LIMIT THEOREMS FOR RANDOM EXPONENTIAL SUMS AND THEIR APPLICATIONS TO INSURANCE AND THE RANDOM ENERGY MODEL

by

Huseyin Erturk

A dissertation submitted to the faculty of
The University of North Carolina at Charlotte
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in
Applied Mathematics

Charlotte

2016

Approved by:

Dr. Stanislav Molchanov

Dr. Isaac Sonin

Dr. Michael Grabchak

Dr. Craig Allan

ABSTRACT

HUSEYIN ERTURK. Limit theorems for random exponential sums and their applications to insurance and the random energy model. (Under the direction of DR. STANISLAV MOLCHANOV)

In this dissertation, we are mainly concerned with the sum of random exponentials, $S_N(t) = \sum_{i=1}^{N(t)} e^{tX_i}$. Here, $t, N(t) \to \infty$ in appropriate form and $\{X_i, i \geq 1\}$ are independent and identically distributed random variables (i.i.d.). Our first goal is to find the limiting distributions of $S_N(t)$ for a new class of random variables. For some classes, such results are known (Ben Arous et al., 2003) [5].

Secondly, we apply these limit theorems to some insurance models and the random energy model (REM) in statistical physics. Specifically for the first case, we give the estimate of the ruin probability in terms of the empirical data. For the REM, we present the analysis of the free energy for a new class of distribution. In some particular cases, we prove the existence of several critical points for the free energy. In some other cases, we prove the absence of phase transitions.

Our results give a new approach to compute the ruin probabilities of insurance portfolios empirically when there is a sequence of insurance portfolios with a custom growth rate of the claim amounts. The second application introduces a simple method to drive the free energy in the case the random variables in the statistical sum can be represented as a function of standard exponential random variables.

The technical tool of this study includes the classical limit theory for the sum of i.i.d. random variables and different asymptotic methods like the Euler-Maclaurin formula and Laplace method (from De Bruijn, 1981) [13].

ACKNOWLEDGMENTS

I would like to express my deep appreciation and gratitude to my advisor, Dr. Stanislav A. Molchanov for his guidance and mentorship all the way from the classes of the PhD program through to the completion of this dissertation.

I would also like to thank my committe members, Dr. Isaac Sonin, Dr. Michael Grabchak and Dr. Craig Allan for their constructive feedback, help, support and time.

I am highly indebted to Dr. Arvin who had been the Graduate Program Coordinator when I was accepted to the PhD program. His support during the qualifiers is appreciated. Also, I would like to thank to the Graduate Committee of 2010 for putting the trust in me during admissions and starting my journey from my home country Turkey to the USA. I highly appreciate helpful revision and editing suggestions from Laura Redden Erturk.

I am grateful to all my professors at UNCC who put their faith in me and urged me to improve.

TABLE OF CONTENTS

CHAPTER 1: INTRODUCTION

1.1 A General Summary

The main object in this paper is the partial sum of exponentials of the form

$$S_N(t) = \sum_{i=1}^{N(t)} e^{tX_i} \tag{1}$$

where the sequence,

$$\{X_1, X_2, ..., X_{N(t)}\}, \tag{2}$$

is composed of i.i.d. random variables. First, we analyze the limiting behavior of this object for different growth rates of N(t) when the sequence (2) is double exponentially distributed (8). In our analysis, we show that the random exponential sum converges to normal distribution or stable distribution under appropriate additive and multiplicative factors of t. After this theoretical analysis, we explore applications of the statistical sum in insurance mathematics and statistical physics.

1.2 Two Particular Applications

The first application of the partial sum of exponentials is from insurance mathematics. Consider a portfolio consisting of N policies with individual risks $\{X_1, ..., X_N\}$ over a given time period and assume that the nonnegative random variables $\{X_1, ..., X_N\}$ are i.i.d. Here the aggregate claim amount can be calculated as $U = \sum_{i=1}^{N} X_i$ and the risk reserve process is given by $R(s) = u + \beta s - U$ where β is the premium rate, s is time and u is the initial reserve. One problem is to estimate the Lundberg bounds which approximate the tail distribution of U, $\overline{F}_U(x) = P(U > x)$. This requires the solution of the Laplace equation (from Rolski et al., 1999: pp. 125-126) [12]

$$m(\gamma) = E(\gamma X) = p^{-1} \tag{3}$$

where p is a small constant. We assume that the solution exists and it is called the adjustment coefficient, γ. Also, the same equation helps us to approximate the ruin probability $\psi(s) := P\left(\min_{s\geq 0} R(s) < 0\right)$ for appropriate p which is essential for insurance companies (from Rolski et al., 1999: pp. 125-126 and pp. 170-171) [12].

In practical applications, γ is estimated using a statistical method and this estimation utilizes the empirical Laplace transform. Hence, we replace $m(\gamma)$ (3) with the empirical Laplace transform. Also, we define p on the right hand side of the Laplace equation as a sequence, p_n. When $n \to \infty$, $p_n \to 0$. Then, we obtain the empirical Laplace equation:

$$\bar{m}_U(\gamma_n) := \frac{1}{N(\gamma_n)} \sum_{i=1}^{N(\gamma_n)} e^{\gamma_n U_i} = p_n^{-1} \,. \tag{4}$$

It means that we have a sequence of adjustment coefficients, γ_n, for a sequence of insurance portfolios, which give a sequence of Lundberg bounds to esimate ruin probabilities from below and above. Our interest is to analyze the asymptotic behaviour of γ_n when n is large. We make use of the exponential sum to develop this estimation procedure. The estimation of γ has been studied by Csörgö and Teugels (1990) [6] where the classical central limit theorem has been used. Our approach is slightly different in the sense that we can control the growth rate of the number of individual risks.

Another application of this study is the REM, which was first introduced by Derrida (1981) [2]. Eisele (1983) [7] demonstrated the phase transitions (non-analiticity) of the free energy in the class of Weibull-type distributions. We will show similar results for the Weibull distribution, relatively heavy-tailed distribution and relatively light-tailed double exponential distribution using order statistics. Also, we will show that there are several critical points for the mixed Weibull distribution.

The REM, introduced by Derrida (1981) [2], describes the system of size n with 2^n energy levels where $E_i = \sqrt{n}X_i$ and $\{ X_i,\ i = 1, 2, ..., N \}$ are i.i.d. random variables following the N(0,1) distribution. Thermodynamics of the system is quantified by the statistical sum, or so-called the partition function. This partition function in Derrida's model has the following form

$$Z_n(\beta) = \sum_{i=1}^{N} e^{\beta A(n) X_i} \tag{5}$$

where $A(n) = \sqrt{n}$ and $\beta > 0$ is the inverse temperature. We use the same statistical sum with different selections of $A(n)$. Derrida defines the free energy by the following formula

$$\chi(\beta) = \lim_{n\to\infty} \frac{logZ_n(\beta)}{n}$$

According to Derrida's results, the free energy is quantified as

$$\chi(\beta) = \begin{cases} \beta^2/2 + \beta_c^2/2, & \text{if } 0 < \beta \leq \beta_c \\ \beta\beta_c, & \text{if } \beta \geq \beta_c \end{cases}$$

where $\beta_c = \sqrt{2log2}$. It is important to note that $\chi(\beta)$ and $\chi'(\beta)$ are continuous but

$\chi''(\beta)$ has a jump, which is called a third order phase transition. $\chi(\beta)$ is convex and continuous. The phase transition introduces the presence of two analytic branches in the free energy. One branch corresponds to the high temperature i.e. $\beta = \frac{1}{kT} < \beta_{critical}$. The second branch corresponds to the low temperature i.e. $\beta \geq \beta_{critical}$.

Derrida's paper was extended in several directions. In Eisele (1983) [7], the results of Derrida (1981) [2] were proven for Weibull-type distributions. Later on, Olivieri and Picco (1984) [3] and also Pastur (1989) [4] rigorously derived the limits. The mathematical justification of this result as well as the theory of limit theorems for the sum $Z_n(\beta)$ was analyzed in detail in the mathematical paper by Bovier, Kurkova and Löwe (2002) [1]. Ben Arous, Bogachev and Molchanov (2003) [5] extended the results to the Weibull/Frechet-type tails. It contains the complete theory of of the limiting distributions for the sum of the random exponentials in the case

$$Z_n(\beta) = \sum_{i=1}^{N} e^{tX_i}$$

$$P\{X_i > a\} = \exp\left\{-\frac{a^{\varrho}L(a)}{\varrho}\right\}$$

where $\varrho \geq 1$ and $L(a)$ is a slowly varying function with additional regulatory properties (from Ben Arous et al., 2003) [5].

The technical tools in Bovier, Kurkova and Löwe (2002) [1] and Ben Arous, Bogachev and Molchanov (2003) [5] are traditional Bahr-Esseen inequality (from Bahr and Esseen, 1965) [8] and the Lyapunov fraction that are used for the proof of the law of large numbers and the central limit theorem. Also, standard methods for the stable distributions are utilized.

We use the methods developed by Ben Arous, Bogachev and Molchanov (2003) [5] for the computation of the free energy. In addition to this methodology, we develop the new approach based on the properties of the variational series of exponential random variables (from Feller, 1971: pp. 17-21) [11]. This approach covers the REM outside of the Weibull-type tail and the Bahr-Esseen inequality. We analyze four types of distributions for the REM: the Weibull, mixed Weibull, light-tailed and heavy-tailed distribution.

CHAPTER 2: STATEMENT OF VARIABLES AND DISTRIBUTIONS

In this chapter, we state the variables and distributions that are used throughout the whole dissertation. All of the sequence of random variables in this study are assumed to be independent identically distributed random variables and we will refer this term as i.i.d.

The Weibull distribution is the most commonly used distribution. The Weibull random variable, X, follows the law

$$1. \quad P(X > x) = \begin{cases} \exp\left\{-\dfrac{x^{\varrho}}{\varrho}\right\}, & \text{if x} \geq 0 \\ 1, & \text{o.w.} \end{cases} \tag{6}$$

where $1 < \varrho < \infty$. Also, we make use of the mixed Weibull distribution

$$2. \quad X = \begin{cases} X_1, & \text{with prob. p and } P(X_1 > x) = \exp\left\{-\dfrac{x^{\varrho}}{\varrho}\right\} \quad (7a) \\ L(n) + X_2, & \text{with prob. 1-p and } P(X_2 > x) = \exp\left\{-\dfrac{x^{\varrho}}{\varrho}\right\} \quad (7b) \end{cases}$$

where n is a large number and $1 < \varrho < \infty$. In the next chapter, we work on the double exponential distribution, which has lighter tails than the Weibull distribution. Ben Arous, Bogachev and Molchanov (2003) [5] analyze the limiting distributions of the random exponential sum (1) when the X_i's in the statistical sum are Weibull-type random variables. We extend this to the double exponential random variable, which has the distribution function

$$3. \quad P(X > x) = \begin{cases} \exp\{1 - e^x\}, & \text{if } x \geq 0 \qquad (8a) \\ 1, & \text{o.w.} \qquad (8b) \end{cases}$$

In addition to the above distributions, we have a relatively heavy-tailed distribution. Corresponding heavy tailed random variable is defined as a function of standard exponential random variables. Heaviness of the tail behavior is relative to the Weibull distribution. The standard exponential distribution and the relatively heavy-tailed random variable are respectively expressed as

$$4. \quad P(Y > x) = \begin{cases} \exp\{-x\}, & \text{if x} \geq 0 \\ 1, & \text{o.w.} \end{cases} \tag{9}$$

$$5. \quad X = \frac{1+Y}{\ln(1+Y)}. \tag{10}$$

Assume that w_X stands for $\operatorname{ess\,sup} X$, and $P(X < w_X) = 1$, which means X is finite with probability 1 and the log-tail distribution for the above distributions is

$$h(x) = -\log P(X > x) \tag{11}$$

where $x \in \mathbb{R}$ and $h(x)$ is non-negative, non-decreasing and right-continuous. From the above information, we can state that $P(X > x) = e^{-h(x)}$ such that $x < w_X$. If h is regularly varying at infinity with the index ϱ, we write $h \in R_\varrho$ where $1 < \varrho < \infty$. It means that for any $\kappa > 0$ we have $h(\kappa x)/h(x) \to \kappa^\varrho$ as $x \to \infty$.

We frequently work with the Laplace transform and we require that $E\left[e^{tX_i}\right] < \infty$ for finite t. The selected distributions above satisfy this condition and a detailed analysis for Laplace transform is given in Appendix B. We introduce the cumulant generating function

$$H(t) = \log E[e^{tX}] \tag{12}$$

where $H(t)$ is well-defined and non-decreasing for any $t \geq 0$. $H(t) \to \infty$ as $t \to \infty$. For the Weibull distribution,

$$\varrho^{'} = \frac{\varrho}{\varrho - 1} \tag{13}$$

is being used as the exponent of the cumulant generating function with the condition that $1 < \varrho^{'} < \infty$. Note that $1 = \frac{1}{\varrho} + \frac{1}{\varrho^{'}}$. It is important to mention that $h \in R_{\varrho}$ implies $H \in R_{\varrho'}$.

