# MULTI-REPRESENTATION OF LINEAR FUNCTIONS FOR PRE-COLLEGE STUDENTS IN A STEM CONTEXT

by

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A dissertation submitted to the faculty of
University of North Carolina at Charlotte
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in
Curriculum and Instruction

Charlotte

2015

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# ABSTRACT

SHAGUFTA YASIN RAJA. Multi-representation of linear functions for pre-college students in a STEM context.(Under the directions of Dr. DAVID PUGALEE)

In this case study, five pre-college students and a teacher were interviewed using an assessment instrument consisting of six problems involving linear functions. The qualitative research study model was inspired by Lesh, Post and Behr's Translation Model (1987). The theoretical framework for the study was based on cognitive and social constructivism. There were three major findings for research question 1. (How do multi-representations help in understanding of linear functions for Pre-College students?). 1. Four out of five students successfully completed tasks related to linear functions and their multi-representations (algebraic, graphic, tabular, pictorial, and contextual).2. Students were limited in their ability to move flexibly among representations of linear functions. 3. Students' faced difficulty with constructing graphs from a linear function expressed as an algebraic equation or presented as word problem. The data for research question 2 (How can students' experiences with graphs of linear functions be characterized?) showed that the students had very limited knowledge about systems of equations and their applications in real life situations. Recommendations based on the study findings include : 1. Teacher professional development and training in line with the common core standards and interdisciplinary constructs for incorporating technology in classroom ((TPACK) is a viable framework). 2. Students difficulty with vocabulary underscores the need for specific literacy supports that conceptual development through text. Such supports require teacher professional development around specific tools to promote

effective student engagement with multiple forms of text.  Future studies should include a focus on the significance of interactions involving teacher's perceptions, knowledge and teaching strategies influencing student achievement and understanding of linear functions.

# DEDICATION

I dedicate my dissertation work to my family and many friends. A special feeling of gratitude to my late parents, father Mohammad Siddique and mother Muzamil Khatoon, whose words of encouragement and push for tenacity ring in my ears. My children Junaid, Amer and Samreen are very special, they have never left my side and endured this long process with me, always offering support and love. I will always appreciate all they have done, for the many hours of proofreading, and for helping me to master the technology skills. I also dedicate this dissertation to Sadaf, Abdul, Atif, Sheena, my husband and many friends who have supported me throughout the process. I dedicate this work and give special love to my grandchildren, Naima, Dean and Isa for being a motivation for me throughout the entire doctorate program.

# ACKNOWLEDGEMENTS

This dissertation would not have been possible without the support of many people in my life. Many thanks to my adviser, David Pugalee, who read my numerous revisions and helped make some sense of the confusion. Also thanks to my committee members, Victor Cifarelli, Anthony Fernandez and Yogandra Kakad for the tremendous assistance, guidance, and moral support. Special thanks to my family, friends and editors for helping with my dissertation writing process. Thanks to the University of North Carolina at Charlotte Graduate School for helping me with the dissertation completion process.

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| | |
|---|---|
| STEM | science technology engineering and math |
| SAT | scholastic achievement test |
| ACT | American college test |
| NEAP | national evaluation of academic progress |
| CCSCOS | common core standard course of study |
| Atlas Ti | name of qualitative software |
| NCTM | national council of teachers in mathematics |
| GPS | global positioning system |
| ELL | English language learner |
| EOC | end of course |
| NCDPI | North Carolina department of public instructions |
| MSEN | math science education network |
| UNC | University North Carolina |

# CHAPTER 1: INTRODUCTION AND PURPOSE OF STUDY

## 1.1 Background Information

The multi-representations of linear functions, such as graphs and symbols, carry great significance in various contexts, including medicine, business, and engineering. The concept of linear functions, which is typically taught in middle school algebra classrooms, is essential for students entering high schools nationwide. After being added to many high schools' graduation requirements, particularly for all college-bound students, algebra has gained momentum in its importance as a mathematics course (McGraner, 2011). Algebra scores are also now used as indicators of success for students pursuing higher education in Science, Technology, Engineering, and Mathematics (STEM)-related fields (Fey, 1989). Likewise, the Scholastic Aptitude Test (SAT), a highly reliable standardized measurement of college readiness, is used in the admissions process at nearly all four-year undergraduate colleges and universities in the United States (College Board, 2011). The SAT is recognized as an objective measure of mathematics and English skills. It is considered to be uniform across all test takers, a fair and valid predictor of college success for students from varied backgrounds. As reported by the College Board, most foreign universities now consider SAT scores of US students; and a combination of the SAT score and high school performance is considered a better predictor of college success than

SAT scores or high school grades alone (Shaw, Kobrin, Patterson, & Matte, 2012).

Unfortunately, for the past three years, averaged SAT scores showed no improvement in the area of mathematics. Jaschik (2013), reporting for "Higher Ed," provided a comparison of math scores over a period of four years (2010-2013). SAT scores were down for the year 2011 by one point in mathematics (from 515 in 2010 to 514 in 2011). In 2012, the overall scores for the SAT dropped more than the previous year, but the mathematics scores stayed the same (514). The 2013 scores remained relatively stagnant at 514, signifying no growth or improvement in mathematics. Despite the greater number of students taking the SAT, the overall scores have shown little or no improvement each year since 2011.

One possible explanation for the lackluster performance is that the large sample size of average-performing students taking the SAT has kept the scores flat. David Coleman, the College Board president (2013), considered these stagnant math scores to be "no news" and issued a call of action for the College Board to motivate more students to take the SAT. An increase in the sample size for students taking the SAT will provide more reliable data from SAT scores used in college admissions. In a recent executive summary report in the New York Times (2013), 43% or less of students who graduated from high school showed readiness for the rigor of college work; this number has remained virtually unchanged for the past five years. The lack of ability to fully understand algebraic problems in linear functions is one of the reasons for incorrect responses on SAT math sections (Welder, 2006). Performance on algebra questions is highly dependent on how the students perceive or understand the graphic representation of functions (Yerushalmy, 2000).

Warren and Pierce (2004) elucidate that although comprehension of linear functions and value of graphs has presented serious challenges in teaching algebra content in high school, algebra curriculums have not changed much in the past century. The difficulty in recognizing linear function vocabulary is considered another major issue (Brenner, 1997). Hart (1981) found that the root cause of the difficulty in recognizing linear functions was the lack of familiarity with vocabulary in math. Brenner, referring to a survey report from Hart (1981) research, noted that 3,000 middle school students found translating a functional relationship from data points to algebraic or symbolic representation are the most difficult concept in linear functions.(Brenner, 1997).

In courses such as algebra and geometry, connections primarily focus on conceptual learning of core algebraic topics using multiple strategies, such as slope of line and rate of change (Dietiker, 2005). A crucial part of laying the foundation for the conceptual understanding of linear functions is based on justifying, generalizing, making connections, reverse thinking, applying, and extending the focus for core objectives (Dietiker, et al., 2005). An example of such strategic learning is that the students generate models or patterns based on previous understanding/perceptions and justify their understanding by drawing a generalized sequence. Making visual representations (e.g., tables, charts, and graphs) helps students synthesize the material that is being presented in order to gain more focus and applied understanding (Monk, 2007).

Linear functions are central topics in mathematics but are relatively understudied in research and development of mathematics education (Hercovics,Wagner, Kierman(Eds), & Kiernan, 1989). Since the 1920s, several researchers have attempted to establish that "without functional thinking there can be no real understanding or appreciation of mathematics," (Berslich, 1928). Berslich also notes work from Hendrick (1922) about functional relations, pointing out that functional relationships occur as connections in real life. Despite the importance of the focus on cognitive understanding of functions for elementary and secondary school levels, most of the research studies cited in NCTM's (1989) Principles and Standards for School Mathematics involved a comparison of teaching and learning of linear functions, instead of focusing on their understanding of concept development. Kaput (1989) also indicated that there are problems which are actually related to cognitive understanding of functions at the elementary and secondary school levels. Kieran & Chalouh (1993) postulated that the ways in which students explain algebraic concepts to themselves are actually the techniques for understanding multiple mathematical representations. These techniques are rarely taught in math classes; therefore, students develop their own ways to make learning algebra meaningful for them.

The End Of Course (EOC) results for the North Carolina Department of Public Instruction (NCDPI) showed that most pre-college students had difficulty solving linear equations involving multiple steps and translating information from symbols to tables or graphs.  This lack of flexibility between representations is expected because of the lack of students' ability to perceive problems from different perspectives and representations. Students' engagement in the classroom and their realization of the importance of linear functions in one or two variables are critically valued objectives by the Common Core State Standards for Mathematics (CCSSO, 2010).

A quarter of century ago, Booth (1989) pointed out that students tend to view algebra as little more than a set of arbitrary manipulative techniques that seem to have little, if any, purpose to them. Booth added that the main purpose of studying algebra for high school students was to learn how to represent general mathematical relationships and procedures while using multiple representations. A wide range of problems can be solved and new logical/arithmetic relationships can be developed based on students' prior knowledge and understanding of different representations. Bednarz (1996) emphasized that the commonly used algebra curriculum heavily focused on simplification and manipulation, rather than the general ideas useful to create the basis of algebra (graphs and their interpretations).

The ultimate challenge faced by pre-college students involves understanding the graphic representation of linear functions in context or making inferences of results based on the data. The increase in technology has broadened the scope of learning for 21$^{st}$-century students (Young, 1993) In addition to the popularity of technology and media resources in educational literacy; children construct their knowledge based on their prior experiences and make inferences in relation to their own perspectives. Thus, in generating a new "register or schema", children's' minds undergo a transformation and then convert the register or schema back to the representation without changing the originality of the object (Williams, 1993). NCTM (2000) considers the representation of linear functions to be an important objective that should be addressed in both middle and high school mathematics. The multi-representations of linear functions such as graphs and symbols are also very important in various contexts including medicine, business, and engineering. For example, the representations of rates of population changes, economic data, business growth, and community resources are often based on graphs and tables. As such, it is clear that students' engagement with functions in the mathematics classroom and their comprehension of the importance of linear functions in society is critically valued. This current study will take into consideration that understanding concepts is a cognitive process that triggers the internal representational system of students, and that multi-representations both enable and limit the learning process (Smith, 2007).

## 1.2 Statement of Problem

The guiding question is: How do multi-representations help pre-college students understand linear functions? Cognitive ability essential for understanding the meaning of word problem, is based on several factors, such as student perspectives or beliefs, previously learned concepts, background information. Understanding a word problem is very crucial for the students. If the information is not fully translated from the narrative form, the misunderstanding will lead to incorrect problem-solving strategies. Related research on cognitive strategic learning is an emerging field for student learning. Mathematical problem-solving techniques and instructional methods help students translate information from words to symbolic representations necessary for successfully solving problems. There has been less attention paid in research as to how cognitive understanding of learning concepts, especially linear functions, can be improved for pre-college students.

## 1.3 Purpose of the Study

Despite the significance placed on algebra in students' education, Chazan and Yerushalmy (2003) reported that US students performed poorly on the National Assessment of Educational Progress (NAEP). Pinchbeck (1991) further highlighted this lackluster performance by noting that 53.8% of a group of freshman college students' responses on remedial intermediate algebra exams were incorrect. Pinchbeck categorizes an alarming 46.2% of these incorrect responses were due to errors caused by lack of prerequisite knowledge. If one reason for low scores is a lack of preparation, it is essential to identify content that is required for algebra. NCTM Standards (1989, 1991, and 2000) emphasized the understanding of linear functions as a core objective in the algebra curriculum.

The importance of pre-college students becoming well versed in multi-representations of linear functions can be understood by considering four definite outcomes towards success in colleges and universities (Lesh & Doer, 2003). First, students with an advanced understanding of linear functions gain an advantage for placement in high-demand areas of STEM education. Second, being competitive, they can easily connect to educational and global marketplaces. Third, people who are well informed and able to navigate through business data with an understanding of graphic representation minimize the risks of being victimized by credit schemes or other scandals aimed at consumers. Fourth, multiple representations ensure that the pedagogy and curriculum will provide an opportunity to be involved, encouraged, and challenged in diverse educational plans. In the past, representations of linear functions have been limited to symbols and equations. Algebraic-based teaching practices incorrectly assumed that students who understand and are successful in problem solving symbolically would be able to understand connections among algebraic solutions and the graphical or tabular representations of linear function problems. According to Kaput (1989), working with graphs engages a person's gestalt, producing the ability to characterize the functional quantitative relation to visual presentations or graphs. Visuals for linear functions are widely used in almost all math, science, and related fields, such as medicine and engineering. In addition to being major components in standard algebra courses, solving linear equations and inequalities and sketching and manipulating graphs, properties of line graphs, and linear systems of equations with one or two variables are major areas for exploration and learning (CCSSO, 2010). The purpose of the current study is to describe the impact of multi-representations on pre-college students' understanding of linear functions and how they characterize linear functions from a STEM perspective.

## 1.4 Research Questions

Using multi-representations such as graphs, charts/tables, and symbolic representations in the context of word problems will enhance pre-college students' understanding of problems involving linear functions. The flexibility of translating and transforming between representations will help the students make inferences of results based on the data. In other words, students will be able to identify the functional relationships as encoded in word problems and be able to do the reverse; that is, to decode in words the functional relationships in the representations.

NCTM (2000) focused on conceptual understanding rather than procedural knowledge or rule-driven computations involving teacher-facilitated group work and discussion of students' conceptual learning activities. *Principles and Standards for School Mathematics* (NCTM, 2000) illustrated the use of graphs, equations, and tables as tools used to investigate the elimination of medicine from the body in a three-part problem. For example, solving an algebra problem is the same as following a step-by-step procedure to administer medication in an athlete's body. The purpose of such activities is to gain a deeper understanding of linear functions through multi-representation of common concepts. In another illustration, the representations for "rates of change" in populations, economy, business growth, and community resources were shown in graphs and tables. In the light of the above examples, it is very important for students to realize the value of and the need to understand linear functions in real-world applications.

The importance of multi-representations of linear functions as an integral component of algebraic understanding has emerged over a period of time. Graphs and other representations have always been used as mathematical activities in the curriculum; however, multi-representations of mathematical functions have seldom been an emphasized concept. Curriculum developers and publishers have described multi-representations as supporting new views for comparing teaching and learning methods with definite outcomes. Minimal attention has been given to the fact that representation of functions involves particular cognitive processes. The meanings extracted from word problems in linear functions are required for constructing representations such as graphs and tables. The lack of expertise in creating a representation for a given problem causes difficulty in making a transition from arithmetic to algebra. An example is the difficulty that beginning algebra students have in connecting graphical and tabular forms of representations to algebraic forms of representation (McCoy, 1994). The difficulty in understanding linear functions leads to misconceptions that impede students' progress in understanding subsequent algebra concepts (Booth, 1989).

Leinhardt (1990), described students' difficulty in understanding algebra concepts because the content is often delivered to them in the form of abstract algebraic terms instead of as concept development activities. Graphs, tables, and equations all can be used to display relationships and functional behavior. In the 1980s, research shifted to developing skills that promoted student understanding of symbol interpretation and math vocabulary, which was the focus of algebra reform movements such as "Algebra for All" (Kieran, 1992). NCTM (2000) standards consider representations of linear functions to be an important objective that should be addressed in both middle and high school mathematics. First, it requires all students in grades 6–12 to generalize patterns, understand the meaning of functions, and use multiple representations for linear functions. Second, students must also be able to understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions.

It should be noted that much research has discussed the difficulties that students encounter while solving word problems (Booth, 1981; Chaiklin, 1989; Clement, 1982; Kieran, 1992; Stacey & Macgregor, 2000). The research also shows that different students demonstrate a distinct level of understanding of word problems with contextual or symbolic representations. Markowitz et al. (1988) commented that students with relatively lower abilities find it easier to handle situations involving functions that are given within a descriptive context/story than those that are only presented with mathematical symbols. Some difficulties are likely due to unfamiliarity with related math vocabulary used for representing functional parameters such as “domain” and “range.” This lack of familiarity may be due to the cultural or sociological background of the students. Welder (2006) added that students from the ninth and tenth grades also struggled with word problems related to constant functions and disconnected graphical representations for functions (e.g., step functions). Moreover, Markowitz et al. (1988) concluded that students have an easier time handling functions that are given in graphical form than those in algebraic format. Welder (2006) also concluded that the development of graphic ability is essential before learning the trends or characteristic behaviors of linear functions. Boulton-Lewis (1999) pointed out a definite procedure that is sometimes followed for teaching linear function in class is based on the assumption that all students start at the same point and follow the same strategic pattern of learning functions. “The child is often expected to start with symbols alone or quantities expressed in word problems, to map the symbols into concrete representations, to perform the operation with the concrete materials, and to map the resulting quantity back into symbols (written or verbal)” (Boulton-Lewis, 1999, p. 5).

It is important to note that if this statement was true, then students working with all representations of linear functions would be able to move flexibly among the representations and feel more comfortable in solving problems. The transformation of symbols into math vocabulary is a major skill for problem-solving, particularly in the high school math curriculum. Recently, Lagrange and his research collaborators indicated in their study that functional dependencies can also be considered as a type of representation that can be included in the category of non-symbolic set of multi-representations (Lagrange & Minh, 2009; this includes pictorial, graphic and verbal representations. Research shows that the process of solving a word problem consists of multistage cognitive phases such as constructing a mental picture and using mathematical representations in words, graphs, tables, and equations (Lester, 1994). The learner uses the mental picture to determine the strategy for finding a solution and, solving a mathematical problem cognitively, thus understanding the meaning of the solution (Brenner, 1997).

Mayer (1989) remarked that the understanding of problems is a cognitive process. Cognitive strategy instruction is an emerging topic in a wide variety of subject domains. It pertains to teaching according to students' background knowledge and their prior experience of translating the word problems into other modes of representations, such as diagrams, pictures, graphs, concrete objects, and symbols. Student engagement in class and their realization of the importance of linear functions in society is critical. Among the challenges faced by Pre-College students' understanding graphic representations of linear functions in context and making inferences of results based on the data. It implies that students will be able to identify the functional relationships as encoded in word problems and decode other representations of functional relationships into words. As described previously in this section, Lesh and Doer (1987) showed the importance of pre-college students becoming well versed in multi-representations of linear functions. This can be supported by four definite outcomes towards college success: placement in high-demand STEM areas, competitiveness in the global marketplace, awareness of graphic misrepresentation of business data, and application of pedagogy and curriculum to become involved, encouraged, and challenged in diverse educational plans.

## 1.5 Rationale

Most research findings support the use of multi-representation for enhancement of student understanding, but some studies show that using multi-representations for linear function problems adds difficulty and confusion for students. A National Research Council (2012) publication explained that the use of scientific representations such as pictures, graphs, and models may facilitate students' problem-solving skills in science and engineering students' conceptual understanding in science and engineering. Some other research studies claimed that the use of multi-representation may be useful for students in STEM disciplines, but that students had the most difficulty in moving among representations or transferring their problem-solving skills across the representations (Nguyen & Sanjay, 2010). Linear functions are the most important concept in science, business, and engineering disciplines; therefore, it is essential for students pursuing STEM careers to develop a thorough understanding of transforming representations. The current study will focus on the issue of the use of multi-representation for solving problems in linear functions in a STEM perspective. The two research questions for the study were carefully selected for the qualitative study with an open-ended interview protocol; that is, research questions were asked with an open mind as to what might be found. This approach fitted with that of grounded theory, and the solution to the problem is strictly based on the responses from the participants.

The two research questions explored in this study were:

1. How do multi-representations help pre-college students understand linear functions?
2. How can students' experiences with graphs of linear functions be characterized?

The study employed a "phenomenological case-study research design" to describe the nature of students' understanding in making sense of multi-representations of linear functions used in engineering, physics, and other STEM contexts – particularly problems related to graphic representations such as rate of change and the corresponding equations of lines. The process of extracting and translating meaning from a word problem is a cognitive process, and the use of multi-representations can be helpful in promoting students' understanding of linear functions. Special attention was given to students' learning that encompasses less analytical meaning and more equal-level visuals as well as nonverbal expressions of contextual understanding of linear functions and concepts. This included the students' perception of the slope of a line as a linear functional representation and the characteristic behavior of linear functions representing various physical or engineering processes.

## 1.6 Summary

The first chapter of the dissertation describes how students' understanding of linear functions has been relatively understudied in mathematics education research. Multi-representation of linear functions can enhance the cognitive ability of pre-college students to understand linear functions, which are particularly relevant in STEM contexts. The difficulty in understanding linear functions leads to misconceptions that impede student progress in subsequent algebra concepts. The understanding of linear functions through multi-representations has the potential to promote several definite outcomes towards success in students, such as student placement, competitive education to global learning standards and pedagogy, and purposefulness of algebra curriculum for learning from the student's own perspective.

The remaining chapters are organized as follows. A literature review in Chapter 2 describes what is currently known about pre-college students' understanding and usage of multiple representations of linear functions related to STEM fields. This chapter also explores how students perceive representations of linear functions and related behavior changes within various representations. Following the literature review, the methodology used in the research is presented in Chapter 3. In this regard, the use of a case study for five students is presented. Qualitative data collection from observations, artifacts, one-on-one interviews, and audio/video recordings within the case study design provide richer and more relevant data. Results and data analysis are presented in Chapter 4. Finally, Chapter 5 provides a summary and discussion of the salient results, concluding remarks, and implications and directions for future study.

# CHAPTER 2: LITERATURE REVIEW

This research study is based on the guiding question: How do multi-representations help in the understanding of linear functions by pre-college students? The students from the 2014 Mathematics and Science Education Network (MSEN) Pre-College Algebra program participated in the study. The MSEN Pre-College program offers enrichment activities and hands-on experiences in Science Technology, Engineering and Mathematical-related disciplines. This chapter is a review of relevant literature in context of multi-representations (such as graphs, tables or symbols) and linear functions in high school algebra courses. The review is followed by the theoretical framework for the current research.

## 2.1Representing Linear Functions

Elementary and middle schools are often focused on developing concepts related to patterns and algebraic notations, whereas high school math courses are more focused on developing an understanding of functions and their shapes. For example, two high school math courses, Algebra 1 and Geometry, are often required for high school graduation. Algebra 1 and Geometry are both essentially based on the concept of linear functions and their behaviors regarding independent variables. Jacob (1995) describes this connection by saying, “The fixed ratio of length of sides of similar triangles give linear functions” and “the use of similar triangles, and the linear functions they give, important in many types of distance estimations" (p.40). Mathematically, the definition of a function may look simple, but a visual, conceptual picture for representation can become a challenge (Baker &Tall, 1992). Algebra 1 deals closely with linear functions and their graphical representations. Generally, algebra students are introduced to three concepts related to functions: simple notations of equations, set diagrams (elements within the same set have similar properties), and the elements in the domain map to the elements in the range. However, representations such as tables, ordered pairs, graphs, and formulas are not quite as commonly understood by learners as representations of linear functions (Akkoe & Tall, 2001). Testing of prerequisite knowledge for learning algebra indicates that the basic terminology and concept of linear functions are often misunderstood or not learned correctly by algebra students (Welder, 2006 ). Many students cannot recognize linear functions because functions are often discussed in the context of quadratic functions and other functional relationships such as inverse function. Very little emphasis is paid to linear functions and their graphical or other representations. Research also shows that students do not identify function as a one-to-one correspondence between variables (Clement, 2001).

The terms "linear functions" and "correspondence" are very closely related. A function is a one-to-one correspondence between the independent (x) variable and the dependent (y) variable, whereas a function is defined as a mapping of the independent variable (x) with the dependent variable (y). For example, the line graph shown below in Figure 2.1, represented by the algebraic equation y = mx, is a one-to-one correspondence and a linear function describing y to be a function of x, the independent variable.

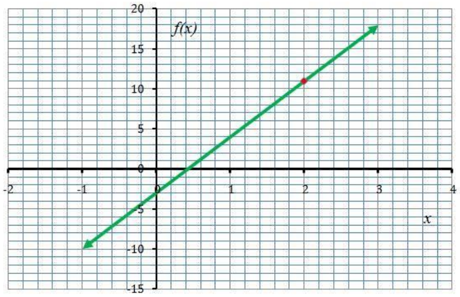



Figure 2.1: One-on-One Correspondence and Linear Function Representation

Linearity is a key concept in mathematics and science education from early childhood education to university level education (Rouche, 1989). Bardini, Pierce, and Stacey (2004) elucidate in their study that experienced mathematics teachers find that students are “switching off” at the point of developing concepts of linear functions in algebra. Schwartz (1991) explains that there are several representations for functions, but the most well-known are symbols and graphs. According to the cognitive characteristics related to symbolic and graphical representations, “they both complement each other to a great extent and offer access to insights into the other representations” of linear function (Schwartz, 2003). Schwartz also described three premises for a linear function. First, the variable serves as the root for expressing a relationship as a function. Second, a function can be expressed in multi-representations such as symbolic, graphical or even tabular forms. Third, an increase in technology improves the flexibility of students to choose the representations of their choice for problem-solving in linear functions.

A major part of instructional time for teaching linear function in most high school algebra courses is spent on transforming and comparing linear functions without recognizing the nature of activities such as simplifying, factoring, expanding, and identifying like terms. Common Core Standards for 8th grade: Mathematics Unpacked Content 2010 elaborates the above standard as, “Students routinely seek patterns or structures to model and solve problems” (p.3). According to the Common Core Standards for 8th grade math, students apply algebraic properties (such as commutative, associative) to generate equivalent expressions for problem-solving. Students examine patterns in tables and graphs to generate equations and describe the relevance. In doing so the students are expected to understand the effects of transformations and establish connections in terms of congruence and similarity (Common Core Standards, 2010). Such activities help students to flexibly move from algebraic/symbolic functional representations and re-express them into equivalent representations such as graphing the linear function. Schwartz (2003) presented the following transformations and comparisons used in finding solutions for linear equalities and inequalities representing symbolic representations (factors, expansions).

- Sketching, graphical: representing graphical representations.
- Modifying symbolic representations with varying coefficients & exponents)
- Modifying graphical representations (translation, dilation, reflection formats)
- Interpretation and use of binary combinations in mathematics (order of operations and representations such as graphical & symbolic)
- Rate of change (graphical and symbolic) accumulation (graphical and symbolic notation).

### 2.1.1 Summary

Both Algebra 1 and Geometry are among the list of pre-requisite math courses for high school graduation in North Carolina. Algebra 1 deals closely with linear functions and their graphic representations. Research shows that the mathematical definition of “functions” seems simple, but in reality, the conceptual meaning is quite challenging. Very little emphasis is paid to linear functions and their graphs or other representations. Many students cannot recognize linear functions because the conceptual meaning of a “function” often discussed in the context of quadratic, exponential, and other functions. Linear functions are one-to-one correspondence closely related and easily described by the algebraic equation $y = mx + c$, where y is a function of independent variable x. A major part of instructional time for teaching linear function in most high school algebra courses is spent on transforming and comparing linear functions without recognizing the nature of activities such as simplifying, factoring, expanding, and identifying like terms. According to the observations of an MSEN Pre-College mathematics teacher, many students give up at the point of developing concepts of linear functions because developing a relationship between independent and dependent variable is challenging for them.

