thus, $m_1(t) \geq e^{\frac{t^{\alpha'}}{\alpha'}-\kappa t+\frac{\alpha'}{2}\ln t+\frac{1}{2}\ln(\frac{2\pi}{\alpha-1})+o(1)}$. For the pth moment of the solution $u(t,0)$, we have

$$\langle u^p(t,x)\rangle = \langle E_0\{\exp(\sum_{i=1}^p \int_0^t \xi(x_s^i))\}\rangle \geq \exp\{\frac{p^{\alpha'}t^{\alpha'}}{\alpha'} - p\kappa t + \frac{\alpha'}{2}\ln t + \frac{1}{2}\ln(\frac{2\pi}{\alpha-1}) + o(1)\}$$

$$= \exp\{\frac{p^{\alpha'}t^{\alpha'}}{\alpha'} - p\kappa t + O(\ln t)\}.$$

(b) Upper estimate of $m_p(t)$. We obtain the following results after applying Hölder's inequality, Jensen's inequality and Fubini's theorem.

$$\langle u^p(t,x)\rangle = \langle (E_0 \exp(\int_0^t \xi(x_s)ds)^p\rangle \leq \langle (E_0 \exp(p\int_0^t \xi(x_s)ds)\rangle = E_0\langle (\exp(p\int_0^t \xi(x_s)ds)\rangle$$

$$\leq \frac{1}{t}\int_0^t ds E_0\langle \exp pt\xi(x_s))\rangle = exp\{\frac{p^{\alpha'}t^{\alpha'}}{\alpha'} + O(\ln t)\}.$$

Combining the lower and upper estimates of the $m_p(t)$ of the solution $u(t,0)$, we get the result. □

Remark 3.2. *Following the method in [Molchanov, 1994], a more precise upper estimate can be proved:*

$$\langle \ln u^p(t,x)\rangle = \frac{p^{\alpha'}t^{\alpha'}}{\alpha'} - p\kappa t + O(\ln t).$$

3.2 The quenched asymptotic property of $u(t,0)$ with Weibull potential $V(x,\omega_m)$:$P\{V(\cdot) > x\} = \exp\{-\frac{x^\alpha}{\alpha}\}$.

For our discussion of the quenched asymptotic properties of the solution $u(t,0)$, we need the following lemma concerning the asymptotics of $\max_{|x|\le n} V(x)$ as $n \to \infty$ for the potential $P\{V(x,\omega_m) > a\} = \exp\{-\frac{a^\alpha}{\alpha}\}$.

Lemma 3.1. *P_m-a.s.,*

$$\max_{|x|\le n} V(x) \underset{n\to\infty}{\sim} (\alpha d \ln n)^{1/\alpha}. \tag{3.3}$$

Proof. Using the Borel-Cantelli lemma for the event $A_x^{\pm} = \{V(x) > (1\pm\epsilon)(\alpha d \ln x)^{1/\alpha}\}$, $|x| = |x_1| + ... + |x_k|$, straightforward calculation proves the lemma. □

Theorem 3.2. *P_m-a.s. for $t\to\infty$,*

$$\limsup_{t\to\infty} \frac{\ln u(t,0)}{t^{\alpha'}} \le \frac{1}{\alpha'},$$

$$\liminf_{t\to\infty} \frac{\ln u(t,0)}{t^{\alpha'}} \ge \frac{1}{\alpha'}\left(\frac{d}{d+\beta}\right)^{\frac{\alpha'}{\alpha}}.$$

Proof. (a) Lower estimate for the quenched asymptotics of $u(t,0)$. To check the lower estimate, let us consider the "almost optimal" trajectory x_s, $s\ge 0$. This trajectory spends time $t\le 1$ at the origin, then jumps to the point $x_0 = x_0(t,\omega_m)$ of the very high local maximum of $V(x,\cdot)$ and stays there until moment t (i.e., time at least $t-1$.)

Assume that $|x_0| \in [R(1-\delta^{'}), R]$ for some R, $R \gg 1$ and $V(x_0) \geq (1-\delta)(\alpha d \ln R)^{1/\alpha}$. Then, for $R \to \infty$,

$$u(t,0) \geq \max_R \; [e^{-\kappa t} \cdot C_2 \cdot \frac{1}{R^{d+\beta}} \exp\{(t-1)(1-\delta)(\alpha d \ln R)^{1/\alpha}]\}$$

$$\geq C_2 e^{-\kappa t} \max_R \; e^{-(d+\beta)\ln R + t(1-\delta^{''})(\alpha d \ln R)^{1/\alpha}} \text{ (because } a(0,x_0) \geq \frac{C}{|x_0|^{d+\beta}}).$$

Putting $x = \ln R$ we find

$$\max_x \; [-(d+\beta)x + \tilde{t}(\alpha d x)^{1/\alpha}], \quad \tilde{t} = (1-\delta^{''})t.$$

The equation for the critical point gives:

$$\tilde{t}d(\alpha d x_0)^{1/\alpha - 1} = d + \beta \quad \Rightarrow x_0 = \frac{1}{\alpha d}(\frac{\tilde{t}d}{d+\beta})^{\alpha'}.$$

The value at the critical point is

$$\tilde{t}^{\alpha'} \frac{1}{\alpha'}(\frac{d}{d+\beta})^{\frac{\alpha'}{\alpha}}.$$

Because $\tilde{t} = (1-\delta^{''})t$ and $\delta^{''}$ is arbitrarily small we have proved the lower estimate

$$\liminf_{t\to\infty} \frac{\ln u(t,0)}{t^{\alpha'}} \geq \frac{1}{\alpha'}(\frac{d}{d+\beta})^{\frac{\alpha'}{\alpha}}.$$

(b) Upper estimate for the quenched asymptotics of $u(t,0)$. Consider $V^+(x,\omega_m) = \max(0, V(x)) \geq V(x,\omega_m)$. Obviously $\int_0^t V^+(x_s)ds \geq \int_0^t V(x_s)ds$, i.e., $\tilde{u}(t,x,\omega_m) \geq u(t,x)$.

It follows from the Kac-Feynman formula that $\tilde{u} \uparrow$ as a function of t for fixed ω_m. It means that for $n \leq t < n+1$,

$$\tilde{u}(n,0) \leq \tilde{u}(t,0) \leq \tilde{u}(n+1,0).$$

But $\forall(\epsilon > 0)$ and $t = n$ or $n+1$,

$$P\{\tilde{u}(t,0) > e^{(1+\epsilon)\frac{t^{\alpha'}}{\alpha'}}\} \leq \frac{E\tilde{u}(t,0)}{e^{(1+\epsilon)\frac{t^{\alpha'}}{\alpha'}}} \leq \frac{E\exp{(tV^+(0))}}{e^{(1+\epsilon)\frac{t^{\alpha'}}{\alpha'}}}.$$

Trivial calculation gives

$$E\exp\left(tV^{+}(0)\right) \sim e^{\frac{t^{\alpha'}}{\alpha'}}$$

and, due to the Borel-Cantelli lemma, $\forall(\epsilon > 0)$

$$\tilde{u}(t,0) \le e^{(1+\epsilon')\frac{t^{\alpha'}}{\alpha'}}, \quad t \ge t_0(\omega),$$

first for integer t, then, due to the monotonicity of $\tilde{u}(t,0)$, for all $t > t_0(\omega)$.

It means that

$$\limsup_{t\to\infty} \frac{\ln u(t,0)}{t} \le \limsup_{t\to\infty} \frac{\ln \tilde{u}(t,0)}{t^{\alpha'}} \le (1+\epsilon)\frac{1}{\alpha'}.$$

Because ϵ can be arbitrarily small we have proved the upper estimate. □

Now we will study models when $P\{V(\cdot) > x\} = \exp\{-\frac{x^\alpha}{\alpha}L(x)\}$, and $L(\cdot)$ is a slowly varying function with some additional regularity assumptions. The lemmas and definitions in Chapter 1 are fundamental to our paper.

To make use of the Tauberian Theorem, we restrict our attention to $L(x)$ that satisfy:

Assumption 3.3. *The function $L(x)$ in (3.2) is slowly varying and $L(x) \in NR_0$, that is,*

$$\frac{xL'(x)}{L(x)} \to 0 \quad (x \to \infty).$$

With assumption 3.3, we see that $\frac{xh'(x)}{h(x)} \to \alpha$ when $x \to \infty$, where $h(x) = \frac{x^\alpha L(x)}{\alpha}$. From lemmas 1.2 and 1.3, $h(x)$ is a normalized regularly varying function and is ultimately increasing.