As a result of these definitions we express the expected value of the random exponential sum (1),

$$E[S_N(t)] = \sum_{i=1}^{N} E[e^{tX_i}] = Ne^{H(t)}. \tag{14}$$

For the REM, the random variables in the statistical sum (5) are expressed as a function of exponential random variables, $Y_1, ..., Y_N$ (9), such that $X_i = f(Y_i)$ (10). This enables us to express the statistical sum in a simplified form and compute the free energy using the Euler-Maclaurin formula and the Laplace Method. The results for the free energy depend on the structure of the distribution, which is specified by $f(Y_i)$, and the selection of $A(n)$. $A(n)$ is an analytic and increasing function of n. For the appropriate selection of $A(n)$, we assume that there exists a p-a.s. limit for the free energy

$$\chi(\beta) \quad := \quad \lim_{n\to\infty} \frac{\log Z_n(\beta)}{n}$$

Also, there are other variables that we will use for various theorems

$$B(t) \qquad = (\lambda t)^t \tag{15}$$

$$A(t) = \begin{cases} E[S_N(t)], & \text{for } 1 < \lambda < 2 \quad \text{(16a)} \\ E[S_N(t) 1_{\{Y \leq \tau\}}], & \text{for } \lambda = 1 \quad \text{(16b)} \\ 0, & \text{for } 0 < \lambda < 1\,. \quad \text{(16c)} \end{cases}$$

CHAPTER 3: LIMIT THEOREMS FOR WEIBULL AND DOUBLE EXPONENTIAL DISTRIBUTIONS

This section is devoted to the convergence of the random exponential sum (1) when X_i's (2) have the Weibull (6) or double exponential distribution (8). A similar analysis has been done for the Weibull distribution by Ben Arous, Bogachev and Molchanov (2003) [5]. We extend this to the double exponential distribution. We look for the range of the exponential rate, λ, on $N(t)$ that gives the necessary and sufficient conditions for the existence of the law of large numbers (LLN), central limit theorem (CLT) and convergence to the stable distribution. Before starting our theorems, we specify the growth rate of $N(t)$. In this chapter, Case 1 refers to the Weibull distribution (6) and Case 2 refers to the double exponential distribution (8). When X_i's have the Weibull-type distribution,

$$N(t) = e^{\lambda H(t)} \tag{17}$$

is being used as the growth rate. $H(t)$ is the cumulant generating function introduced in (12). The asymptotic of $H(t)$, $H_0(t)$, can be found in Appendix A. When X_i's have the double exponential distribution (8), the growth rate is

$$N(t) = e^{\lambda t}. \tag{18}$$

We first prove Lemma 1 which is then used in the proof of the LLN and CLT. In later sections, we prove the LLN, CLT and convergence to the stable distribution.

3.1 Main Lemma

Lemma 1. *Consider the function*

$$v_\lambda(x) \;:=\; \lambda(x-1) - (x^{\varrho'} - x) \quad x \geq 1$$

if $\lambda > \lambda_b$ ($\lambda_b = \lambda_1 = \varrho' - 1$ for Case 1) then there exists $x_0 > 1$ such that $v_\lambda(x) > 0$ for all $x \in (1, x_0)$.

Proof. Note that $v_\lambda(1) = 0$ and $v'_\lambda(x) = \lambda - (\varrho' x^{\varrho'-1} - 1)$ so $v'_\lambda(1) = \lambda - (\varrho' - 1) = \lambda - \lambda_b > 0$ where $\lambda_b = \lambda_1$ for Case 1. Based on the Taylor's formula, $v_\lambda(x) > 0$ for all $x > 1$ sufficiently close to 1. □

3.2 Main Theorems

Theorem 2. *The Law of large numbers (LLN) for different growth rates of $N(t)$,*

$$\frac{S_N(t)}{E[S_N(t)]} \xrightarrow{p} 1. \tag{19}$$

1. *Assume that X_i's (2) in the statistical sum (5) have the Weibull distribution (6). If $\lambda > \varrho' - 1 = \lambda_1$ (17), the LLN holds.*

2. *Assume that X_i's (2) in the statistical sum (5) have the double exponential distribution (8). If $\lambda > 1$ (18), the LLN holds.*

Proof. Set

$$S^*_N(t) = \frac{S_N(t)}{E[S_N(t)]} = \frac{1}{N}\sum_{i=1}^{N} e^{tx_i - H(t)}$$

It is sufficient to show that $\lim_{t\to\infty} E\left|S_N^*(t)-1\right|^r = 0$ for some $r>1$. We first derive

$$E|S_N^*(t)-1|^r = E\left|\frac{\sum_{i=1}^N e^{tx_i-H(t)}}{N} - 1\right|^r$$
$$= E\left|\frac{\sum_{i=1}^N e^{tx_i-H(t)} - 1}{N}\right|^r = N^{-r}E\left|\sum_{i=1}^N e^{tx_i-H(t)} - 1\right|^r.$$

Using the Bahr-Esseen inequality (from Bahr and Esseen, 1965) [8] and

$(x+1)^r \le 2^{r-1}(x^r+1)$ where $(x>0, r\ge 1)$, we get

$$N^{-r}E\left|\sum_{i=1}^N e^{tx_i-H(t)} - 1\right|^r \le 2N^{-r}\sum_{i=1}^N E\left|e^{tx_i-H(t)}-1\right|^r$$
$$\le 2N^{1-r}E\left|e^{tx_i-H(t)}+1\right|^r \le 2N^{1-r}2^{r-1}E\left|e^{rtx_i-rH(t)}+1\right|$$
$$= 2^r N^{1-r}e^{H(rt)-rH(t)} + 2^r N^{1-r}. \tag{20}$$

Case 1: Using $H \in R_{\varrho'}$ and Appendix A,

$$\liminf_{t\to\infty}\left[\frac{(r-1)\log(N)}{H(t)} - \frac{H(rt)}{H(t)} + r\right] = \lambda(r-1) - \left(r^{\varrho'} - r\right) = v_\lambda(r).$$

By Lemma 1, we can choose $r>1$ such that $v_\lambda(r)>0$ when $\lambda > \lambda_1 = \frac{\varrho'}{\varrho} = \varrho' - 1$ and this implies that the right hand side converges to 0.

Case 2: For the double exponential distribution, we use the Bahr-Esseen inequality (from Bahr and Esseen, 1965) [8] to obtain

$$E\left|S_N^*(t)-1\right|^r < 2^r N^{1-r}e^{H(rt)-rH(t)} + 2^r N^{1-r}.$$

The Cumulant generating function $H(t)$ of the double exponential distribution has an asymptotic equivalent of

$$H(t) = t\ln(t) - t + \frac{\ln t}{2} + \underline{\varrho}(1)$$

(Refer to Appendix B for details). Only the first two terms play a role in the proof of the LLN. Using the substitution $r = 1 + \epsilon$ and the limit $\epsilon \to 0^+$, we must have

$$(r-1)\log(N) - H(rt) + rH(t) \cong \epsilon \log(N) - (1+\epsilon)\epsilon t + \epsilon/2(\ln t - 1) > 0$$

for the existence of the LLN which implies that

$$\frac{\log N}{t} = \lambda > 1.$$

□

Theorem 3. *The CLT for different growth rates of $N(t)$,*

$$\frac{S_N(t) - E[S_N(t)]}{(Var[S_N(t)])^{1/2}} \xrightarrow{d} N(0,1). \tag{21}$$

1. *Assume that X_i's (2) in the statistical sum (5) have the Weibull distribution (6). If $\lambda > 2^{\varrho'} \frac{\varrho'}{\varrho} = \lambda_2$ (17), the CLT holds.*

2. *Assume that X_i's (2) in the statistical sum (5) have the double exponential distribution (8). If $\lambda > 2$ (18), the CLT holds.*

Proof. Suppose that $e^{tX_1}, e^{tX_2}, \ldots$ is a sequence of independent random variables, each with a finite expected value and variance. We know from Lemma 4.1 in Ben Arous et al. (2003) [5] that $Var(e^{tX_i}) \cong e^{H(2t)}$ for the Weibull distribution. This asymptotics also holds for the double exponential distribution, which can be proven using the same steps of Lemma 4.1 [5]. Define

$$s_n^2 = \sum_{i=1}^{N(t)} Var(e^{tX_i}) \cong N(t)e^{H(2t)}$$

If for some $\delta > 0$, the Lyapunov's condition

$$\lim_{t\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{N(t)} E\left[\left|e^{tX_i} - E\left(e^{tX_i}\right)\right|^{2+\delta}\right] = 0$$

is satisfied, then $\dfrac{S_N(t) - E[S_N(t)]}{Var[S_N(t)]^{1/2}}$ converges to the standard normal distribution. Using the Lyapunov's condition and the inequality, $(x+1)^r \leq 2^{r-1}(x^r+1)$ where $(x > 0, r \geq 1)$, we obtain

$$\begin{aligned}
&\frac{1}{s_n^{2+\delta}} \sum_{i=1}^{N(t)} E[|e^{tX_i} - E(e^{tX_i})|^{2+\delta}] \\
&\cong N(t)^{-\delta/2} \exp\{H(t)(2+\delta)\} \exp\{-H(2t)(1+\delta/2)\} E\left[\left|\frac{e^{tX_i}}{e^{H(t)}} - 1\right|^{2+\delta}\right] \\
&\leq \exp\{-\ln(N(t))\delta/2 + H(t)(2+\delta) - H(2t)(1+\delta/2)\} \, E\left[\left(\frac{e^{tX_i}}{e^{H(t)}} + 1\right)^{2+\delta}\right] \\
&\leq 2^{1+\delta} \exp\{-\ln(N(t))\delta/2 + H(t)(2+\delta) - H(2t)(1+\delta/2)\} \left[E\left[\frac{e^{t(2+\delta)X_i}}{e^{H(t)(2+\delta)}}\right] + 1\right] \\
&= 2^{1+\delta} \exp\{-\ln(N(t))\delta/2 - H(2t)(1+\delta/2) + H(t(2+\delta))\} (1+o(1)). \qquad (22)
\end{aligned}$$

Case 1: Using $H \in R_{\varrho'}$ and the substitution $r = 1 + \delta/2$,

$$\begin{aligned}
&\liminf_{t\to\infty} \left[\frac{\ln(N(t))\delta/2}{H(t)} + \frac{H(2t)(1+\delta/2)}{H(t)} - \frac{H(t(2+\delta))}{H(t)}\right] \\
&\cong 2^{\varrho'} \left[\frac{\lambda}{2^{\varrho'}}(r-1) - (r^{\varrho'} - r)\right] = 2^{\varrho'} v_{\lambda/2^{\varrho'}}(r)
\end{aligned}$$

for small δ. By Lemma 1, we can choose $r > 1$ such that $v_{\lambda/2^{\varrho'}}(r) > 0$ when $\lambda/2^{\varrho'} > \lambda_1 = \dfrac{\varrho'}{\varrho}$ and this implies that the CLT holds if $\lambda > \lambda_2 = 2^{\varrho'} \dfrac{\varrho'}{\varrho}$.

Case 2: We make use of the inequality that we obtained in (22) and we derive

$$\begin{aligned}
&\frac{1}{s_n^{2+\delta}} \sum_{i=1}^{N(t)} E\left[\left|e^{tX_i} - E\left(e^{tX_i}\right)\right|^{2+\delta}\right] \\
&\leq 2^{r-1} \exp\{-\ln(N(t))\delta/2 - H(2t)(1+\delta/2) + H(t(2+\delta))\} \, (1+o(1))
\end{aligned}$$

where $H(t)$ for the double exponential distribution has an asymptotic equivalent of $H_0(t) = t\ln(t) - t$. Then, the requirement for the CLT is the following condition

$$\liminf_{\substack{t\to\infty \\ \delta\to 0^+}} \left[\ln(N(t))\delta/2 + H(2t)(1+\delta/2) - H(t(2+\delta))\right] > 0$$

This inequality implies that we have the CLT if

$$\liminf_{t\to\infty} \frac{\ln(N)}{t} = \lambda > \liminf_{\delta\to 0^+} \frac{\ln(1+\delta/2)}{\delta/2}(2+\delta) = 2$$

□

Theorem 4. *Conditions for Convergence to an Infinitely Divisible Distribution*

We use the theorem about the weak convergence of sums of independent random variables from Ben Arous et al. (2003) [5] which is also given in a similar form in the book of Petrov (1978) [9]. Suppose that

$$Y_i(t) = \frac{e^{tX_i}}{B(t)} \tag{23}$$

is a sequence of i.i.d. random variables where $B(t)$ is a multiplicative factor. Additionally, we define $A(t)$ as an additive factor. Both $A(t)$ and $B(t)$ are increasing function of t such that $A(t), B(t) \to \infty$ as $t \to \infty$. According to classical theorems on weak convergence of sums of independent random variables, in order that

$$S_N^*(t) = \sum_{i=1}^{N(t)} Y_i(t) - \frac{A(t)}{B(t)} \tag{24}$$

converges to an infinitely divisible law with characteristic function

$$\phi(u) = \exp\left\{ iau - \frac{\sigma^2 u^2}{2} + \int_{|x|>0} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) dL(x) \right\}, \tag{25}$$

it is necessary and sufficient that the following conditions hold:

1. At all continuity points, $L(x)$ *satisfies*

$$L(x) = \begin{cases} \lim_{t\to\infty} NP\{Y \leq x\} & \text{for } x < 0 \\ -\lim_{t\to\infty} NP\{Y > x\} & \text{for } x > 0. \end{cases} \tag{26}$$

2. σ^2 *satisfies*

$$\sigma^2 = \lim_{\tau\to 0^+} \lim_{t\to\infty} \sup NVar[Y 1_{\{Y\leq\tau\}}] = \lim_{\tau\to 0^+} \lim_{t\to\infty} \inf NVar[Y 1_{\{Y\leq\tau\}}] \tag{27}$$

3. For each $\tau > 0$ *the following identity is satisfied,*

$$\begin{aligned} &\lim_{t\to\infty} \left\{ NE[Y 1_{\{Y\leq\tau\}}] - \frac{A(t)}{B(t)} \right\} \\ &= a + \int_0^\tau \frac{x^3}{1+x^2}\, dL(x) - \int_\tau^\infty \frac{x}{1+x^2}\, dL(x). \end{aligned} \tag{28}$$

Here, a *is a constant depending on the distribution function.*

Theorem 5. *Suppose that* X_i*'s in (23) are i.i.d. double exponentially distributed random variables (8). Also, suppose that* $N(t)$ *is defined as in (18) and* λ *satisfies the inequality,* $0 < \lambda < 2$. *Then,*

$$\frac{S_N(t) - A(t)}{B(t)} \xrightarrow{d} F_\lambda \tag{29}$$

for large t *where* $A(t)$ *and* $B(t)$ *are given in (16) and (15) respectively.* F_λ *is an infinitely divisible distribution with the characteristic function,*

$$\phi_\lambda(u) = \exp\left\{ iau + \lambda e \int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{dx}{x^{\lambda+1}} \right\}, \tag{30}$$

where a is given by

$$a = \begin{cases} \dfrac{\lambda e \pi}{2\cos\dfrac{\lambda\pi}{2}} & \text{for } \lambda \neq 1 \\ 0 & \text{for } \lambda = 1. \end{cases} \tag{31}$$

Proof. To prove this theorem, we need to show that the three conditions in Theorem (4) are satisfied.