## 2.2 Multiple Representations of Functions

A significant body of research has been published on the use of multi-representations (graphical, tabular, and algebraic) to develop a better understanding of concepts of linear functions. NCTM (2000) standards emphasize the idea that “representations should be treated as essential elements in supporting students’ understanding of mathematical concepts and relationships; different representations of functions support different ways of thinking and manipulating mathematical objects” (NCTM, 2000, p. 361). This section of research on multi-representations of functions focuses on three major aspects: first, the need for using multi-representations to solve problems in linear functions; second, aspects dealing with the related learning theories and developments of understanding the concepts of linear functions; and finally, the connections between representations and their interpretations. Although 21st-century students find multi-representation and multimedia resources easily available to them, students find it difficult to effectively use them for problem solving in linear functions (Anisworth, 1999). Other research found that some students did not use certain representations because they did not recognize them as viable choices (Hercovics, 1994).

Need for using multi-representations to solve problems in linear functions:

In mathematics, the term "linear function" is defined as a polynomial function of degree zero or one. A linear function is characterized by a constant rate of change for dependent and independent variable quantities and by a graphical representation of a straight line. Another important characteristic is that there is a one-to-one correspondence of variable. It must also be noted that all proportional functions are linear, but not all linear functions are proportional. For example, $y = mx$ is a linear equation that is also a representation of proportional function; but $y = mx + b$ is a linear function that is not proportional, due to the additional quantity b, the y-intercept. It is important for students to have a clear understanding of linear functions before entering higher-level math courses. Kaput (1992) pointed out that the use of multiple representations of linear functions could be helpful in presenting clear and better pictures of concepts for making connections with complex processes.

Crowley and Tall (1999) compared the work of a student who struggled with basic concepts in algebra to another student who was achieving at the same level but had been more successful in higher-level algebra courses. The first student, who used only used algebraic representations, was less successful than the student who employed more than one representation for problem solving. Crowley and Tall postulated that the results differed because students with “diffuse cognitive structures” are less successful than others in college algebra courses (Crowley & Tall, 1999). They described “diffuse cognitive structures” as the physical brain elements characterized by having difficulty in transitioning concepts from one type of representation to the other (Thompson, 1994). There could be several ways to solve a given problem involving linear functions. According to Breslich (1928), Multi-representations call for making a connection between learners’ prior understanding and inspiration for applying a certain method of problem solving. Understanding multi-representational tools allows students to decide which representation is most relevant to them for solving problems involving linear functions and to justify the reason that the selected representation was chosen over another. Dufour-Janvier, Bednarz, and Belanger (1987) commented that expectations about a student’s conceptual learning of linear functions assume that the learner has “grasped” the given representations and that he or she knows the possibilities, the limits, and the effectiveness of each of the representations at hand. In her research, Dugdale (1993) suggested that if the students have learned different representations and methods for problem solving, their ability to deal with new situations would be improved to a great extent. In order for students to use representations effectively, one must examine which representations are most interesting and useful from the point of view of the students. Keller (1998) and McGowan (1999) found that one of the key factors in student success for developmental mathematics is developing skills to move flexibly across multiple representations of functions. Students should not be limited to the use of graphs, tables, symbols, and pictures/drawings but also include emerging technologies such as computer simulations and e-technology.

Yerushalmy and Schwartz (1993) argued that deeper understanding is a direct outcome of multi-representation of functions. Keller and Hirsch (1998) also found that the use of multiple representations in mathematics led to better understanding of mathematical concepts for students with differing backgrounds by addressing the details of complex problems by using small steps. Keller (1998) stated that “multiple representations promote the cognitive linking of representations within themselves.” Research by Hiebert and Carpenter (1992) further supports the idea that flexible and deeper understanding of concepts is easily promoted by using multi-representations.

Despite strong support for students’ use of multiple representations in learning mathematical and algebraic concepts, there have been legitimate concerns about using too many representations since it may cause more confusion than understanding (Dufour-Janvier, 1987). Algebra learners differ in their skills and abilities to face challenges in understanding linear relationships. Students must be challenged according to their academic levels; the same teaching methods and resources should not be used for all students. The representations and resources for motivational challenges must be balanced to provide a positive learning outcome for using multi-representations. Rose (2012) feels that the use of multi-representations sometimes does not provide a detailed picture of a physical situation. For example, the multi-representations of linear functions used in Global Positioning Systems (GPS) can sometimes become confusing to comprehend (Jayal, 2000). Each GPS satellite consists of a clock and a radio transmitter; a receiver on Earth records the transit time of the signal from the satellite. For GPS operation, three such signals are retrieved from three separate satellites. The GPS operation is completed by the triangulation process as shown below in Figure 2.3.

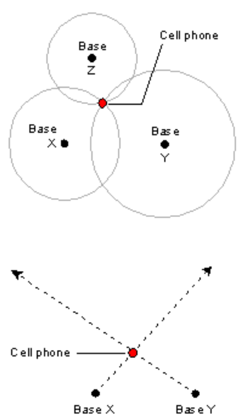



Figure 2.3: GPS Triangulation Process: A modification of figure; “Triangulating a mobile device using Timing Advance” Source: Preliminary experimental results of a GSM. Spirito, M. A., Mattioli, A. G. (1999).

“Triangulation is a process by which the location of a radio transmitter can be determined by measuring either the radial distance, or the direction, of the received signal from two or three different points. Triangulation is also used in cellular communications to pinpoint the geographic position of a user.” (1999, Spirito. A., Mattioli, A. G., p. 2074). Figure 2.3 is an illustration of recording the time delays from transmitters to the receiver. A radio signal traveling at the speed of light is used to determine the distances.

There are three points of intersection for three or more satellites. Observed time delays recorded by the receiver on Earth from three or more global positioning satellites are used to decode the distances between objects. The aspect of time data used to estimate linear-radial distance and the approximation used to overcome the complication of the curved path due to the shape of the Earth is confusing for readers. The radial distance/paths are approximated to be linear for using the matrix equation representing the linear function. The linearized observation data is expressed as $b = Ax + v$. The equation is dimensionally correct where “b” represents a residual observation matrix with dimensions dx1, “d” is the number of linearly independent path lengths, the design matrix A has dimensions of $d \times p$, and the x matrix has dimension of $p \times 1$. The observation errors are stored in matrix v which also has the dimensions of $d \times 1$ (Blewitt, 2001). GPS using the concept of linearity is quite advanced for high school students. Due to easy availability of GPS technology on their phones and other gadgets, it never occurs to them that the math behind this useful invention is system of linear functions and their representations.

The conceptual development of linear functions is dependent on students' understanding of the relationship between patterns, algebraic notations, and graphical representations. Despite the fact that both symbolic and graphical representations complement each other, many students cannot recognize linear functions. Very little emphasis is placed upon the transformations among graphic, symbolic and tabular representations of linear functions. Algebra students solve problems in linear functions using predetermined steps without recognizing the nature and relationships among independent and dependent variables.

### 2.2.1 Summary

The results of a longitudinal research study by Crowley and Tall (2000) showed that students using multi-representations for solving problems in linear functions were more successful than students using only algebraic representations in college mathematics courses. Thompson (1994) indicated that the cause of inability to use multi-representations may be the lack of diffused cognitive structures, a physical brain element that helps in transmission of concepts from one type of representation to the other. Understanding multi-representations enables a student to make a decision to select a representation and justify why it was selected for a particular situation/problem. Keller (1998) and McGowan (1991) found that it is the development of cognitive skills that help in flexibly moving among representations. Students should not be limited to the use of symbolic, algebraic, and graphical representations but they must be encouraged to learn and use modern technology.

## 2.3 Learning theories and development and concepts of linear functions:

A major contribution in research based on the potential use and benefits of multi-representations of linear functions comes from Dufour-Janvier, Bednarz, and Belanger (1987), who suggested that multi-representations were inherently included in mathematical concepts and make mathematics interesting and easier to understand. Kaput (1992) also discussed that the use of multiple representations or notations to present a clearer picture of a concept to algebra students has been overlooked. Leinhardt, (1990) showed that students cannot construct the abstract concept of function without knowing the function in action, and the examples of functions cannot be studied unless a prototype representation is constructed. Thorpe (1989) said that in all meetings for algebra standards, functions are the main topic of discussion for algebra instruction. In the same context, Dubinsky and Harel (1992) and Coney and Wilson (1993) also agreed with Thorpe that functions are the centerpiece of mathematics curricula.

Eisenberg and Leake (1994) expressed concerns about teachers' reluctance to use visual methods to teach linear functions. Students not familiar with visual representations for problem solving showed resistance and unwillingness to use graphical representations for problem solving. Smith (2000) commented that the most important goal of algebra was that the students understand the relationships among tables, graphs, and symbols and determine the advantages or disadvantages of each type of representation for its particular use. Moschkowich. et al. (1995) affirmed that using multi-representations to teach linear functions enhances students' understanding of functions. However, the dilemma for curriculum developers is that different representations are presented and interpreted in different ways. For example, picture drawings are made to represent perceptions and ideas, whereas graphs and formulas represent clusters and relationships. The predefined curriculum for grade level mathematics limits the flexibility of choosing from multi-representations for teaching and learning transcendental concepts, such as functions. The use of multiple representations helps the learner to work with the complexity of the real world problems and at the same time avoid oversimplification of the concept of linear functions (Jonassen, 1994). Mathematical knowledge in the form of internally-represented information develops understanding if the representations are achieved and processed as a structured and cohesive network (Hiebert, 1992). In traditional classrooms, more emphasis is paid to the manipulation of mathematical symbols and skills than on multi-representations (Brenner, 1997). High school mathematics instructional resources utilize tables and graphs for displaying information with little manipulation of representations. In contrast, equations are generally acted upon and manipulated (Kaput, 1989). The Lesh, Behr, and Post (1987) Translational Model for representations of linear functions between student and teacher, and student and curriculum, explains that as a result of manipulations, students begin developing a representations of linear functions as separate entities without rich relationships. Flexibility and representations in and of themselves are important; however, the translations within and between representations are critical for mathematical understanding.

The Principles and Standards for School Mathematics (NCTM, 2000) stated: "Representations should be treated as essential elements in supporting students' approaches, arguments, and understandings to one's self and to others; in recognizing connections among related mathematical concepts; and in applying mathematics to realistic problem situations through modeling." (p. 67). Lesh, Post, and Behr's (1983) model addressed the topic of conceptual and representational instability linked to the transitions in mathematical learning and problem solving. Later in 1987, the Lesh, Post, and Behr's model was refined as shown in Figure 2.1. This model for student conceptual learning shows that the learning process is not a single one-way mapping, but it consists of interconnecting paths such as connection between student and teacher, student and student and student and curriculum. The concrete and realistic word problems from textbooks or other media seldom work for students to solve mathematical problems. Instead, the student's brain works on a series of several representational systems parallel to the different parts of the problem in a given situation. This inherent multitasking capability is developed through prior experiences and learning environments. Interactive classroom environments, teacher presentations, and curriculum materials are a few aspects of the learning environment conducive to student learning.

Concept development is an evolving process that occurs during the course of problem-solving sessions. It is observed that almost none of the problems are exclusively a "symbol-symbol" or "word-word" problem; instead, they comprise of a mixture of mathematical terms, real objects, and spoken words, making each situation a multimodal entity. While explaining the multimodal designs representing classroom discourse, Lim (2011) explained that design include the use of language, gesture, and images thus developing visual, oratorical, and written representations as semiotic modalities. Students' level of comprehension of the problem leads him to select the type of representation or modality best fits in his schema. In a majority of cases, the students use their prior learning to interpret situations into their own perspectives and then try to translate the given information from the problem into representations already known to them; such as pictures, charts, symbols, or others. The path directed towards finding the solutions to problems goes back and forth several times among representations. The student can only reach to a solution if they can flexibly move among multi-representations. The solutions are generated and characterized by many small steps to match the given information to formulate solutions that can be further interpreted and manipulated.

Lesh, Post, and Behr's (1987) Translation Model, as illustrated in Figure 2.4, shows the mesh of pathways for students translating representations, moving flexibly among multi-representations, and finally mapping the results into the same or other multi-representational systems. This Lesh, Post and Behr (1987) interactive model shows that the relationship of real-life problems in linear functions is a translation of pictorial, symbolic, verbal, written, and manipulative understanding of concepts.

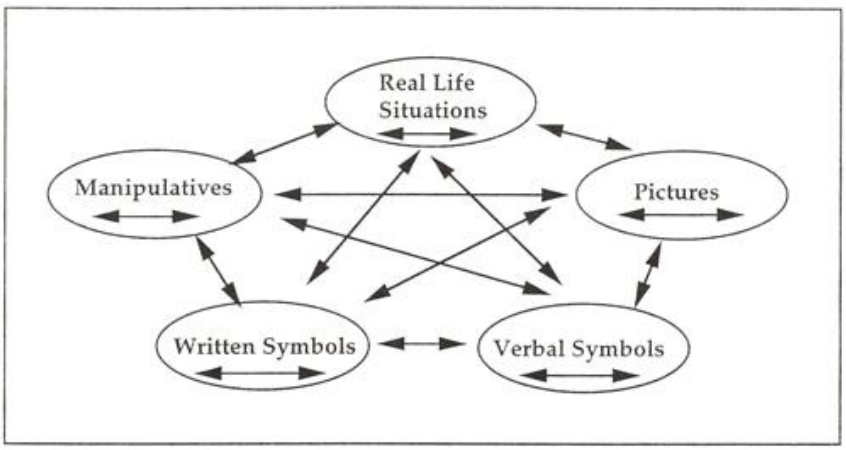



Figure 2.4: Lesh, Post, and Behr's (1987) Translation Model

*Pictures:* Spontaneously drawn pictures and diagrams are powerful learning tools that promote discussions about mathematical ideas to assist in findings about students' understanding of concepts.

*Manipulatives:* Manipulatives are objects that can be touched and moved. They provide opportunities to compare and represent mathematical ideas such as fractions or place values to identify patterns. Manipulatives allow students to build up mental representations and acquire skills in using and modifying the known representations and synthesize new ideas (Davis, 1984; Kilpatrick, 1985; Thompson,1994).

*Spoken Language*: Read aloud expressions for mathematical reasoning provide the students with opportunities, to make parts of their prior knowledge explicit which may have been previously implicit for them.

*Written Symbols:* Both mathematical symbols and words are used by teachers and students to make mathematical ideas comprehensible and connected.

*Relevant Situations:* The relevance of situation is context-dependent but not necessarily connected to real life situations that hold students' interest. Similarly, teachers make connections of representations and mathematical ideas to assess a student's ability to transform a representation of a mathematical idea into another representation.

The curriculum that fails to make connections among representations and instructional techniques, deals with different strengths and weaknesses that each representation offered to the students (Piez, 1997). This type of curriculum supports the deficit model instead of teaching students' development and avoiding the misconceptions. NCTM (1989, 2000) has promoted curricula that foster connections and emphasize multiple representations. Research findings indicate that students enrolled both in secondary and college-level mathematics before calculus is not making connections among representations of functions (Knuth, 2000). Historically, the focus of Algebra 1 has been on symbol manipulation, but current reforms call for algebra instructions that connect ideas, relationships, and patterns in a meaningful way (NCTM, 2000).

### 2.3.1 Summary

Research in linear functions focuses on three important aspects of problem solving: the use of multi-representation of linear function, flexibility to move among representations, and the ability to continue to make connections between representations and interpretations. Lesh, Post, and Behr's model for "Rational Number Project" is a comprehensive approach for integrating multi-representations of linear functions in order to develop a better conceptual understanding in students. The relevance of contextual algebraic concepts are not only connected to real life situation but also hold students' interest in problem-solving.

### 2.4 Connection between representations and their interpretations:

A mathematical text is often interpreted in terms of symbols and signs to represent sequential patterns of text and operations. The order of steps to make the representation meaningful and lead to outcomes in a true mathematical perspective with established rules plays a major role in the learning process. Since different representations emphasize different features of the function concept, the ability to move flexibly among representations is critical so that students will be able to choose the representation that will facilitate their ability to solve a problem efficiently (Dufour-Janvier et al., 1987; Dugdale, 1993; Kaput, 1987; Keller & Hirsch, 1998; Knuth, 2000a; Moschkovich et al., 1993).

Krutetskii (1969) described this type of much-needed flexible thinking as reversibility. He suggested that for a complete picture, the learner must be able to look at the direct connection as well as the reverse process. Such capability does not develop automatically, nor can it be inherited; but it is a developmental process that requires instructions and practice designed to facilitate cognitive learning. McGowan and Tall (2001) also suggested that an inability to move flexibly among representational forms of functions may cause a conceptual rift that would prevent a student from progressing further until that gap was bridged. The two specific kinds of issues related to multi-representations of linear functions are flexibility to move within representations and problem-solving. Knuth (2000) reported that the lack of students' ability to make connections is due to the typical format of problem-solving for mathematical functions (e.g., beginning with algebraic representations and ending with graphical representations). He also pointed out that understanding both directions, algebraic-to-graphic and graphic-to-algebraic, was fundamental for developing the flexibility among the representations. Moschkovich, Schoenfield, and Arcavi (1993) wrote that as students begin to make sense of the domains of representation for linear functions, they do not know what aspects to ignore and what to investigate.

Janvier and colleagues classified representations as "external" and "internal" processes of learning. "External" refers to the symbolic, tabular, visual and graphical representations, and "internal" refers to the cognitive representations, such as mental images formed in their schema Lesh, Post, and Behr's (1987) model shows that the external representations are the mode by which the internal representations of mathematical ideas are connected. The nature of internal representations is such that they cannot be communicated unless expressed in the form of external representations. Limiting oneself to one form representation may restrict the mind, and many internal representations could be left undetected.

### 2.4.1 Summary

Since different representations emphasize different features of the concept of functions, the ability to move flexibly among representations is critical. The student must be able to comprehend the direct connection as well as the reverse process to make sense of the domains of the representations of linear function.

## 2.5 Visual Representation

Linear functions are often represented visually and interpreted by using graphs. However, there are some notable limitations and difficulties faced by students and teachers while using visuals to explain mathematical concepts. Sidoli (2007) identified that it is not known to what extent the visual representation would be helpful for algebraic concept-learning among students. Rosken (2006) defined visualization as an understanding developed via interpretation and the use of visuals such as graphs, pictures, and diagrams reflecting individual perception. Arcavi (2008) found that students with prior experience with symbolic representation find it easy to learn about the characteristics of linear functions regardless of their complexity of the situation presented to them. For others— especially mathematics students—visualization can play a powerful complementary role in engaging and enabling learning through scaffolding processes such as checking for hypothesis, and making predictions.

## 2.6 Graphical Representations

Students must comprehend the relationship between tables, graphs, and equations in order to have a profound understanding of functions (NCTM, 2000). Bell and Janvier (1981) described pictorial distractions as graphs that are judged by visually salient clues, regardless of the underlying meanings. Experts may often be surprised that students who are unfamiliar (or partially familiar) with the underlying concepts see “irrelevancies” that are automatically dismissed (or unnoticed) by the expert’s vision. The perception is shaped by what we know when we are looking at a diagram or graph (Dreyfus, 1994). The graphical representation of linear function as often used in algebra is the plot in the Cartesian coordinates system. The linear behavior and the concept of rate of change, slope, direct and inverse relationships, regression line, and trends are common characteristics used to describe the phenomenon. Without the understanding of the characteristics of graphic representation, it is difficult to express simultaneously-occurring processes found in many STEM perspectives.

## 2.7 Cognitive Learning of Multi-representations

Jacobs and Cleveland (1999) describe some of the inherently possessed qualities necessary to comprehend the use of skills to develop an understanding to analyze and reach the students’ true potential. However, the potential level of concept development varies among individuals. The learning process continues to develop as a natural progression of explorations, experiences, and understanding simultaneously. Cognitive learning is a result of direct or indirect physical experiences that leads to the acquisition of knowledge that matches the individual’s schema. Bruner (1964) described learning-by-discovery as a process that takes place in three stages for the internal organization of previously known concepts. The first stage is the inactive learning level, in which a child needs action on materials to comprehend the concept. In the second stage, the iconic level, the child creates a mental picture for the representation of objects and ideas but cannot manipulate them directly. In the third, or symbolic, level, the child starts strictly manipulating symbols and does not need to manipulate objects. The same idea is expressed as the development and concept of schema. The schema is a diagrammatic (conceptual) representation created in human minds based on prior experience (Gick, 1983). The relationship between cognitive traits and learning styles such as spoken language, verbal expressions and comments helps in student understanding of linear functions. A well-known saying about visual learner, "a picture is truly worth a thousand words." Reflects on students' cognitive understanding of representation (Brisbane, 1911). Bernard and Tall (2001) found that mathematical thinking involves two different kinds of mental activities, which are both referred to as schemes or schemas. A scheme is a sequence of actions that eventually become long-term cognitive links for the learner. The schema is a physical structure in the brain that allows schemes to connect over a period of time and offers a more subtle way of building up mental concepts capable of operating flexibly as cognitive units (Bernard, 2001). In short, research shows that students make use of schemas leading to the expression of mathematical concepts in multi- representations. It also reflects that due to differences in individual schemas, students face difficulties moving flexibly between representations. Janvier (1987) commented that the modes of representations and the students' ability to transform between translations were the main focus in the previous research frameworks used to analyze student understanding. Lately, the trend has changed in combining the Process-Object theory and functional representations (Janvier, 1987; Kaput, 1989)**.** The Process-Object theory, as described by Hartsfield (2013), is a theoretical perspective providing one potential way to describe the development of a student's concept of sampling. Hartsfield (2013) reports, that when a concept is first introduced to students, it is meaningless unless it is repeated and becomes an internalized mental process. This process helps students move mathematical concepts such as linear functions to be condensed and settled in the brain as constructed objects, which comes into action as soon as the words "linear function" displays in students' perception (Kaput,1989). Thompson (1994) also described this construct or object as "self-evaluating expression."

## 2.8 Students' Preference for Representations

Student preference for algebraic representations has been identified as a major influence in their unwillingness to use an alternate representation, even when one was readily available and perhaps easier to use (Knuth, 2000; Pierz & Voxman, 1997). Student reluctance to use an alternate representation has been observed even when the curriculum emphasized multiple representations of functions (Knuth, 2000; Thompson, 1994). Yerushalmy and Schwartz (1993) found that one of the most significant influences on student preferences of representation used to solve a problem was an instructional emphasis on the manipulation of the algebraic representation. Students often viewed graphs as end-products or something extra that was unrelated to the algebraic representation. Traditional curricula emphasized an equation-to-graph direction in teaching functional representations. This method may have hindered students' ability to see the graph as a viable means for solving a problem (Knuth, 2000). Leinhardt et al. (1990) states that the tasks that required translations with the "rule/equation to graph" were routinely presented to students rather than the "graph to equation/rule." Students initially produced a table of values that satisfies the equation (typically written in slope-intercept form) and then plotted the values on a Cartesian coordinate graph. This method of teaching is mathematically straightforward but is dependent on the expertise of the teacher in presenting the concept repeatedly and not expecting student mastery of equation-graph connection to be established in a very short period of time (Schoenfield et al., 1993). Leinhardt et al. (1991) explained the student conceptual learning development in a different perspective: that teachers having shaky mathematical knowledge tend to stay within their comfort zones and do not extend their assessments of student-acquired knowledge beyond the curriculum and standardized tests, resulting in the fact that students master procedures with little or no conceptual understanding. Little evidence challenging this expectation emerges from the individual tasks students usually encounter when solving problems involving linear functions.

All representations are not alike and have varying limitations or strengths in different contexts and directions for solving linear function equations. Keller and Hirsch (1998) pointed out that there are many factors influencing students' preferences for representations. These include student ability level, confidence in symbolic manipulations, experience, perception of the viability of the representation as an option, and the complexity of symbolic information. One significant area found in Pierz's and Voxman's (1997) research was that students had difficulty reading information from graphical representations. Due to a lack of a complete understanding of Cartesian coordinates used in graphing, students tried to avoid graphical representations and their interpretations. The use of graphing calculators in the classroom helps students and teachers overcome the challenge of graphic representations and their interpretations. A teacher can attempt to assess student learning after teaching the material, but the answers to two important questions often need to be further explored: Did the students actually learn? What did the teachers do to develop or impede students' long-term understanding? These are open-ended questions with no right or wrong answer. Mixed opinions have surfaced regarding the regular use of graphing calculators to produce a shift in the type of representation meant for solving linear function problems, but most students still use symbolic manipulations as an efficient method for solving problems. Others found that students using graphing calculators had a better understanding of functions because of the increased practice with representations, and their ability to tie together mathematical ideas, thus resulting in their overall success (Hollar & Norwood, 1999; O'Callaghan, 1998; Van Streun, 2000).

The use of a graphing calculator is required in high school algebra classes. This requirement has caused conflicting issues for the conceptual learning of linear functions. Javier (1987) contended that different cognitive processes are involved in translating from one representation to another in a different order. The direction of the connecting representations can make one representation easier to use than another for different individuals. Students and teachers are often inclined to use the curriculum-recommended notations for linear functions instead of solving problems with other representations of their choice. Yerushalmy (1990) compared the progress of a group of students with access to graphing calculators for all observed lessons with another group of students from a similar setting who were taught algebra in a real-life context using whole-class discussions and interventions. Some major outcomes of the study were that teaching students through visuals (graphs) or computations (symbols, etc.) indicated that emphasizing multiple representations does not necessarily motivate students to adapt the use of multi-representations for problem-solving involving linear functions. The best results are seen for teaching directly through strategies and then combining visuals and computational methods for solving and crosschecking algebraic problems. Yerushalmy (1990) and Goldenberg (1987) also found that algebra concepts were flexible and inventive for using multiple representations. Student perceptions of functions were based on three aspects: identification and classification of families of functions, connections between inclination and slope of graphs as a rate of change, and distinction between the properties of a function and its visual representation. The core concept as defined by Thompson (1994) is the introduction of verbal representations in formal or everyday language. Image generation becomes quite complicated if many different representations are presented to students at one time. The reason could be the fact that an instructional presentation by teacher is effective if each of its components is connected to the prototype examples developed in the class. Bakar and Tall (1992) explained that the development of function is not possible for learners unless he or she has experienced and developed a prototype example under the same conditions as applicable to the functions itself.

## 2.9 Theoretical Framework

The theoretical framework for the study was based on cognitive and social constructivism. It incorporated student interviews to get at the cognitive thinking process and to find if the use of multi-representation helped the MSEN Pre-College students gain a better understanding of linear functions. Artifacts and the interview process provided three sets of data for each student: (1) written response to the interview task sheet with six problems related to linear functions and their multi-representations, (2) students' verbal responses to interview questions, and (3) visual expressions stored as audio-video recordings. As previously discussed, Lesh, Post, and Behr's (1987) Translation Model provides a viable model for the current study. The translation model explains the students' prior learning and is linked to comprehension in their perspective, translated into a homogeneous representation that makes sense.