This assumption is not completely sufficient for our analysis. To prove results similar to theorems 3.1 and 3.2, we need the additional technical

Assumption 3.4. *$L(x)$ in (3.2) satisfies*

$$L(xL^\alpha(x))/L(x) \to 1 \ (x \to \infty) \ locally\ uniformly\ in\ \alpha \in R.$$

We will use assumption 3.4 to control the critical point x_0 in the application of the Laplace method. Assumption 3.4 is fulfilled for all " standard" slowly varying functions, for example, $L(x) = \ln^{\beta}(2+x)$, $\beta \in R^1$, $L(x) = \ln\ln^{\gamma}(x+4)$, $\gamma \in R^1$ and $L(x) = \exp(\ln^{\beta}(2+x))$, $0 < \beta < \frac{1}{2}$. The function $L(x) = \exp(\ln^{\beta}(2+x))$, $\frac{1}{2} < \beta < 1$, is slowly varying, but elementary calculations give $\lim\limits_{x\to\infty} \frac{L(xL(x))}{L(x)} = +\infty$, i.e., assumption 3.4 restricts the growth of $L(x)$.

The theorem 1.2 is related to the sufficient condition for assumption 3.4.

3.3 The annealed asymptotic property of $u(t,0)$ with potential $V(x,\omega_m)$: $P\{V(\cdot) > x\} = \exp\{-\frac{x^\alpha}{\alpha}L(x)\}$.

The following theorem gives the annealed asymptotics of $u(t,x)$ for the potential $P\{V(x,\omega_m) > x\} = \exp\{-\frac{x^\alpha}{\alpha}L(x)\}$.

Theorem 3.5. *Under assumptions* 3.3 *and* 3.4, *for every* $p \in N$ *and* $t \to \infty$,

$$\langle e^{ptV(0)}\rangle e^{-\kappa pt} \le \langle u^p(t,x)\rangle \le \langle e^{ptV(0)}\rangle$$

and

$$\lim_{t\to\infty} \frac{\ln\langle u^p(t,x)\rangle}{t^{\alpha'}} L^{\frac{1}{\alpha-1}}(t^{\frac{1}{\alpha-1}}) = \frac{p^{\alpha'}}{\alpha'}, \quad \textit{where } \alpha' \textit{ satisfies } \frac{1}{\alpha} + \frac{1}{\alpha'} = 1.$$

Proof. The first moment of the solution $u(t,0)$ with tail probability $P\{V(\cdot) > x\} = \exp\{-h(x)\}$, where $h(x) = \frac{x^\alpha}{\alpha}L(x)$, is

$$m_1(t) = \langle u(t,0)\rangle \ge \langle e^{tV(0)}\rangle e^{-\kappa t}. \tag{3.4}$$

Using integration by parts, we get

$$\langle e^{tV(0)}\rangle = 1 + t\int_0^\infty e^{tx} P\{V(0) > x\}dx = 1 + t\int_0^\infty e^{tx - \frac{x^\alpha}{\alpha}L(x)}dx,$$

i.e. we have to evaluate asymptotically

$$I(t) = \int_0^\infty e^{tx - h(x)}dx.$$

The natural idea is to apply the Laplace method. One can do this under the additional assumption that $L(x) \in C^2_{loc}$ and $\frac{x^2 L''(x)}{\alpha L(x)} \to 0$. On the level of the logarithmical asymptotics, however, the initial assumption 3.4 is sufficient.

Fix the point $x_0 = (\frac{t}{L(t^{\frac{1}{\alpha-1}})})^{\frac{1}{\alpha-1}}$ (which is not exactly the extreme for $tx - h(x)$) and divide $[0, \infty)$ into the following intervals:

$$\Delta_{-1} = [0, \frac{1}{2}x_0),\ \Delta_0 = [\frac{1}{2}x_0, 2x_0),\ \dots,\ \Delta_n = [2^n x_0, 2^{n+1}x_0),\ \dots$$

Since $x_0(t) \to \infty$ and L is slowly varying, there exists a function $\delta = \delta(t) \to 0$, $t \to \infty$ such that for $x \in \Delta_n$, $n \geq 0$, we have

$$(1-\delta)L(2^n x_0) \leq L(x) \leq (1+\delta)L(2^n x_0).$$

Finally, the exponent $tx - h(x)$ is increasing on $[0, \frac{1}{2}x_0)$ and decreasing on $[2x_0, \infty)$ as the function of x.

Consider the integral $I_0(t) = \int_{\Delta_0} e^{tx-h(x)}dx$, then

$$\int_{\Delta_0} e^{tx-(1+\delta)\frac{x^\alpha}{a}L(x_0)}dx \leq I_0 \leq \int_{\Delta_0} e^{tx-(1-\delta)\frac{x^\alpha}{a}L(x_0)}dx.$$

The critical points here are $x_\pm(t) = \left(\frac{t}{L(x_0)}\right)^{\frac{1}{\alpha-1}} \frac{1}{(1\mp\delta)^{\frac{1}{\alpha-1}}}$. The usual Laplace method (L now is constant) gives

$$\ln I_0 \sim \frac{t^{\frac{\alpha}{\alpha-1}}}{L^{\frac{1}{\alpha-1}}(x_0)}(1 - \frac{1}{\alpha}).$$

On Δ_{-1} we can use a very rough estimate:

$$I_{-1}(t) = \int_{\Delta_{-1}} e^{tx-h(x)}dx \leq |\Delta_{-1}| \cdot e^{t\cdot\frac{x_0}{2} - h(\frac{x_0}{2})}$$

$$\leq \frac{x_0}{2} e^{\frac{t^{\alpha/(\alpha-1)}}{L^{1/(\alpha-1)}(x_0)}(\frac{1}{2} - (\frac{1}{2})^\alpha \cdot \frac{1}{\alpha} \frac{L(\frac{x_0}{2})}{L(x_0)})}$$

and $I_{-1}(t)$ is exponentially smaller than $I_0(t)$.

Similarly, for $n \geq 1$,

$$
\begin{aligned}
I_n(t) &= \int_{\Delta_n} e^{tx-h(x)}dx \leq |\Delta_n| \cdot e^{t\cdot\frac{2^n x_0}{2} - h(2^n x_0)} \\
&\leq 2^n x_0 \cdot e^{\frac{t^{\alpha/(\alpha-1)}}{L^{1/(\alpha-1)}(x_0)}(2^n - \frac{2^{\alpha n}}{\alpha}\cdot\frac{L(2^n x_0)}{L(x_0)}\cdot\frac{L^{\alpha/(\alpha-1)}(x_0)}{L^{\alpha/(\alpha-1)}(t^{1/(\alpha-1)})})} \qquad (3.5) \\
&\leq 2^n x_0 \cdot e^{\frac{t^{\alpha/(\alpha-1)}}{L^{1/(\alpha-1)}(x_0)}(2^n - \frac{2^{(\alpha-\delta)n}}{\alpha})}.
\end{aligned}
$$

In (3.5) we use assumption 3.4, which provides that $L(x_0)/L(t^{1/(\alpha-1)}) \to 1$.

Again, $\sum_{n\geq 1} I_n(t)$ is exponentially smaller than I_0. Finally,

$$
\ln I(t) \underset{t\to\infty}{\sim} \frac{t^{\frac{\alpha}{\alpha-1}}}{L^{\frac{1}{\alpha-1}}(x_0)}(1-\frac{1}{\alpha}) \sim \frac{t^{\alpha'}}{\alpha'}\frac{1}{L^{\frac{1}{\alpha-1}}(t^{\frac{1}{\alpha-1}})}.
$$

In the last step, we use assumption 3.4 again. □

Remark 3.3. *Under the additional assumption that $L(x) \in C^2_{loc}$ and $\frac{x^2 L''(x)}{\alpha L(x)} \to 0$, we can also use Laplace's method to prove the theorem.*

The first moment of the solution $u(t,0)$ for tail probability $P\{V(\cdot) > x\} = \exp\{-h(x)\}$, where $h(x) = \frac{x^\alpha}{\alpha}L(x)$, is

$$
m_1(t) = \langle u(t,0)\rangle \geq \langle e^{tV(0)}\rangle e^{-\kappa t},
$$

$$
\langle e^{tV(0)}\rangle = \int_0^\infty e^{tx-h(x)}h'(x)dx.
$$

With assumptions 3.3 and 3.4 of the normalized regularly varying function, the function $h(x)$ is ultimately increasing. So the point x_0 that maximized the $tx - h(x)$, or $tx - \frac{x^\alpha}{\alpha}L(x)$ is unique. The above equation can be written as

$$
\begin{aligned}
\langle e^{tV(0)}\rangle &= \int_0^{x_0-\epsilon} e^{tx-\frac{x^\alpha}{\alpha}L(x)}h'(x)dx + \int_{x_0-\epsilon}^{x_0+\epsilon} e^{tx-\frac{x^\alpha}{\alpha}L(x)}h'(x)dx \\
&\quad + \int_{x_0+\epsilon}^{\infty} e^{tx-\frac{x^\alpha}{\alpha}L(x)}h'(x)dx, \qquad (3.6)
\end{aligned}
$$

in which the first and third items tend to 0.