1. For selected $B(t) = (\lambda t)^t$ (15), the function $L(x)$ (26) is given by

$$L(x) = \begin{cases} \lim_{t\to\infty} NP\{Y \leq x\} = 0 & \text{for } x < 0 \\ -\lim_{t\to\infty} NP\{Y > x\} = -x^{-\lambda}e & \text{for } x > 0. \end{cases} \tag{32}$$

where Y is given in (23). Because $Y \geq 0$, $L(x) = 0$ holds in the case $x < 0$. Assume that $x > 0$. By using (15), (18) and (23) we obtain

$$\begin{aligned} &NP\{Y(t) > x)\} = e^{\lambda t} P\left\{\frac{e^{tX}}{B(t)} > x\right\} = e^{\lambda t} P\left\{X > \frac{\ln x + \ln B(t)}{t}\right\} \\ &\cong \exp\left\{1 + \lambda t - \exp\left(\frac{\ln x + \ln B(t)}{t}\right)\right\} \cong \exp\left\{1 + \lambda t - \left(1 + \frac{\ln x}{t}\right)\lambda t\right\} \\ &= -x^{-\lambda} e \end{aligned}$$

for large t, which shows that (32) holds.

2. We claim that (27) holds and $\sigma^2 = 0$ for all $\lambda \in (0, 2)$. Since

$$0 \leq Var\left[Y 1_{\{Y\leq\tau\}}\right] \leq E\left[Y^2 1_{\{Y\leq\tau\}}\right],$$

we just need to prove that

$$\sigma^2 = \lim_{\tau\to 0^+} \lim_{t\to\infty} NE\left[Y^2 1_{\{Y\leq\tau\}}\right] = 0\,. \tag{33}$$

We introduce a common variable which will be used throughout this theorem,

$$\eta(t,\tau) = \frac{\ln B(t) + \ln \tau}{t} \tag{34}$$

Using (15), (18) and (34) for any $\tau > 0$,

$$NE\left[Y^2 1_{\{Y\leq\tau\}}\right] \cong N(t)E\left[\frac{e^{2tX}}{B^2(t)} 1_{\{X\leq\eta(t,\tau)\}}\right]$$
$$\cong \frac{N(t)e}{B^2(t)} \int_0^{+\infty} \exp\left\{2tx + x - e^x\right\} 1_{\{x\leq\eta(t,\tau)\}}\, dx$$

We use the substitution $x = y + \ln(2t+1)$ (18), Appendix B.2 and B.3 which give

$$C(t) \int_{-\ln(2t+1)}^{+\infty} \exp\left\{(2t+1)(y - e^y)\right\} 1_{\{y\leq\eta(t,\tau)-\ln(2t+1)\}}\, dy$$
$$= C(t) \int_{-\ln(2t+1)}^{K} \exp\left\{(2t+1)(y - e^y)\right\}\, dy$$
$$= C(t) \exp\left\{(2t+1)(K - e^K)\right\} \frac{1}{(2t+1) \mid g'(K) \mid} \tag{35}$$

where

$$C(t) = \frac{N(t)e}{B^2(t)} \exp\{(2t+1)\ln(2t+1)\}$$
$$K = \ln(\lambda/2) + \frac{\ln \tau}{t} - \ln\left(1 + \frac{1}{2t}\right) \tag{36}$$

The substitution of K (36) into (35) gives us (Refer to Appendix B.4, for details)

$$NE\left[Y^2 1_{\{Y\leq\tau\}}\right] \cong \frac{\lambda\tau^{2-\lambda}e}{|g'(K)|}$$

where $|g'(K)| \cong 1 - \lambda/2 > 0$ when t is large. Then

$$\sigma^2 = \lim_{\tau\to 0^+}\lim_{t\to\infty} NE[Y^2 1_{\{Y\leq\tau\}}] = \lim_{\tau\to 0^+} \frac{\lambda\tau^{2-\lambda}e}{|g'(K)|} = \lim_{\tau\to 0^+} \frac{\lambda\tau^{2-\lambda}e}{1-\lambda/2} = 0$$

3. When $\lambda \in (0,2)$, the limit,

$$D_\lambda(\tau) = \lim_{t\to\infty} \left\{ NE[Y1_{\{Y\le\tau\}}] - \frac{A(t)}{B(t)} \right\},$$

exists for each $\tau > 0$ where A(t) and B(t) are given in (16), (15) respectively. Then $D_\lambda(\tau)$ can be expressed

$$D_\lambda(\tau) = \begin{cases} \dfrac{\lambda e}{1-\lambda}\tau^{1-\lambda} & \text{for } \lambda \ne 1 \\ e\ln\tau & \text{for } \lambda = 1. \end{cases} \tag{37}$$

3a.) Assume that $\lambda \in (0,1)$. Then $A(t) = 0$ (16c). Using the substitution $x = y + \ln(t+1)$, (34) and Appendix B.2

$$\begin{aligned} NE\left[Y1_{\{Y\le\tau\}}\right] &\cong \frac{N(t)e}{B(t)} \int_0^{+\infty} \exp\left\{tx + x - e^X\right\} 1_{\{x\le\eta(t,\tau)\}}\, dx \\ &= D(t) \int_{-\ln(t+1)}^{+\infty} \exp\left\{(t+1)(y-e^y)\right\} 1_{\{y\le\eta(t,\tau)-\ln(t+1)\}}\, dy \\ &= D(t) \int_{-\ln(t+1)}^{K} \exp\left\{(t+1)(y-e^y)\right\}\, dy \\ &= D(t) \exp\left\{(t+1)(K-e^K)\right\} \frac{1}{(t+1)\mid g'(K)\mid} \end{aligned} \tag{38}$$

where

$$D(t) = \frac{N(t)e}{B(t)} \exp\{(t+1)\ln(t+1)\} \tag{39}$$

$$K = \ln(\lambda) + \frac{\ln\tau}{t} - \ln\left(1 + \frac{1}{t}\right) \tag{40}$$

The substitution of K (40) into (38) gives us (Refer to Appendix B.4, B.5 for details)

$$NE\left[Y1_{\{Y\le\tau\}}\right] \cong \frac{\lambda\tau^{1-\lambda}e}{|g'(K)|} \cong \frac{\lambda\tau^{1-\lambda}e}{1-\lambda}$$

when t is large.

3b.) Assume that $\lambda \in (1,2)$. Also, $A(t) = E[S_N(t)]$ (16a). Using the substitution $x = y + \ln(t+1)$ (39) and Appendix B.3

$$\begin{aligned}
&NE[Y1_{\{Y\leq\tau\}}] - \frac{A(t)}{B(t)} \\
&\cong \frac{N(t)e}{B(t)} \int_0^{+\infty} \exp\left\{tx + x - e^X\right\} 1_{\{x>\eta(t,\tau)\}}\, dx \\
&= D(t) \int_{-\ln(t+1)}^{+\infty} \exp\left\{(t+1)(y - e^y)\right\} 1_{\{y>\eta(t,\tau)-\ln(t+1)\}}\, dy \\
&= D(t) \int_K^{\infty} \exp\left\{(t+1)(y - e^y)\right\}\, dy \\
&= D(t) \exp\left\{(t+1)(K - e^K)\right\} \frac{1}{(t+1) \mid g'(K) \mid}
\end{aligned} \tag{41}$$

where

$$K = \ln(\lambda) + \frac{\ln\tau}{t} - \ln\left(1 + \frac{1}{t}\right) > 0 \tag{42}$$

for large t. Substitution of K (42) into (41) gives us (Refer to Appendix B.4 and B.6 for details)

$$NE\left[Y1_{\{Y\leq\tau\}}\right] - \frac{A(t)}{B(t)} \cong \frac{\lambda\tau^{1-\lambda}e}{|g'(K)|} \cong \frac{\lambda\tau^{1-\lambda}e}{1-\lambda}$$

when t is large.

3c.) Assume that $\lambda = 1$ and $\tau > 1$ for definiteness. Also, $A(t) = E[S_N(t)1_{\{Y\leq 1\}}]$

(16b). Using N(t) (18) and B(t) (15),

$$\begin{aligned}
&NE[Y1_{\{Y\le\tau\}}] - \frac{A(t)}{B(t)} \\
&\cong \frac{N(t)e}{B(t)} \int_0^{+\infty} \exp\left\{tx + x - e^X\right\} [1_{\{x\le\eta(t,\tau)\}} - 1_{\{x\le\ln B(t)/t\}}]\,dx \\
&= \frac{N(t)e}{B(t)} \int_{\ln B(t)/t}^{\eta(t,\tau)} \exp\left\{tx + x - e^X\right\}\,dx \\
&\cong \frac{N(t)e}{B(t)} \frac{\ln\tau}{t} \exp\left\{tK + K - e^K\right\} \qquad (43)
\end{aligned}$$

where

$$K = \ln t + \frac{\ln\tau}{t} \qquad (44)$$

for large t. Substitution of K (44) into (43) gives us

$$NE\left[Y1_{\{Y\le\tau\}}\right] - \frac{A(t)}{B(t)} \cong e\ln\tau$$

when t is large.

3d.) The parameter a defined in (31) satisfies the identity (28) with $L(x)$ specified by (32),

$$\begin{aligned}
D_\lambda(\tau) \quad &= \lim_{t\to\infty}\left\{NE[Y1_{\{Y\le\tau\}}] - \frac{A(t)}{B(t)}\right\} \\
&\cong a + \int_0^\tau \frac{ex^{2-\lambda}}{1+x^2}\,dx - \int_\tau^\infty \frac{ex^{-\lambda}}{1+x^2}\,dx \qquad (45)
\end{aligned}$$

where $D_\lambda(\tau)$ is given by (37).

Assume that $\lambda \in (0,1)$. It is known that

$$\int_0^\tau \frac{x^{2-\lambda}}{1+x^2}\,d(x) = \frac{\tau^{1-\lambda}}{1-\lambda} - \int_0^\tau \frac{x^{-\lambda}}{1+x^2}\,dx \qquad (46)$$

Using (32) and (31), the equation (45) results in

$$\frac{\pi}{2cos\left(\frac{\lambda\pi}{2}\right)} = \int_0^\infty \frac{x^{-\lambda}}{1+x^2}\,dx$$

which is true from Gradshteyn and Ryzhik (1994) [10].

When $\lambda \in (1,2)$, it is known that

$$\int_\tau^\infty \frac{x^\lambda}{1+x^2}\,d(x) = \frac{\tau^{1-\lambda}}{\lambda-1} - \int_\tau^\infty \frac{x^{2-\lambda}}{1+x^2}\,dx \tag{47}$$

Using (32) and (31), equation (45) turns out to be

$$\frac{\pi}{2cos\left(\frac{\lambda\pi}{2}\right)} + \int_0^\infty \frac{x^{2-\lambda}}{1+x^2}\,dx = 0,$$

which is true again from Gradshteyn and Ryzhik [10].

For $\lambda = 1$, equation (45) has the form

$$\ln\tau = \int_0^\tau \frac{x}{1+x^2}\,dx + \int_\tau^\infty \frac{1}{(1+x^2)x}\,dx$$

The integral on the right can be computed using calculus as

$$\frac{1}{2}\ln\left(1+x^2\right)\Big|_0^\tau + \frac{1}{2}\ln\left(\frac{x^2}{1+x^2}\right)\Big|_\tau^\infty = \ln\tau$$

This completes the proof. □

Theorem 6. *The charachteristic function* ϕ_λ *determined by Theorem 5 corresponds to a stable probability law with the exponent* $\lambda \in (0,2)$ *and the skewness parameter*

$\beta = 1$ and can be represented in canonical form by

$$\phi_\lambda(u) = \begin{cases} \exp\left\{-\Gamma(1-\lambda)e \mid u \mid^\lambda \exp\left(-\frac{i\pi\lambda}{2}\, sgn(u)\right)\right\} & \textit{for } \lambda \in (0,1) \\ \exp\left\{\frac{\Gamma(2-\lambda)}{\lambda - 1} e \mid u \mid^\lambda \exp\left(-\frac{i\pi\lambda}{2}\, sgn(u)\right)\right\} & \textit{for } \lambda \in (1,2) \\ \exp\left\{iu(1-\gamma)e - \frac{\pi e}{2} \mid u \mid \left(1 + i\, sgn(u)\, \frac{2}{\pi} \ln \mid u \mid\right)\right\} & \textit{for } \lambda = 1 \end{cases} \tag{48}$$

where $\Gamma(s) = \int_\tau^\infty x^{s-1} e^{-x}\, dx$ is the gamma function, $sgn(u) : u/ \mid u \mid$ for $u \neq 0$ and $sgn(u) := 0$, and $\gamma = 0.5772...$ is the Euler constant. The proof of this theorem can be found in the paper by Ben Arous et al. (2003) [5].

CHAPTER 4: APPLICATION: STATISTICAL ESTIMATION OF THE LUNDBERG ROOT USING THE EMPIRICAL LAPLACE TRANSFORM

Many applications in insurance mathematics are related to compound distributions and their corresponding ruin probabilities. The ruin probability of an insurance portfolio is one of the major concerns of an insurance company and it depends on the tail behavior of the insurance portfolio.