The concept of social constructivism is derived from the work of Vygotsky, who considered conceptual learning to be a social activity. Social construct is identified as a perspective behavior that is social rather than intellectual behavior. Vygotsky's concept of a cognitive and social construct is further expounded upon by Berger and Luckman (1996), who argue that the concept of social construct is directly related to the conceptual learning development in a social environment. Social cognition is thus understood to be a mode of encoding and processing information in the human brain in terms of psychological reasoning or representations of concepts. In the preface section of Piaget's early books, Vygotsky (1986) theorized "the true direction of development of thinking is not from individual to social but from social to individual" (p. 36).

## 2.10 Summary and Introduction to Chapter 3

The topic of multi-representations of linear functions is not new to the research community. It had been frequently addressed in context of strengths and weaknesses to find solutions for algebra problems. This chapter of literature review uncovered many important aspects that had not been researched such as the use of multi-representations for simultaneous processes occurring in real life situations. It had been established through students' 2011 SAT scores for linear functions and their ability to apply conceptual understanding in real-life situations. NCTM (2000) standards emphasize the idea that, "Representations should be treated as essential elements in supporting students' understanding of mathematical concepts and relationships; different representations of functions support different ways of thinking and manipulating mathematical objects" (NCTM, 2000, p. 361). It is important to focus on three important facts: 1. Explore the need of using multi-representations to solve linear functions problems and their characteristics. 2. Learn from prior research and theories regarding the developments of understanding of the concepts of linear functions. 3. Connections between representation and their interpretations for students' successful real-life experiences with linear functions.

Although multi-representation and multimedia resources are easily available for 21st century students, they find it difficult to use them effectively for problem-solving in linear functions (Anisworth, 1999). Other research found that some students did not use a certain representation because they did not recognize the viable choice (Hercovics, 1994). Leinhardt et al. (1991) hold teachers and their instructional strategies responsible for students’ lack of understanding and confidence in using multi-representations. They believe that if the teachers have shaky mathematical knowledge, they tend to stay within their comfort zones and try to deliver curriculum objectives and prepare students for standardized tests. Such practices result in the fact that students master procedures with little or no conceptual understanding or flexibility to move among multiple representations. Unfortunately, this affects the individual student’s understanding for solving problems involving linear functions.

The literature review also directed the reader’s attention to cognitive learning development as a natural progression of explorations, experiences, and understanding simultaneously. Cognitive learning is a result of direct or indirect physical experiences that leads to the acquisition of knowledge that matches the individual’s schema. The schema is a diagrammatic (conceptual) representation created in human minds based on prior experience (Gick, 1983). Cognitive scientists have established that our brains generally convert written words into their spoken equivalents and process them in the same way that spoken words are processed. Written words are therefore not equivalent to real visual information. Bernard and Tall (2001) established that a sound understanding of concepts is only possible through mental activities that are stored in the cognitive units called schemas. Students not exposed to multi-representations of linear functions get deprived of the development of such units.

Post, Lesh and Behr’s (1987) Translation Model is a close estimate for development of human cognitive and social constructs and a useful theoretical lens to answer the two research questions:

1. How do multi-representations help pre-college students understand linear functions?
2. How can students’ experiences with graphs of linear functions be characterized?

The major components of the model are pictorial, manipulative, written, verbal/spoken, and symbolic representations to understand real-life problems related to linear functions. The current research study will focus on exploring students’ understanding of graphical, symbolic, algebraic, and tabular representations of linear functions being used flexibly and reversibly for conceptual development in high school algebra classes.

The following chapter, “Methodology,” will explain the method chosen to study multi-representation of linear functions using multiple case studies within the proposed theoretical framework to organize and explain qualitative data collected from student and teacher interviews, artifacts, and student assessment results.

# CHAPTER 3: METHODOLOGY

## 3.1 Introduction

The literature review in Chapter 2 provided information regarding how students think about linear functions and their characteristics such as slope, intercepts, graphs and other representations. The rationale for this current research study is to understand how multi-representations help students' understanding of linear functions and their characteristics based on the cognitive and social constructivism framework mentioned in Chapter 2. Although previous research provides valuable insights into student comprehension of linear functions in algebra, more research is needed to explore the use of multi-representations. Very little has been investigated about the long-term understanding and retention of concepts in linear functions for college-bound students that are already motivated to learn. Such students may still show lack of a clear understanding of linear functions and flexibility to make transitions among multi-representations in higher-level math courses and STEM careers. Rigorous mathematics acts as a gateway to higher education and more lucrative careers (Moses & Cobb, 2001; NCTM, 2000). Merely teaching algebra to students to perform well on standardized and achievement tests ("content") is not enough for solving problems with real-life situations ("context") and professional careers (medicine, engineering, technology, etc.).

This research study will explore the use of multi-representations for the content and context of linear functions used by MSEN Pre-College students and their teachers. Five case studies will be developed using written tasks and interviews. The first part of the interview will provide general information about the participants, including their experiences in mathematics and their overall opinions regarding linear functions. The rest of interview will be focused on the assessment tasks and exploring each of the students' conceptual understanding.

## 3.1 Background

Kruth (2000) identified factors influencing conceptual understanding of linear functions among students as the lack of motivation and availability of learning resources. MSEN Pre-College is an optional enrichment program offered on Saturdays and three weeks in the summer. The program is open to all middle and high school students maintaining a 3.0 GPA who show interest in science and math education, verified by a letter of recommendation from their school. The self-motivation of MSEN Pre-College students is evident from the extra efforts made by the students to attend an optional algebra program. The students' classwork records show that despite above-average GPAs, they still face difficulties in conceptual understanding of linear functions. Welder (2006) referred to students' incorrect responses on standardized tests as due to the lack of understanding of algebraic problems in students' perspective. Yerushalmy also considers that "performance on algebra items is highly dependent on how the students perceive or understand the graphic representation of functions"

(Yerushalmy, 2000). The current study tends to find that if the multi-representations of linear functions help students' gain a conceptual understanding and allows them to move flexibly between representations.

Behr, Post and Lesh (1987) emphasized that besides the knowledge of multi- -representations, the translations and transformations within representations are highly important. In the same context of students' understanding functions and their flexibility to move among multiple representations, Clement (2001) observed that a majority of students expect an algebraic formula representing functions instead of other representations. He continues to point out that students are successful in answering questions for the prototype functions discussed in class or in textbooks. For example, students are taught that $y = x^2$ is a function and $x = y^2$ is not. However, they face difficulty in recognizing the reasoning for functional characteristics if the given functions were not taught in class or described in their textbooks.

Test scores are often considered to be the measures of student comprehension. Very rarely is their depth of comprehension found essential for successful completion of an algebra curriculum. Among standardized test questions, it is common that the students are asked to check if a given relationship is a function or not. Most high school curricula teach the "vertical line test" to be the only way to check for functional behaviors and leave out the concept of one-on-one correspondence. Tall (1996) found that, despite the fact that the concept of functions is very basic and is taught in early grades using patterns, it is the most difficult concept for students to understand. The concept-development process begins with the mental creation of images related to a given problem and then moves to looking at their characteristics. The gap between creating mental images and applying mathematical concepts involving linear functions becomes obvious when represented differently from what students have encountered in the classrooms (Montgomery, S. N. & Groat, L. N., 1998).

## 3.2 Sampling Process

In qualitative research studies, sampling is a deliberate process to include participants according to the theoretical needs of the study and in accordance to fixed criteria set for the qualitative inquiry. Sampling is crucial for the attainment of validity of the research process. Since the number of students in the MSEN Pre-College algebra class is very small, a “convenience sampling” method will be used for the research phase of the study. Convenience sampling is selecting the sample by including participants who are readily available and who meet the study criteria.

All students in the MSEN Pre-College algebra class were eligible to participate in the study. The selection criteria are:

1. The student and teacher must be part of the MSEN Pre-College Program.
2. Each of the participants must have signed the consent or parental agreement form for the study.
3. Parental permission for audio and video recording is on file at the MSEN Pre-College office.
4. The age of students in the study must be 14 years or older.
5. The math requirement for the participants is to have completed the 2012 approved Common Core Algebra Course Strand F-4 (Interpret functions that arise in applications in terms of the context) or to have at least completed the unit on linear functions in the algebra class.

The MSEN Pre-College population is diverse, with 88% African-American, 8% White, 2% Asian/Indian, and 2% Hispanic students. All students in the MSEN Pre-College Algebra class will be invited to participate in the study. Only students meeting the criteria for participation will be used to construct case studies. Individual students participating in the study will be given pseudonyms for recording interviews. Written work for the assessment tasks instrument, the original documents with students' real first and last names, will be shredded as soon as the study is completed. Audio-video records will also be destroyed as soon as the transcription of interviews with pseudonyms is completed. The only records used for data analysis will be student and teacher interview transcriptions, student work on assessment task instruments, and the researcher's notes for observations. The study will take place at UNC Charlotte. Participation in the study will be totally voluntary and no monetary or other compensation will be provided to students or algebra teachers in the research study.

## 3.3 Research Model

The proposed case study research model is appropriate for describing the development of students' conceptual understanding in linear functions using multi-representations. The representations under consideration are graphs (pictorial), symbols (algebraic), verbal (interview record), manipulatives, and the written responses obtained from the assessment task. This research model is very similar to the original Lesh (1983) translation model for "Rational Number Concepts" and to the Lesh, Post and Behr (1987) "Translation Theoretical Model." The details of this model were described in Chapter 2.

The Lesh (1983) model was first designed to study three critical aspects for students learning fractions in elementary school. The three goals were: understanding and recognizing multi-representations, the ability to flexibly move among representations, and translating concepts between different representations (Figure 3-1).

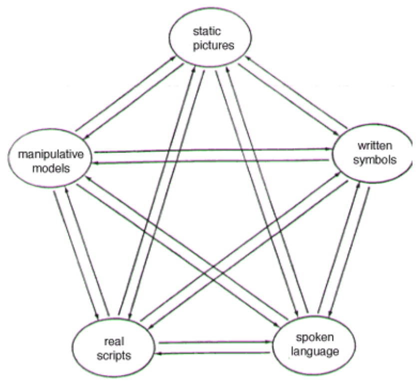



Figure 3.1. Original Lesh Model (1983).

The Lesh, Behr, and Post model (1987) further expanded the Lesh (1983) model and used it for the understanding of fractions for middle school students.  The modified model explicitly looked at the ability to translate within and between written symbols, spoken language, pictures, models, and context. The model shown below (Figure 3.2) is a slight adaptation from the one first introduced by Lesh, Post and Behr in 1987.

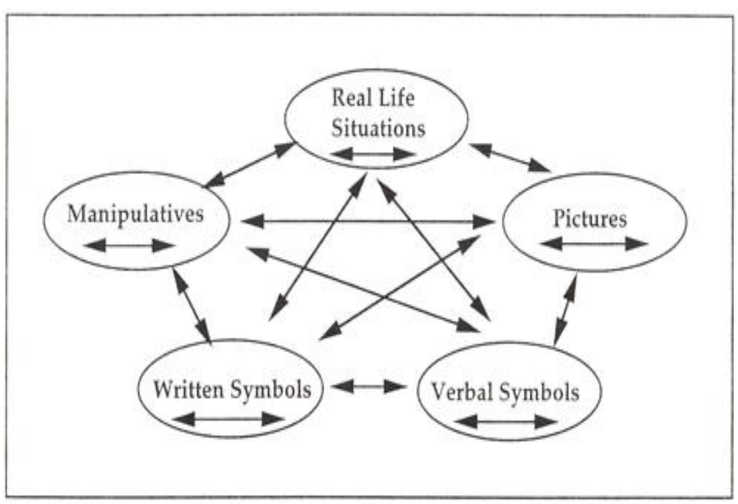


Figure 3.2 A slight adaptation from the one first introduced by Lesh, Post, and Behr (1987)

The model in Figure 3.2 is still seen to be valid for mathematics beyond elementary and middle school courses such as linear functions in high school algebra. Emphasis is added by using tables, graphs, and algebraic representations to pictorial and symbolic representations. The conceptual and contextual knowledge is explored by the students' written work and the use of manipulatives. The "Multiple Representation Web" in Figure 3.3 is a good example of highlighting the connection between multiple representations (Dietiker, et al. 2005).

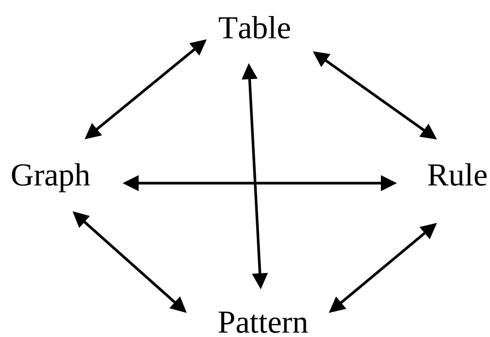



Figure 3.3 Multiple Representation Web adapted from Whitney (2010)

The Lesh, Behr, and Post Model (1987) also builds on the same idea of student conceptual learning experience as a transformation among five representations (picture, verbal, written, and manipulative) to conceptualize, characterize, and comprehend real life situations.

## 3.4 Interview tasks assessment instrument

According to Fowler (2002), structured interviews provide a thorough discussion for each interview question and the ability of the researcher to extend the interview discussion in many relevant directions. A disadvantage of using structured interviews would be if participants could not understand the questions. In such situations, the interviewer is limited to providing only a few previously-scripted explanations or definitions and asking the participants to do their best. The interviewer is not expected to explain the question beyond repeating the question. The assessment instrument (Appendix D) for the current study consists of six interview tasks/problems. The tasks were designed to characterize linear functions and to elicit information such as the students' ability to move flexibly among multi-representations. The assessment instrument is adapted and modified from the doctoral dissertation of Whitney (Whitney, 2010). Four out of the six interview tasks are based on representations (tables, graphs, context, and equations) and the remaining two are conceptual problems in linear functions. The interview tasks were reviewed for clarity and validity by a qualitative math research specialist at UNC Charlotte. She is an assistant professor and a math specialist. The math specialist agreed to review the tasks to verify that they are aligned with the research questions for the current study and the course objectives for linear functions based on the North Carolina High School Standard Course of Study. There was 100% agreement from the math specialist in terms of the validity of the tasks and the corresponding objectives.

Table 3-1 briefly describes the objectives for interview assessment tasks.

Table 3.1 Interview task objectives

| Task | Description of Objective | Multi-representations/ characteristics of linear functions |
|---|---|---|
| 1 | The task is recognizing the pattern and looking for a constant rate of change in a definite sequence for linear functions. It will also assess if students are able to translate directly between a linear context and an equation. | Patterns, contextual<br>Algebraic<br><br>Rate of change<br>Flexibility to move among representations |

| | | |
|---|---|---|
| 2 | This task will determine if students are able to assess the justification of representing the graph and equation interchangeably for the same equation. Probing interview questions will help to get in-depth of understanding of students about multi-representations | Graphical and Algebraic Interchangeability for representations<br>Conceptual understanding |
| 3 | This task will indicate the ability of students to find the rate of change in different representations. How are the independent and dependent variables related to and coordinated with each other? | Table, graph, contextual, algebraic, and visual<br>Positive and negative rate of change/slope<br>Dependent and independent variables |
| 4 | The task will help the researcher find the connection of the student learning experience with contextual and mathematical formulation of linear equations | Contextual, Algebraic, and Contextual and mathematical formulation of linear equations |
| 5 | This task will assess if the students are able to visually picture and compare equations and describe their similarities and differences. Interview questions will probe to assess if they can translate an equation into words | Visual, pictorial, algebraic<br>Compare and contrast characteristics<br>Flexibility to translate from equation to context |
| 6 | The expectation for assessment for this task is that the student be able to comprehend and make inferences about the rate of change or slope of each function (drawn in red and blue) and system of equations demonstrating simultaneous behaviors at the point of intersection. What is happening at the point of intersection, and what physical paradigm is observed from these graphic representations? | Graphic algebraic, and visual<br><br>Rate of change, slope, simultaneous behaviors at the point of intersection, comprehension, meaning of a solution to a system of linear functions. |

The initial assessment instrument shown in Figure 3-4 was designed to provide verbal instructions to participants for each task. The math specialist suggested that delivering verbal directions at the time of the interview for problem solving is not advisable because it may not be clearly understood by the student. Therefore, she suggested including written explicit instructions to avoid any unforeseen variables for data collection.

Figure 3.4 Initial Assessment Instrument

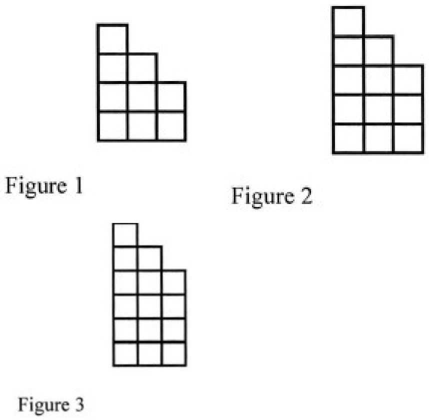
*Linear function Problems Interview Tasks*

*Directions: you will see one card at a time with a problem or pictures. Answer the questions asked by researcher and show your work in the space provided for each problem (box)*

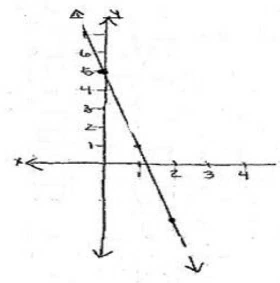
Name: ___________________ Period: _______ Date: __________

Please use the space inside the box to show your work

Task1

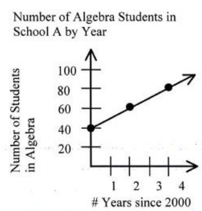



Task2



Task3

School A:




School B:

| Year | Population of School B |
|---|---|
| 2000 | 32 |
| 2001 | 60 |
| 2003 | 116 |

School C:

y = 120 – 30x

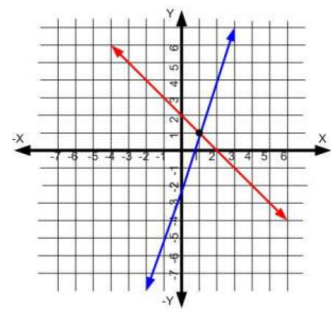
Where x=number of years since 2000, and y = the number of algebra students at school C

School D:

In 2000 School D had 27 algebra students. In 2003 school D had 72 students enrolled in Algebra.

Task4

Solve the equation.. What is the rate of change
of this linear function?
-(-x - 5) = y – 2

| | |
|---|---|
| Task5<br><br>Jose left the White House and drove toward the recycling plant at an average speed of 40 km/h. Rob left some time later driving in the same direction at an average speed of 48km/h. After driving for five hours Rob caught up with Jose. How long did Jose drive before Rob caught up | Explain the characteristics of the following representation for the system of equations<br>y = 3x - 2, in blue and y = -x +2 Task6<br> |

3.5 Modifications in tasks assessment instrument

The major modifications made in the initial assessment instrument were changes in the format of the questions asked and making the assessment instrument more user friendly. Each objective/problem statement was broken down into smaller parts, based on the expectation that the parts would be answered in order. Other modifications included specific written instructions to be followed for each task and better pictorial and graphical representations. Secondly, a pre-study test with three to five students from the same algebra class (same classroom environment and teacher) was conducted. The pre-study assessment consisted of the same task/problem statements as the interview research instrument. A pre-study was conducted for all students present in the algebra class while the selected participants for the research study were temporarily sent to another classroom for a different math activity. The purpose of the pre-study was to confirm the clarity of the assessment task instrument. The informational data collected from the pre-study will not be used as part of the case studies, but it was used for the refinement of the assessment tasks, as explained in the next section.

## 3.6 Pre-Study outcomes and modifications

In the pre-study, students were asked to write the problem statement in their own words, clearly stating what is given and what is being asked in the task/problem (Pre-Study task sheet, Appendix C). Four out of five of the pre-study participants had no difficulty in comprehending what was given and what was being asked in the pre-study task instrument. Two students had difficulty with language expressions in some tasks. Specifically, one student was not sure what was meant by “intercept” in Task 2b. A second student, an English language learner, had difficulty comprehending what was meant by “Assuming population rate of change is linear” in Task 3. He was also confused by the statement, “How many and what type of solution is expected for the given graphic representing blue and red lines intersecting at a point?” and the phrase “given graph” in Task 6.

In order to help with clarification of task statements, the word “intercept” in Task 2 is defined in parentheses as “where the graph of an equation crosses the axis.” For Task 3, an explanation for the comment was added as “the graph representing population is always a straight line.” For Task 6, the word “given graph” was changed to the “graph shown below.” The feedback from the pre-study was used to simplify the language of the problems in order to be more understandable for diverse learners. The modified assessment task sheet is given in Figure 3-5. Table 3-2 represents a quick overview of the type of representations included in each task and the expected outcomes of student work.

Figure 3.5 Modified assessment instrument

Task Sheet Algebra 1 Linear functions

Please show all your work

Task 1

Look at the following pattern and make picture to represent Fig 0, Fig 4 and Fig 5. Can you find a rule that will determine the total # of squares in any pattern like the one used for this task?

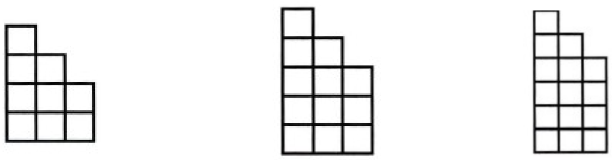


Figure 1 Figure 2 Figure 3

Rule:

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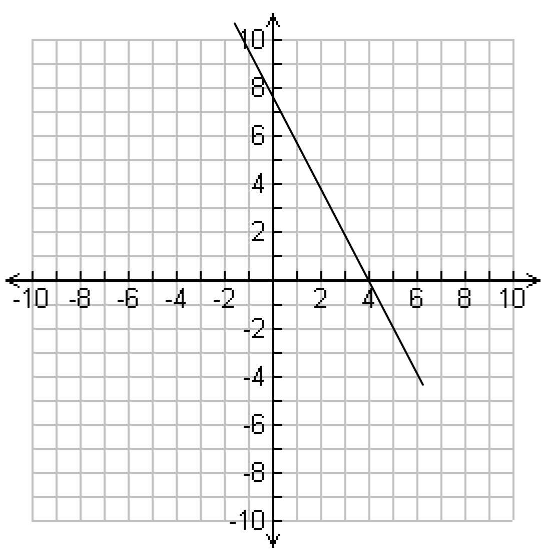
Task 2

Explain what is being represented in the given graphic representation?




a. True/ False y= 3x – 4 is the equation of the line represented in the graph. Give reasons for your answer.

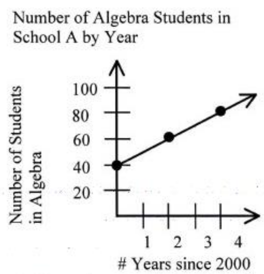
b. What part of the line in the graph represents the rate of change

c. What are the slope and intercepts of the line in the graph? What are the slope and intercepts ((where the graph of an equation crosses the axis) of the line represented by the equation y= 3x -4.

Task 3

Below, you will see the number of algebra students at four different schools, School A, B, C and D. Assuming  population rate of change is linear (the graph representing population is always a straight line) answer the following questions.

School A:




School B:

| Year | Population of School B |
|---|---|
| 2000 | 32 |
| 2001 | 60 |
| 2003 | 116 |

School C:

y = 120 – 30x

Where x=number of years since 2000, and y = the number of algebra students at school C

School D:

In 2000 School D had 27 algebra students. In 2003 school D had 72 students enrolled in Algebra.

a. Determine the rate of change in population for each school starting year 2000 until 2005.

School A

School B

School C

School D

b. What is the total population of students in each school in the year 2005?

School A

School B

School

School D

c. What is the predicted population in 2015?

School A

School B

School C

School D

d. Draw a graph to represent each of the school algebra student population in School B, C and D.

e.

School A

School B

School C

School D

f. Explain in own words about the answer to the problem

Task 4

a. Transform the equation in standard form.

$$-(-x - 5) = y - 2$$

b. What is the rate of change of the linear equation $-(-x - 5) = y - 2$?

c. Represent the given equation as a graph.

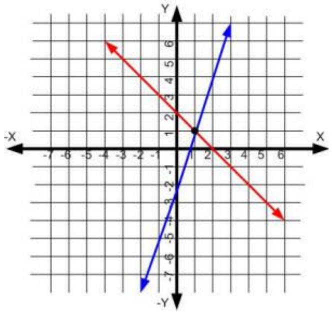
Task 5

Jose left the White House and drove toward the recycling plant at an average speed of 40 km/h. Rob left some time later driving in the same direction at an average speed of 48km/h. After driving for five hours Rob caught up with Jose. How long did Jose drive before Rob caught up

Task 6

a. How many and what type of solution is expected for the graph shown below representing blue and red lines intersecting at a point?
   # of solutions ______________
   Type of solution ____________

b. Write the slope and intercepts, of the following representation for the system of equations. $y = 3x - 2$, in Blue and $y = -x + 2$ in Red



Blue line
Slope _________ X intercept =_______ Y intercept =____________.

Red line
Slope _________ X intercept =_______ Y intercept =____________

c. Write a real life word problem (similar to task 5)

Table 3-2.Brief description of interview tasks

| Task | Given / Brief Description | Representation | Expectation |
|---|---|---|---|
| 1. | Three figures (Fig 1, Fig 2, Fig3) | Pictorial representation | Draw the next two figures Fig 4, Fig 5 and Fig 0 Find the rule in Algebraic representation. |
| 2 | A graph for a linear function<br>An equation for a linear function. | Graphical<br>Algebraic<br>Symbolic | Compare the two representations and characterize the two linear function<br>Find rate of change and show flexibility to move among representations |
| 3 | Growth in population of algebra students in four schools with the same demographics.<br>Linear functions<br>Multi-representations | School A graphic<br>School B table<br>School C symbolic/algebraic<br>School 4 contextual | Characterize Conceptual understanding<br>Flexibility to move among representations<br>Transformation of |

| | | | |
|---|---|---|---|
| | | | multi-<br>representations |
| 4 | Algebraic equation<br>(distributive property, order of operations) | Algebraic<br>X and y intercepts | Transformation among representations<br>Equation of linear function<br>Graphic representation |
| 5 | Word problem with boundary conditions or constraints for problem solving in solving linear function | Contextual | Pictorial<br>Graphic<br>Algebraic<br>Flexibility to transform among multi-representations. |
| 6 | Red and Blue line graphs<br>System of linear functions<br>Real life word problem | Algebraic<br>Graphic<br>Word problem | Number and significance of system of linear functions<br>Conceptual & cognitive thinking<br>Contextual<br>Prior knowledge |

## 3.7 Data Collection

"All is data," said Glaser (2002); meaning everything that is explored from research settings, including firsthand information obtained from participants' interviews or gathered from other data sources or literature reviews. Data collection is dependent on the researcher's interests and use for interpretation. The current research model will consist of independent case studies using Lesh, Behr and Post's (1987) translation model. This model has shown to be very effective for answering the research questions for the current study. A multiple case study design will be used to gain an in-depth understanding of real-life situation problems related to linear functions. A multiple case study design is less vulnerable than a single case study design in regards to the generalizability of the findings. Some strategies from grounded theory will also be used for data collection and analysis of the case study. The advantage of using grounded theory methods are many, such as gathering empirical data leading to emerging fields or perspectives, and guiding interests to sensitizing concepts and disciplinary perspectives. The sensitizing concepts also provide a place to start or end the interview conversations employed in this study. This allows for increased flexibility while simultaneously obtaining more focus on what is happening with the students' understanding. The empirical data obtained from open-ended interview questions may also contain some new ideas for future research. Grounded theory strategies are also known to establish the best fit between the initial findings and the emerging or recurrent ideas. Glaser and Strauss (1967, 1978) emphasize that the basic social process in the field is sometimes discovered during interviews. They also add that social processes remain invisible until a change happens. For example, the students' choice of words and actions to respond when they were asked questions in the interview.