For convenience of notation, we define $g(x) = tx - h(x)$ *and notice*

$$g''(x_0) = -(\alpha - 1)x_0^{\alpha-2}L(x_0) < 0. \tag{3.7}$$

By Laplace's method for the second term of (3.6) *and also considering* (3.7)*, we get*

$$\int_{x_0-\epsilon}^{x_0+\epsilon} e^{tx-\frac{x^\alpha}{\alpha}L(x)}h'(x)dx \sim Ce^{tx_0-\frac{x_0^\alpha}{\alpha}L(x_0)}, \tag{3.8}$$

where $C = h'(x_0)\int_{-\epsilon}^{\epsilon} e^{g''(x)x^2}dx$ *and* x_0 *satisfies* $g'(x_0) = 0$*, that is,*

$$t - x_0^{\alpha-1}L(x_0)(1 + \frac{x_0 L'(x_0)}{\alpha L(x_0)}) = 0.$$

As $t \to \infty$*,* $x_0 \to \infty$*, so, by assumption* 3.3*, the above equation is equivalent to*

$$t = x_0^{\alpha-1}L(x_0) \text{ or } x_0 = (\frac{t}{L(x_0)})^{\frac{1}{\alpha-1}}.$$

To solve the iteration function for x_0*, we set the initial value of* x_0 *as*

$$x_0^{(0)} = t^{\frac{1}{\alpha-1}}.$$

With theorem 1.2*, lemma* 1.4 *and assumption* 3.4*, we get the solution* x_0 *by iteration as*

$$x_0 = (\frac{t}{L(t^{\frac{1}{\alpha-1}})})^{\frac{1}{\alpha-1}}(1 + o(1)) \tag{3.9}$$

and by lemma 1.1 *and theorem* 1.2*,*

$$L(x_0) = L(t^{\frac{1}{\alpha-1}})(1 + o(1)). \tag{3.10}$$

With (3.7)*,* (3.9) *and* (3.10)*,* (3.8) *is*

$$\langle e^{tV(0)} \rangle \sim Ce^{\frac{t^{\alpha'}}{\alpha'}\frac{1}{L^{\frac{1}{\alpha-1}}(t^{\frac{1}{\alpha-1}})}(1+o(1))},$$

where $C = h'(x_0)\int_{-\epsilon}^{\epsilon} e^{g''(x_0)x^2}dx$*. With* (3.7)*,* (3.9) *and the form of* $h'(x_0)$*, we can obtain further that*

$$C \sim e^{O(\ln t)}.$$

Now we can follow the proof of theorem 3.1.

3.4 The quenched asymptotic property of $u(t,0)$ with potential $V(x,\omega_m)$: $P\{V(\cdot) > x\} = \exp\{-\frac{x^\alpha}{\alpha} L(x)\}$.

For the quenched asymptotics of $u(t,x)$, $t \to \infty$, as in the previous section we need the asymptotics of $\max\limits_{|x|\le n} V(x)$ similar to lemma 3.1 .

Lemma 3.2. *If function $L(x)$ satisfies assumption* 3.4*, P_m-a.s., then*

$$\max_{|x|\le n} V(x) \underset{n\to\infty}{\sim} \left(\frac{\alpha d \ln n}{L(\ln^{1/\alpha} n)}\right)^{1/\alpha}.$$

Proof. (a) The lower estimate for $V(x)$. For notational convenience, we define $\beta(x) = \frac{\alpha d \ln |x|}{L(\ln^{1/\alpha}|x|)}$. Consider the events $A_{x,\delta} = \{\omega_m : V(x) > (1+\delta)\beta^{1/\alpha}(x)\}$. Then, for any $\delta > 0$,

$$\begin{aligned} P(A_{x,\delta}) &= \exp\{-(1+\delta')\cdot\beta(x)\cdot\frac{1}{\alpha}L(\beta^{1/\alpha}(x))\} \\ &= \frac{1}{|x|^{d(1+\delta')(1+o(1))}}. \end{aligned}$$

The last step follows from assumption 3.4.

Because $\sum\limits_x \frac{1}{|x|^{d(1+\delta')(1+o(1))}} < \infty$, we have that for $|x| \ge C(\delta,\omega_m)$, $V(x) \le (1+\delta)\beta^{1/\alpha}(x)$, due to the Borel-Cantelli lemma.

(b) The upper estimate for $V(x)$. We define $\beta(R_n) = \frac{\alpha d \ln R_n}{L(\ln^{1/\alpha} R_n)}$. Let us split Z^d into Γ_n: $\Gamma_n = \{\vec{x} : R_n < \|\vec{x}\|_\infty \le R_{n+1}\}$, where $R_n = (1+\gamma)^n$ and $\gamma > 0$, and consider the event $B_{n,\delta} = \{\omega_m : \max\limits_{\Gamma_n} V(x) < (1-\delta)\beta^{1/\alpha}(R_n)\}$. The events $B_{n,\delta}$ are independent for different n, and due to independence

$$\begin{aligned} P\{B_{n,\delta}\} &= P^{|\Gamma_n|}\{V(x) < (1-\delta)\beta^{1/\alpha}(R_n)\} = \left(1 - P\{V(x) \ge (1-\delta)\beta^{1/\alpha}(R_n)\}\right)^{|\Gamma_n|} \\ &\sim e^{-|\Gamma_n| P\{V(x) > (1-\delta)\beta^{1/\alpha}(R_n)\}}. \end{aligned}$$

But with $|\Gamma_n| \sim \gamma R_n \cdot (2R_n)^{d-1}$ and $P\{V(x) > (1-\delta)\beta^{1/\alpha}(R_n)\} \sim \frac{c}{R_n^{d(1-\delta')}}$,

$$P\{B_{n,\delta}\} \sim e^{-\gamma R_n \cdot (2R_n)^{d-1} \cdot \frac{c}{R_n^{d(1-\delta')}}} \leq e^{-c\gamma R_n^{\delta' d}}, \quad \sum_n P\{B_{n,\delta}\} < \infty.$$

So, for $n > n_0(\omega)$, $\max\limits_{\Gamma_n} V(x) \geq (1-\delta)\beta^{1/\alpha}(R_n)$. Thus, lemma 3.2 is proved. □

Let us give some illustrations.