First, we present a section about finding the upper and lower bounds of the tail probability in the case of compound distributions from the book by Rolski et al. (1999: pp. 125-131 and 170-171) [12]. Here, the tail probability is approximated using Lundberg bounds and the corresponding adjustment coefficient. Assume that the solution of the following Lundberg equation exists

$$L(\gamma) = \int_0^\infty e^{\gamma x}\, dF(x) = \frac{1}{p}\,. \tag{49}$$

This solution is called as the adjustment coefficient. The question is how to estimate the unknown solution of this equation. Csorgo and Teugels (1990) [6] answered this question by developing an estimation procedure using the empirical Laplace transform.

The first section motivates our study about estimation of Lundberg type bounds. Then, we use the empirical Laplace transform and customize the growth rate of the number of individual risks, i.e. the number of claims. The number of individual risks is a function of the adjustment coefficient.

4.1 Geometric Compounds

This section is mainly taken from the book of Rolski et al. (1999: pp. 125-131) [12] and it is followed by the Lundberg root estimation.

Consider a portfolio consisting of infinitely many policies with individual risks $\{X_1, X_2, ...\}$ over a given time period. Assume that the non-negative random variables $\{X_1, X_2, ...\}$ are i.i.d. Weibull-type random variables (6) with distribution function F_X. First, we investigate the asymptotic behavior of the tail probability $\overline{F}_U(x) = P(U > x)$ of the compound $U = \sum_{i=1}^{N} X_i$ when $N \to \infty$. Here, N has a geometric distribution with a parameter $p \in (0, 1)$. We are able to determine this by finding the upper and lower Lundberg bounds, but in practical applications, we do not know the form of $L(t)$ precisely. Only a sample version of $L(t)$ or $\overline{L}_N(t)$ is known for an insurance company. This is defined as the empirical Laplace transform:

$$\overline{L}_N(\gamma) = \frac{1}{N}\sum_{j=1}^{N}(e^{\gamma X_j}). \tag{50}$$

The compound geometric distribution F_U is given by

$$F_U(x) = \sum_{i=0}^{\infty}(1-p)p^i\, F_X^{*i}(x) \tag{51}$$

Writing the first summand in (51) separately, we get

$$F_U = (1-p)\delta_0 + pF_X * F_U \tag{52}$$

where

$$\delta_0 = \delta_0(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

This is called the defective renewal equation or transient renewal equation. Replacing the distribution F_U on the right hand side of (52) by the term $(1-p)\,\delta_0 + p\,F_X * F_U$ and iterating this procedure, we get

$$F_U(x) = \lim_{n\to\infty} F_n(x) \tag{53}$$

for $x \geq 0$ where F_n is defined as

$$F_n = (1-p)\delta_0 + p\,F_X * F_{n-1} \tag{54}$$

for all $n \geq 1$ and F_0 is an arbitrary initial distribution on $\mathbb{R}_+$. Additionally, assume that

$$L(\gamma) = \int_0^\infty e^{\gamma x}\,dF_X(x) = \frac{1}{p} \tag{55}$$

has a solution where p is the parameter of the geometric distribution and $F_X(x)$ is the distribution function of the Weibull distributed individual risks, $\{X_1, X_2, ...\}$. γ is the adjustment coefficient here. Let $x_0 = \sup\{x : F_U(x) < 1\}$. Then the following theorem gives us the Lundberg bounds based on the existence of the adjustment coefficient.

Theorem 7. *If X is a geometric compound with characteristics (p, F_X) such that*

(55) admits a positive solution ,γ, then

$$a_- e^{-\gamma x} \leq \overline{F}_U(x) \leq a_+ e^{-\gamma x}$$

where $x \geq 0$ and

$$a_- = \inf_{x\in[0,x_0)} \frac{e^{\gamma x}\overline{F}_X(x)}{\int_x^\infty e^{\gamma y}\, dF_X(y)}$$
$$a_+ = \sup_{x\in[0,x_0)} \frac{e^{\gamma x}\overline{F}_X(x)}{\int_x^\infty e^{\gamma y}\, dF_X(y)} \tag{56}$$

Proof. To find the upper bound in (56) we aim to find an initial distribution F_0 such that the corresponding distribution F_1 defined in (54) for $n = 1$ satisfies

$$F_1(x) \geq F_0(x) \tag{57}$$

for $x \geq 0$. Then $F_X(x) * F_1(x) \geq F_X(x) * F_0(x)$ for $x \geq 0$ and by induction, $F_{n+1}(x) \geq F_n(x)$ for all $x \geq 0$ and $n \in \mathbb{N}$. This means that

$$\overline{F}_U(x) \leq \overline{F}_0(x) \tag{58}$$

for $x \geq 0$. Let $F_0(x) = 1 - a\, e^{-\gamma x} = (1-a)\delta_0(x) + a\, G(x)$ where $a \in (0,1]$ is some constant and $G(x) = 1 - e^{-\gamma x}$. Inserting this into (54) we obtain

$$\begin{aligned} F_1(x) \quad &= 1 - p + p\left((1-a)F_X(x) + a\int_0^x G(x-y)\, dF_X(y)\right) \\ &= 1 - p + p\left(F_X(x) - a\int_0^x e^{-\gamma(x-y)}\, dF_X(y)\right) \end{aligned}$$

for all $x \geq 0$. Since we want to arrive at (57) we look for a such that

$$1 - p + p\left(F_X(x) - a\int_0^x e^{-\gamma(x-y)}\,dF_X(y)\right) \geq 1 - ae^{\gamma x} \tag{59}$$

This inequality can be simplified to

$$a\left(1 - p\int_0^x e^{\gamma y}\,dF_X(y)\right) \geq pe^{\gamma x}\,\overline{F}_X(x)$$

which is trivial for $x \geq x_0$ using

$$1 = p\int_0^\infty e^{\gamma y}\,dF_X(y) = p\int_0^x e^{\gamma y}\,dF_X(y) + p\int_x^\infty e^{\gamma y}\,dF_X(y)$$

Then (59) is equivalent to

$$ap\int_x^\infty e^{\gamma y}\,dF_X(y) \geq pe^{\gamma x}\overline{F}_X(x) \tag{60}$$

Setting $a_+ = \sup_{x\in[0,x_0)} \frac{e^{\gamma x}\overline{F}_X(x)}{\int_x^\infty e^{\gamma y}\,dF_X(y)}$, we get (57) and consequently (58). The upper bound follows and the lower bound can be driven similarly. □

4.2 Estimation of the Adjustment Coefficient

When we find the bounds to the tail probability in the previous section, we assume the existence of a solution to the Lundberg equation (55), the adjustment coefficient. In this section, we study an estimation problem which is motivated by the previous section. Here, the empirical Laplace transform and the central limit theorem are applied to convert the Laplace equation (55) into an estimation problem.

We made use of the Laplace transform for Lundberg bounds in Theorem 7. It is assumed that the Laplace transform $L(\gamma)$ exists in an open neighborhood of the origin, $I = (-\infty, \sigma)$, where σ is the abscissa of convergence of $L(\gamma)$. $L(\gamma)$ is arbitrarily many times differentiable in I. Also, $L(\gamma)$ is an increasing convex function on I and has non-negative random variables.

In this section, we introduce additional assumptions. We assume that we have a sequence of insurance portfolios $\{0, 1, 2, ..., n, ...\}$ and that the n'th portfolio has N_n individual risks. Individual risks $\{X_1, X_2, ..., X_{N_n}\}$ follow the Weibull law with the parameter $\varrho > 1$ and have a distribution function F_X (6). They are i.i.d. random variables. N_n is defined as:

$$N_n = N_n(\gamma) = e^{\lambda H(\gamma)}, \tag{61}$$

where $H(\gamma) = \dfrac{\gamma^{\varrho'}}{\varrho'}$ for the Weibull distribution, when γ is large and λ is a constant.

Our goal is to find the solution of the following equation for varying p_n

$$L(\gamma) = \int_0^\infty e^{\gamma x}\, dF(x) = \frac{1}{p_n}. \tag{62}$$

To ensure that there is a solution, we select p_n small enough. It means that we select

$$p_n \cong exp\{-c\gamma^{\varrho'}\}$$

with constant and big enough c. We can state that $p_n \to 0$ as $n \to \infty$.

We define each solution of (62) as t_n. Note that $t_n \to \infty$ when $n \to \infty$ and t_n is the real Lundberg root of the estimation problem. In real applications, we only have a sample of information and we don't have a precise form of $L(\gamma)$. Hence, we replace the Laplace transform with the empirical Laplace transform and call the following equation the n'th empirical Lundberg equation

$$\overline{L}_n(\gamma) = \frac{1}{N_n(\gamma)} \sum_{j=1}^{N_n(\gamma)} (e^{\gamma X_j}) = \frac{1}{p_n}\,. \tag{63}$$

There is a sequence of solutions for the empirical Lundberg equations with an appropriate decay factor p_n. We define it as a sequence, τ_n.

Based on the above definitions, we have the following array scheme containing i.i.d. individual risks

$$X_1, X_2, ..., X_{N_1}$$

$$X_1, X_2, ..., X_{N_2}$$

$$............................$$

$$X_1, X_2, ..., X_{N_n}$$

$$............................$$

Each line refers to a portfolio, which is composed of individual risks. There is a sequence of real solutions, t_n, to the Lundberg Equations

$$L(t_n) = E(e^{t_n X_j}) = \int_0^\infty e^{t_n x}\, dF_X(x) = \frac{1}{p_n}. \tag{64}$$

On the other hand, we have a sequence of adjustment coefficients, τ_n and they are solutions to the equations

$$\overline{L}_n(\tau_n) = \frac{1}{N_n(\tau_n)} \sum_{j=1}^{N_n(\tau_n)} (e^{\tau_n X_j}) = \int_0^\infty e^{\tau_n x}\, dF_{N_n(\tau_n)}(x) = \frac{1}{p_n} \tag{65}$$

where

$$F_{N_n(\tau_n)}(x) \;=\; \frac{1}{N_n(\tau_n)}\#\{1 \le j \le N_n : X_j \le x\}$$

is the empirical distribution function of the sample. Here, $\overline{L}_n(\tau_n)$ is a random analytic function for all values of τ_n. We obtain a sample based estimator of the adjustment coefficient using the empirical Laplace transform.

We introduce some expressions to be used in limits.

$$W_{N_n}(\tau_n) = \sum_{j=1}^{N_n(\tau_n)} \{e^{\tau_n X_j} - E(e^{\tau_n X_j})\} \tag{66}$$

Also, one term Taylor series expansion gives us the identity

$$\exp(\tau_n X_j) = \exp(t_n X_j) + (\tau_n - t_n)X_j \exp(\tau_n(j)X_j) \tag{67}$$

where $\tau_n(j)$ satisfies the inequalities $min(\tau_n, t_n) \le \tau_n(j) \le \max(\tau_n, t_n)$. We also use the following abbreviations:

$$S_{N_n}(\tau_n) = \sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n X_j} \tag{68}$$

where $\tau_n(j)$ is determined by the above equations.

We estimate t_n by solving the equation $\overline{L}_{N_n}(\tau_n) = 1/p_n$ and $L(t_n) = 1/p_n$ when p_n is small. Combining the two equations results in

$$\begin{aligned} 0 \quad &= N_n(\tau_n)\left(\overline{L}_n(\tau_n) - L(t_n)\right) \\ &= \sum_{j=1}^{N_n(\tau_n)} \left(e^{\tau_n X_j} - E(e^{t_n X_j})\right). \end{aligned}$$

Applying the Taylor series approximation from (67) gives us

$$\sum_{j=1}^{N_n(\tau_n)} (e^{\tau_n X_j} - E(e^{t_n X_j})) = \sum_{j=1}^{N_n(\tau_n)} (e^{t_n X_j} - E(e^{t_n X_j})) + (\tau_n - t_n) \sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n(j) X_j} = 0.$$

Also, rearrangement of terms leads to

$$t_n - \tau_n = \frac{\sum_{j=1}^{N_n(\tau_n)} (e^{t_n X_j} - E(e^{t_n X_j}))}{\sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n(j) X_j}}. \tag{69}$$

Assume that the growth rate for the number of claims satisfies

$$\lambda > \lambda_2 = 2^{\varrho'} \frac{\varrho'}{\varrho} \tag{70}$$

Then, the LLN holds from Theorem 2 and we can write

$$\frac{\frac{1}{N_n(t_n)} \sum_{j=1}^{N_n(t_n)} e^{t_n X_j}}{E[e^{t_n X_j}]} \xrightarrow{p} 1, \tag{71}$$

$$\frac{\frac{1}{N_n(\tau_n)} \sum_{j=1}^{N_n(\tau_n)} e^{\tau_n X_j}}{E[e^{\tau_n X_j}]} \xrightarrow{p} 1. \tag{72}$$

where the denominator in (71) and the nominator in (72) are equal to $1/p_n$. This implies the asymptotic equivalence of $E[e^{\tau_n X_j}] \cong exp\{\tau_n^{\varrho}/\varrho\}$ and $E[e^{t_n X_j}] \cong exp\{t_n^{\varrho}/\varrho\}$ in probability for the Weibull distribution. Using this fact and the expressions (66) and (68), we can approximate the ratio in (69) as

$$t_n - \tau_n = \frac{\sum_{j=1}^{N_n(\tau_n)} (e^{t_n X_j} - E(e^{t_n X_j}))}{\sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n(j) X_j}} \cong \frac{\sum_{j=1}^{N_n(\tau_n)} (e^{\tau_n X_j} - E(e^{\tau_n X_j}))}{\sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n X_j}} = \frac{W_{N_n}(\tau_n)}{S_{N_n}(\tau_n)} \tag{73}$$

Theorem 3 and condition (70) imply that

$$\frac{\sum_{j=1}^{N_n(\tau_n)} (e^{\tau_n X_j} - E(e^{\tau_n X_j}))}{\left(Var\left[\sum_{j=1}^{N_n(\tau_n)} e^{\tau_n X_j}\right]\right)^{1/2}} \xrightarrow{d} N(0,1) \tag{74}$$