A combination of written responses and interviews will serve to provide evidence towards the two research questions for the study. The algebra classroom teacher's interview will be conducted to better understand the students' academic background and the instructional environment. The interview questions range from a loosely guided exploration of topics to structured questions. The participants' reflections are expected to provide in-depth data for participants' perceptions, impressions and concepts under study. The raw data collected from the sources of data will be written responses for assessment tasks (pictorial/graphs, contextual, algebraic and charts), and teacher and student interviews. The participants will be assigned pseudonyms for safeguarding privacy, and the interviews will be transcribed later for data analysis purposes.

Interview questions for students and the teacher were written as open-ended and conversational, thus allowing the participants' firsthand views, ideas and impressions to be explored. The first few questions will be intended to get a broad perspective of participants' interests and levels of confidence. More focused questions will then be asked to invite a more detailed discussion of linear functions and their representations. Using open-ended and nonjudgmental questions will encourage some unanticipated responses, generating emerging stories or concepts. The use of key words will help the interviewer collect more data from the participants. For example, the terms "easy" or "hard" used in an interview question allows for an open response, encouraging participants to describe their experiences and conceptions. It is very important that interview participants connect the relevant experiences as they reflect on interview questions. The participants become more engaged and responsive to answer questions with commonly used terms, such as problems being "easy" or "hard." Such questions leave more room for the interviewer to ask for in-depth information, and the participant can share his or her point of view and support for his or her response. For example, Question # 46 in the student interview protocol states, "There are three different representations shown in the story problem Task 3. Which representation is easiest for you to understand? Which one is the hardest for you to understand?" For this particular type of question, the researcher can ask many more probing questions. Along with other questions, the researcher can ask the interviewees why they have answered in certain ways, whether they can give reasons for their answers, and if they can elaborate or better describe their answers. Such prompts will help the interviewer observe and record the students' expressions, comfort levels, prior experiences, knowledge, and learning objectives as first-hand data. Charmaz (2002a, p. 33), emphasized "Look for the 'ums' and 'you know's' and then explore what they indicate." In the same context, Charmaz said it is essential to record the expressions, signal words, critical moments representing struggles, and articulations of personal experiences by the participants (Charmaz, 2006). On the part of the researcher, it is important to be aware of the key points identified above, to shape research questions, make inquiries, and gather useful data. It is also critical to keep a well-defined direction for obtaining data aligned with the research questions for the study. For example, student interview protocol Question # 35 states "Tell me more about this representation, what is the characteristic of this graph meaning, how does y changes when we change x?" These type of questions assist the interviewer in learning more about conceptual understanding rather than making assumptions about what is meant by the participants.

## 3.8 Data Analysis Procedures

In a qualitative research case study, data collection and data analysis are not exclusive or independent of each other; in fact, both are simultaneously-occurring processes. For example, during the interview process to collect data for the case studies, the researcher will allow the students to have a flow in their thoughts and reconstruct some of the interview protocol questions. The current case study data analysis employs a recursive and inferential process. It will consist of discussion of individual case studies with some strategies borrowed from grounded theory, such as transcribing interviews and coding the interview data for each case study, categorizing findings, and selecting themes related to the data collected and the two research questions. The next section will explain the methods that will be used for case study data analysis.

### 3.9 Transcribing Students' Interviews

The representation of audible and visual data in written form is the first essential part of qualitative analysis. The purpose of transcribing interview data is to have a deeper understanding of students' conceptual learning. It seems to be a straightforward task, but it is an interactive act rather than a technical procedure. It is impossible to interpret all human interactions and complexity of data, which may lead to unanticipated outcomes from the interviews. The technical skills for transcribing involves making judgments to include or exclude non-verbal dimensions of the interview such as which of the non-verbal parts of interview can be ignored or repeatedly stated. Transcribing also involves interpreting data; for example, if the participant said, "I don't know" or "I don't, no." For the current study about multi-representations, participating student interviews will be transcribed word-by-word for each individual case. Each research study requires a different level of detail and representation. This study's transcriptions include facial expressions, body movements, and verbal statements for student interviews.

### 3.10 Qualitative Data Coding

A commonly-used first step for qualitative research data analysis is coding the data. Qualitative coding is a process of naming a data segment, ideas and concepts to define, compare, interpret and share them with others. After the interview data is transcribed, it will be broken down to pieces to examine student concepts closely, giving names and codes, to compare relations, similarities and differences relating to student understanding for the same task. “Coding is a pivoting link between collecting data and developing an emergent thought to explain the data” (Charmaz, 2006, p.46). The current study will use open coding for labeling concepts. In other words, defining and developing categories based on what the interviewee might mean (as extracted from the transcription). The codes will closely adhere to the data to show the actions and disclosures of student perspectives. The codes will guide the learning process to start making sense of the data and understanding the participants’ standpoints and the rationale for why these actions were taken (i.e., student work/response).

The grounded theory strategy used for this multi-case study consists of two main phases.

1. Initial coding phase: includes clear reading of data, naming each word, line, and segment to extract useful information related to the purpose of the study (understanding of multi-representations of linear functions). During initial coding, the idea is to remain open to all possible theoretical directions indicated by the data, and codes are chosen from the content or sometimes decided by the researcher. This practice is called “open coding” and “in vivo coding.” The difference between open coding and in vivo coding is that the in vivo coding preserves the participants’ meaning of their points of view and actions, while open coding can be a word, sentence, or phrase used to identify action. For example, in the current study, some concepts from reading interview data such as patterns, definitions, sequences, meaning of terms, schooling, and confusion are open in vivo codes. There can be many types of open coding; the current study will use only line-

by-line open coding. (The interview transcript will be read line-by-line, every sentence and even word by word and an abstract label or name given directly from the content). This process of going through the data line-by-line and assigning codes is called line-by-line coding, which can be very helpful in building concepts and categories. Open codes are provisional, comparative, and grounded in data (Charmaz, 2006, p. 44). It is provisional because many ideas may be available for analytic discussion and can be reworded for better understanding and capture the meaning and actions. Initial codes are comparative because they can be compared within the current data and to the previously-collected data in the study. The initial open codes can be great in number (several pages of codes), but they are likely to be helpful for uncovering the minutiae embedded in the data. The open coding helps to explore the gaps and find holes leading to unanticipated directions in data analysis. For the current study, a code book will be made to keep all the codes and their definitions in one place. Later, it will be used to find the relevance of codes to the research questions, and for verification purposes, it will be shared with a graduate student to see if he or she can confirm the applicability for the focused coding phase.

2. Focused coding phase: The initial codes are used to sort, synthesize, integrate, and organize data to promote data analysis. It pinpoints and develops the most salient categories in big chunks of data to follow the subsequent steps for data analysis. Sorting implies the separation of events or objects with common characteristics and grouping them as categories, which display differentiating properties. It is not unusual to find overlapping events that can be placed in either category; the researcher has to make a decision as to which is more relevant to the initial codes. The names of the categories

may be different from the in vivo coding or open coding. For example, the category for the open codes “patterns, definition, sequence, change” can be “rate of change” or “functional behavior,” which contain many open codes that express the scope of findings within the group. Synthesizing also implies putting the findings together in categories. The categories are used to construct themes related to the two research questions for the current study. The qualitative data analysis will be completed by using the software Atlas TI for selected themes. The interview video transcriptions will be uploaded using the Atlas TI 7 software. The coding and selecting themes tools from Atlas TI 7 will be used to analyze the data and compare outcomes among participants. The software can also be used to represent the results as graphics or semantic diagrams.

Qualitative case studies are often questioned by researchers in terms of the transferability and reliability of the small number of participants and specific individual participants. Transferability of data implies that it can be used in other situations and provide similar outcomes. Stake and Denscombe (1998) suggested that although each case may be unique, it still represents a broader group, thus confirming the transferability of the qualitative case study. The last part of the current data analysis is the check for transferability. This will be done by putting thick data descriptions from students’ records into the Atlas TI 7 program and extracting results to reflect transferability for the current study. Reliability of the data will be verified by sharing the codebooks and software coding, using a triangulation process and member check with participating students. I will also use peer debriefing with colleagues at CSTEM. The memos generated from Atlas TI 7 will be helpful to support dependability of the data.

## 3.11 Summary and Introduction to Chapter 4

Chapter 3 provided the methodology for the data collection procedures for the study. The sources of data will be the students' written responses for assessment tasks (pictorial/graphs, contextual, algebraic and charts) and teacher and student interviews. The participants will be assigned pseudonyms for safeguarding privacy. The interviews will be transcribed for each individual case study for data analysis purposes. Some strategies will be borrowed from grounded theory for initial and focused coding of contextual data. Atlas TI 7 software will be used for categorizing codes and creating themes related to the two research questions for the study.

Chapter 4 will provide findings from the data analysis, the individual interviews, and students' written responses for the six tasks. The outcome of the data analysis of the current study will be valuable to find the similarities and differences in using multirepresentations among five students. The results will help to reflect upon the level of conceptual understanding of linear functions in pre-college students and applications in STEM disciplines The study will expand to explain the barriers that students encounter as they continue using multi-representations for linear functions. It is also important to look for any significant difficulties faced by students in flexibly moving across representations and relationships among student perceptions.

# CHAPTER 4: DATA COLLECTION AND FINDINGS

Research question one seeks to characterize students' understanding of linear functions and their ability to build connections among multi-representations. As a part of the data collection process, students' perceptions and background information were also recorded providing information on indirect influences on their understanding of linear functions.

## 4.1 Research Question One

How do multi-representations help Pre-College students understand linear functions?

## 4.2 Research Findings

There were three major findings regarding question one.

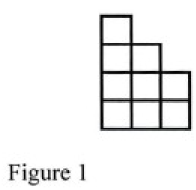
1. Four out of five students successfully completed tasks related to linear functions and their multi-representations (algebraic, graphic, tabular, pictorial, and contextual).
2. Students were limited in their ability to move flexibly among representations of linear functions.
3. Students faced difficulty with constructing graphs from a linear function expressed as an algebraic equation or presented in word problem.

## 4.3 Discussion of Finding 1

1. Four out of five students successfully completed tasks related to linear functions and their multi-representations (algebraic, graphic, tabular, pictorial, and contextual). The interview tasks for the current study were designed to probe students' prior understanding

of algebraic, graphic, tabular, pictorial and contextual representations of linear functions and their ability to flexibly move among various representations. Students' responses for Interview Task 1 provided information about their prior experiences. Task 1 asked participants to find a rule for the rate of change of successive patterns for a given set of figures and predict the linear behavior of the function.

Statement of Task 1: Look at the following pattern and make pictures to represent Fig 0, Fig 4 and Fig 5. Can you find a rule that will determine the total # of squares in any pattern like the given figures?)




Figures 4.1: Pattern 1

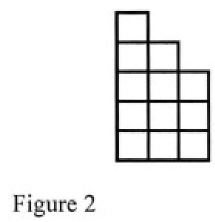



Figure 4.2: Pattern 2

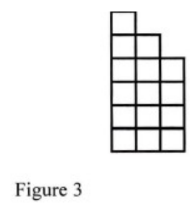



Figure 4.3: Pattern 3

Although the students' verbal responses did not show a clear understanding of rate of change and growth of patterns, their written responses showed that the students were familiar with the recursive behavior of the patterns. The students worked on task 1 by looking at patterns and finding a rule for constant rate of change in different ways. One student, out of the five participants, showed that there was a direct relationship between the dimensions of the given patterns and the figure numbers. This student expressed it as a rule for the growth of the pattern. Four out of five participants were confused about finding the rule for the rate of change of the given pictorial representation of a linear function (given as patterns). For example, when Aaron looked at the shapes as squares and lines in the given representations, he wrote a sequence as "6, 9, 12, 15…."

He found the sequence to be a recursive pattern of adding 3 in order to get to the next pictorial representation. He wrote the rule as “adding 3”; then he realized the rule could be different and changed it to y = 3x. Edward demonstrated confusion, though he continued to attempt identifying a rule. First, he wrote the sequence as 6 + 3, 9 + 3, 12 + 3; then he wrote “Figure 6 = 24 and (x (5)) +3 or 6(5) +3 =33 and (10(5) +3 = 53”. He found that it did not match the given pattern and tried to find his mistake. After some struggle he gave up and said, “I know it, but I cannot express it at this time.” He then said, “Honestly, I know the pattern but I don’t know how to write it.” For the pictorial representation, Edward traced the boundary for Figure. 3, and then he drew outlines for Figure. 4 and Figure. 5, that was slightly bigger than the drawing for Figure. 3.

Fig 4.4 shows Edward’s visual perceptions before constructing the tile patterns.

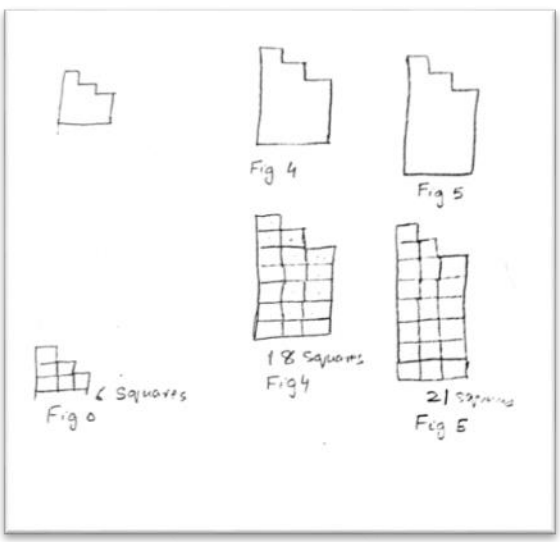



Figure 4.4 Copy of Edward’s work for task 1

Next, he filled the outline with smaller-sized squares. He also wrote his rule as “This involves adding three to each figure to find the next.” Likewise, Logan tried but was unable to make Figure 5; he stopped after making Figure 0 and Figure 4. Logan wrote his rule as “$n_5$ = 3F + 6” and explained it as 3 times the figure number plus 6. Logan was not able to explain the reason for multiplying by 3 and adding 6. He looked at the figures from 1 to 3 for a few minutes and then made the following list:

| | |
|---|---|
| 0. | 6 |
| 1. | 9 |
| 2. | 12 |
| 3. | 15 |
| 4. | 18 |
| 5. | --- |

Kader and Julie counted the number of square tiles in each figure and used mental math to write a rule for the progression. They first made pictures for figure 4 and 5 and then went back to draw a picture for figure 0. Julie used mental math, meaning that she did the calculations in her head using the math facts from to her memory, and made an estimate for the answer to the question. Julie formulated an equation for the rule as y = x + 3 (adding 3). She then used the trial and error method to justify it (a method to substitute random values for variables that satisfy the given equation). She changed her rule to “y = x,” tried to write the sequence, and found that it did not match the given pattern. Julie also returned to the same rule (y = x + 3) and wrote the arithmetic sequence as 9, 12, 15, 18, 21. She calculated Fig. 0 = 6 by assumption and said, “the way it works with the rule x + 3 is that Fig. 0 = 6, Fig.1 = 6+3 = 9, Fig. 2 = 9 + 3 = 12, Fig.3 = 12 + 3 = 15, Fig. 4 = 15 + 3 = 18, and Fig. 5 = 18 + 3 = 21.” The researcher asked her why she did not begin with x = 0 for Fig 0, x = 1 for Fig.1, and so on, and Julie answered “because it works this way!”

Kader explained that the number of square tiles helped him determine the constant and find a rule for the given pattern (see figure 4.2). Kader calculated a rate of change for the pattern. He used mathematical reasoning to investigate the rate of change. He wrote two ordered pairs (1, 9) and (2, 12), then wrote (12 - 9) / (2 - 1) = 3. Next he wrote y = 3x +

6. For x = 0: he calculated y = 6, verified the equation for x = 1, y = 9, and then he made a table. From the table, he calculated the number of squares for Fig. 0, Fig. 4, and Fig.5 and drew pictures for each one of them. Kader circled his rule: y = 3x + 6. Figure 4.5 below represent Kader's written work sample.

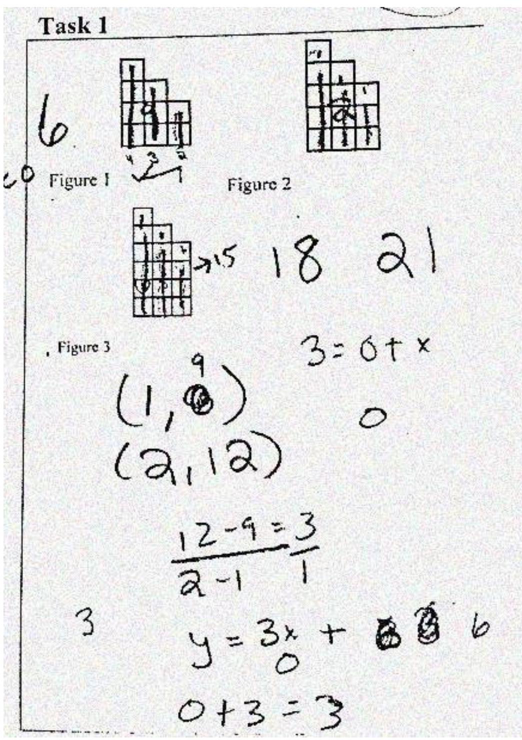
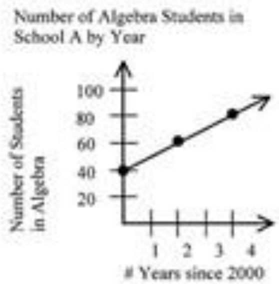



Figure 4.5 Copy of Kader's Task 1

Interestingly for task 3, the algebra student population for four schools was represented by multi-representations of linear functions. The problem shows four representations of three similar linear functions. Find which school has the fastest rate of change?

Task 3: Below, you will see the number of algebra students at four different schools, School A, B, C and D. Assuming population rate of change is linear answer the following questions.

School A:




School B:

| Year | Population of School B |
|---|---|
| 2000 | 32 |
| 2001 | 60 |
| 2003 | 116 |

School C:

$y = 120 - 30x$

Where x=number of years since 2000, and y = the number of algebra students at school C

School D:

In 2000 School D had 27 algebra students. In 2003 school D had 72 students enrolled in Algebra.

a. Determine the rate of change in population for each school starting year 2000 until 2005.

b. What is the total population of students in each school in the year 2005?

c. What is the predicted population in 2015?

d. Draw a graph to represent each of the school algebra student population in School B, C and D.

e. Explain in own words about the answer to the problem

Figure 4.6 Assessment Instrument Task 3

All five participating students recognized the recursive behavior of linear functions. In order to find the linear functional behavior of each representation, the common approach was to look for the rate of change of the function, but the students showed a lack of in-depth understanding of each representation (algebraic, contextual, graphic, and table). All five students were able to calculate the rate of change, but did not identify it as a rule for the growth and decay of the linear function. Kader and Edward showed awareness of the independent and dependent variables as an essential part of any linear function. Julie made her graph on a graphing calculator for the algebraic and contextual representations. The graphing calculator uses TI-85 software for problem solving. Julie displayed the graphic representations of linear function but her interview responses did not show conceptual understanding of the relationship between a linear function and its graphic representation.

In considering Task 5, Aaron used mental math to find the time taken by the two riders moving at different speeds to meet at a certain point, but his diagram was not consistent with the work shown in context of task 5 (Figure 4.7).

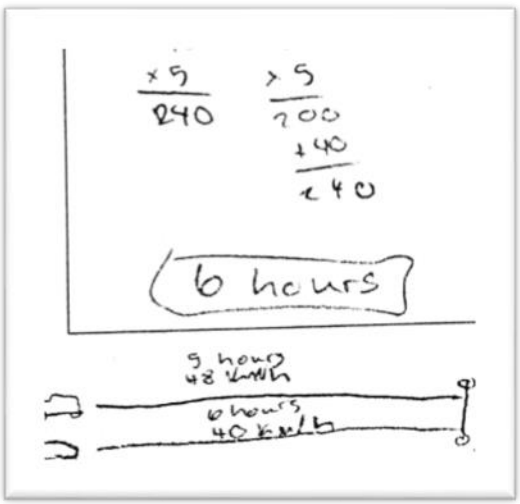



Figure 4. 7 Aaron's work on task 5

Logan made a table to find the distance covered by the two riders to meet at a point. Julie used mental math, but the final answer was inconsistent with what was asked in the problem.

"Jose drove one hour extra to catch up with Rob," said Kader. The question asked was; "How long did Jose drive before Rob caught up?"

Kader: Okay, Jose left the white house and drove to the recycling plant at an average of 40 kilometers per hour. Rob left sometime later, driving in the same direction at an average speed of 48 kilometers per hour. After driving for 5 hours, Rob caught up with Jose. How long did Jose drive before Rob caught up?

Researcher: Okay, tell me how are you going to work on this problem?

Kader: The first step that I would do is that I would first draw out this equation. So I would just label the J as Jose, so she is going at 40 KMH.

Researcher: Mhm.

Kader: And then Rob here started driving later at 48 KMH. And then, she has already been driving, how long will it be before these two equal each other?

Researcher: Okay.

Kader: So what I did was that since it's saying after driving for 5 hours, Rob caught up with her. So you had to find out how much distance Rob has driven for 5 hours, so since we know that he is doing 48 kilometers per hour, and we do 48x5=240, so after 5 hours, Rob has driven 240 KMH.

Researcher: Okay.

Kader: And we know that Jose is also equaling the same thing because they caught up. They are breaking even. We know that 240, which can take the place of Rob's speed is equal to 40J. So we can divide both sides by 40 to isolate J, you find out the Jose has rode for 6 hours before Rob. We can write (shows on paper) (6, 240) and (5, 240).

Kader used vector representations. Kader determined the coordinates to be (6,240) and (5, 240). Using mental math, he wrote a proportion to compare the time and distance traveled (240/40 = 40J/40), reduced it to 240 = 40J, and found that Jose traveled 6 hours to catch up with Rob. He did not provide the justification for using 240 as the distance covered or any comments about the other rider, with Jose going at a speed of 48 km/h. Kader did not explain the mental math processes he used to find how their times compared. This may reflect on the students' lack of confidence in justifying the problem-solving method used for linear functions. Figure 4.8 shows Kader's performance on Task 4.

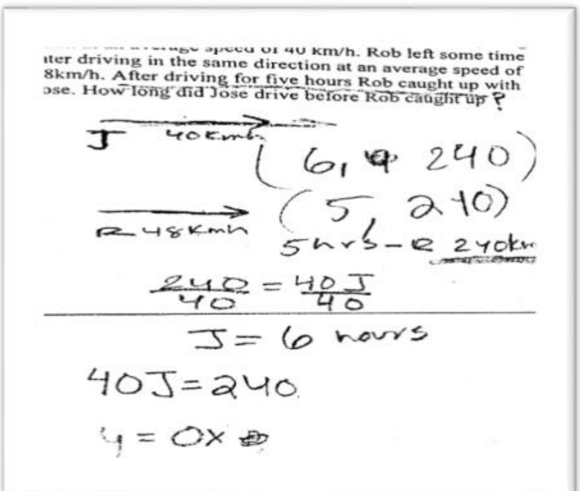



Figure 4. 8 Kader's work for Task 4

Edward commented for task 4; "No, it cannot be solved because there are two different variables (speed and time) for two different people."

## 4.4 Summary Finding 1:

The results from the computer simulation using qualitative analysis software, Atlas Ti-7, showed that on average, about half of the students were not successful in demonstrating conceptual understanding of multi-representations of linear functions.

Additionally, the collection of interview response data regarding students' contextual understanding of linear functions is still not sufficient to confirm the students' ability to transform and flexibly move among different representations. Student performance for task 1 demonstrated evidence for their understanding of linear functions and the related characteristics. The results showed that most of the participants lacked a clear understanding of rate of change and growth patterns. They also had difficulty in flexibly transforming pictures to equations. All five students observed that the figures were changing yet staying the same for pictorial growth. Three out of five students relied on the number of squares and mental math skills, while the other two students considered the dimensions of the figures as well. All but one student were able to show that there is a direct relationship between one of the dimensions and the figure number. The students were familiar with the visual representation for the growth pattern but found it difficult to define a rule for the rate of change in the patterns. They were not able to make a connection for the changing number of tiles in successive figures and to relate the pictorial representation to contextual or algebraic representations.

## 4.5 Discussion of Findings 2

Students showed limited ability to move flexibly among representations of linear functions.

The results from Atlas Ti -7 data analysis showed that less than one fifth of the students' responses demonstrated an understanding of multi-representations of linear functions. Additionally, four out of five of students showed limited understanding of positive and negative rates of change. Also two out of five students' written responses reflected lack of flexibility in moving among representations. Analysis of the interview data showed that it was difficult for four out of five students to express a word problem using graphic or algebraic representations, and only one out of five students showed in depth knowledge of the rate of change of linear functions. The majority of the students were able to interpret and make inferences from graphic representations but faced difficulty in translating linear functions to multi-representations.

Interview responses to Tasks 1, 2, 3 and 4 provided evidence of the students' difficulty to flexibly move among multi-representations such as patterns, tables, graphs, contextual and algebraic representations of linear functions. In Task 1, Kader translated the linear context to the linear function as $y = 3x + 3$. Comparing the equation and the given patterns, he found a deficit of three tiles for each consecutive pattern; then he changed his function to $y = 3x + 6$. All five students tried to verify the algebraic representation using the individual rules they found to represent the sequence for the pattern. None of the students talked about the fact that the changing figures showed a recursive rate of change, representing a continuous linear function behavior. Out of the five students, only Kader recognized that the given patterns represented a linear function in progression and that Figure 0 represented the initial condition as a constant, Figure 0 = 6.