Corollary 3.1. *If $L(x) = \ln^\beta x$, P_m-a.s.:*

$$\max_{|x|\leq n} V(x) \underset{n\to\infty}{\sim} \left(\frac{\alpha d \ln n}{\ln \ln^\beta n}\right)^{1/\alpha} \alpha^{\beta/\alpha}.$$

Corollary 3.2. *If $L(x) = \ln \ln^\beta x$, P_m-a.s.:*

$$\max_{|x|\leq n} V(x) \underset{n\to\infty}{\sim} \left(\frac{\alpha d \ln n}{\ln \ln \ln^\beta n}\right)^{1/\alpha}.$$

Corollary 3.3. *If $L(x) = e^{\ln^\beta x}$, $0 < \beta < \frac{1}{2}$, P_m-a.s.:*

$$\max_{|x|\leq n} V(x) \underset{n\to\infty}{\sim} \left(\frac{\alpha d \ln n}{e^{(\frac{\ln \ln n}{\alpha})^\beta}}\right)^{1/\alpha}.$$

Proposition 3.1. *If $L(x) = e^{\sqrt{\ln x}}$, with P_m-a.s.,*

$$\max_{|x|\leq n} V(x) \underset{n\to\infty}{\sim} e^{\frac{1}{2\alpha^2}} \left(\frac{\alpha d \ln n}{e^{\sqrt{\frac{\ln \ln n}{\alpha}}}}\right)^{1/\alpha}.$$

Proof. Consider the events $A_{x,\delta} = \left\{\omega_m : V(x) > (1+\delta) e^{\frac{1}{2\alpha^2}} \left(\frac{\alpha d \ln |x|}{e^{\sqrt{\frac{\ln \ln |x|}{\alpha}}}}\right)^{1/\alpha}\right\}$. Then, for any $\delta > 0$,

$$\begin{aligned} P(A_{x,\delta}) &= \exp\left\{-(1+\delta') d \ln |x| \left(1 + O\left(\frac{1}{\sqrt{\ln \ln |x|}}\right)\right)\right\} \\ &= \frac{1}{|x|^{d(1+\delta')\left(1+O\left(\frac{1}{\sqrt{\ln \ln |x|}}\right)\right)}}. \end{aligned}$$

Since $\sum\limits_x \frac{1}{|x|^{d(1+\delta')\left(1+O\left(\frac{1}{\sqrt{\ln\ln|x|}}\right)\right)}} < \infty$, we have that for $|x| \geq C(\delta, \omega_m)$, $V(x) \leq (1+\delta) e^{\frac{1}{2\alpha^2}} \left(\frac{\alpha d \ln |x|}{e^{\sqrt{\frac{\ln \ln |x|}{\alpha}}}}\right)^{1/\alpha}$, due to the Borel-Cantelli lemma.

Similarly we can prove the upper limit of $V(\cdot)$. Proposition 3.1 is proved. $\square$

Remark 3.4. *Assumption* 3.4 *is very important in establishing lemma* 3.2*. For example, with the function* $L(x) = e^{\ln^{1/\beta} x}, \beta \geq 2$*, the* $\max_{|x|\leq n} V(x)$ *are different for the different ranges* $\beta > 2$ *and* $\beta = 2$ *as shown in corollary* 3.3 *and proposition* 3.1*. The difference in* $\max_{|x|\leq n} V(x)$ *arises in that* $L(x) = e^{\ln^{1/\beta} x}$ *satisfies assumption* 3.4 *when* $\beta > 2$ *and does not when* $\beta = 2$*.*

Using lemma 3.2, we can obtain the quenched asymptotics of $u(t,x)$ for the potential $P\{V(x,\omega_m) > x\} = \exp\{-\frac{x^\alpha}{\alpha}L(x)\}$.

Theorem 3.6. *Under assumptions* 3.3 *and* 3.4*,* P_m*-a.s., for* $t \to \infty$*:*

$$\limsup_{t\to\infty} \frac{\ln u(t,0)}{t^{\alpha'}} L^{\frac{1}{\alpha-1}}(t^{\frac{1}{\alpha-1}}) \leq \frac{1}{\alpha'},$$

$$\liminf_{t\to\infty} \frac{\ln u(t,0)}{t^{\alpha'}} L^{\frac{1}{\alpha-1}}(t^{\frac{1}{\alpha-1}}) \geq \frac{1}{\alpha'}(\frac{d}{d+\beta})^{\frac{\alpha'}{\alpha}}, \quad \textit{where } \alpha' \textit{ satisfies } \frac{1}{\alpha} + \frac{1}{\alpha'} = 1.$$

Proof. (a) The lower estimate for the quenched asymptotics of $u(t,0)$. Assume that $|x_0| \in [R(1-\delta'), R]$ for some R, $R \gg 1$ and $V(x_0) \geq (1-\delta)\left(\frac{\alpha d \ln R}{L(\ln^{\frac{1}{\alpha}} R)}\right)^{1/\alpha}$. Then, for $R \to \infty$,

$$u(t,0) \geq \max_R \, [e^{-\kappa t} \cdot C_2 \cdot \frac{1}{R^{d+\beta}} exp\{(t-1)(1-\delta)\left(\frac{\alpha d \ln R}{L(\ln^{\frac{1}{\alpha}} R)}\right)^{1/\alpha}]\}$$

$$\geq C_2 e^{-\kappa t} \max_R \, e^{-(d+\beta)\ln R + t(1-\delta'')\left(\frac{\alpha d \ln R}{L(\ln^{\frac{1}{\alpha}} R)}\right)^{1/\alpha}}.$$

Putting $x = \ln^{\frac{1}{\alpha}} R$ we find

$$\max_x \, [-(d+\beta)x^\alpha + \tilde{t}\frac{(\alpha d)^{1/\alpha} x}{L^{\frac{1}{\alpha}}(x)}], \quad \tilde{t} = (1-\delta'')t.$$

The equation for the critical point is:

$$\hat{t}\frac{1 - \frac{x_0 L'(x_0)}{\alpha L(x_0)}}{L^{\frac{1}{\alpha}}(x_0)} - \alpha(d+\beta)x_0^{\alpha-1} = 0,$$

where $\hat{t} = \tilde{t}(\alpha d)^{1/\alpha}$. Under assumption 3.4

$$L^{\frac{1}{\alpha}}(x_0) = \frac{\hat{t}}{\alpha(d+\beta)x_0^{\alpha-1}},$$

or

$$x_0 = (\frac{\bar{t}}{L^{\frac{1}{\alpha}}(x_0)})^{\frac{1}{\alpha-1}}, \tag{3.11}$$

where $\bar{t} = \frac{\hat{t}}{\alpha(d+\beta)}$.

We use the iterative method to solve equation (3.11) by setting the initial point $x_0 = \bar{t}^{\frac{1}{\alpha-1}}$. We obtain

$$x_0 \sim \frac{\bar{t}^{\frac{1}{\alpha-1}}}{L^{\frac{1}{\alpha}\frac{1}{\alpha-1}}(\bar{t}^{\frac{1}{\alpha-1}})}.$$

The value at the critical point is

$$\bar{t}^{\alpha'}\frac{1}{\alpha'}(\frac{d}{d+\beta})^{\frac{\alpha'}{\alpha}}\frac{1}{L^{\frac{1}{\alpha-1}}(\bar{t}^{\frac{1}{\alpha-1}})}.$$

Because δ'' is arbitrarily small, we have proved the lower estimate:

$$\liminf_{t\to\infty}\frac{\ln u(t,0)}{t^{\alpha'}}L^{\frac{1}{\alpha-1}}(t^{\frac{1}{\alpha-1}}) \geq \frac{1}{\alpha'}(\frac{d}{d+\beta})^{\frac{\alpha'}{\alpha}}.$$

(b) The upper estimate for the quenched asymptotics of $u(t,0)$. Applying the upper estimate of the quenched asymptotics of $u(t,0)$, theorem 3.2 and the result in theorem 3.5, we have

$$\limsup_{t\to\infty}\frac{\ln u(t,0)}{t^{\alpha'}}L^{\frac{1}{\alpha-1}}(t^{\frac{1}{\alpha-1}}) \leq \frac{1}{\alpha'}.$$

Combining the lower and upper estimates of the quenched asymptotics of $u(t,0)$, we obtain the result. $\square$

Let us emphasize that we did not find the exact quenched asymptotics but only the upper and lower estimates of the same order. To find the true asymptotics we need a better understanding of the a.s. behavior of the underlying random walk $x(t)$, $t \to \infty$ (upper and lower functions, etc.). In the 1-d case the situation is clear, but the multidimensional case is more difficult.

CHAPTER 4: MOSER'S PROBLEM

It is well known that the position of the maximal term in any sequence of r.v.s is not a stopping time. Can one, however, guess the maximal term using the Markov strategy? The answer to this question (in the case of i.i.d. r.v.s X_i, $i = 1, 2, ..., n$) is closely related to the following optimal stopping problem.

Let $\{X_i, i \geq 1\}$ be non-negative i.i.d random variables with a continuous distribution function $F(x)$ and $EX < \infty$. We want to calculate

$$S_n = \max_{\tau \leq n} EX_\tau, \tag{4.1}$$

where τ is the stopping time, i.e. $I_{\{\tau \leq k\}} = \varphi(X_1, ..., X_k)$. The particular case when the $X_i, i \geq 1$, are uniformly distributed on $[0, 1]$ is known as Moser's problem. When the distribution of X_k is known but not uniform on the interval $[0, 1]$, we call the problem the generalized Moser's problem. The optimal stopping time τ_M can be defined in terms of the thresholds $h_n > h_{n-1} > ... > h_1 = 0$. Here, $\tau_M = \{\min k : X_k > h_{n-k+1}\}$.