We know from Lemma 4.1 (Ben Arous et al. 2003: p. 16)[5] that $Var(e^{tX_i}) \cong e^{H(2t)}$ for the Weibull distribution. Because $\varrho' > 1$ in (13), cross terms have lower degree and we get asymptotic equivalency

$$Var\left[\sum_{j=1}^{N_n(\tau_n)} e^{\tau_n X_j}\right] \cong N_n(\tau_n) e^{H(2\tau_n)}.$$

The asymptotics above gives us the equivalence of the nominator $t_n - \tau_n$, $W_{N_n}(\tau_n)$, as

$$\begin{aligned} W_{N_n}(\tau_n) &= \sum_{j=1}^{N_n(\tau_n)} (e^{\tau_n X_j} - E(e^{\tau_n X_j})) \\ &\cong Var\left[\sum_{j=1}^{N_n(\tau_n)} e^{\tau_n X_j}\right]^{1/2} N(0,1) \cong N_n(\tau_n)^{1/2} e^{H(2\tau_n)/2} N(0,1). \end{aligned} \tag{75}$$

The asymptotic of $S_{N_n}(\tau_n)$ is driven in Appendix A as

$$\begin{aligned} S_{N_n}(\tau_n) &= \sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n X_j} \cong N_n(\tau_n) E[X_j e^{\tau_n X_j}] \\ &\cong N_n(\tau_n) \exp\left\{\frac{\tau_n^{\varrho'}}{\varrho'}\right\} = N_n(\tau_n) \exp\{H(\tau_n)\} \end{aligned} \tag{76}$$

Here, we have the LLN because of the condition (70). As a result of these limits, the asymptotic confidence interval for t_n is

$$\lim_{n\to\infty} P\left\{\tau_n - z_{\alpha/2}\, J(\tau_n) \le t_n \le \tau_n + z_{\alpha/2}\, J(\tau_n)\right\}$$
$$= 1 - \alpha \tag{77}$$

where $\phi(z_{\alpha/2}) = 1 - \alpha/2$ for $0 < \alpha < 1$ and

$$J(\tau_n) = N_n(\tau_n)^{-1/2} \exp\{H(2\tau_n)/2 - H(\tau_n)\}$$

.

CHAPTER 5: APPLICATION: THE REM MODEL

The free energy was driven using concepts of convergence in probability in papers by Eisele (1983) [7] and Bovier et al. (2002) [1]. However ,this computation required long derivations. Hence, we develop a different approach using order statistics, the Euler-Maclaurin series and the Laplace method which simplifies the process. In the first part, we introduce variables for our computations. Then, we derive the free energy for the the Weibull distribution using the limiting distributions similar to the paper by Ben Arous et al. (2003) [5]. Then, we develop the new approach using order statistics. The free energy is calculated for the Weibull, relatively heavy-tailed (10) and relatively light-tailed (8) distributions using this method. Once the statistical sum is represented in terms of exponential random variables, deriving the free energy is quite straight forward.

5.1 Variable Definitions

Assume that $\{\, X_i,\ i = 1, 2, ..., N \,\}$ are i.i.d. random variables. We already defined the free energy in Chapter 1 as

$$\chi(\beta) \qquad = \lim_{n\to\infty} \frac{logZ_n(\beta)}{n} \tag{78}$$

where

$$Z_n(\beta) = \sum_{i=1}^{N(n)} e^{\beta A(n) X_i} \tag{79}$$

is the statistical sum or partition function and β is strictly positive. For simplicity, we assume that

$$N = [e^n] \tag{80}$$

$$\ln N = n + \underline{\underline{O}}\left(e^{-n}\right)$$

$A(n)$ in the statistical sum is selected in such a way that the free energy converges. For different distributions, we will select the proper growth factor for $A(n)$.

5.2 The REM Using Limit Theorems for the Weibull Distribution

Assume that $\{X_i,\, i = 1, 2, ..., N\}$ are i.i.d. random variables with the Weibull distribution and we select

$$A(n) = n^{1/\varrho^{'}} \tag{81}$$

as the proper growth factor where $\varrho^{'}$ is introduced in (13) for the Weibull distribution.

The cumulant generating function for the Weibull is

$$H(t) = logE[e^{tX}] \cong \frac{t^{\varrho^{'}}}{\varrho^{'}} \tag{82}$$

for large t. $H(t)$ is well defined, non-decreasing and $H(t)/t \to \infty$ as $t \to \infty$. Also, $A(n)$ is an increasing funtion of n. As a result of these definitions, we express the expected value of the statistical sum for large n as

$$E[Z_n(\beta)] = \sum_{i=1}^{[e^n]} E\left[e^{\beta A(n) X_i}\right] \cong [e^n] exp\left\{\frac{\beta^{\varrho^{'}} n}{\varrho^{'}}\right\} \tag{83}$$

5.2.1 Main Theorems

Theorem 8. *The Law of Large Numbers for the statistical sum*

Let $\ln E[e^{\beta A(n) X_i}] = H(\beta A(n))$. *For sufficiently small* ϵ, *if*

$$n\epsilon - H((1+\epsilon)\beta A(n)) + (1+\epsilon)H(\beta A(n))] > 0$$

then we have

$$\frac{Z_n(\beta)}{E[Z_n(\beta)]} \xrightarrow{p} 1 \tag{84}$$

as $n \to \infty$.

Proof. Set t= $\beta A(n)$. Also, define

$$Z_n^*(\beta) = \frac{Z_n(\beta)}{E[Z_n(\beta)]} = \frac{1}{N}\sum_{i=1}^{N} e^{tx_i - H(t)} \tag{85}$$

We have to prove that $Z_n^*(\beta) \xrightarrow{p} 0$ as $n \to \infty$. It is sufficient to show that

$$\lim_{n\to\infty} E|Z_n^*(\beta) - 1|^r = 0$$

for some $r > 1$. Using the Bahr-Esseen inequality and

$$(x+1)^r \leq 2^{r-1}(x^r + 1)$$

where $(x > 0, r \geq 1)$, we obtained in the proof of the LLN, Theorem 2

$$E\,|Z_n^*(\beta) - 1|^r \leq 2^r N^{1-r} e^{H(rt) - rH(t)} + 2^r N^{1-r} \tag{86}$$

For the existence of the limit, we must have $\lim_{n\to\infty} E|Z_n^*(\beta) - 1|^r = 0$. Substituting $t = \beta A(n)$, $r = 1+\epsilon$, $N = [e^n]$ (80) to the right hand side of the inequality (86), we obtain the condition

$$\liminf_{\substack{t\to\infty \\ \epsilon\to 0^+}} \left[n\epsilon - H((1+\epsilon)\beta A(n)) + (1+\epsilon)H(\beta A(n))\right] \quad > \quad 0$$

for the existence of the LLN. □

Theorem 9. *Assume that X_i's are i.i.d. Weibull-type random variables and we have the following conditions*

$$M_{1,N}(n) = \max\left(e^{\beta A(n) X_i}\right) \tag{87}$$

for $i = 1, ..., N = [e^n]$ and $\beta A(n) = \beta n^{1/\varrho'}$. Then

$$\begin{aligned} P\left(\frac{M_{1,N}(n)}{B(n)} < x\right) \quad &\to \quad K(x) = e^{-x^{-\alpha}} \\ \log M_{1,N}(n) \quad &\cong \quad \ln B(n) \end{aligned}$$

for large n where $\ln B(n) = \beta\varrho^{1/\varrho} n$.

Proof. Let's call $A = \beta A(n)$, which implies that $n = \dfrac{A^{\varrho'}}{\beta^{\varrho'}}$. Then

$$\begin{aligned} P\left(\frac{M_{1,N}(n)}{B(n)} < x\right) \quad &= \quad \left[P\left(\frac{e^{AX_i}}{B(n)} < x\right)\right]^N \\ &= \quad \left[P\left(X_i < \frac{\ln(B(n)x)}{A}\right)\right]^N \\ &= \quad \left[1 - exp\left\{-\frac{A^{-\varrho}\ln^{\varrho}(B(n)x)}{\varrho}\right\}\right]^N \\ &\cong \quad exp\left\{-exp\left\{\ln N - \frac{A^{-\varrho}\ln^{\varrho}(B(n)x)}{\varrho}\right\}\right\} . \end{aligned}$$

The asymptotic of the exponent can be computed using the binomial formula

$$\begin{aligned} \ln N - \frac{A^{-\varrho}\ln^{\varrho}(B(n)x)}{\varrho} \quad &= \quad n - n^{1-\varrho}\beta^{-\varrho}\ln^{\varrho} B(n)\left(1 + \varrho\frac{\ln x}{\ln B(n)}\right)/\varrho \\ &= \quad n - n^{1-\varrho}\beta^{-\varrho}\ln^{\varrho} B(n)/\varrho + n^{1-\varrho}\beta^{-\varrho}\ln^{\varrho-1} B(n)\ln x \end{aligned}$$

Plugging $\ln B(n) = \beta\varrho^{1/\varrho}n$ into the above equation , we obtain

$$P\left(\frac{M_{1,N}(n)}{B(n)} < x\right) \to -\frac{\varrho^{1/\varrho'}}{\beta}\ln x$$

Then, we can state that $logM_{1,N}(n) \cong \ln B(n)$ for large n. □

5.2.2 The Computation of the Random Energy

Lemma 10. *Assume that we have a sequence of i.i.d. Weibull-type random variables $X_1, ..., X_N$ (6). When we select $\beta A(n) = \beta n^{1/\varrho'}$, the statistical sum satisfies the LLN for $0 < \beta < \varrho^{1/\varrho'} = \beta_{critical}$. Also, the free energy can be quantified by the following formula in this interval*

$$\chi(\beta) \quad := \quad 1 + \frac{\beta^{\varrho'}}{\varrho'} .$$

Proof. In Appendix B, we have proven that the moment generating function satisfies $H(\beta A(n)) = H(\beta n^{1/\varrho'}) = nf(\beta) + o(n)$ for large n. Using the equivalent of $H(t)$, we obtain that

$$H(\beta n^{1/\varrho'}) \cong \frac{(\beta n^{1/\varrho'})^{\varrho'}}{\varrho'} = \frac{\beta^{\varrho'} n}{\varrho'} . \tag{88}$$

Using Theorem 8, we must have the following condition for the LLN when ϵ is small

$$n\epsilon - H((1+\epsilon)\beta A(n)) + (1+\epsilon)H(\beta A(n)) > 0 .$$

The binomial formula implies that

$$n\epsilon - H((1+\epsilon)\beta A(n)) + (1+\epsilon)H(\beta A(n)) = n\epsilon - (1+\epsilon)^{\varrho'}\frac{\beta^{\varrho'} n}{\varrho'} + (1+\epsilon)\frac{\beta^{\varrho'} n}{\varrho'}$$
$$\cong n\epsilon - \left(1+\epsilon\varrho'\right)\frac{\beta^{\varrho'} n}{\varrho'} + (1+\epsilon)\frac{\beta^{\varrho'} n}{\varrho'} = n\epsilon - \epsilon\left(\varrho' - 1\right)\frac{\beta^{\varrho'} n}{\varrho'} \tag{89}$$

for small ϵ. (89) should be positive for the LLN, which implies that β must satisfy the inequality $0 < \beta < \varrho^{1/\varrho'}$. Also, we formulated the statistical sum in (83). When the LLN holds,

$$\chi(\beta) = \lim_{n\to\infty} \frac{logZ_n(\beta)}{n} = 1 + f(\beta) = 1 + \frac{\beta^{\varrho'}}{\varrho'}$$

□

Theorem 11. *(Ben Arous et al., 2013: p. 48) [5] When the LLN is not satisfied which means $\beta \geq \beta_{critical}$,*

$$\frac{\ln M_{1,N}(n)}{\ln Z_N(\beta)} \xrightarrow{p} 1$$

as $n \to \infty$. Here, $M_{1,N}(n) = \max(e^{\beta A(n) X_i}, i = 1, ..., N = [e^n])$, $\beta A(n)$ is an increasing function of n and X_i are i.i.d. Weibull-type random variables.

Proof. The proof of this theorem can be found in the paper by Ben Arous et al. (2013: p. 48) [5]. □

Using Theorem 9 and Theorem 11, we can state that

$$\chi(\beta) := \lim_{n\to\infty} \frac{\ln Z_n(\beta)}{n} = \lim_{n\to\infty} \frac{\ln M_{1,N}(n)}{n}$$

$$= \lim_{n\to\infty} \frac{\ln B(n)}{n} = \beta \varrho^{1/\varrho} \qquad \text{if } \beta \geq \beta_{critical} = \varrho^{1/\varrho'}$$

Combining this result and Lemma 10, the free energy can be calculated as follows:

$$\chi(\beta) = \begin{cases} 1 + \dfrac{\beta^{\varrho'}}{\varrho'}, & \text{if } \beta < \varrho^{1/\varrho'} = \beta_{critical} \\ \beta \varrho^{1/\varrho}, & \text{if } \beta \geq \beta_{critical} \end{cases}$$

This result is obtained using convergence in probability concepts. In the next section, we introduce the method of order statistics.

5.3 The REM Using Order Statistics

We compute the free energy using order statistics. The central assumption in this section is that the random variables in the statistical sum (79) can be expressed as an increasing function of standard exponentially distributed random variables.