In Task 2, a graphic representation (see Figure 4.9 and an algebraic equation ($y= 3x -4$) representing linear functions were given. Task 2 provided an opportunity for students' to demonstrate an in-depth understanding about the positive and negative rate of change and the connection between multi-representations. The students were asked to compare the two representations and, using their knowledge of rate of change and intercepts, determine whether the two representations were for the same function or for two different functions. The two representations were continuous and infinite but had different slopes and x-, y- intercepts. Representation 1 was a graph as shown in Fig 4.9 and the representation 2 was the algebraic equation y = 3x - 4

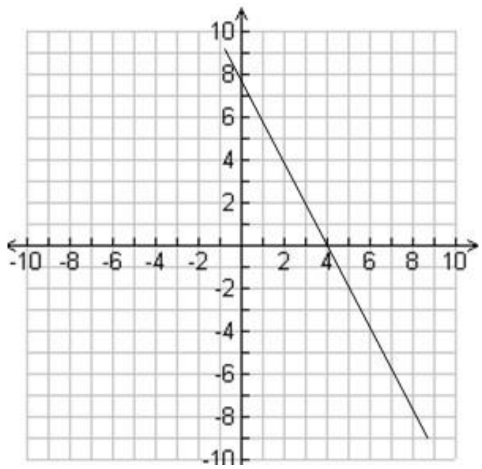



Figure 4.9 Task 2 graphic representations

All five students were able to recognize the y-intercept of the graphic representation, but finding the slope from the graph was a challenge for three out of five students. On the other hand, two students found it easy to find the slope of a linear function as the coefficient of x in the algebraic representation. Aaron, Edward and Logan explained each representation separately. Aaron spent 5-7 minutes thinking and then wrote, "I think it is negative for the given graph because it is going down." Other than Aaron, no student talked about the negative slope to represent 'going down' behavior. For the given algebraic representation y = 3x – 4, Aaron explained that "the intercept is -4, but the given graph has a slope of negative 1 and the slope of algebraic representation is positive 3x." He copied the graph on his paper and worked on task 2 graph. Figure 4. 10 shows Aarons work for task 2.

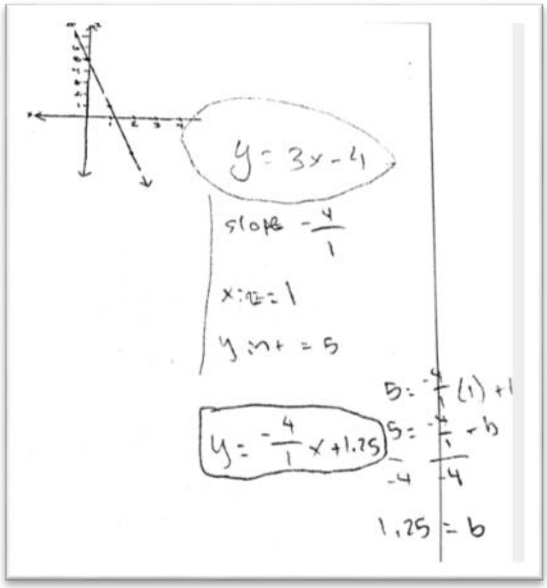



Figure 4.10 Aaron's work on task 2

Aaron found the slope for the graphic representation to be -4 /1 and the x intercept to be 1. For the y intercept, he substituted the value of y from the graph y = mx = 5. For x the equation was transformed to 5= (-4/1) x + b. Solving for "x," he found b = 1.75 and wrote the equation for the graph as y = (-4/1) x +1.75.

Two out of five students knew how to write the slope intercept form of the equation, but they had difficulty writing the slope intercept form equation for linear functions. An example of student work for task 2 is shown in Figure 4.11

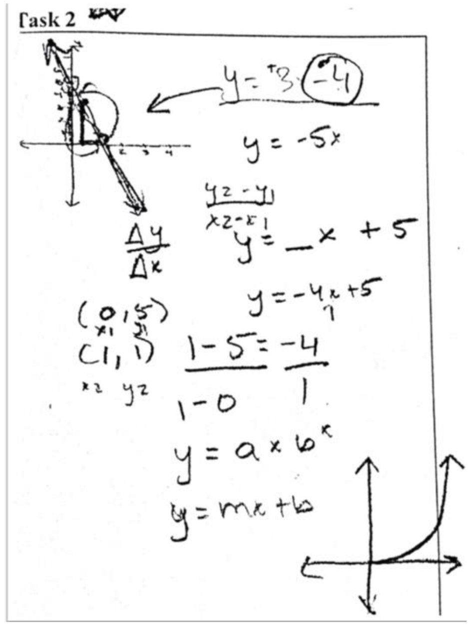



Figure 4.11 Student work: moving from graphic to algebraic representation

Edward found that the y-intercept was positive in the graph but was negative in the algebraic representation, the slope of the given graph = -2, x intercept = 4, and y intercept = 8. For the given algebraic function, he found the slope to be -3, x-intercept to be 3, and y intercept to be -4, but his graphic representation was different, the linear function equation showed the x intercept as +1, y intercept as -2, and the slope -2. As shown in figure 4-12.

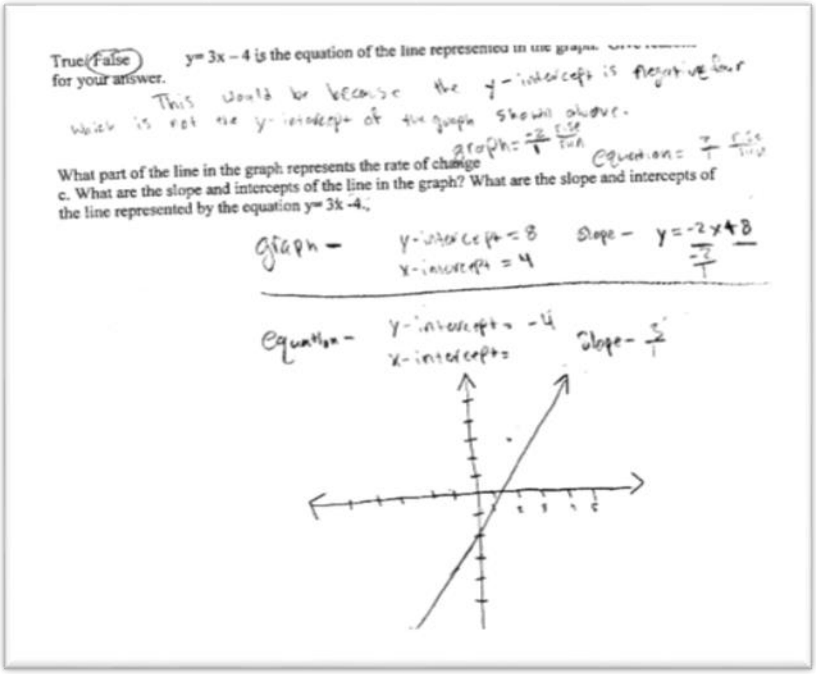
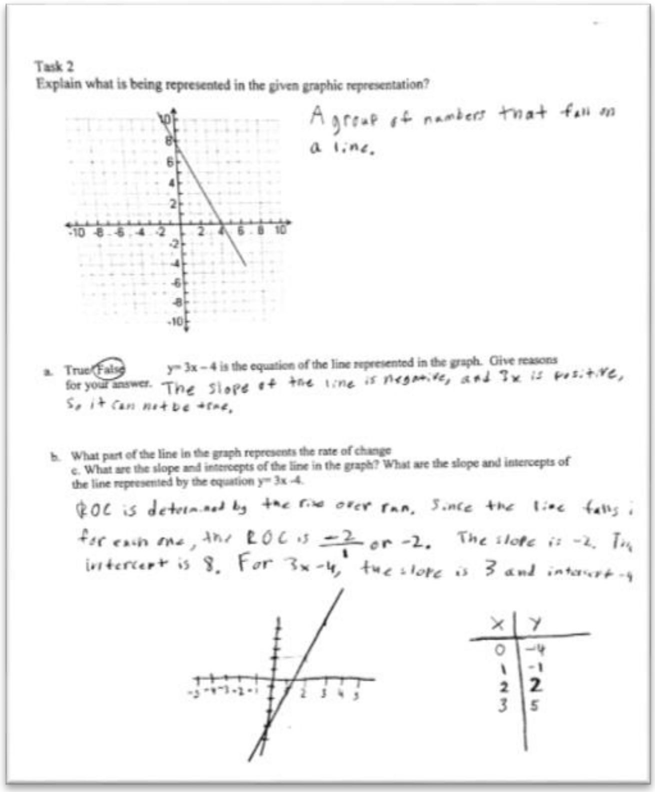



Figure 4.12 Edward’s work on task 2

Edward showed the slope is -2 for the graph, but he was not able to explain, verbally or symbolically, why the slope has a negative sign in front of it. Logan showed a clear concept of slope and y-intercept for a given linear function but he said he did not know the x-intercept. Logan calculated the slope using the definition of slope $m = (y_2 - y_1) / (x_2 - x_1) = -2$ with y intercept = 8 from the given graph. Figure 4.12 represents Logan’s written work for Task 2. Logan wrote the coefficient of x as slope =3 and y intercept =-4 for the given algebraic representation.


Task 2

Explain what is being represented in the given graphic representation?

A group of numbers that fall on a line.

a. True/False   y= 3x – 4 is the equation of the line represented in the graph. Give reasons for your answer. The slope of the line is negative, and 3x is positive, so it can not be true.

b. What part of the line in the graph represents the rate of change

c. What are the slope and intercepts of the line in the graph? What are the slope and intercepts of the line represented by the equation y= 3x -4.

ROC is determined by the rise over run. Since the line falls 2 for each one, the ROC is -2/1 or -2. The slope is -2. The intercept is 8. For 3x-4, the slope is 3 and intercept -4

| x | y |
|---|---|
| 0 | -4 |
| 1 | -1 |
| 2 | 2 |
| 3 | 5 |

Figure 4.13 Logan's written work for Task 2

Julie looked at the task for 3 minutes, then wrote the two ordered pairs as (0, 5) and (1, 1) and created two algebraic representation for the graph as "y – 5 = 5 (x - 1) and y = 5x + 1." She concluded that slope was 5/1 for the graphic representation. For the given linear function equation, she drew a graph with y-intercept -4 and x-intercept 3. Julie defined slope as $m = (y^2 - y^1) / (x^2 – x^1)$ and explained that "for every y going up by 5 units, the change on x axis is one unit," but she wrote the slope for algebraic representation as -4/1.

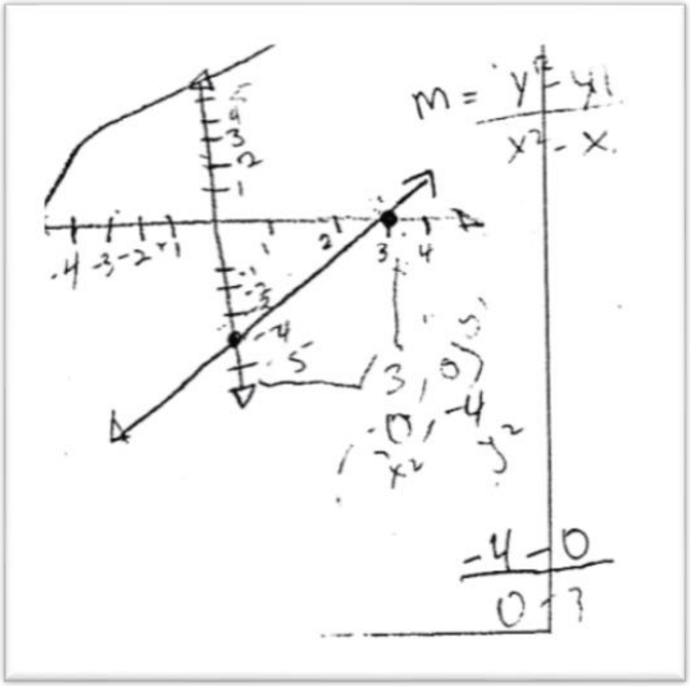


Figure 4.14 Julie’s written work for Task 2

Kader used a different approach and constructed a table of values for task 2. Kader reasoned for the difference in the graphic and algebraic representations, that the y-intercepts were different. Kader did not comment on the slopes being different. Kader made a careless mistake in writing the equation as y = 3x +4 instead of the given equation in the problem as y = 3x - 4. He corrected his mistake and made a new graph for y= 3x – 4 as shown in figure 4.15.

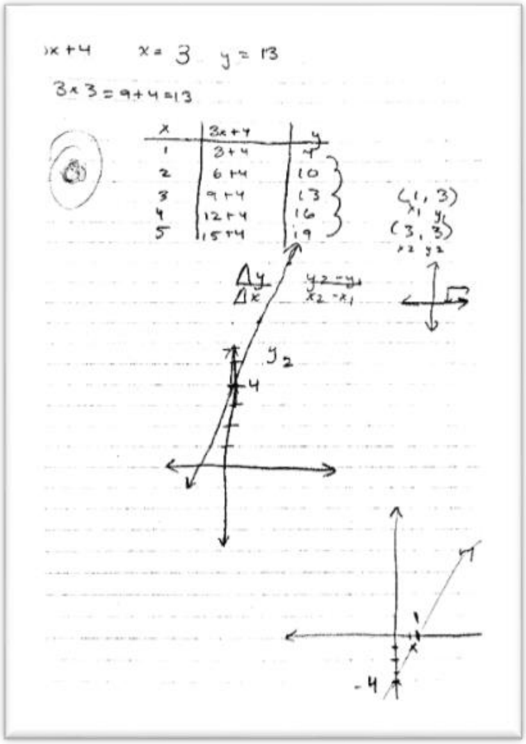


Figure 4.15 Kader's representation for y = 3x + 4

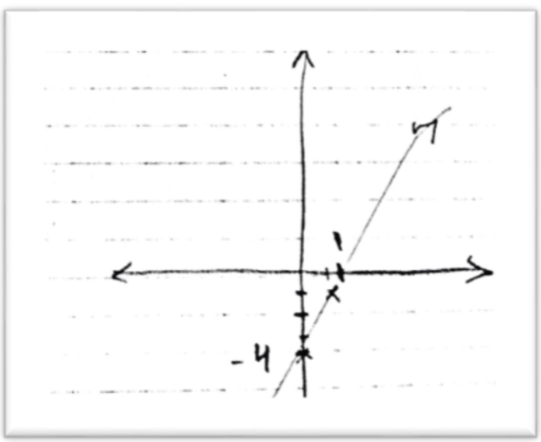


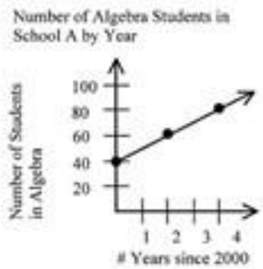
Figure 4.16 Kader's representation for y = 3x – 4

Task 3, further demonstrated that there was a significant lack of in depth understanding of linear functions, their representations, and their ability to transition from one type of representation to another.

Figure 4.17: Task 3

Task 3: Below, you will see the number of algebra students at four different schools, School A, B, C and D. Assuming population rate of change is linear answer the following questions.

School A:




School B:

| Year | Population of School B |
|---|---|
| 2000 | 32 |
| 2001 | 60 |
| 2003 | 116 |

School C:

y = 120 – 30x

Where x=number of years since 2000, and y = the number of algebra students at school C

School D:

In 2000 School D had 27 algebra students. In 2003 school D had 72 students enrolled in Algebra.

a. Determine the rate of change in population for each school starting year 2000 until 2005.

b. What is the total population of students in each school in the year 2005?

c. What is the predicted population in 2015?

d. Draw a graph to represent each of the school algebra student population in School B, C and D.

e. Explain in own words about the answer to the problem

Figure 4.17 Task 3

The student work samples for Task 3 are shown in Figures 4.18 and 4.19.

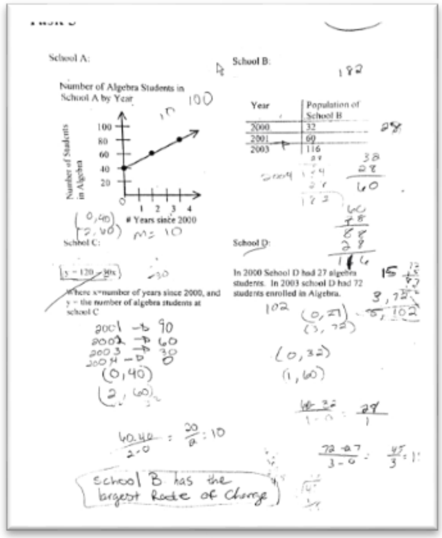



Figure 4.18: Student work for Task 3

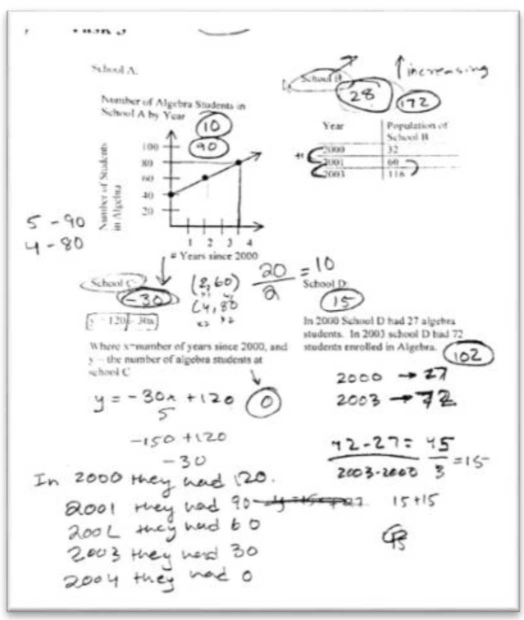



Figure 4.19: Student work for Task 3

All students were confused when they found a negative rate of change for the student population in School C. Logan and Edward, drew the graphs for schools B, C, and D but insisted that the negative slope could be ignored because the number of students cannot be negative in real life. Figure 4.13 and 4.14 show students' written responses.

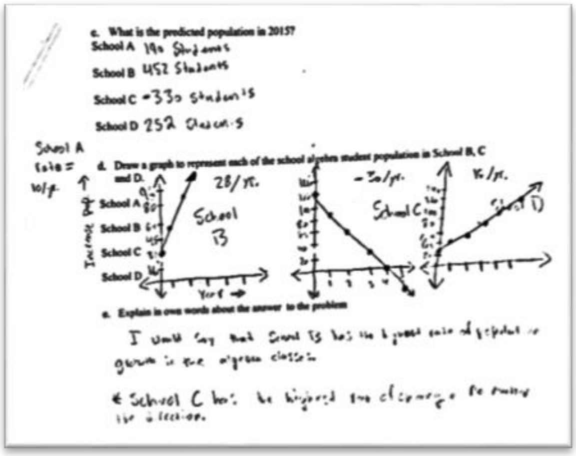



Figure 4.20 shows graphic representation task3 part d by Logan

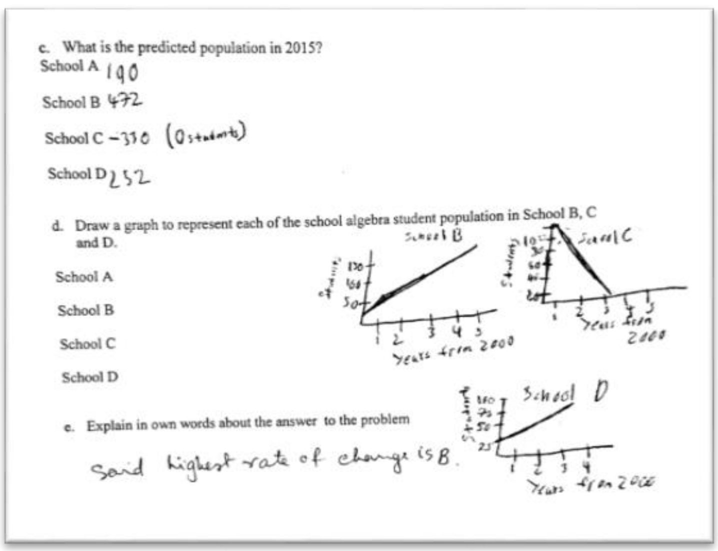



Figure 4.21 Edward's Task 3 part d graphic representations

Despite some errors for choosing scale, three students showed an understanding of trend and graphic representation of a linear function given as an algebraic equation. The rationale for Task 4 was that changing the way a function is represented does not affect the nature or type of the function, but different forms or representations highlight different characteristics of a given function. In Task 4 a linear function in two variables in non-standard form was given. The students were expected to transform the given equation into standard form, derive the rate of change, and transform the linear functions from algebraic (equation) to graphic (visual) representations. Two out of five students demonstrated their conceptual understanding.

Task 5 presents the following scenario: Jose left the White House and drove toward the recycling plant at an average speed of 40 km/h. Rob left some time later driving in the same direction at an average speed of 48km/h. After driving for five hours Rob caught up with Jose. How long did Jose drive before Rob caught up?

Each student used a different approach to solve the problem to this real-life situation. Edward withdrew from working on Task 5, saying that the students in his class were expected to use only the representations taught in class and that he was not taught to make graphic representations for a word problem.

Kader successfully moved from contextual to graphic representations and made inferences from a constructed vector diagram. Another student drew a picture representing the cars and the path followed, but he did not show that “Rob” (the name from the problem description) started later than “Jose” (the name from the problem description). Kader, another student, went a step ahead of the remaining four students because he used an advanced mathematical approach for Task 5. He used vector representation and notation to explain the situation of the two runners, who started at two different times and at different speeds, in the problem. Vectors generally represent physical contexts and real-life situations in STEM-related fields. Kader successfully moved from contextual to other representations and made inferences from his vector diagram. Two students relied on a calculator and hesitated to show their work on paper.

Edward excused himself from working on Task 5. He recognized the problem to be about ratios and proportions but did not realize that he was looking for the linear behavior of the function. Edward said "Ah ratios! Honestly I do not know ratios—proportions, I don't know how to work on this problem. I am not comfortable with ratios and proportions." The last part of Task 5 asked students to write a word problem similar to the Task 5 problem and related to linear functions. Edward and Logan did not opt to writing similar problems for linear functions. Julie, Aaron and Kader wrote their word problems with their solutions for linear relationship. As presented below the students struggled to find solutions for their own word problems related to linear functions.

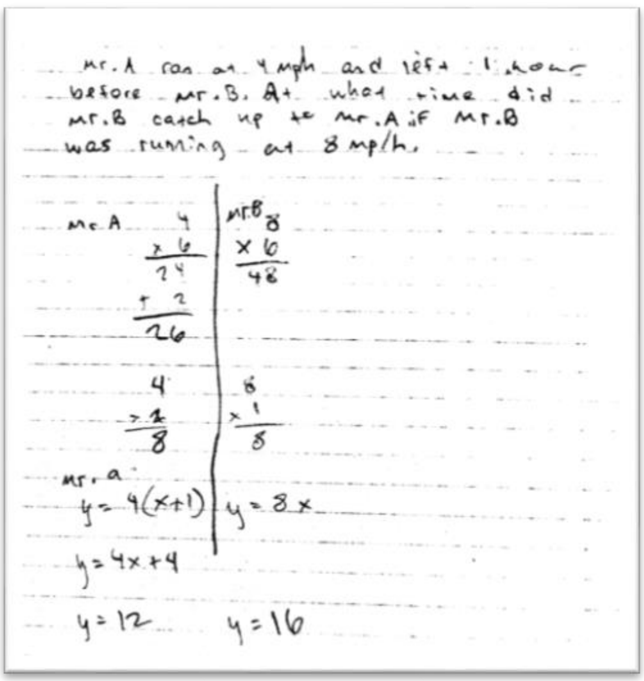
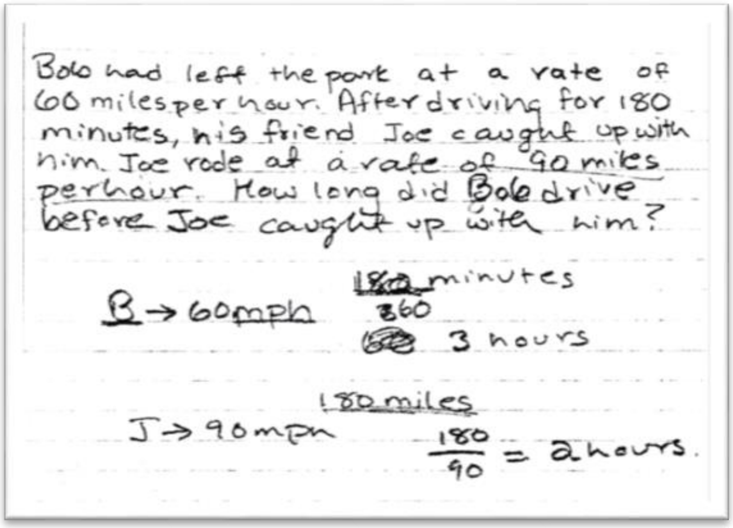



Figure 4.22 Julie's word problem



Bob had left the park at a rate of 60 milesper hour. After driving for 180 minutes, his friend Joe caught up with him. Joe rode at a rate of 90 miles perhour. How long did Bob drive before Joe caught up with him?

B → 60mph

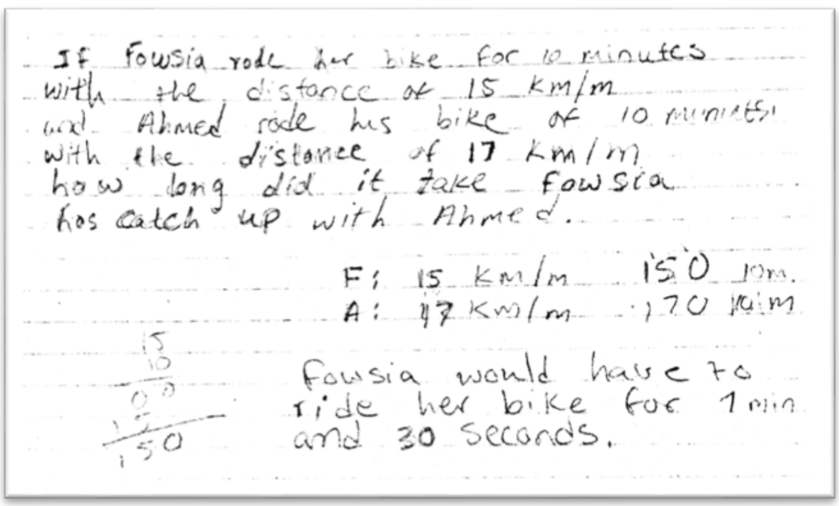
180 minutes
360
3 hours

180 miles

J → 90mph

$\frac{180}{90}$ = 2hours.

Figure 4.23 Kader’s word problem



If fowsia rode her bike for 10 minutes with the distance of 15 km/m and Ahmed rode his bike of 10 minutes with the distance of 17 km/m how long did it take fowsia has catch up with Ahmed.

F: 15 km/m 150 10m.
A: 17 km/m 170 10m

15
10
150

fowsia would have to ride her bike for 1 min and 30 seconds.