In Moser's problem, a decision maker wants to maximize the expected value of X_τ. Moser's problem is closely related to the well-known secretary problem, the goal of which is to maximize the probability that the decision maker obtains $M_n = \max\limits_{1 \leq i \leq n} X_i$ or to minimize the rank of X_τ in the variational sequence $X_{(1)} > X_{(2)} > ... > X_{(n)}$. Although considerable attention has been devoted to the secretary problem and Moser's problem, little attention has been given previously to the connection between these problems. In our paper, we will illustrate the connection between Moser's problem and the secretary problem with full information. For convenience of expression, we consider the random variables X_i with $i = 0, 1, ..., n$ instead of $i = 1, ..., n$. Correspondingly, the optimal rule is to stop when $X_k > h_{n-k}$ instead of when $X_k > h_{n-k+1}$.

We first mention several results concernng the secretary problem and Moser's problem. Bruss (2005) and Ferguson (1989a) present detailed surveys about the secretary problem and Moser's problem. Generally, there are four kinds of secretary problems:
(I) The classical secretary problem. A decision maker sequentially observes the relative rank of X_k without any information about $X_1, ..., X_n$. The decision maker wants to maximize $P\{X_{\tau_s} = M_n, M_n = \max\limits_{1 \le i \le n} X_i\}$, where τ_s is the optimal stopping time for this problem. The famous result for the classical secretary problem is that $\lim\limits_{n\to\infty} P\{X_{\tau_s} = M_n, M_n = \max\limits_{1\le i\le n} X_i\} = \frac{1}{e}$. See Dynkin and Yushkevich (1969) for details.
(II) The secretary problem with full information. The situation is as in problem (I) except that the decision maker knows the distribution of X_k and that the X_i are i.i.d. random variables. The optimal probability is $\pi_n = \max\limits_{\tau_f} P\{X_{\tau_f} = M_n, M_n = \max\limits_{1\le i\le n} X_i\}$, where τ_f is the optimal stopping time for this problem. J. Gilbert and F. Mosteller (1966) obtained the following results: (a) $\lim\limits_{n\to\infty} \pi_n = e^{-c} + (e^c - c - 1)\int_1^\infty x^{-1}e^{-cx}dx \simeq 0.5801$, and (b) the optimal probability π_n is independent of the distribution function and is strictly decreasing in n. Here, $c \simeq 0.804352$ is the solution to $\sum_{j=1}^{\infty} \frac{c^j}{j!j} = 1$. See Gilbert and Mosteller (1966) and Samuels (1991) for details.
(III) The expected rank problem without information. In this problem, the object of the decision maker is to minimize the rank of X_{τ_r}, where τ_r is the stopping time, but he has no information about the distribution of X_k. Chow(1964) obtained an optimal limit value for problem (III) of 3.8695.
(IV) The expected rank problem with information. The decision maker wants to minimize the rank of X_{τ_r} with full knowledge about the distribution of X_k. Problem (IV) is referred to as Robbin's problem.

Moser's problem is different from every form of the secretary problem. The secretary problems have (for i.i.d. r.v.s) a non-parametric nature. The answers in the secretary problem are dimensionless. The answers in Moser's problem depend essentially on the distribution of the r.v.s X_i, $i = 1, ..., n$, and have the same dimension as the X_i.

Due to Moser, the optimal strategy for the optimization of $EX_\tau, \tau \leq n$ depends on the optimal thresholds h_{n-k+1} and $\tau_M = \{\min\ k : X_k > h_{n-k+1}\}$. For random variables with the uniform distribution on [0,1], the asymptotic result for h_n is $h_n = \frac{1}{n+\ln n+c} = \frac{1}{n} - \frac{\ln n}{n^2} + ..., \quad c \simeq 1.76799$.

Karlin (1962) estimated the thresholds h_n for the standard exponential distribution as $h_n = \ln n + o(1)$ using $h_{n+1} = h_n + e^{-h_n}$. The corresponding equation was discussed in Guttman (1960) for the normal distribution and in J. Gilbert and F. Mosteller (1966) for the inverse power distribution .

The main contribution of our paper is the consideration not only of exponential, power and other simple tails of the distributions but also of general regular tails containing the slowly varying function L (with some minor technical restrictions).

We will derive formulas for the thresholds h_k when the tail probability has a Weibull type distribution, that is,

$$P\{X > x\} = e^{-\frac{x^\alpha}{\alpha} L(x)}, \quad \alpha > 0. \tag{4.2}$$

The exponential distribution $P\{X > x\} = e^{-x} I_{x \geq 0}$ belongs to this class.

We will discuss the asymptotic properties of h_n for two additional cases:

A. β-type distribution. Here $X \in [0, 1]$ and $P\{X > 1 - x\} = x^\alpha L(x), x \to 0$, and $L(x)$ is a slowly varying function with additional technical restrictions. Case $\alpha = 1$ corresponds to Moser's models.

B. Heavy tails case $X \geq 1$, $P\{X > x\} = \frac{1}{x^\alpha L(x)}, \alpha > 1$. (We need $\alpha > 1$ to guarantee that $EX < \infty$). The function $L(x)$ here is slowly varying either for $x \to 0$ or $x \to \infty$.

For a β-type distribution and the heavy tails case, we only give asymptotic properties of h_n with $L(x) = \ln^\beta x$ instead of the general result for the slowly varying function $L(x)$.

In the last section, we will study the connection between Moser's problem and the secretary problem with full information: If a decision maker has full information about the random variable X_k and follows the optimal rule for Moser's problem, what is the $P\{X_{\tau_M} = M_n, M_n = \max_{1\le i\le n} X_i\}$? We use π_n to represent this probability. Here, τ_M is the optimal stopping time for Moser's problem. We get the limit for π_n when X_k has the standard exponential distribution or the uniform distribution on the interval $[0,1]$. In the following context, the stopping time τ refers to the optimal stopping time τ_M for Moser's problem.

4.1 Equations for h_n and S_n

4.1.1 The recursive relationship for h_n

Let us first derive the recursive relationship for h_n. In Moser's problem it is known that the optimal stopping time τ can be defined in term of the thresholds $h_n > h_{n-1} > \ldots > h_1 > h_0 = 0$. Here, $\tau = \{\min k : X_k > h_{n-k+1}\}$. There is an obvious recursive procedure for the calculation of h_n:

$$S_n = \max_h \left[\int_h^\infty x dF(x) + S_{n-1} \int_0^h dF(x) \right] = \max_h \left[\int_h^\infty x dF(x) + \int_0^h S_{n-1} dF(x) \right]$$

$$= \int_0^\infty x dF(x) + \max_h \int_0^h (S_{n-1} - x) dF(x),$$

$$\text{while} \quad \int_0^h (S_{n-1} - x) dF(x) = \int_0^{S_{n-1}} (S_{n-1} - x) dF(x) + \int_{S_{n-1}}^h (S_{n-1} - x) dF(x). \quad (4.3)$$

From (4.3), we see that $\int_0^h (S_{n-1} - x) dF(x)$ increases if $h < S_{n-1}$ and $\int_0^h (S_{n-1} - x) dF(x)$ decreases if $h > S_{n-1}$, i.e. S_n obtains its maximum value when

$$h = h_n = S_{n-1}, \quad (4.4)$$

$$h_{n+1} = h_n + \int_{h_n}^\infty (x - h_n) \cdot dF(x) = h_n + \int_{h_n}^\infty P(X > x) dx. \quad (4.5)$$

Formulas (4.4) and (4.5) reduce the problem of optimal stopping to the analysis of the iteration of an appropriate monotone function $h_{n+1} = H(h_n)$, $H(h) = h + \int_h^\infty P(X > x)dx$.

$L(x)$ in (4.2) is a slowly varying function with additional technical restrictions.

Assumption 4.1. *The function $L(x)$ in (4.2) is slowly varying and $L(x) \in NR_0$, that is,*

$$\frac{xL'(x)}{L(x)} \to 0 \quad (x \to \infty).$$

With assumption 4.1, we see that $\frac{xh'(x)}{h(x)} \to \alpha$ when $x \to \infty$, where $h(x) = \frac{x^\alpha L(x)}{\alpha}$. From lemmas 1.2 and 1.3, $h(x)$ is a normalized regularly varying function and is ultimately increasing.