5.3.1 Formulation of the Statistical Sum

We introduce exponential random variables that will be rearranged in the statistical sum. Let

$$\{Y_1, Y_2, ..., Y_i, ..., Y_N\} \tag{90}$$

$$P\{Y_i > x\} = \begin{cases} e^{-x}, & \text{if x} \geq 0 \\ 1, & \text{o.w.} \end{cases}$$

such that $X_i = f(Y_i)$ where f is a monotone increasing function of standard exponentially distributed random variables, Y_i. Also, we reorder the sequence in (90) to obtain the variational sequence of the sample

$$Y_{(1)} > Y_{(2)} > .... > Y_{(i)} > ... > Y_{(N)} \,. \tag{91}$$

We make use of a proposition from Feller Volume 2 (see Feller, 1971: p. 19) [11] to express each ordered random variable in (91) and we obtain

$$\begin{aligned}
Y_{(1)} &= W_1 + \frac{W_2}{2} + \dots\dots\dots\dots + \frac{W_N}{N} \\
Y_{(2)} &= \frac{W_2}{2} + \dots\dots\dots\dots + \frac{W_N}{N} \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
Y_{(i)} &= \frac{W_i}{i} + \dots + \frac{W_N}{N} \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
Y_{(N)} &= \frac{W_N}{N}
\end{aligned} \tag{92}$$

where $\{W_1, W_2, ..., W_i, ..., W_N\}$ is another set of i.i.d. standard exponential random variables.

This helps us to derive the partition function in terms of standard exponential random variables:

$$\begin{aligned}
Z_n(\beta) &= \sum_{i=1}^{N} e^{\beta A(n) X_i} = \sum_{i=1}^{N} e^{\beta A(n) f(Y_i)} \\
&= \sum_{i=1}^{N} exp\left\{\beta A(n) f\left(\frac{W_i}{i} + ... + \frac{W_N}{N}\right)\right\}
\end{aligned} \tag{93}$$

To be able to simplify the above expression, $\frac{W_i}{i} + ... + \frac{W_N}{N}$, we prove three propositions to obtain its asymptotic equivalent in the following section.

Proposition 12. *Suppose that* $\{Y_1, Y_2, ..., Y_l, ..., Y_N\}$ *are standard exponentially distributed random variables. Then,* $M_{Y_N} = \max(Y_1, Y_2, ..., Y_N) - \ln N$ *converges to the standard Gumbel distribution as* $N \to \infty$.

Proof. Let $F(x) = 1 - e^{-x}$ for $x \in [0, \infty)$. When $x \in \mathbb{R}$, the cumulative distribution function of M_{Y_N} can be expressed as

$$\begin{aligned} P\{M_{Y_N} \leq x\} &= P\{\max\{Y_1, Y_2..., Y_N\} \leq x + \ln N\} \\ &= F^N(x + \ln N) \\ &= \{1 - exp\{-x - \ln N\}\}^N \end{aligned}$$

which converges to $exp\{-e^{-x}\}$ as $N \to \infty$. □

Proposition 13. *Let $\{W_1, W_2, ..., W_l, ..., W_N\}$ be a set of i.i.d. standard exponential random variables. Then,*

$$\sum_{l=i}^{\infty} \frac{W_l - 1}{l}$$

follows Gumbel distribution such that

$$P\left\{\sum_{l=i}^{\infty} \frac{W_l - 1}{l} \leq x\right\} = exp\left\{-e^{-x+\gamma}\right\}$$

where γ is the Euler constant.

Proof. Let $M_{Y_N} = \max(Y_1, Y_2, ..., Y_N) - \ln N$ where $\{Y_1, Y_2, ..., Y_l, ..., Y_N\}$ are i.i.d. standard exponentially distributed random variables. Using the representation in (92), we can write for any $x \in \mathbb{R}$ that

$$\begin{aligned} &P\{M_{Y_N} \leq x\} \\ &= P\left\{\max(Y_1, Y_2, ..., Y_N) - 1 - \frac{1}{2} - ... - \frac{1}{N} \leq x + \ln N - 1 - \frac{1}{2} - ... - \frac{1}{N}\right\} \\ &= P\left\{\sum_{l=i}^{N} \frac{W_l - 1}{l} \leq x + \ln N - \left(1 + \frac{1}{2} + ... + \frac{1}{l} + ... + \frac{1}{N}\right)\right\} \end{aligned}$$

converges to a standard Gumbel distribution as $N \to \infty$, which was proven in Proposition 12. Note that

$$\gamma = \lim_{N\to\infty}\left[\ln N - \left(1 + \frac{1}{2} + ... + \frac{1}{l} + ... + \frac{1}{N}\right)\right]$$

where γ is the Euler constant. Then we can drive the the distribution function as follows

$$P\left\{\sum_{l=i}^{\infty}\frac{W_l - 1}{l} \leq x\right\} = exp\left\{-e^{-x+\gamma}\right\}$$

Note that $E\left[\sum_{l=i}^{\infty}\frac{W_l - 1}{l}\right] = 0$ and $Var\left(\sum_{l=i}^{\infty}\frac{W_l - 1}{l}\right) = \frac{\Pi^2}{6}$ □

Proposition 14. *Let* $\{W_1, W_2, ..., W_i, ..., W_N\}$ *be a set of i.i.d. standard exponential random variables. Then the summation,* $\frac{W_i}{i} + ... + \frac{W_N}{N}$, *can be approximated by*

$$\frac{W_i}{i} + ... + \frac{W_N}{N} \cong \ln N - \ln i + \sum_{l=i}^{N}\frac{W_l - 1}{l} = \ln N - \ln i + \underline{\underline{O}}(1) \tag{94}$$

when N is large.

Proof. From the Euler-Maclaurin formula, we get the following approximation for large N

$$\sum_{l=i}^{N}\frac{1}{l} = \int_i^N \frac{1}{x}\,dx + \underline{\underline{O}}(1) = \ln N - \ln i + \underline{\underline{O}}\left(\frac{1}{N}\right) + \underline{\underline{O}}\left(\frac{1}{i}\right) \tag{95}$$

Using the result above,

$$\frac{W_i}{i} + ... + \frac{W_N}{N} = \sum_{l=i}^{N}\frac{1}{l} + \sum_{l=i}^{N}\frac{W_l - 1}{l} = \ln N - \ln i + \sum_{l=i}^{N}\frac{W_l - 1}{l} + \overline{\overline{o}}(1) \tag{96}$$

when N and i are large. Also, Kolmogorov's two series theorem implies that the series $\sum_{l=i}^{N}\frac{W_l - 1}{l}$ is convergent as $\sum_{l=i}^{N} Var\left(\frac{W_l - 1}{l}\right)$ and $\sum_{l=i}^{N} E\left[\frac{W_l - 1}{l}\right]$ are convergent. Also, Proposition 13 states that $\sum_{l=i}^{N}\frac{W_l - 1}{l}$ converges to a Gumbel distribution. This proves the approximation (94).

□

By substituting (94) into (93), the statistical sum is expressed as follows:

$$\begin{aligned} Z_n(\beta) &= \sum_{i=1}^{N} exp\left\{\beta A(n) f\left(\frac{W_i}{i} + ... + \frac{W_N}{N}\right)\right\} \\ &= \sum_{i=1}^{N} exp\left\{\beta A(n) f\left(\ln N - \ln i + \underline{\underline{O}}(1)\right)\right\} \end{aligned} \quad (97)$$

5.3.2 Computation of Limits

In this section, we compute the free energy for i.i.d. random variables in the statistical sum, (79), which are functions of standard exponential random variables, such that $X_i = f(Y_i)$ (90). We make use of the simplified statistical sum formula (97) and obtain the asymptotic behavior of the free energy. At the very end, we show two phase transitions for the mixed Weibull distribution.

5.3.3 The Weibull-Type Distribution

$X_i = f(Y_i) = Y_i^{1/\varrho}\varrho^{1/\varrho}$ and X_i's are i.i.d. random variables because Y_i's are i.i.d standard exponential random variables as in (90). It is apparent that X_i's have the Weibull distribution

$$P\{X_i > a\} = P\left\{Y_i > \frac{a^\varrho}{\varrho}\right\} = exp\left\{-\frac{a^\varrho}{\varrho}\right\} \quad (98)$$

where $a \geq 0$. Also, we select $A(n) = n^{1/\varrho'}$ in the statistical sum (79). The statistical sum can be expressed as

$$\begin{aligned} Z_n(\beta) &= \sum_{i=1}^{N} exp\{\beta A(n) X_i\} \\ &= \sum_{i=1}^{N} exp\left\{\beta n^{1/\varrho'} \varrho^{1/\varrho} \left(n - \ln i + \underline{\underline{O}}(1)\right)^{1/\varrho}\right\} \end{aligned} \tag{99}$$

by using (80) and (94). Also, the Euler-MacLaurin series gives us the approximate integral of this series

$$\begin{aligned} &\sum_{i=1}^{N} exp\left\{\beta n^{1/\varrho'} \varrho^{1/\varrho}(n - \ln i + \underline{\underline{O}}(1))^{1/\varrho}\right\} \\ &= \int_1^N exp\left\{\beta \varrho^{1/\varrho} n^{1/\varrho'} (n - \ln x + \underline{\underline{O}}(1))^{1/\varrho}\right\} dx + \underline{\underline{O}}(exp\left\{\beta \varrho^{1/\varrho} n\right\}) \end{aligned} \tag{100}$$

Note that for some $c > 0$, we can find bounds on $Z_N(\beta)$, such that

$$\sum_{i=1}^{N_1} exp\left\{\beta n^{1/\varrho'} \varrho^{1/\varrho} \left(n - \ln i - c\right)^{1/\varrho}\right\} < Z_N(\beta) < \sum_{i=1}^{N} exp\left\{\beta n^{1/\varrho'} \varrho^{1/\varrho} \left(n - \ln i + c\right)^{1/\varrho}\right\}$$

where $N_1 = [e^{n-c}]$. The integral in (100) is computed by replacing $\underline{\underline{O}}(1)$ with c in Appendix A using the Laplace method. This helps us find the lower and upper bounds of $Z_N(\beta)$. As they only differ by a constant multiplier, these constant multipliers cancel out in the limit so as to give the free energy as

$$\chi(\beta) = \lim_{n \to \infty} \frac{log Z_n(\beta)}{n} = \begin{cases} 1 + \dfrac{\beta^{\varrho'}}{\varrho'}, & \text{if } 0 < \beta < \beta_c = \varrho^{1/\varrho'}, \\ \beta \varrho^{1/\varrho}, & \text{if } \beta \geq \beta_c \quad . \end{cases}$$

Note that $\chi(\beta_c) = \varrho$ and $\chi'(\beta_c) = 1$

5.3.4 Relatively Heavy-Tailed Distribution

Let $x_i = f(Y_i) = \dfrac{1+Y_i}{\ln(1+Y_i)}$, where Y_i s are i.i.d standard exponential random variables (90). Note that these random variables have heavier tails than the Weibull distribution.

$$P\{X_i > a\} = P\left\{\frac{1+Y_i}{\ln(1+Y_i)} > a\right\} = exp\left\{-a\ln a - a\ln\ln a + \overline{\overline{O}}(1)\right\} \tag{101}$$

We select $A(n) = \ln n$ in the statistical sum (79). By using (80) and (94), the asymptotic of the statistical sum can be expressed as

$$\begin{aligned} Z_n(\beta) &= \sum_{i=1}^{N} exp\{\beta A(n)X_i\} \cong \sum_{i=1}^{N} exp\left\{\beta \ln n \frac{1+n-\ln i}{\ln(1+n-\ln i)}\right\} \\ &\cong \sum_{i=1}^{N} exp\left\{\beta\left(1+n-\ln i\right)\right\} = e^{\beta n}\sum_{i=1}^{N}\frac{e^{\beta}}{i^{\beta}} \end{aligned} \tag{102}$$

The sequence of the sums, $\sum_{i=1}^{N}\frac{e^{\beta}}{i^{\beta}}$, converges to the finite limit, $\sum_{i=1}^{\infty}\frac{e^{\beta}}{i^{\beta}}$ iff $\beta > 1$. When $\beta < 1$, we use the Euler-MacLaurin series to approximate the asymptotic of the the series in terms of an integral. It gives us

$$\sum_{i=1}^{N}\frac{e^{\beta}}{i^{\beta}} \cong e^{\beta}\int_{1}^{N}\frac{1}{x^{\beta}}\,dx \cong e^{\beta}\,\frac{exp\{n(1-\beta)\}}{1-\beta} \tag{103}$$

When $\beta = 1$, we again use the Euler-MacLaurin series to approximate the asymptotic of the series in terms of an integral. It gives us

$$\sum_{i=1}^{N}\frac{e^{\beta}}{i} \cong e\int_{1}^{N}\frac{1}{x}\,dx \cong e\,\ln N \tag{104}$$

Then, the free energy is given as

$$\chi(\beta) = \lim_{n\to\infty} \frac{logZ_n(\beta)}{n} = \begin{cases} 1, & \text{if } 0 < \beta \le \beta_c = 1, \\ \beta, & \text{if } \beta > \beta_c \quad . \end{cases}$$

Also, note that $\chi(\beta_c) = 1$, $\chi'(\beta_c) = 1$ and they are continuous.

5.3.5 The Double Exponential Distribution

Let $X_i = f(Y_i) = \ln Y_i$ where Y_i s are i.i.d random variables with the standard exponential distribution (90). Note that these random variables have lighter tails than the Weibull distribution.

$$P\{X_i > a\} = P\left\{\ln Y_i > a\right\} = exp\left\{-e^a\right\} \tag{105}$$

for $a \geq 0$. We select $A(n) = \dfrac{n}{\ln n}$ in the statistical sum (79). By using (80) and (94), the asymptotic of the statistical sum can be expressed as

$$Z_n(\beta) = \sum_{i=1}^{N} exp\{\beta A(n) X_i\} \cong \sum_{i=1}^{N} exp\left\{\beta \frac{n}{\ln n} \ln(n - \ln i)\right\}$$

We express the upper and lower bounds of $\chi_n(\beta) = \dfrac{logZ_n(\beta)}{n}$ for large n in the following inequality

$$\frac{logN_1}{n} + \frac{\beta \frac{n}{\ln n} \ln(n - logN_1)}{n} < \frac{logZ_n(\beta)}{n} < \frac{logN}{n} + \frac{\beta \frac{n}{\ln n} \ln(n)}{n} \tag{106}$$

where $N_1 = [e^{\lambda n}]$ for $\lambda < 1$. Simplification of this inequality gives us

$$\lambda + \beta \frac{n + \ln(1-\lambda)}{n} < \frac{logZ_n(\beta)}{n} < 1 + \beta \tag{107}$$

for large n. Because λ is arbitrarily close to 1, we deduce that

$$\lim_{n\to\infty} \frac{logZ_n(\beta)}{n} = 1 + \beta$$

for any $\beta > 0$.