Figure 4.24 Aaron word problem

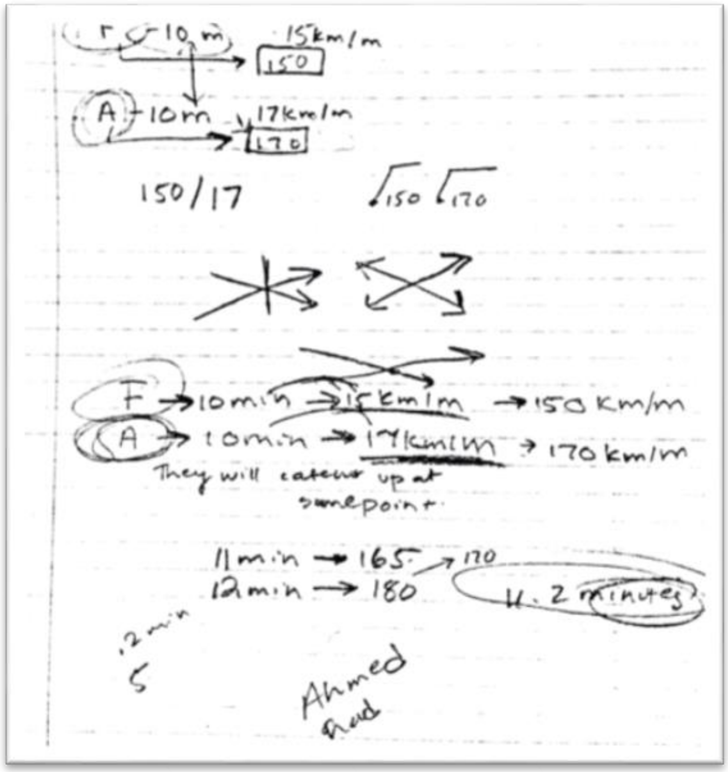



Figure 4.25 Kader’s solution to Aaron’s word problem

Task 6 provided further evidence that the students’ level of comprehension and their ability to make inferences about the rate of change/slope of a linear function and the concept of slope and y- intercept was weak. Kader showed in his written response, Figure 4.23 that there was a solution at point (1, 1) for the two intersecting graphs of linear functions but he did not specify it to be the only real solution to the problem. He also wrote his comment that “not perpendicular, not parallel – 1 solution, not imaginary”. It seems from his comments that he has an idea that if the lines are not parallel in a system of functions, it is expected to have one solution. It also shows that the student was relying on his visual memory and not thinking conceptually about the system of functions represented in task 6.

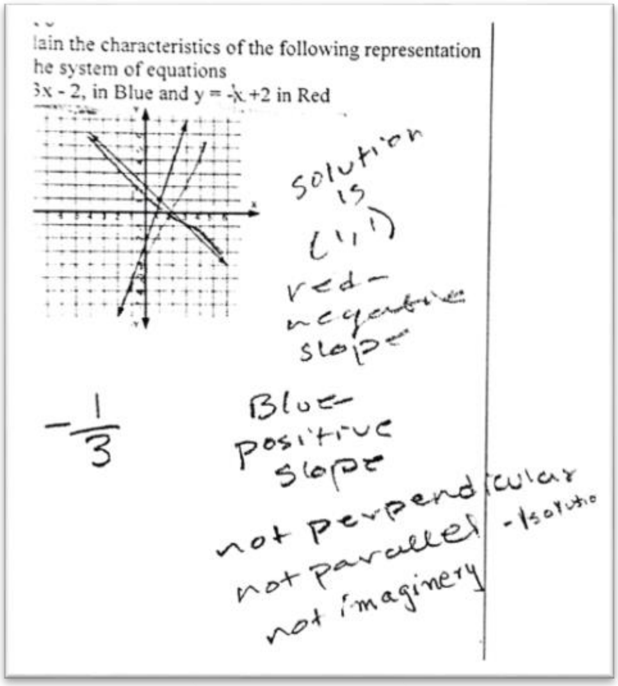



Figure 4.26 Kader’s work sample for task 6

Logan, Figure 4.24, had a similar comment “Intersecting at equality” for the types of solution for the system of linear functions in task 6. He was able to find the slope and intercepts for each individual function (represented as blue and red line).

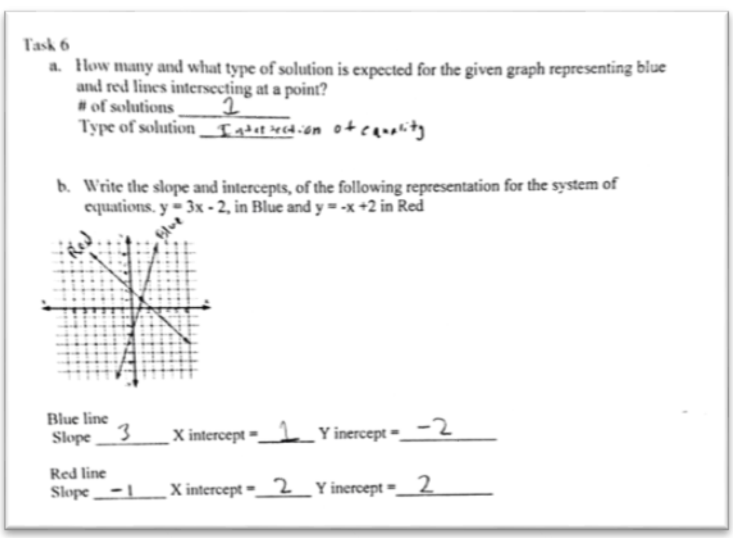
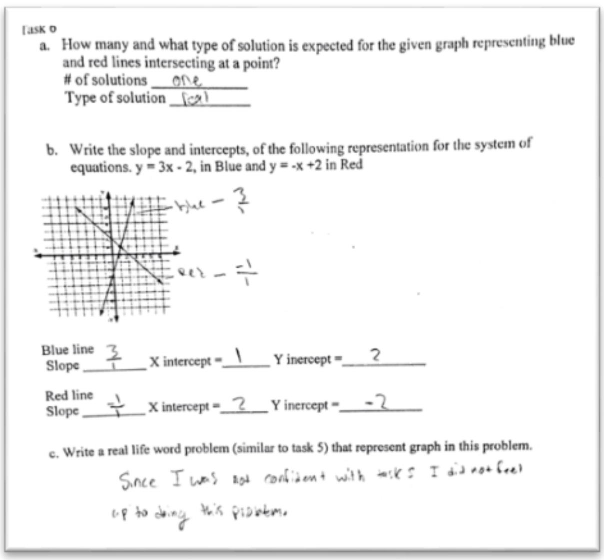



Figure 4.27 Logan’s work sample for task 6

Edward found that there is one real solution for the given system of functions in task 6, but his illustration on the graph was incorrect. He interchanged the slopes for red and blue lines representing linear functions in the given task. Edward showed his understanding for algebraic representation but it seemed like he made a guess for graphic representations. A similar situation was also noted from his interview responses in task 2 ("I am not good at drawings and graphs" Edward interview response to question #2).


Task 6

a. How many and what type of solution is expected for the given graph representing blue and red lines intersecting at a point?
\# of solutions one
Type of solution real

b. Write the slope and intercepts, of the following representation for the system of equations. y = 3x - 2, in Blue and y = -x +2 in Red



Blue line Slope 3/1 X intercept = 1 Y inercept = 2

Red line Slope -1/1 X intercept = 2 Y inercept = -2

c. Write a real life word problem (similar to task 5) that represent graph in this problem.

Since I was not confident with task 5 I did not feel up to doing this problem.

Figure 4.28 Edward's work sample for task 6

## 4.6 Summary Findings 2

All five students faced difficulty translating among graphic and algebraic representations, as well as a lack of understanding about the nature of positive and negative rate of change. Comparing the given graphic representations with the corresponding algebraic equation showed that the students were more familiar with the trends of graphs with positive slopes. Three out of five students were able to retrieve information from algebraic representations and make graphic representations, but the students were unable to flexibly move among the two representations in responding to the task. The students were familiar with the definition of slope but three out of five students did not use the correct notation; they used m= $(y^2-y^1)/(x^2-x^1)$ instead of m = $(y_2-y_1)/(x_2-x_1)$. Only one student showed in-depth knowledge of rate of change of linear functions through the response to the task. Interestingly, students from the same school classroom made the same mistakes and misconceptions about concepts related to linear functions, perhaps emphasizing the influence of teacher instruction on conceptual understanding. Four out of five students recognized the rate of change to be rise over run for the graphic representation, and the coefficient of the independent variable to be the slope or rate of change in the algebraic representation.

## 4.7 Discussion Finding 3

Students faced difficulty in constructing graphs from a linear function expressed as an algebraic equation or as word problem.

Translating a linear function into a graphic reorientation was a challenging concept for most of the participants. All five students used algebraic method for solving problems in linear functions. When asked about their preference choosing algebraic method over graphic method for solving problems in linear functions; a student commented, “My teacher used graphs in a separate topic in class but never showed its relation to an equation.”

Task 4 was presented as follows:

a. Transform the equation in standard form - (-x - 5) = y - 2

b. What is the rate of change of the linear equation - (-x – 5) = y -2?

c. Represent the given equation as a graph.

Figure 4. 19 showed Kader's work for task 4. He reduced the equation to standard form and then constructed a table to find the rate of change for the linear function. He found rate of change to be 1 and found the intercepts as x= -7 and y= 7. Kader made a graph but limited it only in second quadrant (marked off the extension of the line). When asked that why he marked off the extended line, Kader said "then it will be beyond y=7 which is not right"

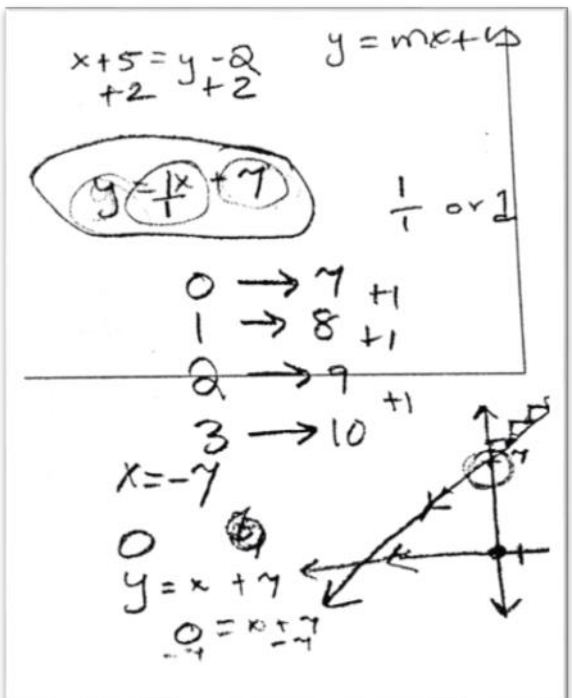



Figure 4.29 Kader's work on task 4

Edward worked on task 4 using mental math. He easily transformed the equation into standard form and reported the rate of change is 1. He did not give the values of intercepts worked on the calculator, copied the representation of graph without any scale indicated on the x and y axis. When he was asked to explain the graphic representation, he simply wrote "It is still going up and the x-intercept is negative".

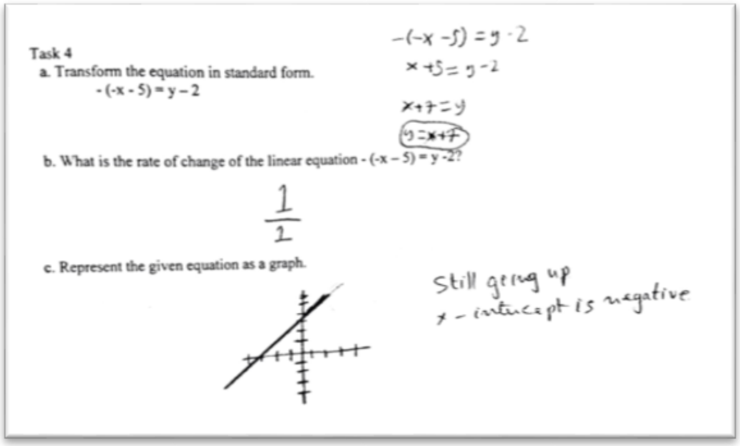



Figure 4.30 Edward’s work on Task 4

Logan also completed the first two parts of task 4 but for the graph he used the y intercept. He used mental math and drew the line. When he was asked to explain his graphic representations and what the x intercept is, he said; “this is it and the function has no x intercept.”

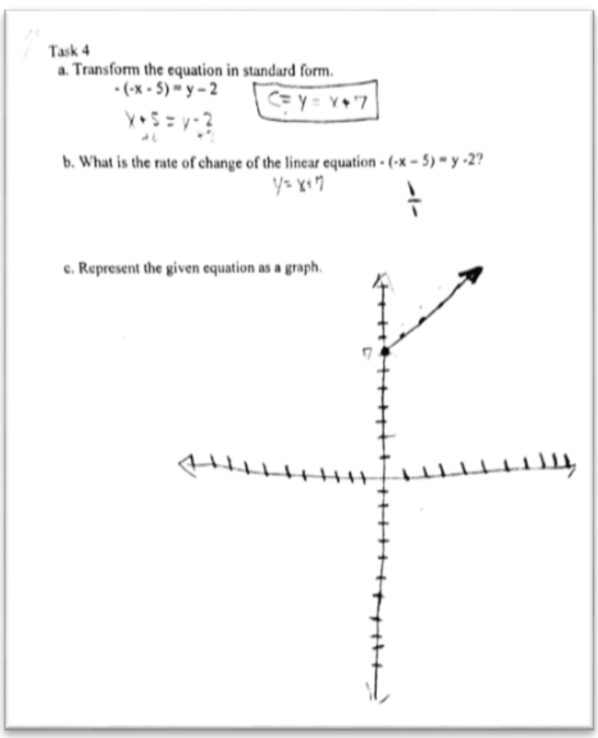



Figure 4.31 Logan’s work on task 4

Julie forgot to distribute the negative sign in the algebraic representation of task 5 (Y-2 = -(x-5)). Therefore, her calculations showed the y-intercept to be negative (-7) instead of positive. She found that slope =1 but did not make the graph. She said, “There is insufficient information to make a graphic representation.” Aaron was also hesitant to make the graphic representation for the transformed linear representation (task 5). Edward and Logan said they were not sure about the characteristics for the linear functions and Logan commented “still going up and x intercept is negative.”

Concerning students understanding of the graphic representation and flexibility to move among representations, referring back to figure 4.11, Kader’s work on task 2 also shed some light on students’ need to develop more skills and an in-depth understanding of these concepts. In task 2 students were a given a graph and they were asked to determine the characteristics of linear function and compare it with the given algebraic representation. Kader chose the two points on graph as (0, 5) and (1, 1). Using the definition of slope, she found it to be -4/1. Comparing it with a standard equation of line, he found the equation for the graph as y= -4x +5, thus proving that the given algebraic function y= 3x -4 is not representing the graph for linear function. When he was asked to explain the solution; Kader wrote an exponential function, $y = a. b^{x}$ and tried to establish correspondence with the standard form of linear functions y = mx +b.

Julie used the coefficient of x and the y intercept to represent slope for the given algebraic representation in task 2**.** For the given equation y =3x -4 she wrote the slope to be = -4/-3 and then she made a random graph and found its slope to be 3. She seemed to be confused and trying to re-call something from memory. In one corner of the page, she wrote an equation of line for a different problem instead of her own construction. When she was asked to explain Julie said she was not ready and excused herself form being further interviewed on task 2.

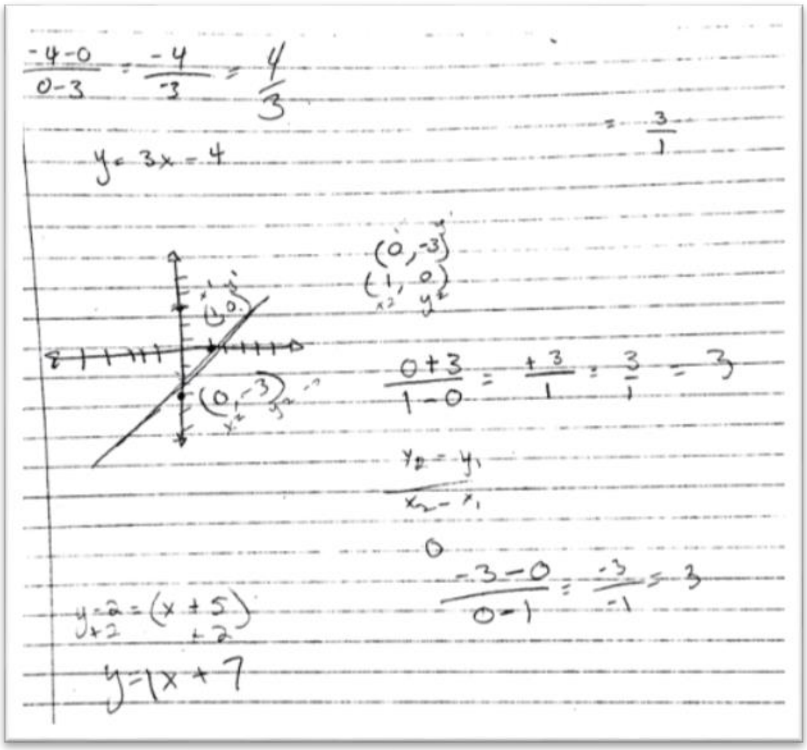



Figure 4.32 Julie's work on task 2

In Task 3, all students successfully found the rate of change in tabular and graphic representations, but they faced difficulty in the problem when it was represented as word problem. Only one out of five student attempted "part e" to explain the result from the graphic representations for task 3.

## 4.8 Summary Findings 3

Kader, Edward and Logan showed understanding of the standard algebraic representation of linear functions but explaining graphic representations in a functional perspective was not clear to them. All five students treated graphic representations as picture which inclines in a certain way, which can be found by using scientific calculators. Kader made a table to put values in the calculator to get the graph. The concept of intercept was also less clear. They treated it as point where the line cuts the axis with no physical meaning attached to it. The students found the rate of change or slope by using a mathematical formula. None of the students' representations showed a one-on-one correspondence or a clear concept of functions. Kader insisted to terminate his graph at the y intercept, while Julie had confusion for comparing linear and exponential graphs. The inability of students to explain linear functions and their representations could have been a result of students' misconceptions, teacher instructions and lack of familiarity of the language of mathematics.

## 4.9 Summary of Results for Research Question One

The three findings for research question one indicated that the students' lack an understanding of multi-representations of linear functions. Students' responses for the written tasks and interview questions showed a limited understanding of multi-representations of linear functions. The students' knowledge of multi-representations such as graphs, tables and algebraic statements were evident but their ability to make inference and move flexibly among representations was limited. Students' prior knowledge of the growth of patterns leading to finding the rate of change was insufficient. Four out of five students had difficulty in writing a rule for the growth of pattern similar to an algebraic representation of linear functions. Only one out of five students was able to show a direct relationship between the rates of growth of pattern leading to the slope intercept form of linear function. For task 3 all five students struggled to comprehend the concept of negative slope of linear function. They were not able to conceptualize that the negative slope indicate the reverse behavior of a function and in particular for task 3 it meant a rate for the decrease in student population. Initially all five students considered negative slope to represent something non existing, one student changed his mind after prompted with questions from the researcher. All five students had demonstrated that they could recall the mathematical definition of slope, m= $(y_2 - y_1)/(x_2 - x_1)$ and that slope = Rise over Run but they did not know the exact meaning and implications of slope of a linear function. Three out of five used the incorrect notation for the mathematical definition of slope (m= $(y^2-y^1)/(x^2-x^1)$ instead of m= $(y_2 - y_1)/(x_2 - x_1)$). The students also had difficulty in differentiating between slope and y- intercept for the graphic representation of linear functions. Three out of five students used graphing calculators to construct graphs. The students copied the trace of the line representing linear function from the calculator to paper without a complete understanding of the nature and characteristics of the graph. None of the students showed one on one correspondence represented by linear functions.

## 4.10 Research Question Two

How can students' experiences with graphs of linear functions be characterized?

Table 4.1 below presents the description of the tasks and their relationship in providing data for research question 2.

Table 4.1

| Characteristics | Tasks and Objectives | Description |
|---|---|---|
| Rate of change<br>/slope | Task1/Objective 1<br>Task 1 Objective 2<br>Task 3 Objective 2 | • *Recognizing the patterns and finding contextual rules for the sequence*<br>• *Determining a constant rate of change*<br>• *The ability of students to find positive and negative rate of change of the function* |

| | | |
|---|---|---|
| Independent and dependent variables | Task 2 Objective 3<br><br>Task 3 Objective 3<br><br>Task 6 Objective 2<br><br>Task 5 Objective 2 | • *Explore the students' in-depth understanding of the characteristics of linear functions*<br>• *The students' understanding of the relationship between independent and dependent variables*<br>• *Characteristics of linear functions represented by system of equations. Flexibility to translate one type of representation into another and be able to make inferences from the representations of linear functions* |
| X and Y intercepts | Task 2 Objective 2<br><br>Task 4/Objective 1<br><br>Task 4 Objective 2 | • *Explore the students' in-depth understanding of the characteristics of linear functions*<br>• *Investigate students' understanding of the characteristics of linear functions*<br>• *Transforming from contextual to graphical representation of linear function* |

## 4.11 Findings for Research Question Two

There were two major findings for research question 2.

1. The students were able to make inferences from graphic representation of linear functions, but they were not able to explain the effects of change in independent variable on dependent variables for word problems.
2. Students had difficulty differentiating between slope and x and y intercepts for a given representation.

## 4.12 Research Question 2 Finding 1:

The students were able to make inferences from a graphic representation of linear functions, but their concept of the role of independent and dependent variables was not clear. Three out of five students were unable to identify the relationship between independent and dependent variables for the linear function. All students showed their knowledge of the mathematical definition of slope but they were not able to connect the significance of slope with dependent and independent variables.

In Task 3 Part (d), the students were expected to infer the student population for each school in the year 2015. The change in population was a direct outcome of the dependence of a linear function on the corresponding variable. All students except Aaron used graphing calculators and predicted the student population in the four schools; while Aaron used algebraic expressions for each school: School A, y=13x (actual slope from graph was 10 not 13), School B, y = 28x, School C, y = -30x+120 and School D, y = 15x. He did not use the population for the year 2000 as the y intercept for any of the schools except School C. When he was asked about it, Aaron replied, “Because it was an algebraic problem given like that”. Logan used the coefficient of x (slope) as y-intercept to make the graphs.

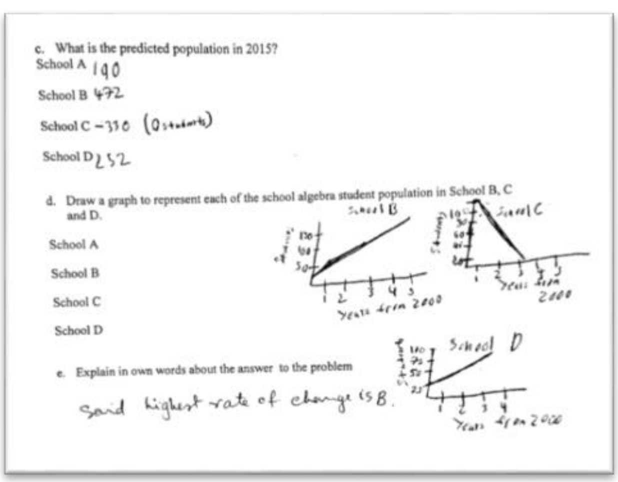


Figure 4.33 Logan's sample work

Logan used coefficient of x (slope) as y intercept in his graph. Similar graphs were produced by Kader and Edward for task 3 part d. From the graphic representations and related interview discussion, Logan, Kader, and Edward did not had a clear concept of correspondence among independent and dependent variables for linear functions. Student's knowledge of how to transform equations to graphs was very limited. Aaron was not sure what the characteristics of linear functions were. Julie concluded, "There is insufficient information to make a graphic representation." Logan completed all the steps and drew a graph, but he was not sure about the characteristics for the linear functions and commented, "still going up and x intercept is negative".

4.13 Research Question 2 Finding 2:

Students had difficulty differentiating between slope and x and y intercepts for a given representation.

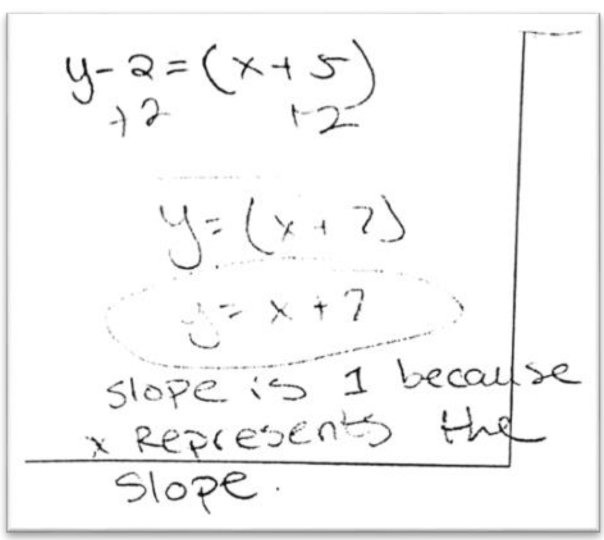



Figure 4.34 Aaron's representation for Task 4

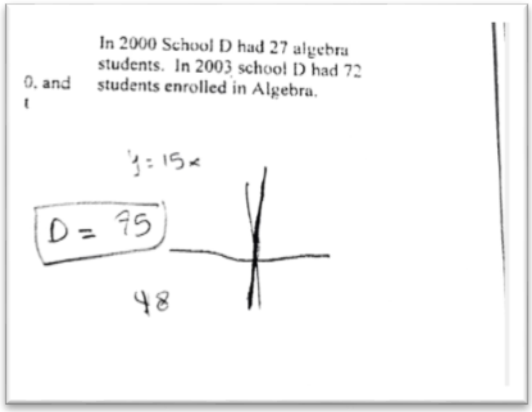



Figure 4.35: Student graph for linear function with y-intercept = 0 and slope = 15

Figure 4.34 and 4.35 shows the student's lack of understanding of the concept of y-intercept and slope of the linear function representing student population in task 3 school D. The graph for the linear function y= 15x shows what direct proportionality for the graph showing the rate of increase in student population as 15. Hence making the total population in 5 years to be 75, but the student wrote total population of student for school D after 5 years to be 95 students.

In fact the student had forgotten to use initial population of student in year 2000 as the y intercept. With this being said the linear function equation for school D was $y = 15x + 27$. Predicting the total population in 5years will be $y = 75 + 27 = 102$ students. One of the difficulties faced by the participants was to find the x-intercept. Kader used graphic representation and solved the linear function using the definition of x-intercept ($y = 0$ for x-intercept). Logan found the x-intercept by extending the slope intercept representation of linear function in task 2 and 4. Edward did not find the x-intercept for the representation. In Task 6 students were expected to identify the type of solution for the system of linear function (Equations: $y = 3x - 2$ in Blue and $y = -x + 2$ in Red) and show the characteristics for both functions, write a similar situation (contextual form) in words, and explain the linear function behaviors in a physical paradigm.