This assumption is not completely sufficient for our analysis. We need the following additional technical assumption.

Assumption 4.2. *$L(x)$ in (4.2) satisfies*

$$L(xL^\beta(x))/L(x) \to 1 \ (x \to \infty) \ locally \ uniformly \ in \ \beta \in R.$$

Theorem 1.2 provides sufficient conditions for assumption 4.2.

4.1.2 Asymptotic results for h_n for upper tail distributions with slowly varying function $L(x)$

Theorem 4.3. *Suppose we have a r.v. with cdf $1 - F(x) = e^{-\frac{x^\alpha}{\alpha}L(x)}$, in which $\alpha > 0$ and $L(x)$ satisfies assumptions 4.1 and 4.2, with convergence rate $r(x)$, i.e.,*

$$L(xL^\beta(x)) \underset{x\to\infty}{\longrightarrow} L(x)(1 + r(x)), \ locally \ uniformly \ in \ \beta \in R,$$

where $r(x) \neq 0$ and $r(x) \to 0$ when $x \to \infty$. Then

(a) When $r(x) \cdot O\left(\frac{\ln n}{\ln\ln n}\right) = 1$, the threshold

$$h_n = \left(\frac{\alpha \ln n}{L((\alpha \ln n)^{1/\alpha})}\right)^{1/\alpha} (1 + O(\frac{\ln\ln n}{\ln n})).$$

(b) When $r(x) \cdot o\left(\frac{\ln n}{\ln \ln n}\right) = 1$, the threshold

$$h_n = \left(\frac{\alpha \ln n}{L((\alpha \ln n)^{1/\alpha})}\right)^{1/\alpha} (1 + O(r(n))).$$

Proof. Step 1: Let us find recursive relationships for h_n and $1/h_n$. An iterative relationship for h_n can be derived as follows:

$$h_{n+1} = h_n + \int_{h_n}^{\infty} P(X > x)dx = h_n + \int_{h_n}^{\infty} e^{-\frac{x^\alpha}{\alpha}L(x)}dx = h_n + \left(\int_{h_n}^{h_n+\epsilon} + \int_{h_n+\epsilon}^{\infty}\right) e^{-\frac{x^\alpha}{\alpha}L(x)}dx. \tag{4.6}$$

By the Laplace method, $\int_{h_n+\epsilon}^{\infty} e^{-\frac{x^\alpha}{\alpha}L(x)}dx$ tends to 0 as $h_n \to \infty$. Thus,

$$h_{n+1} = h_n - \frac{e^{-\frac{h_n^\alpha}{\alpha}L(h_n)}}{(-\frac{x^\alpha}{\alpha}L(x))'|_{x=h_n}} (1 + o(1)) = h_n + C\frac{e^{-\frac{h_n^\alpha}{\alpha}L(h_n)}}{h_n^{\alpha-1}L(h_n)} (1 + o(1)).$$

The iterative relationship can be rewritten as :

$$h_{n+1}^{-1} = h_n^{-1} - C\frac{e^{-\frac{h_n^\alpha}{\alpha}L(h_n)}}{h_n^{\alpha+1}L(h_n)} (1 + o(1)).$$

Let $x_n = \frac{1}{h_n}$, then

$$x_{n+1} = x_n - \frac{Cx_n^{\alpha+1}e^{-\frac{L\left(\frac{1}{x_n}\right)}{\alpha x_n^\alpha}}}{L\left(\frac{1}{x_n}\right)}(1 + o(1)). \tag{4.7}$$

With iterative relationship (4.7), we see that $\exists a_0$ such that the function

$$f(x) = x - \frac{Cx^{\alpha+1}e^{-\frac{L\left(\frac{1}{x}\right)}{\alpha x^\alpha}}}{L\left(\frac{1}{x}\right)}(1 + o(1))$$

is monotonically increasing on the interval $(0, a_0]$.

Step 2: Case (a): $r(x) \cdot O\left(\frac{\ln n}{\ln \ln n}\right) = 1$. Let us prove the following claim:

There exist positive K and N such that for all $n > N$ the following is true: For all x in the interval $0 < x < \left(\frac{\alpha \ln n}{L((\alpha \ln n)^{1/\alpha})}\right)^{-1/\alpha} \left(1 + K\frac{\ln \ln n}{\ln n}\right) < a_0$, we have that $0 < x - \frac{Cx^{\alpha+1}e^{-\frac{L(\frac{1}{x})}{\alpha x^\alpha}}}{L(\frac{1}{x})}(1 + o(1)) < \left(\frac{\alpha \ln(n+1)}{L((\alpha \ln(n+1))^{1/\alpha})}\right)^{-1/\alpha} (1 + K\frac{\ln \ln(n+1)}{\ln(n+1)})$.

In order to show this, put $\lambda_n = \left(\frac{\alpha \ln n}{L\left((\alpha \ln n)^{1/\alpha}\right)}\right)^{-1/\alpha}\left(1 + K\frac{\ln\ln n}{\ln n}\right)$. Then,

$$\lambda_{n+1} = \left(\frac{\alpha\ln(n+1)}{L((\alpha\ln(n+1))^{1/\alpha})}\right)^{-1/\alpha}\left(1+K\frac{\ln\ln(n+1)}{\ln(n+1)}\right) = \lambda_n - (\frac{1}{n})^{1+(\frac{1}{\alpha}+1)\frac{\ln\ln n}{\ln n}+o(\frac{\ln\ln n}{\ln n})}.$$

$$L\left(\frac{1}{\lambda_n}\right) = L\left(\frac{(\alpha\ln n)^{\frac{1}{\alpha}}}{L^{\frac{1}{\alpha}}((\alpha\ln n)^{\frac{1}{\alpha}})}(1-\frac{K\ln\ln n}{\ln n})\right) = L((\alpha\ln n)^{\frac{1}{\alpha}})\left(1+O\left(\frac{\ln\ln n}{\ln n}\right)\right),$$

or $\exists M_1, M_2$, such that $L((\alpha\ln n)^{\frac{1}{\alpha}})\left(1+M_1\frac{\ln\ln n}{\ln n}\right) \le L\left(\frac{1}{\lambda_n}\right) \le L((\alpha\ln n)^{\frac{1}{\alpha}})\left(1+M_2\frac{\ln\ln n}{\ln n}\right)$.

Similarly, we have that

$$\frac{\lambda_n^{\alpha+1}}{L(\frac{1}{\lambda_n})} = \left(\frac{1}{n}\right)^{(1+\frac{1}{\alpha})\frac{\ln\ln n}{\ln n}+o(\frac{\ln\ln n}{\ln n})},$$

$$e^{-\frac{L(\frac{1}{\lambda_n})}{\alpha\lambda_n^{\alpha}}} \ge e^{-\ln n(1-(\alpha K-M_2)\frac{\ln\ln n}{\ln n}+o(\frac{\ln\ln n}{\ln n}))} = \left(\frac{1}{n}\right)^{1-(\alpha K-M_2)\frac{\ln\ln n}{\ln n}+o(\frac{\ln\ln n}{\ln n})}.$$

Then

$$\lambda_{n+1} - \lambda_n + \frac{C\lambda_n^{\alpha+1}e^{-\frac{L(\frac{1}{\lambda_n})}{\alpha\lambda_n^{\alpha}}}}{L(\frac{1}{\lambda_n})}(1+o(1))$$
$$\ge (\frac{1}{n})^{1+(M_2+\frac{1}{\alpha}+1-\alpha K)\frac{\ln\ln n}{\ln n}+o(\frac{\ln\ln n}{\ln n})} - (\frac{1}{n})^{1+(\frac{1}{\alpha}+1)\frac{\ln\ln n}{\ln n}+o(\frac{\ln\ln n}{\ln n})}.$$

From here we can see that when n is big enough and K is positive, it is true that $\lambda_{n+1} > \lambda_n - \frac{C\lambda_n^{\alpha+1}e^{-\frac{L(\frac{1}{\lambda_n})}{\alpha\lambda_n^{\alpha}}}}{L(\frac{1}{\lambda_n})}(1+o(1))$.