5.4 The Mixed Weibull Distribution

We repeat the experiment of selecting mixed Weibull-type random variables. In this experiment, we either choose the Weibull-type random variable with probability p or the shifted Weibull-type random variable with probability $q = 1 - p$. As a result of this experiment, the random variables in the statistical sum (93) can be expressed as

$$X = \begin{cases} Y_1, & \text{with probability p and } P(Y_1 > x) = exp\left\{-\dfrac{x^\varrho}{\varrho}\right\} \\ an^{1/\varrho} + \sigma Y_2, & \text{with probability q and } P(Y_2 > x) = exp\left\{-\dfrac{x^\varrho}{\varrho}\right\} \end{cases}$$

Also, assume that we repeat this experiment $N = [e^n]$ times. We obtain v_n Weibull and $N - v_n$ shifted Weibull random variables. Such a mixed distribution has the following interpretation: Out of N i.i.d. random variables, $\{X_1, X_2, ..., X_i, ...X_N\}$, in the statistical sum, the set of indexes is the union of v_n successes, $Y_1^j, j = 1, 2, ..., v_n$, and $N - v_n$ failures, $an^{1/\varrho} + \sigma Y_2^k$, $k = 1, 2, ..., N - v_n$. Here, v_n has the binomial distribution $B(N, p)$. Also, $A(n) = n^{1/\varrho'}$. Then, we can express the statistical sum

(93) as follows.

$$
\begin{aligned}
Z_n(\beta) &= \sum_{i=1}^{N} exp\left\{\beta A(n) X_i\right\} \\
&= \sum_{j=1}^{v^n} exp\left\{\beta A(n) Y_1^j\right\} + \sum_{k=1}^{N-v_n} exp\left\{\beta A(n)\left(an^{1/\varrho} + \sigma Y_2^k\right)\right\} \\
&= \sum_{j=1}^{v^n} exp\left\{\beta A(n) Y_1^j\right\} + exp\left\{a\beta n\right\} \sum_{k=1}^{N-v_n} exp\left\{\beta A(n)\sigma Y_2^k\right\} \\
&= Z_n^1(\beta) + Z_n^2(\beta)
\end{aligned}
$$

It means that the exponent in the sum varies depending on the result of the experiment. If the Weibull-type sample is selected in a single draw, the exponent is $\beta A(n) = \beta n^{1/\varrho'}$. If the shifted Weibull-type sample is selected in the same single draw, the exponent is $\sigma\beta A(n) = \beta n^{1/\varrho'}\sigma$. When σ is different than 1, this gives two critical points in the free energy.

For the mixed Weibull case, we have v_n successes as a result of random sampling. We assume that v_n, Y_1^j, Y_2^k are independent. In the previous section 5.3.3, we obtained the free energy for the Weibull case. We can still make use of this section's results for Y_1^j's. For Y_2^k's, we simply use the same formulation from 5.3.3 by replacing β with $\sigma\beta$. Using the independence of the random variables, it can be stated that

$$
\begin{aligned}
Z_n^1(\beta) &= \sum_{i=1}^{v_n} e^{\beta n^{1/\varrho'} X_i} \cong \sum_{i=1}^{p[e^n]} e^{\beta n^{1/\varrho'} X_i} \\
&\cong \begin{cases} exp\left\{\left(1 + \dfrac{\beta^{\varrho'}}{\varrho'}\right) n\right\}, & \text{if } \beta < \varrho^{1/\varrho'} = \beta_{critical_1} \\ exp\left\{\beta \varrho^{1/\varrho} n\right\}, & \text{if } \beta \geq \beta_{critical_1} \end{cases}
\end{aligned}
$$

In the case of shifted samples, we obtain:

$$
\begin{aligned}
Z_n^2(\beta) &= e^{a\beta n}\sum_{i=1}^{[e^n]-v_n} e^{\beta n^{1/\varrho'}\sigma X_i} \cong e^{a\beta n}\sum_{i=1}^{(1-p)[e^n]} e^{\beta\sigma n^{1/\varrho'}X_i} \\
&\cong \begin{cases} exp\left\{\left(1+a\beta+\dfrac{(\beta\sigma)^{\varrho'}}{\varrho'}\right)n\right\}, & \text{if } \beta < \dfrac{\varrho^{1/\varrho'}}{\sigma} = \beta_{critical_2} \\ exp\left\{\left(a\beta+\beta\sigma\varrho^{1/\varrho}\right)n\right\}, & \text{if } \beta \geq \beta_{critical_2} \end{cases}
\end{aligned}
$$

To be able to calculate the free energy, $\chi(\beta) = \lim_{n\to\infty}\dfrac{\ln Z_n(\beta)}{n}$, we make use of the following inequality

$$
\begin{aligned}
\max(Z_n^1(\beta), Z_n^2(\beta)) &< Z_n(\beta) < 2\max(Z_n^1(\beta), Z_n^2(\beta)) \\
\ln\max(Z_n^1(\beta), Z_n^2(\beta)) &< \ln Z_n(\beta) < \ln 2 + \ln\max(Z_n^1(\beta), Z_n^2(\beta)) \\
\lim_{n\to\infty}\frac{\ln\max(Z_n^1(\beta), Z_n^2(\beta))}{n} &< \lim_{n\to\infty}\frac{\ln Z_n(\beta)}{n} < \lim_{n\to\infty}\frac{\ln 2 + \ln\max(Z_n^1(\beta), Z_n^2(\beta))}{n}
\end{aligned}
$$

which implies that

$$
\lim_{n\to\infty}\frac{\ln Z_n(\beta)}{n} = \lim_{n\to\infty}\frac{\ln\,\max(Z_n^1(\beta), Z_n^2(\beta))}{n}
$$

For large n, when $\sigma > 1$

$$
\begin{aligned}
\chi(\beta) &= \lim_{n\to\infty}\frac{\ln Z_n(\beta)}{n} \\
&\cong \begin{cases} 1+\max\left(\dfrac{\beta^{\varrho'}}{\varrho'}, a\beta+\dfrac{(\beta\sigma)^{\varrho'}}{\varrho'}\right), & \text{if } \beta < \dfrac{\varrho^{1/\varrho'}}{\sigma} = \beta_{critical_2} \\ \max(1+\dfrac{\beta^{\varrho'}}{\varrho'}, a\beta+\beta\sigma\varrho^{1/\varrho}), & \text{if } \beta_{critical_2} \leq \beta < \varrho^{1/\varrho'} = \beta_{critical_1} \\ \beta\max(\varrho^{1/\varrho}, a+\sigma\varrho^{1/\varrho}), & \text{if } \beta_{critical_1} \leq \beta \end{cases}
\end{aligned}
$$

For large n, when $\sigma < 1$

$$
\begin{aligned}
\chi(\beta) &= \lim_{n\to\infty} \frac{\ln Z_n(\beta)}{n} \\
&\cong \begin{cases}
1 + \max\left(\dfrac{\beta^{\varrho'}}{\varrho'}\, ,\, a\beta + \dfrac{(\beta\sigma)^{\varrho'}}{\varrho'}\right), & \text{if } \beta < \beta_{critical_1} \\
\max\left(\varrho^{1/\varrho}, 1 + a\beta + \dfrac{(\beta\sigma)^{\varrho'}}{\varrho'}\right), & \text{if } \beta_{critical_1} \leq \beta < \beta_{critical_2} \\
\beta \max\left(\varrho^{1/\varrho}, a + \sigma\varrho^{1/\varrho}\right), & \text{if } \beta_{critical_2} \leq \beta
\end{cases}
\end{aligned}
$$

CHAPTER 6: CONCLUSION

We studied the limit theorems for the sum of random exponentials and their applications. The LLN, the CLT and the convergence to the stable distribution under additive and multiplicative factors have been analyzed for a new class of distribution: the double exponential distribution. This can be extended to other families of distributions. In terms of application, we reviewed the ruin probability estimation of insurance portfolios using a sample based approach. The LLN and the CLT are the main limit theorems. Another application is the REM from the statistical physics. We derived the free energy for the REM through a different methodology when the random variables in the partial sum could be expressed as an increasing function of standard exponential random variables.

Classical studies on the ruin probability estimation of insurance portfolios have been done based on the assumption that the number of claims goes to infinity without any dependence to a parameter. In the fourth chapter, our approach is more controlled by defining the number of claims as a function of the adjustment coefficient and then sending it to infinity. The sample based ruin probability estimates should give better estimates on the insurance portfolios. Because this study is done from a theoretical perspective and there is no empirical data, we do not know how precise our method would be in practice compared to the existing estimation methods. This can be addressed in an empirical paper with real life insurance data. If this approach is tested and yields consistent results with the market data, the insurance companies could use a more accurate approach based on our results. Also, we assumed that the claims have the Weibull-type distribution. This can be reformulated to different class of distributions depending on the claim distributions.

We worked with probability limits for the REM in the fifth chapter. On the other hand, our new approach was about the representation of random variables in terms of standard exponential random variables and the application to the REM. Our contribution in this subject is in terms of the simplification of the computation for the free energy. Also, this method can be applied to more general class of distributions.

REFERENCES

[1] Bovier A., Kurkova I. and Löwe M. 2002: Fluctuations of the free energy in the REM and the p-spin SK Model. Ann. Probab., **30**, pp. 605-651.

[2] Derrida B. 1981: Random Energy Model: An exactly solvable model of disordered models. Phys. Rev. B., **24**, pp. 2613-2626.

[3] Olivieri E. and Picco P. 1984: On the existence of thermodynamics for the Random Energy Model. Comm. Math. Phys., **96**, pp. 125-144.

[4] Pastur L. A. 1989: A limit theorem for sums of exponentials. Mat. Zametki, **46**, No. 3, pp. 50-57; English Translation: Math. Notes, **46**, pp. 712-716.

[5] Ben Arous G., Bogachev L. V. and Molchanov S. A. 2003: Limit theorems for random exponentials. Preprint NI 03078-IGS, Isaac Newton Institute for Math Sci, **89**, pp 1-62.

[6] Csörgö Sándor and Teugels Jef L. 1990: Empirical Laplace Transform and Approximation of Compound Distributions. Journal of Applied Probability, **27**, pp. 88-101.

[7] Eisele T. 1983: On a third-order phase transition. Comm. Math. Phys., **90**, pp. 125-159.

[8] Bengt von Bahr and Carl-Gustav Esseen 1965: Inequalities for the rth Absolute Moment of a Sum of Random Variables, $1 \leq r \leq 2$, **36**, pp. 299-303.

[9] Petrov V. V. 1978: Sums of Independent Random Variables. Berlin, Heidelberg and New York: Springer-Verlag.

[10] Gradshteyn I. S. and Ryzhik I. M. 1994: Table of Integrals, Series and Products, 5th ed. (A. Jeffrey, ed.). Boston: Academic Press.

[11] Feller William. 1971:An introduction to Probability Theory and its Applications Volume 2, 2nd ed. New York, London, Sydney and Toronto: John Wiley and Sons.

[12] Rolski T., Schmidli H., Schmidt V. and Teugels J. 1999: Stochastic Processes for Insurance and Finance. Chichester, New York, Weinheim, Brisbane, Singapore and Toronto: John Wiley and Sons.

[13] De Bruijn N.G. 1981: Asymptotic Methods in Analysis. New York: Dover Publications.

APPENDIX A: THE ASYMPTOTIC BEHAVIOR OF THE WEIBULL INTEGRAL FOR THE REM

Integral for the Weibull Random Variable

We claim that the following integral's asymptotic equivalent for large n is as follows

$$\begin{aligned}\ln I(\beta) &= \ln \int_1^N \exp\left\{\beta \varrho^{1/\varrho} n^{1/\varrho'} (n - \ln x + c)^{1/\varrho}\right\} dx \\ &= \begin{cases} n\left(1 + \dfrac{\beta \varrho'}{\varrho'}\right) + \bar{\bar{o}}(n), & \text{if } 0 < \beta < \beta_c = \varrho^{1/\varrho'}, \\ n\beta\varrho^{1/\varrho} + \bar{\bar{o}}(1), & \text{if } \beta \geq \beta_c. \end{cases}\end{aligned} \tag{108}$$

where N is defined in (80).

Proof. Let $y = \ln x - c$ and $y = nz$. These substitutions provide us with:

$$\begin{aligned}&\int_1^N \exp\left\{\beta \varrho^{1/\varrho} n^{1/\varrho'} (n - \ln x + c)^{1/\varrho}\right\} dx \\ &= ne^c \int_0^1 \exp\left\{n\left(z + \beta\varrho^{1/\varrho}(1-z)^{1/\varrho}\right)\right\} dx\end{aligned}$$

where we define $g(z) = z + \beta\varrho^{1/\varrho}(1-z)^{1/\varrho}$ and it follows that

$$g'(z) = 1 - \frac{\beta}{\varrho^{1/\varrho'}(1-z)^{1/\varrho'}}$$

Then, the conditions below are satisfied:

If $\beta < \varrho^{1/\varrho'} = \beta_c$, then $g(z)$ has a maximum at $z_l = 1 - \dfrac{\beta\varrho'}{\varrho}$.

If $\beta \geq \varrho^{1/\varrho'}$ then $g(z)$ has a maximum at $z_l = 0$, as our integration region is restricted to $(0, 1)$. Then, the asymptotic of the integral can be driven using the Laplace Method.