Task 6: a. How many and what type of solution is expected for the given graph representing blue and red lines intersecting at a point?
# of solutions _____________

Type of solution ___________

b. Write the slope and intercepts, of the following representation for the system of equations. y = 3x - 2, in Blue and y = -x +2 in Red

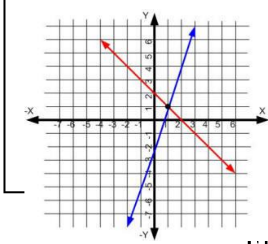


Figure 4.36: Task 6

The two equations given for task 6 were y = 3x - 2 for Blue and y = -x +2 for Red line graph. Interview discussion with Aaron reflected that he was not able to differentiate between slope and y intercept. A short interview segment is given below:

Researcher: What is the Y-intercept of blue line?

Aaron: Intercept for blue?

Researcher: Yes, Y-intercept for blue line.

Aaron: It’s three…

Researcher: What is it?

Aaron: It’s one.

Researcher: Y-intercept for blue line, you say is one?

Aaron: Yes.

Researcher: Okay. And what is the Y-intercept for the red line?

Aaron: Zero.

Researcher: Okay, and what is the slope for the red line?

Aaron: The slope of the red line is negative one.

Researcher: Okay, slope for the blue line?

Aaron: It’s three

Aaron did not comment on the type of solution for the given system of linear functions. Edward found one real solution. Logan also had one solution, but did not state whether it was real or not. When he was asked about the nature of the solution, he was not able to identify it as real and said, “intersection at equality.” Kader and Julie stated that there was one real solution to the given system of linear functions. Three out of five students were able to match the graphical representations with the given algebraic representations. Four out of five students were not able to identify the types of solution and make inferences to describe the meaning of system of linear functions in a physical perspective. Since the student understanding of the types of solutions for a system of linear functions is beyond the focus of this study, it will be discussed in a future research.

### 4.14 Summary of Findings for Research Question 2

All five students perceived linear functions as an abstract mathematical representation with slope and intercepts which were expressed as parts of equation, graph or as word problem. None of the students characterized linear functions as one on one correspondence of independent dependent variables. All students were able to identify the characteristics as rate of change and intercepts for an algebraic representation of linear functions. Their knowledge of the linear function was limited to an algebra concept which was not expanded to the overarching concept of linear functions and their characteristics.

The five students in the study relied on finding the slope, x- and y- intercepts from the graphing calculators without considering the concepts of the characteristics of the function. Task 4 was a relatively straightforward mathematical concept. The students were asked to find the characteristics of a linear representation. Most students located the slope and intercepts, but it was difficult for them to explain the concept in a physical perspective. At this stage, the student used the words “rate of change” and “slope” interchangeably; they were quite familiar with the y-intercept, but the concepts of slope being represented by the coefficient of x and of finding x and y intercepts mathematically were not clear. The major difficulty for three out of five students was drawing the graph using a slope intercept form of equation. The students were not confident in making a graphic representation. The discussion for task 4 shows evidence related to the second research question for this study: that most of the students were unable to find the characteristics of linear functions.

# CHAPTER 5: SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

## 5.1 Purpose of the study (revisited)

The current qualitative study was designed to describe how multi-representations help in the understanding of linear functions by MSEN Pre-College students, and how students' experiences with graphs of linear functions can be characterized. The study employed interviews about the six written tasks as mathematical tools for assessment of students' conceptual understanding. The multiple case study design was used to explore conceptual understanding related to linear functions and finding solutions to real life problems in STEM fields.

This chapter begins with a brief description of the Translation Model proposed by Lesh, Post and Behr. It will be followed by the discussion of the findings of the two research questions. Finally limitations of the study, implications and recommendations for future research will be discussed. Lesh, Post and Behr Transitional Model (1987) provide a useful lens for understanding the findings from the current study.

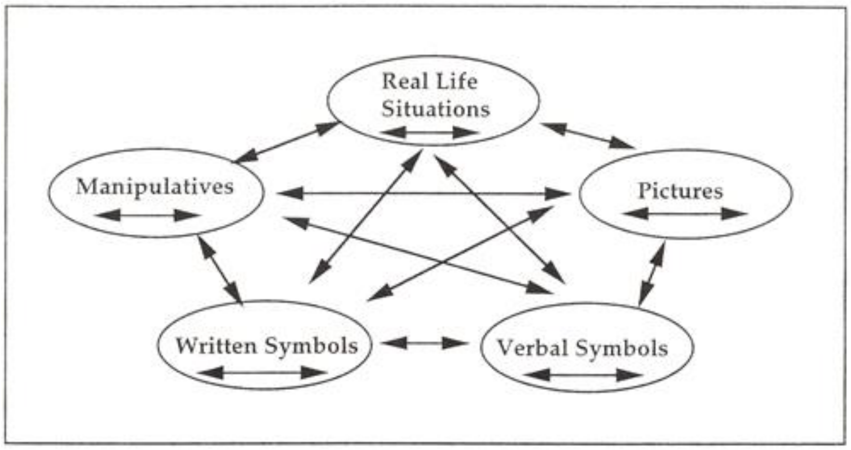



Fig 5.1: Lesh, Post and Behr (1987) Translation Model

The Lesh, Post and Behr Translation Model (1987) informs the current study. All parts of the model are interconnected or influence each other in an indirect way. The research findings for the current study can be best explained through the lens from this model, as described in chapter 2. In order for students to understand linear functions it is very important to place them in a perspective, something that can be done by presenting real life situation problems to students. Within that perspective the student can visualize through pictures/graphs and manipulatives, provided the student is familiar with the math vocabulary. The multi representations of linear functions such as graph (picture) algebraic (manipulative), word problems (written) and classroom presentations (verbal), and the flexibility to move among them, essentially help a student develop conceptual understanding of linear functions as posited through this model.

## 5.2 Summary of major findings of the study (from Chapter 4)

There were three major finding regarding research question one for the study.

1. Four out of five students successfully completed tasks related to linear functions and their multi-representations (algebraic, graphic, tabular, pictorial, and contextual).

2. Students were limited in their ability to move flexibly among representations of linear functions.

3. Students faced difficulty with constructing graphs from a linear function expressed as an algebraic equation or presented in word problem.

5.3 Discussion for Research Findings Questions: 1

How do multi-representations help pre-college students understand linear functions?

The interview tasks instrument used in the study focused on four representations of linear functions: algebraic, graphic/picture, table, and contextual (word). Three main points surfaced from the findings for research question 1 discussed in chapter 4. First, although the students recognized the recursive behavior for rate of change of linear functions yet their understanding of rate of change and slope was not clear. The students used their visual and numeric counting skills to show the growth of pattern representing linear function in task 1. Logan made the outline for the picture to show growth and then filled in the shape with square tiles. Julie used her memory and visual skills (looked at the figures and wrote an algebraic expression) to calculate the number of tiles for next figure and stated her rule as $y=3x$. She did not account for the starting value of the function.

Drawing pictures and by understanding their parts can be secret for gaining in depth conceptual understanding to solve problems in linear functions. It is a kind of a road map that provides a lot of information for interpreting and reaching to a reasonable solution. Graphic representation is an excellent example of pictures for linear functions as viewed in the perspective of Lesh, Post and Behr (1987) Translation model.

Kader talked about the incremental change of the recursive behavior of the pattern but he still did not recognize it as a linear function. It was also noticed in task 2 that the students did not identify rate of change to be equivalent to slope. Julie and Aaron said "slope is a term used in the equation of line, y = mx +b, and rate of change is a general concept for change". In task 3 all students were confused about the negative rate of change in population. They did not look at negative sign to represent decrease in population. Kader said "negative numbers are imaginary therefore the changing population for school "C" can be ignored". The interview data and students' written responses showed students studied and used multi-representations as separate entities. The students' got confused when they were asked to make connections among representations. All students in the current research study demonstrated knowledge of algebra vocabulary for the characteristics of linear functions, but they were unable to comprehend the meaning of slope and intercepts for pictures and graphs in real life perspectives. Using the lens of theoretical framework, Lesh Post and Behr (1987) Translation Model shows: for a successful learning experience it is important that real life linear function problems were taught with an interactive relation among multi-representations, the types of representations described in Lesh, Post and Behr model are manipulative, pictures, verbal, written symbols in real life problems.

Students are taught functions as a part of required algebra course objectives only.. According to a student's response for the interview question: how often does the students use graphic representations in his class? Aaron said "we learned graphs in a separate unit not in linear functions," "my teachers show graphs on the board sometimes to let us know the answer can be verified by using another method". This teaching practice can be understood that the teachers are following a deficit teaching model, meaning that students are taught according to what is needed for learning missing concepts for success in grade level test offered once a year in one setting, and not for success in careers or future applications of linear functions. By just showing a graphic representation does not help students understanding of linear functions and their characteristics. The experience would have been helpful if the students themselves make graphic representations for problems solved algebraically or in other representations.

Second, moving flexibly among representations was difficult for most of the students in the study. It is very important to know that use of representations does not change the nature or type of the function, it simply displays the situation in a different perspective. Linear functions are presented and interpreted in several different ways. For instance, picture drawings are made to represent short cuts, individual ways of perceiving concepts and ideas, whereas graphs and formulas represent algebraic terms cluster together in relationships. Mathematically, the definition of a function may look simple, but a visual, conceptual picture for representation can become a challenge (Baker &Tall, 1992). Moschkowich. et al. (1995) affirmed that using multi-representations to teach linear functions enhances students' understanding of functions. One may speculate that standard published curriculum limits the flexibility of choosing from multi-representations and impedes teaching and learning fundamental concepts, such as functions.

In task 2, the students were able to find the slope for graph and algebraic representations but two out of five students were not able to differentiate the slope from y intercept and also identified "3x" as slope instead of "3". The students had a visual concept of slope to be rise over run, "if it goes up it is positive and when comes down it is negative". It was observed from student work such as shown in figure 5.2, and 5.3, that they had memorized the shape and trend but did not know the meaning of positive and negative slope in real life perspective. The students did not show an understanding and importance of choosing a scale to make a graphic representation. Other students in the study used graphing calculators to look at the graphs. Figure 4.30 and Figure 4.31 student work to show graphic representation of an algebraic equation (y = x + 7). Task 5 and 6 seek for students' conceptual understanding of functions and their real type of solutions. Logan drew a picture for task 5 shown in figure 4.7, which did not represent the actual problem. Logan had difficulty in moving from contextual to pictorial representation. He made two parallel and equal lines labeled with different speeds; while the problem described that one student left earlier than the other, at a lower speed (Figure 4.7).

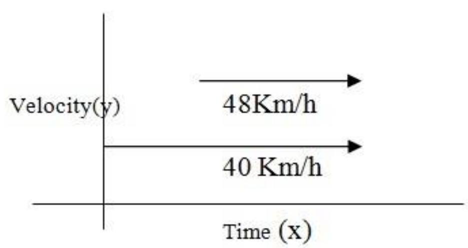



Figure 5.2: Expected student representation for task5

Similarly, the student felt the same hardship for translating graphic representation into algebraic representation for linear function as shown in figure 5.2.(copy of figure 4.11) The figure 5.2 clearly shows that the student knew how slope is defined as change in y over change in x and the algebraic formula for slope of a linear function; but it was not clear whether the student knew about the role of independent and dependent variable for linear function. The student wrote an algebraic form of the equation for the graph and the memorized the equation of a line but then he circled a segment of graph and showed it as an exponential function saying "the linear equation of line can be represented as a function like this". The student showed no conceptual understanding of the fact that the equation of line in fact represents a functional relationship and the one-on-one correspondence also defines linear functions. Although multi-representations and multimedia resources are easily available to students, they still find it difficult to effectively use multi-representations for problem solving in linear functions (Anisworth, 1999).

Third the concept of ratios and proportions as the basis of functional behavior was not clearly understood by the students. Task 3 and 5 are examples of problems describing the role of ratios and proportions in solving problems for linear functions. Task 3 described the scenario for four different school populations changing over a period of five years. Besides other details of the problem the students were asked to predict the total student population in each school after five years. Referring to figures 4.18 and 4.19 in chapter 4, the students did not understood task 3 as the linear function problem in terms of problem in ratios and proportions.

As a part of progression of leaning trajectories in math, ratios and proportions are introduced and taught in middle school math classes before getting to linear functions and their representations (NCTM, 2000). Problem solving in ratios and proportion is a very ancient concept since the time of Euclid (300BC) that opens doors into the concept of linear functions (Madden, J.J. 2008). Basic learning of linear functions is associated with patterns rates of change (task 1) leading into comparing (ratios) and then recognizing, creating and representing as proportions involving mental activity to map ideas (functions). Students' work in task 3 reflects that they had some background information about a constant change, ratios and proportions but they were not able to apply the prior knowledge to the given situation of linear function. They relied on their skills for working with sequence by adding a constant number to the previous value and get the next value of the function in the sequence. This also shows that their process of building concepts at different stages in school math learning was not developed or sustained for future applications.

Most of the students in the group used calculators for task 3 for finding the total population of student for each school in five years without knowing that the calculator operations are also based on the concept of ratios and proportions for example measurements in a certain unit system, currency conversions, making scale drawings, finding rate of change, slope, plotting points for a graphic representation, identify the characteristics of proportional and non-proportional relationships by analyzing a table, equation, and graphs, and problems dealing with percent by using visual and numerical representations of percent.

Students written responses for Task 5 also reflected the students' lack of understanding of ratios and proportions as the basic concept of linear functions. (Statement of Task 5: Jose left the White House and drove toward the recycling plant at an average speed of 40 km/h. Rob left some time later driving in the same direction at an average speed of 48km/h. After driving for five hours Rob caught up with Jose. How long did Jose drive before Rob caught up?)

A student Edward ($10^{th}$ grade) in the study expressed his inability to work on problem connected to the concept of ratios and proportion because in his opinion he was not taught the concept in school. This was a very discouraging remark because it reflected the lack of building concepts of ratios, proportions and functions in general over many years (middle and high school). Another student work for task 5 is shown in Figure 4.8 the information on this sample figure shows that the student was good in retrieving the information from word problem and putting it as ordered pairs. In his effort to show his work he wrote the proportion as 240/40 = 40 J/40, J =6 hours, then 40J = 240.

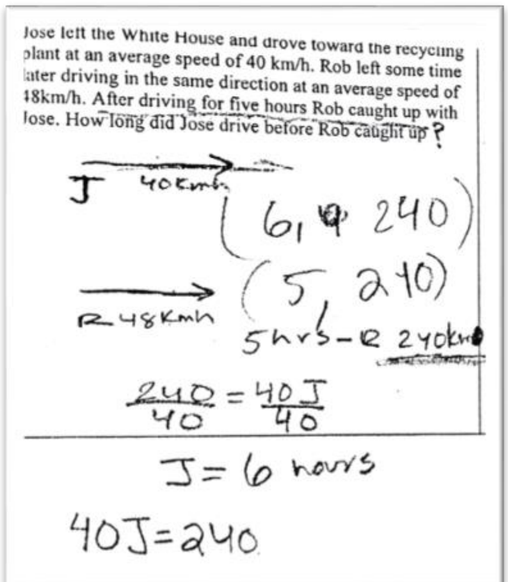



Figure 5.3 Copy of Figure 4.8 Student's work on Task 5

What is wrong with is this picture? Is a serious issue which can be further discussed in next research. For the current study findings, it shows that the student may have the information stored in his brain (schema) and he can process it mentally but cannot express in writing. This issue can be clearly understood in the light of Lesh Post and Behr (1987) Translation Model. The student was not able to comprehend the problem and its solution for a real life situation because the connecting links for multi-representations : written, verbal, manipulative and pictorial/visual were not establish over a period of time to develop and build conceptual learning. Also, Dufour-Janvier, Bednarz, and Belanger (1987) informed in his research that multi-representations were inherently included in mathematical concepts to make mathematics interesting and easier to understand. Thus the literature review, theoretical framework and Lesh, Post and Behr (1987) Model, supports the finding for research question one that multi-representations helps in students' understanding of linear functions.

5.4 Discussion for Research Findings Question 2

How can students' experiences with graphs of linear functions be characterized? There were two major findings for research question 2 (from Chapter 4).

3. The students were able to make inferences from graphic representation of linear functions, but they were not able to explain the effects of change in the independent variable on dependent variables for word problems.
4. Students had difficulty differentiating between slope and x and y intercepts for a given representation.

Graphic represenattions for linear fucntions was not a new concept to students in algebra classroom. All students in the current study had a prior knowledge about the definitions of the charateristics of linear functions such as slope, intercepts and the equation of line y= mx + b. Students successfully made inferences from the graphs (Task 3) that the trends of graphic representation show the increase or decrease in the value of the linear function. One student use the analagy of goin up the hill and going down the hill for direct and inverse proportionality graphs. The students were not able to go to apply the inferences and justify their answer. The concepts and definitions of the characteristics of linear functions were provided to the students in class but very little evidence of conceptual understnding was noted.

## 5.5 Discussion for Research Question Two

There are two very important points to discuss about findings from research question two.

First, some misconceptions about linear functions were observed from the written and interview responses.Students were not able to relate linear functions to real life situations and they did not show a complete understanding to the concept of independent and dependent variables. Julie and Logan had a misunderstanding about linear functions that all linear functions graph pass through the origin. Julie wrote the algebraic representation for linear function as $y= x +3$ where 3 is the y intercept to represent a linear function. A reason for such misconception can be due to lack of conceptual understanding or teacher's instructions. Aaron commented for the algebraic representation for task 4 (Figure 5. 10) that "slope 1 because x represents the slope".

The conceptual meaning of the linear function representing school population changed due to the shape of graph and student not having a clear understanding of characteristics of linear function. For task 3, student work shown in Figures 4.20 and 4.21 the students' inability to comprehend the meaning of slope and making inferences from real life situations are shown. Both students showed that the population of students in school C, after 5 years will be -330 students, Student in Figure 4.20 also wrote "0 students" besides it. They both concluded that the highest of change is school B. Neither of the two explained the meaning of negative slope of a linear function in a real life situation.

Referring to the Lesh, Post and Behr (1987) Translation model the classroom teaching and learning experience of student must establish connections among real life problems and students thinking skills reflected in the form of written, verbal and pictorial demonstration (manipulative). The student only understood the negative on number line negative numerals instead of thinking in a scientific or real life perspective. Also, as mentioned earlier by a student that the graphs were presented to students in class as another approach to problem solving instead of allowing the students to have firsthand experience with graphical representations for problems in linear functions. This is a broken connection (Lesh, Post and Behr (1978) Model) that may leave many unanswered questions in students' mind causing a lack of understanding or misconceptions for students.

Keller and Hirsch (1998) identified in their research that multi-representations promote in depth learning processes and cognitive connections. The research relates to the current study in the sense that if the classroom teaching does not emphasize on the characteristics of linear function and their applications with the correct meanings in different perspectives the cognitive thinking process are blocked/ distorted (misconceptions are generated) instead of being facilitated. In Lesh Post and Behr (1987) Model, the role of verbal representation is very important for clear understanding, it helps to communicate students finding in a way that they do not feel uncomfortable, ask questions and clarify misconceptions before they become long lasting.

Secondly, use of technology by students without having an opportunity to learn about the mathematical steps processed in calculating instruments such as calculators and computers.

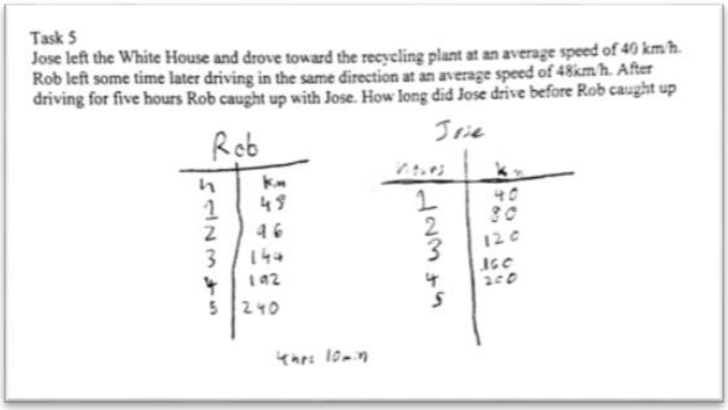



Figure 5.4: Student use of table to solve a word problem

Another important connection in Lesh, Post and Behr (1987) Translation model for cognitive understanding of linear function is written responses for solving problems in real life situations. The student work shown in Figure 5.4 shows that the student preferred to use T- Chart or table to find the time at which the two people travelling at different speeds will meet? The student made a table using correct rate of change and stopped at distances travelled in 5hours each. The student got confused and put in numbers in graphing calculator Ti-85, and then wrote the answer that both will meet after 4 hours and 10 minutes. He was asked by researcher if this is his final answer and he confirmed 4 hours and 10 minutes and showed in the calculator that it was his final answer (6 hrs. was the correct answer) to the word problem. What is the reason for the discrepancy? The researcher took the calculator from him and found that the calculator setting was at scientific and radian, also the student put in 220/ 48 to get the time , the answer was 4.56, he then explained that time cannot be in decimal therefore he divided 0.56 by 60 and concluded the answer was 4 hours and 10 minutes. Amazing, the student was not aware of calculator settings and tried to find a reasonable answer to the problem. In response to a question during interview the student said that graphing calculators are required for algebra class, he had only one lesson for graphing calculator when the teacher showed the class, how to make a table and graph for x and y coordinates. Another important fact to be noted here is that the student did not think critically about the result (4 hrs. and 10 minutes) to be less than 5hrs, the time taken by fast driver Rob who also started later than Jose (slow driver. How can it be possible that they both met before 5hrs? This is because student was not using the verbal/ spoken language representation (from Lesh, Post and Behr Model).

It is a matter of concern that the students have access to technology but they are not provided enough exposure to its operational functions. Calculators and computer are most commonly used by teachers and students in math classrooms but rarely provided with opportunity to learn about the features and processes involved in problem solving. The evidence provided above in Figure 5.14 must not be ignored. The student was not aware of how to check the mode for settings to normal, float and degree setting to provide answer in decimal notation.  Moreover he had a misconception for converting notations. Little knowledge is a dangerous thing. A worse situation may happen if the teacher is not trained for technology use and he /she is unable to assist students in his/her class. Technical, pedagogical assessment of content knowledge (TPACK) model is a dynamic model for interdisciplinary teacher training and professional experiences leading to which type of technology are more useful for teachers and how they are accessible to teachers. The next important step is to provide opportunity for proper use of its operations instead of just displaying or teacher demonstrations of how the linear function operations are conducted. This is in line with the Lesh, Post and Behr (1987) Translation model. The calculators provide manipulative experience, written and verbal experience to develop a picture / graph for the linear function provided the settings are correct.

Research question 2, how the students characterizes linear functions can be further elaborated by expanding to the system of linear functions. Task 6 in the study seeks for the students' understanding of the relationship between independent and dependent variables and the characteristics of linear functions represented by system of equations. Students showed very limited knowledge about the system of equations and their applications in real life situation. It can be considered as an excellent example to demonstrate Lesh, Post and Behr (1987) Translation model in action. Considering the two linear functions in task 6 representing two graphs of two physical processes for example the case of two billiard balls coming from two different directions (physical situation), there are several possibilities: the ball can pass by each other without touching , inconsistent and no real solution representing a physical situation :"no collision" next the lines intersect, one real solution, independent, consistent function and finally the lines overlap, inconsistent, infinitely many solutions and physical situation balls can rub against each other and go together.

The system of linear functions sometimes called simultaneous functions can also represent two independent processes happening at the same time. Student characterizes linear functions as representing physical situations with one or more real and imaginary solutions and behaviors depending on rate of change. Task 5 in the study also described a similar situation. Several examples for system of linear functions can be quoted that exactly match the use of linear functions in medicine and engineering. Solving system of linear functions with more than two functions, using the matrices function on calculators or computer can be used. Complicated cases provide written, verbal, manipulatives and pictorial representations are closer with Lesh, Post and Behr (1987) Translation model and TPACK model. Smith (2007) in Chapter 2, discussed the importance of student drawing for understanding the given situation in the problem. He supports the idea that pictorial representations help in the thought process and encourages students to connect meanings to their representations. The current study also showed with examples that the students were trying to make pictures in tasks 1, 3 and 5 but faced hardship to transform from pictorial to algebraic or other representations.

## 5.6 Implications, recommendations and future study

### Implications of the study

In this section the implication of the finding of the study will be presented followed by recommendations and implications for future research ideas.

The study contributes to the body of knowledge in areas of the use of multi-representations with flexibility to move among different representations for linear functions. The research findings for the study raised several implications related to student learning and teaching. Three main implications are discussed below:

1. Students' perceptions of linear functions and multi-representations of linear functions
2. High school algebra curriculum
3. Teacher perception, content knowledge and professional development.

1. The findings related to the student perceptions about linear functions and their representations need clarification. Some students' perceived functions can be either quadratic or exponential but not linear. The students were not able to find a one on one

correspondence between independent and dependent variables describing linear functions. The finding also directed the readers' attention to the findings that the students in depth understanding of graphs, written and verbal representations is a reason for the students' inability to characterize linear functions. The difficulty in flexibly transforming pictures to equations was one of the issues interfering with the student learning process.

The use of multiple resources and technology is encouraged in algebra classrooms but due to lack of opportunity to use technology with correct guidance and knowledge of helpful features of the device can sometimes be damaging to the students' conceptual learning. The students' misconceptions about negative slope of function in real life situations a major concern which needs to be addressed with both teachers and students. The lack of knowledge of the negative sign representing the reverse operation, severely impacts the students interpretations and making inferences from multi-representations. Real life situations as shown in Lesh, Post and Behr (1987) Translation model can be completely understood if discussed with representations in written symbols, pictures, verbal/spoken language and hand on/experience with manipulatives. The students not familiar with relevance of linear functions and their representation in real life situations face difficulty in making connections and developing critical thinking skills. Almost all process in real life situations are simultaneous and it is often hard to separate them without restricting some of the variables in the ongoing processes.

Task 5 and 6 were designed to explore students' understanding of system of linear functions and its applications in real life situation. The finding for task 5 and 6 implied that the students were unable to see linear functions in a bigger picture. The reason for this might be that students are taught linear function strictly to focus on individual representation and from the perspective of being successful in the course of study instead of a broader perspective of success in life careers.