Case (b): $r(x)\cdot o\left(\frac{\ln n}{\ln\ln n}\right) = 1$. The proof is similar to that of case (a) except for the following changes:

$$L\left(\frac{1}{\lambda_n}\right) = L((\alpha\ln n)^{\frac{1}{\alpha}})\left(1+r(n)\right), \quad \lambda_n = (\frac{\alpha\ln n}{L((\alpha\ln n)^{1/\alpha})})^{-1/\alpha}(1+K\cdot r(n)),$$

$$e^{-\frac{L(\frac{1}{\lambda_n})}{\alpha\lambda_n^{\alpha}}} \ge \left(\frac{1}{n}\right)^{1-(\alpha K-M_2)r(n)+O(\frac{\ln\ln n}{\ln n})},$$

$$\lambda_{n+1} - \lambda_n + \frac{C\lambda_n^{\alpha+1}e^{-\frac{L(\frac{1}{\lambda_n})}{\alpha\lambda_n^{\alpha}}}}{L(\frac{1}{\lambda_n})}(1+o(1)) \ge (\frac{1}{n})^{1-(\alpha K-M_2)r(n)+O(\frac{\ln\ln n}{\ln n})} - (\frac{1}{n})^{1+O(\frac{\ln\ln n}{\ln n})}.$$

Thus, the claim of step 2 is proved. The proof of case (b) is similar to that of case

(a) for the following steps, so we only write the proof for case (a).

Step 3: Let us find the upper estimate for x_n.

Let k be chosen such that $0 < x_k < (\frac{\alpha \ln N}{L((\alpha \ln N)^{1/\alpha})})^{-1/\alpha}(1 + K\frac{\ln\ln N}{\ln N})$; this is possible because x_n tends to 0. Then, with step 2, we can prove by induction that $0 < x_{k+m} < (\frac{\alpha \ln(N+m)}{L((\alpha \ln(N+m))^{1/\alpha})})^{-1/\alpha}(1 + K\frac{\ln\ln(N+m)}{\ln(N+m)})$. We get the upper estimate for x_n as $x_n < (\frac{\alpha \ln n}{L((\alpha \ln n)^{1/\alpha})})^{-1/\alpha}(1 + K\frac{\ln\ln n}{\ln n})$ as n becomes sufficiently large.

Step 4: Let us find the lower estimate for x_n.

In a similar fashion, we can find the lower bound for x_n as $x_n > (\frac{\alpha \ln n}{L((\alpha \ln n)^{1/\alpha})})^{-1/\alpha}(1+ K\frac{\ln\ln n}{\ln n})$ as n becomes sufficiently large.

Step 5: From the results for x_n, we get that $h_n = \left(\frac{\alpha \ln n}{L((\alpha \ln n)^{1/\alpha})}\right)^{1/\alpha}(1+O(\frac{\ln\ln n}{\ln n}))$. $\square$

Remark 4.1. *In theorem 4.3, we present results only up to order $\frac{\ln\ln n}{\ln n}$. If $r(n) = o(\frac{\ln\ln n}{\ln n})$, then we have to expand the recursive relationship (4.7) to higher orders to get results. Technically the proof of theorem 4.3 shows all the details.*

With $L(x) = \ln^\beta x$, the rate of convergence $r(x) = \frac{\ln\ln\ln n}{\ln\ln n}$. By theorem 4.3, we get:

Corollary 4.1. *For a r.v. with CDF $1 - F(x) = e^{-\frac{x^\alpha}{\alpha}\ln^\beta x}$, in which $\alpha > 0$, the thresholds $h_n = C_0 \ln^{\frac{1}{\alpha}} n \ln^{-\frac{\beta}{\alpha}} \ln n(1 + O(\frac{\ln\ln\ln n}{\ln\ln n}))$, where $C_0 = \alpha^{\frac{1+\beta}{\alpha}}$.*

For a β-type distribution and the heavy tails case, we only give the asymptotic properties of h_n with $L(x) = \ln^\beta x$ instead of the general result for the slowly varying function $L(x)$.

Proposition 4.1. *For a r.v. with CDF $1 - F(x) \sim C(A - x)^\alpha \ln^\beta \frac{1}{A-x}$, $\alpha > 0$, the thresholds $A - h_n \sim C_0 n^{-\frac{1}{\alpha}} \ln^{-\frac{\beta}{\alpha}} n$, where $C_0 = (\frac{(\alpha+1)\alpha^{\beta-1}}{C})^{\frac{1}{\alpha}}$.*

Proposition 4.2. *For a r.v. with CDF $1 - F(x) = \frac{C}{x^\alpha \ln^\beta x}$, the thresholds $h_n \sim C_0 n^{\frac{1}{\alpha}} \ln^{-\frac{\beta}{\alpha}} n$, where $C_0 = (\frac{(\alpha-1)\alpha^{-\beta-1}}{C})^{\frac{1}{\alpha}}$.*

4.2 The connection between the generalized Moser's problem and the secretary problem with information

In this section we will prove one particular result on the connection between the generalized Moser's problem and the secretary problem. Assume that random variable X has an exponential distribution with *pdf* $f(x) = e^{-x}$.

With *pdf* $f(x) = e^{-x}$, the slowly varying function $L(x) = 1$, with convergence rate $r(x) = 0$. So we can not apply theorem 4.3. We need the following lemma to obtain the asymptotic results for h_n.

Lemma 4.1. *If random variable X has an exponential distribution with pdf $f(x) = e^{-x}$, the thresholds $h_n = \ln n + \frac{\ln n}{2n} + o(\frac{\ln n}{n^3})$ as $n \to \infty$.*

Proof. : For an exponential random variable X with *pdf* $f(x) = e^{-1}$, we obtain the following iterative relationship for the thresholds:

$$h_{n+1} = h_n + e^{-h_n}. \tag{4.8}$$

After setting $h_n = \ln n + \frac{\ln n}{2n} \cdot (1 + z_n)$ and plugging the form of h_n into (4.8), we have

$$z_{n+1} = \frac{1}{1 + \frac{1}{(n-1)\cdot \ln n}} \cdot z_n + o(\frac{1}{n^2}), \;\; or \;\; z_{n+k} = (\prod_{m=n}^{k} \frac{1}{1 + \frac{1}{(m-1)\cdot \ln m}}) \cdot z_n + o(\frac{1}{n^2}).$$

For $\sum_{m=1}^{\infty} \frac{1}{m \cdot \ln(m+1)} \to \infty$, $\prod_{m=n}^{k} \frac{1}{1+\frac{1}{(m-1)\cdot \ln m}} \to 0$. So, we have $z_n = o(\frac{1}{n^2})$ as $n \to \infty$. Finally, we get that $h_n = \ln n + \frac{\ln n}{2n} + o(\frac{\ln n}{n^3})$ *as* $n \to \infty$. □

Theorem 4.4. *If $X_0, X_1, \ldots, X_n$ are i.i.d. exponential random variables with pdf $f(x) = e^{-x} I_{x \geq 0}$, then $\lim_{n\to\infty} P\{X_\tau = M_n, M_n = \max_{0 \leq i \leq n} X_i\} = (e-2)\int_1^{\infty} \frac{e^{-x}}{x} dx + \frac{1}{e}$. Here τ is the optimal stopping time for the generalized Moser's problem $\max_{\tau \leq n} EX_\tau$.*

Proof. There are two cases for events $\{X_\tau = M_n, M_n = \max_{0 \leq i \leq n} X_i\}$: Case I: $X_\tau < h_n$ and Case II: $h_n < X_\tau < \infty$.

Case I: When $X_\tau < h_n$, the range of X_τ is $\bigcup_i [h_i, h_{i+1})$, $i = 0, 1, 2, ..., n-1$.

If stopping time $\tau =$ q with $h_{n-p} < X_\tau$ (or X_q) $< h_{n-p+1}$, it is necessary that

(1) X_i be less than its threshold h_{n-i} for $i = p, p+1, ..., q-1$.

(2) $h_{n-p} < X_q < h_{n-p+1}$. It is obvious that X_q is greater than its corresponding thresholds h_{n-q}.

(3) $X_i < X_q$ if $i < p$ or $i > q$. In the following discussion, we use $\alpha = \frac{p}{n}$, $\beta = \frac{q}{n}$, where $p, q = 0, 1, ..., n$, to simplify the notation. It is clear that $0 \leq \alpha \leq \beta \leq 1$.