It is expressed as:

$$\ln I(\beta) = \begin{cases} n\left(1 + \dfrac{\beta^{\varrho'}}{\varrho'}\right) + \overline{\overline{o}}(n), & \text{if } 0 < \beta < \beta_c = \varrho^{1/\varrho'}, \\ n\beta\varrho^{1/\varrho} + \overline{\overline{o}}(n), & \text{if } \beta \geq \beta_c. \end{cases}$$

□

Another Integral with the Weibull Distribution and the LLN

Suppose that X_j's have the Weibull distribution (6). We are interested in finding the asymptotic equivalent of

$$S_{N_n}(\tau_n) = \sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n X_j} \tag{109}$$

for large τ_n, which is used in (73). First, we prove the asymptotic equivalence of integral $E[X_j e^{\tau_n X_j}]$, then the LLN. $N_n(\tau_n)$ is given in (17).

Using the substitution $x = t^{\varrho'-1}y$, we obtain

$$\begin{aligned} E[X^r e^{rtX}] &= \int_0^{+\infty} X^r e^{rtX} f_X(x)\, d(x) \\ &= t^{r\varrho'+\varrho'-r} \int_0^{+\infty} y^{\varrho-1+r} \exp\left\{ rt^{\varrho'}\left(y - \frac{y^\varrho}{r\varrho}\right)\right\} d(y) \end{aligned}$$

Also, $g(y) = y - \dfrac{y^\varrho}{r\varrho}$ has a maximum at $y = r^{1/(\varrho-1)}$ and the following conditions are satisfied:

$$\begin{aligned} g(r^{1/(\varrho-1)}) &= r^{1/(\varrho-1)} - \frac{r^{\varrho/(\varrho-1)}}{r\varrho} = r^{\varrho'-1}/\varrho' \\ g'(r^{1/(\varrho-1)}) &= 0 \\ g''(r^{1/(\varrho-1)}) &= -(\varrho-1)r^{1/[(\varrho-1)(\varrho-2)]}/r < 0 \end{aligned}$$

Then we can find the asymptotic expression using the Laplace method:

$$\log E[X^r e^{rtX}] = \frac{(rt)^{\varrho'}}{\varrho'} + \textit{smaller terms}$$

as $t \to \infty$. In the limit, we can see that $\log E[Xe^{tX}] \cong t^{\varrho'}/\varrho'$ for large t.

Theorem 15. *The Law of large numbers (LLN),*

$$\frac{S_{N_N}(\tau_n)}{E[S_N(\tau_n)]} \xrightarrow{p} 1. \tag{110}$$

Assume that X_j's in the statistical sum (109) have the Weibull distribution (6).

If $\lambda > \varrho' - 1 = \lambda_1$ (17), the LLN holds.

Proof. Set $lnE[X_j e^{\tau_n X_j}] = M(\tau_n)$ and

$$S^*_{N_n}(\tau_n) = \frac{S_{N_n}(\tau_n)}{E[S_{N_n}(\tau_n)]} = \frac{1}{N_n(\tau_n)} \sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n X_j - M(\tau_n)}$$

It is sufficient to show that $\lim_{t\to\infty} E\left|S^*_{N_n}(\tau_n) - 1\right|^r = 0$ for some $r > 1$.

$$\begin{aligned} E|S^*_{N_n}(\tau_n) - 1|^r &= E\left|\frac{\sum_{j=1}^{N_n(\tau_n)} X_j e^{\tau_n X_j - M(\tau_n)}}{N_n(\tau_n)} - 1\right|^r \\ = E\left|\frac{\sum_{j=1}^{N_n(\tau_n)} \left[X_j e^{\tau_n X_j - M(\tau_n)} - 1\right]}{N_n(\tau_n)}\right|^r &= N_n(\tau_n)^{-r} E\left|\sum_{j=1}^{N_n(\tau_n)} \left[X_j e^{\tau_n X_j - M(\tau_n)} - 1\right]\right|^r \end{aligned}$$

Using the Bahr-Esseen inequality and $(x+1)^r \le 2^{r-1}(x^r + 1)$, where $(x > 0, r \ge 1)$,

$$\begin{aligned} N_n(\tau_n)^{-r} E\left|\sum_{j=1}^{N_n(\tau_n)} \left[X_j e^{\tau_n X_j - M(\tau_n)} - 1\right]\right|^r &\le 2N_n(\tau_n)^{-r} \sum_{j=1}^{N_n(\tau_n)} E\left|X_j e^{\tau_n X_j - M(\tau_n)} - 1\right|^r \\ \le 2N_n(\tau_n)^{1-r} E\left|X_j e^{\tau_n X_j - M(\tau_n)} + 1\right|^r &\le 2N_n(\tau_n)^{1-r} 2^{r-1} E\left|X_j^r e^{r\tau_n X_j - rM(\tau_n)} + 1\right| \\ < 2^r N_n(\tau_n)^{1-r} E\left(X_j^r e^{r\tau_n X_j}\right) e^{-rM(\tau_n)} + 2^r N_n(\tau_n)^{1-r} \end{aligned} \tag{111}$$

Then

$$\liminf_{n\to\infty}\left[\frac{(r-1)\log N_n(\tau_n)}{M(\tau_n)}-\frac{E\left(X_j^r e^{r\tau_n X_j}\right)}{M(\tau_n)}+r\right]=\lambda(r-1)-\left(r^{\varrho'}-r\right)=v_\lambda(r)$$

By Lemma 1, we can choose $r>1$ such that $v_\lambda(r)>0$ when $\lambda>\lambda_1=\dfrac{\varrho'}{\varrho}=\varrho'-1$ and this implies that the right hand side converges to 0. □

This concludes that

$$\begin{aligned}S_{N_n}(\tau_n)&=\sum_{j=1}^{N_n(\tau_n)}X_j e^{\tau_n X_j}\cong N_n(\tau_n)E[X_j e^{\tau_n X_j}]\\&\cong N_n(\tau_n)\exp\left\{\frac{\tau_n^{\varrho'}}{\varrho'}\right\}=N_n(\tau_n)\exp\{H(\tau_n)\}\end{aligned}\tag{112}$$

in probability for large n and $H(\tau_n)$ is the cumulant generating function of the Weibull distribution.

APPENDIX B: APPLICATION OF THE LAPLACE METHOD TO WEIBULL AND DOUBLE EXPONENTIAL DISTRIBUTIONS

The Weibull Distribution

Suppose that X has the Weibull distribution (6). We are interested in finding the asymptotic equivalent of the cumulant generating function (12). Using the substitution $x = t^{\varrho'-1}y$, we obtain

$$\begin{aligned} E[e^{tX}] &= \int_0^{+\infty} e^{tX} f_X(x)\, d(x) \\ &= t^{\varrho'} \int_0^{+\infty} y^{\varrho-1} \exp\left\{ t^{\varrho'} \left(y - \frac{y^{\varrho}}{\varrho} \right) \right\} d(y)\,. \end{aligned}$$

Also, $g(y) = y - \frac{y^{\varrho}}{\varrho}$ has a maximum at $y = 1$ and the following conditions are satisfied:

$$\begin{aligned} g(1) &= 1 - \frac{1}{\varrho} \\ g'(1) &= 0 \\ g''(1) &= -\varrho + 1 < 0\,. \end{aligned}$$

Then, we can find the asymptotic expression using the Laplace method:

$$H(t) = \log E[e^{tX}] = \frac{t^{\varrho'}}{\varrho'} + \frac{\varrho'}{2}\log(t) + \frac{1}{2}\log(\frac{2\pi}{\varrho - 1}) + o(1)$$

as $t \to \infty$. In the limit we can see that $H_0(t) = t^{\varrho'}/\varrho'$.

The Double Exponential Distribution

1. We calculate the cumulant generating function (12) for large t, when X has the double exponential distribution (8). The density of this distribution is expressed

as

$$f_X(x) = \exp\{1 + x - e^x\}$$

for $x > 0$. Use the substitution $x = y + \ln(t+1)$ to obtain

$$E[e^{tX}] = \int_0^{+\infty} e^{tX} f_X(x)\, d(x)$$
$$= \exp\{1 + \ln(t+1) + t\ln(t+1)\} \int_{-\ln(t+1)}^{+\infty} \exp\{(t+1)(y - e^y)\}\; dy$$

Also, $g(y) := y - e^y$ has a maximum at $y = 0$ from $g^{'}(y) = 1 - e^y = 0$. Then we can apply the Laplace method to obtain:

$$H(t) \;=\; \log E[e^{tX}] \;\cong\; t\ln(t) - t + \frac{\ln t}{2} + \textit{smaller terms} \quad \textit{as } t \to \infty$$

2. We evaluate integrals of type

$$\int_{-M_1(t)}^{+\infty} \exp\{M_2(t)(y - e^y)\}\, 1_{\{y \le K\}}\, dy$$

using the Laplace transform, where $K < 0$ is the maximum point of the region of integration and $M_1(t), M_2(t) \to \infty$ as $t \to \infty$. For large t, this integral is equivalent to

$$\int_{-M_1(t)}^{K} \exp\{M_2(t)(y - e^y)\}\; dy$$

Because $g(y) = y - e^y$ has a maximum at $y = 0$, $g'(y) > 0$ for negative y and the maximum point is outside of the interval of integration, the major contribution to the integral comes from the neighborhood of the boundary point K. Then, the

Laplace method gives us

$$\int_{-M_1(t)}^{K} \exp\{M_2(t)(y-e^y)\}\, dy \cong \exp\left\{M_2(t)(K-e^K)\right\} \frac{1}{M_2(t) \mid g'(K) \mid}$$

3. We evaluate integrals of type

$$\int_0^{+\infty} \exp\{M_2(t)(y-e^y)\}\, 1_{\{y>K\}}\, dy$$

using the Laplace transform, where $K > 0$ is the maximum point of the region of integration and $M_2(t) \to \infty$ as $t \to \infty$. For large t, this integral is equivalent to

$$\int_K^{\infty} \exp\{M_2(t)(y-e^y)\}\, dy$$

Because $g(y) = y - e^y$ has a maximum at $y = 0$, $g'(y) < 0$ for positive y and the maximum point is outside of the interval of integration, the major contribution to the integral comes from the neighborhood of the boundary point K. Then, the Laplace method gives us

$$\int_K^{\infty} \exp\{M_2(t)(y-e^y)\}\, dy \cong \exp\left\{M_2(t)(K-e^K)\right\} \frac{1}{M_2(t) \mid g'(K) \mid}$$

4. We derive the asymptotic equivalent of $K = \dfrac{\ln B(t) + \ln\tau}{t} - \ln(at+1)$ for large t, by (15),

$$\begin{aligned} K &= \frac{\ln B(t) + \ln\tau}{t} - \ln(at+1) = \ln\lambda + \ln t + \frac{\ln\tau}{t} - \ln(at+1) \\ &\cong \ln(\lambda/a) + \frac{\ln\tau}{t} - \ln\left(1 + \frac{1}{at}\right). \end{aligned}$$

5. We simplify $NE[Y^a 1_{\{Y\le\tau\}}]$ using the result (35)

$$
\begin{aligned}
&NE[Y^a 1_{\{Y\le\tau\}}] \\
&= \frac{N(t)}{B^a(t)} \exp\{(at+1)\ln(at+1)\} \exp\left\{(at+1)(K-e^K)\right\} \frac{e}{(at+1) \mid g'(K) \mid}
\end{aligned}
$$

Assume that $\lambda/a < 1$. Using (15), (18), and $K = \ln(\lambda/a) + \dfrac{\ln\tau}{t} - \ln\left(1+\dfrac{1}{at}\right)$,

we obtain

$$
\begin{aligned}
NE[Y^a 1_{\{Y\le\tau\}}] &= e^{\lambda t}\frac{(at+1)^{at+1}}{(\lambda t)^{at}} \exp\left\{(at+1)(K-e^K)\right\} \frac{e}{(at+1) \mid g'(K) \mid} \\
&= aet\left(\frac{a^a e^\lambda}{\lambda^a}\right)^t \exp\left\{(at+1)(K-e^K)\right\} \frac{e}{(at+1) \mid g'(K) \mid} \\
&= aet\left(\frac{a^a e^\lambda}{\lambda^a}\right)^t \left(\frac{\lambda}{a}\right)^{at+1} \tau^{a-\lambda} e^{-1-t\lambda} \frac{e}{(at+1)|g'(K)|} \\
&= \frac{\lambda\tau^{a-\lambda}e}{|g'(K)|}
\end{aligned}
$$

where $g(y) = y - \exp\{y\}$ and $|g'(K)| \cong 1 - \lambda/a$.

6. We then simplify $NE[Y 1_{\{Y\le\tau\}}]$ using the result (41)

$$
\begin{aligned}
&NE[Y 1_{\{Y\le\tau\}}] - \frac{A(t)}{B(t)} \\
&= \frac{N(t)}{B(t)} \exp\{(t+1)\ln(t+1)\} \exp\left\{(t+1)(K-e^K)\right\} \frac{e}{(t+1) \mid g'(K) \mid}
\end{aligned}
$$

where A(t) is given in (16a) for $0 < \lambda < 1$. Using (15), (18), and

$$
K = \ln\lambda + \frac{\ln\tau}{t} - \ln\left(1+\frac{1}{t}\right)
$$

we obtain

$$
\begin{aligned}
&NE[Y1_{\{Y>\tau\}}] = e^{\lambda t}\frac{(t+1)^{t+1}}{(\lambda t)^t} \exp\left\{(t+1)(K-e^K)\right\} \frac{e}{(t+1) \mid g'(K) \mid} \\
&= et\left(\frac{e^\lambda}{\lambda}\right)^t \exp\left\{(t+1)(K-e^K)\right\} \frac{e}{(t+1) \mid g'(K) \mid} \\
&= et\left(\frac{e^\lambda}{\lambda}\right)^t \lambda^{t+1}\tau^{1-\lambda}e^{-1-t\lambda}\frac{e}{(t+1)|g'(K)|} \\
&= \frac{\lambda\tau^{1-\lambda}e}{|g'(K)|}
\end{aligned}
$$

where $g(y) = y - \exp\{y\}$ and $|g'(K)| \cong 1 - \lambda$.