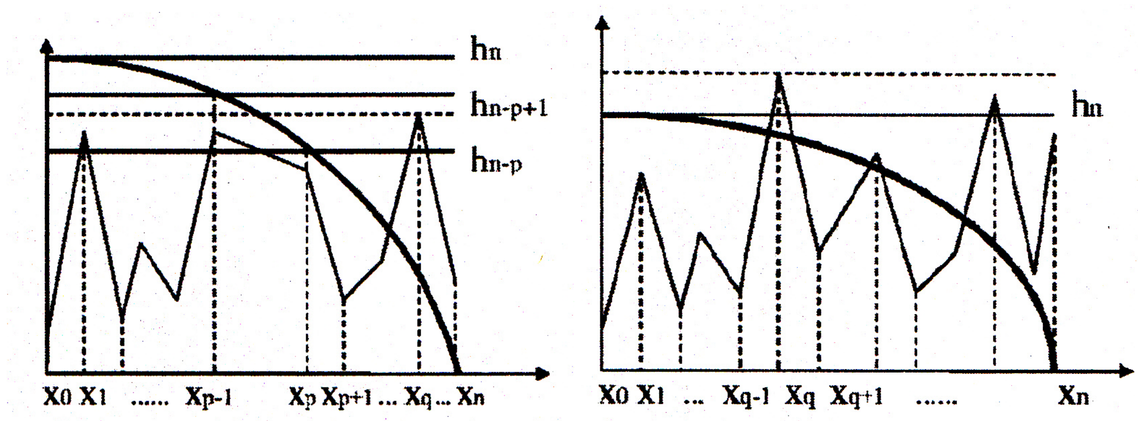

Figure 4.1: $X_\tau < h_n$ (left) and $h_n \leq X_\tau < \infty$ (right).

The Figure 1 (left) shows the idea behind the calculation for the $P\{X_\tau = M_n, M_n = \max_{0\leq i\leq n} X_i, X_\tau < h_n\}$.

$$P\{X_\tau = M_n, M_n = \max_{0\leq i\leq n} X_i, X_\tau < h_n\}$$

$$= \sum_{0\leq\beta\leq1} P\{X_\tau = M_n, M_n = \max_{0\leq i\leq n} X_i, h_{n-\beta n} \leq X_\tau < h_{n-\beta n+1}\}$$

$$= \sum_{0\leq\beta\leq1} \sum_{\beta\leq\alpha\leq1} P\{X_\tau = M_n, M_n = \max_{0\leq i\leq n} X_i, h_{n-\beta n} \leq X_\tau < h_{n-\beta n+1} \;\; and \;\; \tau = \alpha n\},$$

$$while \; P\{X_\tau = M_n, M_n = \max_{0\leq i\leq n} X_i, h_{n-\beta n} \leq X_\tau < h_{n-\beta n+1} \;\; and \;\; \tau = \alpha n\}$$

$$= \sum_{0\leq\beta\leq1} \sum_{\beta\leq\alpha\leq1} \int_{h_{n-\beta n}}^{h_{n-\beta n+1}} \{\prod_{i=\beta n}^{\alpha n} P(X_i < h_{n-i})\} \cdot F^{\beta n+n-\alpha n}(x) dF(x).$$

For an exponentially distributed random variable X_i with *pdf* $f(x) = e^{-x} I_{x\geq0}$ we have that

$$P(X_i < h_{n-i}) = 1 - e^{-h_{n-i}} = e^{-e^{-h_{n-i}}} + O(e^{-2h_{n-i}}) = e^{h_{n-i}-h_{n-i+1}} + O(e^{-2h_{n-i}})$$

$$= \frac{n-i}{n-i+1}(1+O(\frac{\ln n}{n^2})),$$

$$\prod_{i=\beta n}^{\alpha n} P(X_i < h_{n-i}) = \frac{1-\alpha}{1-\beta}(1+O(\frac{\ln n}{n^2})).$$

If $h_{n-\beta n} \le x < h_{n-\beta n+1}$, $F(x) = e^{-\frac{1}{n-\beta n}}(1+O(\frac{\ln n}{n^2}))$, $F^{\beta n+n-\alpha n}(x) = e^{-\frac{1-\alpha+\beta}{1-\beta}}(1+O(\frac{\ln n}{n}))$,

$$h_{n-\beta n+1} - h_{n-\beta n} = \frac{1}{n-n\beta}(1+O(\frac{1}{n})), \text{ and } dF(x) = \frac{1}{n-n\beta}(1+O(\frac{\ln n}{n^2}))d\beta.$$

From the above approximation, the probability

$$P\{X_\tau = M_n, M_n = \max_{0\le i\le n} X_i, h_{n-\beta n} \le X_\tau < h_{n-\beta n+1} \quad and \quad \tau = \alpha n\}$$

$$= \frac{1}{n^2}\frac{1-\alpha}{(1-\beta)^3}e^{-\frac{1-\alpha+\beta}{1-\beta}}(1+O(\frac{\ln n}{n})).$$

When n goes to infinity, the above probability can be written in double integral form as

$$\lim_{n\to\infty} P\{X_\tau = M_n, M_n = \max_{0\le i\le n} X_i, X_\tau < h_n\} = \int_0^1\int_\beta^1 \frac{1-\alpha}{(1-\beta)^3}e^{-\frac{1-\alpha+\beta}{1-\beta}}d\alpha d\beta$$

$$= (e-2)\int_1^\infty \frac{e^{-x}}{x}dx \simeq 0.1576.$$

Case II: If stopping time $\tau = q$ with $h_n \le X_\tau($ or $X_i) < \infty$, it is necessary that (1) X_j be less than its threshold h_{n-j} for $j < i$; (2)$X_i > h_n$; (3) $X_i > X_j$ if $j > i$. Figure 1 (right) shows the idea behind the calculation for the $P\{X_\tau = M_n, M_n = \max_{0\le i\le n} X_i, h_n \le X_\tau < \infty\}$.

$$P\{X_\tau = M_n, M_n = \max_{0\le i\le n} X_i, h_n \le X_\tau < \infty\}$$

$$= \int_{h_n}^\infty F^{(n-1)}(x)dF(x) + \sum_{i=2}^n \prod_{j=n}^i F(h_j)\int_{h_n}^\infty F^{(i-2)}(x)dF(x).$$

To calculate $P\{X_\tau = M_n, M_n = \max\limits_{0\le i\le n} X_i, h_n \le X_\tau < \infty\}$, we use the results:

$$F(h_n) = 1 - \frac{1}{n} + o(\frac{1}{n^2}) \text{ and } \prod_{j=n}^{i} F(h_j) = \frac{i-1}{n} + o(\frac{1}{n}).$$

So, $\lim\limits_{n\to\infty} P\{X_\tau = M_n, M_n = \max\limits_{0\le i\le n} X_i, h_n \le X_\tau < \infty\} = \lim\limits_{n\to\infty} 1 - \{(1-\frac{1}{n}) - (1-\frac{1}{n})^{n+1}\} = \frac{1}{e}$.

Finally, we obtain the result:

$$\lim_{n\to\infty} P\{X_\tau = M_n, M_n = \max_{0\le i\le n} X_i\} = (e-2)\int_1^\infty \frac{e^{-x}}{x}dx + \frac{1}{e} \simeq 0.5255.$$

$\square$

Theorem 4.5. *If X_0, X_1, ... , X_n are i.i.d. uniform random variables with pdf $f(x) = I_{0\le x\le 1}(x)$, then $\lim\limits_{n\to\infty} P\{X_\tau = M_n, M_n = \max\limits_{0\le i\le n} X_i\} = \frac{1}{2}(e^2-5)\int_2^\infty \frac{e^{-x}}{x}dx + \frac{1}{4} + \frac{3}{4e^2}$. Here τ is the optimal stopping time for the generalized Moser's problem $\max\limits_{\tau\le n} EX_\tau$.*

Remark 4.2. *Under the conditions $X_\tau \le h_n$ or $h_n \le X_\tau \le 1$, we obtain the following results: $\lim\limits_{n\to\infty} P\{X_\tau = M_n, M_n = \max\limits_{0\le i\le n} X_i, X_\tau \le h_n\} = \frac{1}{2}(e^2-5)\int_2^\infty \frac{e^{-x}}{x}dx \simeq 0.0584$, $\lim\limits_{n\to\infty} P\{X_\tau = M_n, M_n = \max\limits_{0\le i\le n} X_i, h_n \le X_\tau \le 1\} = \frac{1}{4} + \frac{3}{4e^2} \simeq 0.3515 < 1/e$, and $\lim\limits_{n\to\infty} P\{X_\tau = M_n, M_n = \max\limits_{0\le i\le n} X_i\} \simeq 0.4099$.*

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