2. Very little emphasis is paid in curriculums and instructions to clearly teach algebra students that a linear function is a one on one correspondence not just an algebraic sentence. It is important to note that the students taught in class and the text books followed the drill pattern to teach students the rules and problem solving skills; for each representation separately without making a connection among them or understanding the dynamic behavior or linear function in action. Akkoc and Tall (2003), Clement (2001) explained that the students' experience to see linear function action is limited reduces their ability to recognize the variations of functions. Schwartz (2003) reported that the student learn functions with three premises, the variable, representation and the flexibility to choose the representation. The students could not recognize linear functions because the concept of function is often discussed in the context of quadratic, exponential and other functions.

3. The teachers of the participating students taught one representation at a time to practice but did not teach them how the same problem can be worked on with a different representation and how does multi-representations of function in making inferences if needed. The students were given mathematical definitions and formulas such as slope $m = (y_2 – y_1)/(x_2 –x_1)$, and provided demonstrations for how to make graphs, pictures and use technology instead of following up on the ideas and allowing students an opportunity to learn the concepts using multi-representations for linear functions. Some serious students' misconceptions about types of functions, their characteristic, real life applications and inferences from graphic representations were also part of the findings.

This implies a concern about the teacher's up to date content knowledge and training to teach 21st century students in schools The new method adopted for teacher training and interactive classrooms must be required in all schools.

The lack of understanding of multi-representations becomes a major issue for students, when a contextual problem or algebraic form of linear function is expressed as table or graph. The study finds students having difficulty in flexibly moving among multi-representations. It is important to note that a function can be really presented in many representations but symbolic/algebraic and graphical complement each other Schwartz (1991).

In order to further strengthen the study implications, an algebra teacher of students in the study was interviewed. The interview was conducted to know the teacher's perspective of student learning and her reflection regarding her own experience of teaching in public schools. Overall the teacher's perspective was that the students can benefit from the use of multi-representations of linear function but due to limited time, class size, minimal administrative support and lack of professional development the teachers are not teaching multi-representation for all units in algebra curriculum. This implied that the problem of student learning of multi-representations will persist if the issues faced by teachers in public schools are not addressed.

Regarding her own experience in teaching at public schools, she smiled and said, "public school in this State are like dumping ground, there are too many different levels of students and inclusion students are added to class without providing any training for teachers for it". Besides lack of adequate training for teaching students in a 21st century classrooms, she added, that there are too many objectives in algebra course to be covered within a year and the class size is very large therefore it is difficult for her to allow time for students to practice with multi-representations for algebra problems.

It is very important for students to know the significance and the use of multi-representations in all STEM fields and more. The ability to move among representations and making inferences from graphs enhances their scope of success in academic placement with increased value in their future careers. As observed in the study students were able to understand the patterns but they were not able express until they wrote it on paper or draw the figures. Drawing pictures is a natural instinct that reflects upon the individual cognitive thinking and connecting the meanings of drawing is a systematic act performed by the brain element called "schema". A discussion of schema is beyond the scope of the study therefore it will be discussed in a future study.

In the past century research we can find evidence for emphasis on Algebraic representation. Students and teachers were trained for skills and fast paced learning practices such as how many algebra problems can a student's solve correctly in shortest time and the focus was just on the solution of problem. Such practices and skill drills did improve students' reflexes and challenged the brain for quick reactions but did not helped with critical thinking or reasoning processes for long term understanding. A student in the study remarked that: "I loved math very much but not anymore because algebra is hard" Another student said "I am very good in my math class, I am a top student, I know it but I cannot write or explain". The student comments should not be ignored because they are students who claimed to be one of the best in their classes. Moreover the selected students in the study show a GPA of 3.0 or above on their school report card. Although the student participants in the study had different cultural and linguistic backgrounds and coming from three different schools in the district, none of them were listed as academically at risk students, meaning their performance in class was satisfactory.

It also indicates the problem of using multiple choice test items used for determining student level of accomplishments in math is wrongly representing student ability for problem solving (students could have made a lucky guess for the answer ). Most of the students in the study showed their first attempt for solving problems by use visual and mental math skills but due to misunderstanding or lack of conceptual understanding of linear functions they got confused. To overcome the difficulty some of them used calculators and tried to work with random numbers and operation to find a reasonable answer for them, others tried to guess and wrote previously memorized facts or definitions to help solve the problem. This behavior leads to check for teacher education and training for teaching to diverse students' population. Questions like "was the teacher aware of the relationship among fundamental concepts of ration proportion to linear functions?" Or "Did the teacher taught students about one on one corresponding and mapping, concept of domain and range etc. in class?" Did the student have firsthand experience in working with multi-representations of linear functions?" and "how the student learning was assessed". These are just few questions that can help in finding about the difficulties faced by students and their progress in learning linear functions.

With the introduction of common core standards, a light is seen at the end of the tunnel that might help in the students' global learning process. The key shifts in curriculum if correctly imposed can make it easy to find the places and areas where the problem begins for each individual child. Teacher content knowledge, pedagogy and up to date training in the use of skills, technology and other community resources, along with adequate evaluation methods is major hope for a successful implementation of common core practices. Current research such as TPACK is focused on this goal for student achievement directly linked with teacher professional development.

Learning is a cognitive and social activity must be viewed in perspective. As mentioned in the previous paragraphs the students' comments were simply reflecting their abstract learning of math concepts. IT was very difficult for them to analyze the situation for problem solving in a relevant or real life perspective. In Task 3 (refer to Chapter 4) all students were unable to extract the meaning of negative slope for a data for student population in a real life perspective. After some probing questions and discussion outside the interview protocol two of five students realized the meaning of negative slope representing a decrease in population instead of considering it an imaginary number with no significance.

Another situation discussed for students' performance for task 5 story problems of two people walking in the same direction but starting at two different times and with different speeds, the students were asked to find the time when they meet each other. It was observed that the students looked at an abstract math word problem, they used some visualization to draw a picture or chart but did not attempt to see their results in a real life perspective. The student work in the figure 5.4 show a correct approach for problem solving but the student did not looked at it in perspective that the two students in real life situation cannot meet before the time taken by the fast driver to cover equal distance (five hours). This result showed that the student did not verify his solution to the given problem in a perspective.

A lot of previous research had been published about students understanding of linear functions and their representations. In most of the research articles the emphasis is only on one type of representation. The findings from the study also imply that students' concept development cannot be complete unless students are taught multi- representation as a part of a complete learning model which includes application to relevant real life situations, pictures, spoken language, written symbols and hands on activities (Lesh, Post and Behr (1987) Translation model).

## 5.7 Limitations of the Study

There were four major limitations for the study that prevented it to be generalized for all algebra student population in North Carolina.

1. Sample size: Convenient sampling method was used for the study, meaning students which matched the selection criteria and brought in their parent permission slips on time. Only five students were interviewed for the study, which is a very small sample size to generalize the results for a larger population.
2. Frequency of interviews: The students were interviewed only once. The day of interview might not be a good day for a certain participant.
3. Teacher instructions: Other students in the class were not observed or interviewed for comparing their understanding of instructions from their Math classroom teachers.
4. Students outside the study participants: The students interviewed in the study met the criteria of an overall GPA of 3.0 and submitting the signed permission slip. However the students outside the study may have a better conceptual understanding of multi-representations of linear functions.

## 5.8 Recommendations

An old saying is "practice makes perfect" is commonly misunderstood as drill work instead it implies that working on similar problem with different situations and representations to promote understanding and skills. 21st century students have technology at their fingertips. They use many electronic devices for communication and in classrooms. It is very important that the students and teachers are trained for the use of graphing calculators and the mathematical processes used for problem solving. Calculators have many feature which are more than just working with number operations, the settings and built in features may help the students to find answers to the questions but no conceptual learning will take place unless other representations such as written, verbal or manipulative are used in conjunctions to the pictures and calculations. A recommendation from the study all math teachers must allow opportunity and time for students to learn concepts and their applications before the use of technology in classroom.

The idea of teaching algebra and linear function seems simple and straight forward but in reality for conceptual understanding it is a major challenge. The students must be familiar with the concept of function, its dynamics and characteristics. The adoption of new common core standards seems to be a positive step in this direction. The common core standards are a set of goals and objectives required for each grade level. In order to support the students in each State, it does not restrict the use of specific resources, strategies or training for teachers. IT is left up to the State and the district system to decide which resources and materials can be appropriate for their student population. It also favors the idea of providing individualized training for teachers working inclusion students in their classrooms. Most importantly the standards provide clear achievement level indicators along the way to the goal of college and career readiness for all students.

Another implication of the findings from the current study is that the students did not show coherence in their mathematics learning experience. A student commented that he was not taught ratios and proportions in school therefore he is not comfortable working on word problem for linear function. Another said their teacher did not teach graphs in algebra class the students use calculators for making graphs. The teacher excused herself for not teaching multi-representation in class because of larger class size. These implied issues can be resolved by taking advantage of the Common Core Standards which provides definite “Key Shifts” for grade level specific math standards. It helps the administrators and district officials for teacher training and having students’ accountability for teaching and learning. For example in 6$^{th}$ grade students must master skills in ratios and proportions and early algebraic expressions and equations. In 7$^{th}$ grade, Ratios and proportional relationships, and arithmetic of rational numbers and in 8$^{th}$ grade student master concepts in Linear algebra and linear functions and then high school they get into algebra topics and more

### 5.9 Implication to Future study ideas

Mathematics is not a list of disconnected topics, tricks, or mnemonics; it is a coherent body of knowledge made up of interconnected concepts (Common Core Standards). The current research was focused on students’ understanding of linear function for a short time interval (one semester). Future studies may look into the effects of teacher training on student learning. The study can be made longitudinal to spread over a period of two to three years to find out how the initial use of multi-representation impact their future learning. Another popular topic for future research can be: how does a substantial amount of professional development for teachers help in better understanding of linear functions? The study can also be followed up and reviewed to see if the students from the study pursue higher-level math courses. Another case study can be designed for comparing student learning in the light of cultural diversity. A similar study in future can address the issues of linguistic barrier and issues with math vocabulary affecting student learning of linear functions.

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# APPENDIX A: STUDENT INTERVIEW PROTOCOL

Interview questions for case study (open ended)

1. I have told you about my background. Can you tell me some things about you?
2. How long you have been in MSEN Program Do you like being in the MSEN Pre-College program?
3. What are the things you like the most in school?
4. What else can you tell me about your school?
5. Can you tell me what do you like the most in math?
6. Your class is studying linear functions in this unit can you explain to me what are linear functions
7. What is the difference between a function and an expression?
8. Why do we need to learn linear functions?
9. Do you feel comfortable doing word problems in linear equations?
10. What are some methods used in your class to solve linear equations?
11. Did you make graphs to represent linear functions?
12. You learnt about following patterns and making tables before. Do you sometimes use tables and patterns to solve linear functions?
13. If you are given an equation like y =3x +4 which method would to prefer to solve this equation?
14. Today I will show you six different problems, one at a time. These problems are related to linear equations. I want you to take your time and draw, write and talk to me about the problem as if you are explaining to a friend who missed the lesson in class. I am not looking for the how you understand the problem and whatever method you are comfortable to use for solving. It does not matter if the answer is 100% correct or incorrect. Think aloud as you work and do the best you can.
15. Do you have any question about the procedure? You can always stop me and ask questions during the process.
16. (task 1) There are three figures shown on this card labeled as 1, 2 and 3. Can you draw figure 0, 4 and 5?
17. (student draw the figures) How do you see the figures changing ?
18. (if student is struggling) How many tiles are in fig 1, then in fig 2 and then in fig 3 how many you think are added every time what is you guess for Fig 4 and then fig 5? Now go back and tell me how many were there in fig 0 if we have to draw it?
19. Is there a pattern or a rule?
20. What is the pattern or rule?
21. Is the size of fig growing?
22. Is it growing at the same rate?
23. What do you call this change? (expected answer is changing at a constant rate)
24. How can you check to see if your rule works?

25. (students should give the rule to be times 3 plus 6) why is it 3 times x?
26. What about 6?
27. Can you show me how it works?
28. Can you write an equation for this rule?
29. (student should write y =3x +6) How does this works?
30. What if x is 4 how many tiles do I have in the figure?
31. What is y?
32. Will the value for y change in your equation if I change x to a different value to say 10?
33. Now let us look at card 2. What is shown by this graph?
34. What are the x and y intercept of the linear function shown in this representation
35. Tell me more about this representation, what is the characteristic of this graph meaning how does y change when we change x?
36. Can you write an equation for the function represented by line?
37. Would it help if you make a table?
38. How would this look like if you represent the function as a tiles pattern?
39. Hint: Is this also representing a rate of change or not?
40. (If Yes) what is that rate of change? How do you describe it ?
41. How many equations can you write for this graph?
42. Task 3. What are you looking for in this problem?
43. Does this problem talk about any rate of change?
44. (If Yes) Is it growth or decay?
45. Is it positive or negative growth?
46. There are three different representations shown in the story problem? Which representation is easy to understand? Which is the hardest to understand?
47. What type of graph will you make for School D in the problem?
48. How is problem C is different from problem D?
49. Which representation you chose to work out problem for school C? Why?
50. Which representation you use for problem D? Why?
51. In your words explain what are your findings for each school?
52. How does their rate of change compare?
53. Task 4. What is the first step that you will use to solve this algebraic equation?
54. Tell me about each part of the final equation you got?
55. Can you solve this equation? What is the answer for x and y?
56. (If Yes) explain how?
57. (If No) why?
58. Can we draw a graph for it?
59. Now can you find the value for independent and dependent variables?
60. What type of function is this?
61. Is it different from the one we found in Task 1?
62. Task 5. What is the first step to solve this problem?
63. Show all your steps to solve the word problem?
64. (if the student make a picture drawing, notice the direction and context clue for representation and steps followed)
65. If the student get the correct answer of 6hours driving time for Jose to meet Robert) explain how you got the answer?
66. Could you have done the problem without making the diagram?
67. How is the graphic representation of this problem helpful?

68. Can you make up similar word problem to solve for how late did Robert start after Jose had left the White house?
69. Task 6 Explain the characteristics of red and blue graphs?
70. What is the slope of blue line and what is the slope of red line?
71. What are the x and y intercept for each line?
72. What is the point of intersection?
73. In your own words explain the physical meaning of the point of intersection ?
74. Is this the only way to solve linear equations?
75. What other representations are used to solve system of equations?

Thank you for your time.

# APPENDIX B: TEACHER'S INTERVIEW QUESTIONS PART 1 & PART 2 (OPEN ENDED)

## Background

1. I have told you about my background. Can you tell me about your educational background and other academic expertise or any relevant information about yourself?
2. How long you have been teaching math in MSEN Program
3. Do you like being in the program?
4. Does the UNC Charlotte pre-College program for Algebra 1 challenge all students in class?
5. What can you tell about your students' level of understanding in Algebra1 class?
6. What do you enjoy to teach the most in high school math curriculum for Algebra1?
7. Does your school provide enough resources for teachers to teach linear functions in Algebra1?
8. What do you expect about your students to know from prior knowledge of middle school math curriculum?

## Experience in linear functions

1. Tell me about your experience in teaching linear functions to high school Algebra1students?
2. What are some of the main difficulties you had experienced in the past for student understanding of linear functions?
3. How do you plan a lesson for a typical class of students learning linear function?
4. Are all students in class are between the ages of 14-16?
5. Do you notice a difference in levels of learning among different age students or gender?
6. Teachers use different representations in class for solving same linear function, which representation of linear functions in your opinion is most comfortably used by many of your students?
7. How would you compare your assignment expectations for students using different representations?
8. Which is your most preferable representation for linear functions? (Graphs, Table etc.)
9. How do you think your students liked to make graphs?
10. In your opinion what percent of your class will prefer to use algebraic symbolic approach for solving linear functions?
11. What percent of students you think can make easy transition from graphs to symbolic and vice versa?
12. If you show your class a graphic representation will they be able to find inferences from the graph?

13. Do you expect that all your students to make a table before making a graph?
14. If you pick two points on the graph for linear function, do you think they can explain the trend and other characteristics of the linear function?
15. What were your assessment criteria?
16. Is it the representation or is it the strategy that helps them understand linear functions better? Or both?
17. Do you expect the students to feel confident in presenting their visuals for problem solving?
18. Or more students feel confident to present their calculations of linear functions?
19. What do you think their comfort level will be when going from one type of representation to another?
20. According to common core standards which representation is more appropriate to meet the learning objectives for high school Algebra 1?

**Teaching experience**

1. Did you enjoy the interactive teaching?
2. How do you promote the learning and concept development for math concepts in students?
3. Do you anticipate that use of multi-representations for linear functions will promote better learning and higher scores for end of course exams for Algebra1 students in your class?
4. Which of the following types of activities are helpful in building and learning new concepts in linear functions? Activities based on prior knowledge of linear functions, visuals, graphic representations, math games using symbols, concept maps and tabular representations.
5. What can you say about the rigor and student engagement in your class?
6. Which activities you expect to bring in fun to your class for learning linear functions?

**Closing**

1. What are you expectations about the impact of multi-representation of linear functions on student engagement and conceptual development?
2. What are your plans to incorporate multi-representations in your classroom and teaching?
3. Do you need extra resources or make adjustments in your lesson planning to accomplish the goal to use multi-representations to teach Linear equations.
4. Do you want to add something to our conversation?

Thank you for your time.

APPENDIX B: (Continued) PART 2
(After unit for Linear Functions is completed)

**Student performance and teacher expectations:**

**1.** In the first interview before the beginning of the unit on linear functions we talked about your expectations from students in your Algebra 1 class, what do you think if your students have met your expectations?

**2.** In your class, students were assigned questions/problems using different representations for solving problems related to linear functions, which representation was used the most by the students?

**3.** Why do you think the majority of students used “this”(the one had been pointed out by teacher in question 2) representation?

4. What are some of the main difficulties you had experienced for student understanding of linear functions in your class?
5. Did you notice a difference in levels of learning among different age students or gender of your students?
6. What difference did you find in each of the three students’ performance? (three selected for the study)
7. How do you think the students liked to find new ways of working out problems in linear functions?
8. Is it the assignment or is it the strategy that helps them understand linear functions better? Or both?
9. How would you compare your assignment expectations for students using different representations?
10. What was your assessment criterion?
11. Does your assessment show a difference in achievement level for students using multi-representations than those using only one type of representation for problem solving?
12. Which representation was preferred by most of your students: graph, table, pictorial or symbolic?
13. Why do you think the students liked to make graphs?
14. Approximately what percent of your class preferred to use algebraic symbolic approach for solving linear functions?
15. What percent of students you think can make easy transition from graphs to symbolic and vice versa?
16. When you showed your students in class a graphic representation of a linear function were they able to interpret and find inferences from the graph?
17. Did you find that all your students make a table before making a graph?
18. When you pick two points on the graph for linear function, were they able to explain the trend and other characteristics of the linear function?
19. Did your students felt confident in presenting their visuals for representing solutions to linear function problems?
20. Or more students felt confident in presenting their calculations of linear functions?

21. What do you estimate to be their comfort level in transitioning from one type of representation to another?
22. Did you find most students worked well with graphs as required by the common core standards for learning linear functions in Algebra 1 for high school students?

**Teaching experience:**

23. Do you think that your students benefited from multi-representations of linear functions?
24. Did you enjoy the interactive teaching?
25. How do you promote the learning and concept development for math concepts in students?
26. Do you anticipate that use of multi-representations for linear functions will promote better learning and higher scores for end of course exams for Algebra 1 students in your class?
27. Which of the following types of activities are more helpful than others in building and learning new concepts in linear functions? (Activities based on prior knowledge of linear functions, visuals, graphic representations, and math games using symbols, concept maps and tabular representations).
28. What can you say about the rigor and student engagement in your class?
29. How do you think learning multi-representations of linear functions will impact the students in their future careers in medicine and industry etc, and being successful in higher education?

**Closing**

30. What are you expectations about the impact of multi-representation of linear functions on student engagement and conceptual development?
31. What are your plans to incorporate more multi-representations in your classroom and teaching?
32. Do you need extra resources or make adjustments in your lesson planning to accomplish the goal to use multi-representations to teach Linear equations.
33. Do you want to add something to our conversation?
Thank you for time.

# APPENDIX C: INITIAL ASSESSMENT INSTRUMENT

*Linear function Problems Interview Tasks*
*Directions: you will see one card at a time with a problem or pictures. Answer the questions asked by researcher and show your work in the space provided for each problem (box)*

Name: ___________________ Period: _______ Date: __________

Please use the space inside the box to show your work

Task1

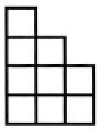


Figure 1

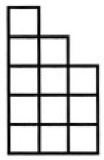


Figure 2

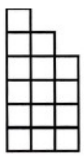
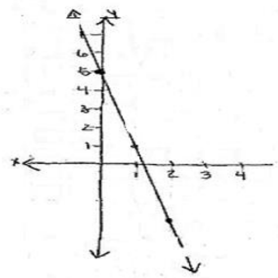


Figure 3

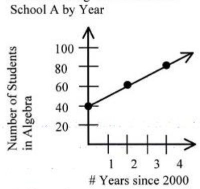
Task2




Task3

School A:

Number of Algebra Students in School A by Year




School B:

| Year | Population of School B |
|---|---|
| 2000 | 32 |
| 2001 | 60 |
| 2003 | 116 |

School C:

y = 120 – 30x

Where x=number of years since 2000, and y = the number of algebra students at school C

School D:

In 2000 School D had 27 algebra students. In 2003 school D had 72 students enrolled in Algebra.

Task4

Solve the equation.. What is the rate of change of this linear function?

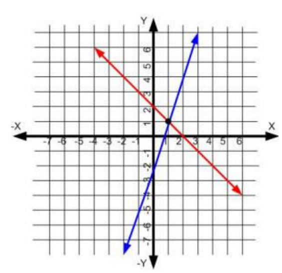
$-(-x - 5) = y - 2$

Task5

Jose left the White House and drove toward the recycling plant at an average speed of 40 km/h. Rob left some time later driving in the same direction at an average speed of 48km/h. After driving for five hours Rob caught up with Jose. How long did Jose drive before Rob caught up

Task6

Explain the characteristics of the following representation for the system of equations $y = 3x - 2$, in blue and $y = -x + 2$ in red



APPENDIX D: INTERVIEW TASKS INSTRUMENT

**Task Sheet** **Algebra 1 Linear functions**

**Please show all your work**

Task 1

Look at the following pattern and make picture to represent Fig 0, Fig 4 and Fig 5. Can you find a rule that will determine the total # of squares in any pattern like the one used for this task?

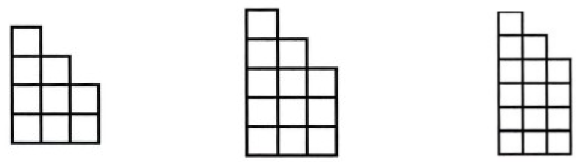
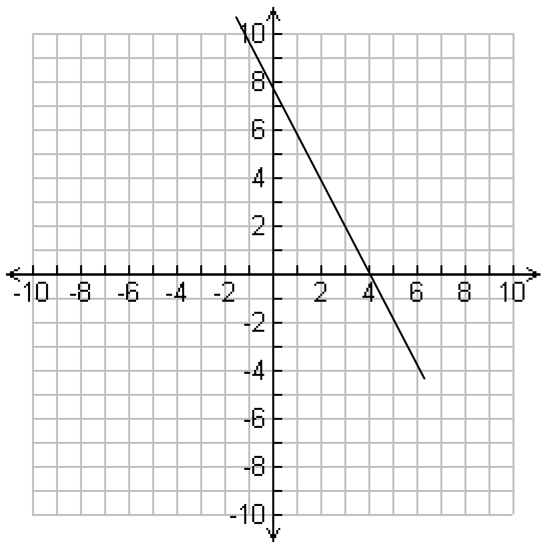


Figure 1 Figure 2 Figure 3

Rule: ______________________________________________________________

Task 2

Explain what is being represented in the given graphic representation?



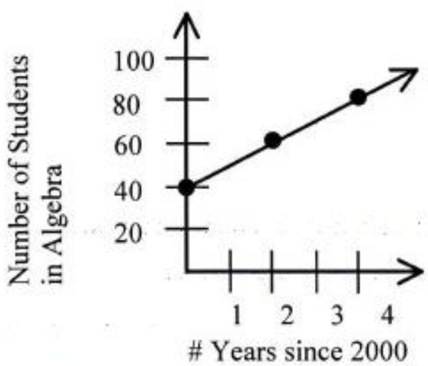

c. True/ False y= 3x – 4 is the equation of the line represented in the graph. Give reasons for your answer.

d. What part of the line in the graph represents the rate of change
e. What are the slope and intercepts of the line in the graph? What are the slope and intercepts of the line represented by the equation y= 3x -4.

Task 3
Below, you will see the number of algebra students at four different schools, School A, B, C and D. Assuming population rate of change is linear answer the following questions.

School A:

Number of Algebra Students in School A by Year




School B:

| Year | Population of School B |
|---|---|
| 2000 | 32 |
| 2001 | 60 |
| 2003 | 116 |

School C:

$y = 120 - 30x$

Where x=number of years since 2000, and y = the number of algebra students at school C

School D:

In 2000 School D had 27 algebra students. In 2003 school D had 72 students enrolled in Algebra.

a. Determine the rate of change in population for each school starting year 2000 until 2005.

School A

School B

School C

School D

b. What is the total population of students in each school in the year 2005?
School A

School B

School C

School D

c. What is the predicted population in 2015?
School A

School B

School C

School D

d. Draw a graph to represent each of the school algebra student population in School B, C and D.

School A

School B

School C

School D

e. Explain in own words about the answer to the problem

Task 4

a. Transform the equation in standard form.

- (-x - 5) = y – 2

b. What is the rate of change of the linear equation - (-x – 5) = y -2?

c. Represent the given equation as a graph.

Task 5

Jose left the White House and drove toward the recycling plant at an average speed of 40 km/h. Rob left some time later driving in the same direction at an average speed of 48km/h. After driving for five hours Rob caught up with Jose. How long did Jose drive before Rob caught up

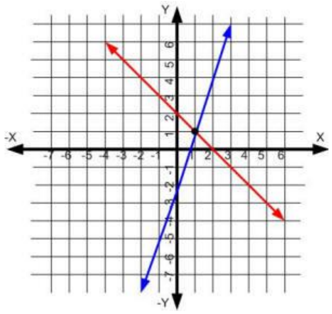
Task 6

a. How many and what type of solution is expected for the given graph representing blue and red lines intersecting at a point?

   # of solutions _____________

   Type of solution ___________

b. Write the slope and intercepts, of the following representation for the system of equations. y = 3x - 2, in Blue and y = -x +2 in Red



Blue line
Slope _________ X intercept =_______ Y inercept =____________

Red line
Slope _________ X intercept =_______ Y inercept =____________

c. Write a real life word problem (similar to task 5) that represent graph in this problem.

College of Education Department of Middle, Secondar, and K-12 Education.
(Parental permission form to participate in the study)

# APPENDIX E: PARENTAL CONSENT FORM

Date: _______________

Dear Parent or Guardian:

I am a doctoral student in the department of Curriculum and Instructions at University of North Carolina Charlotte. My research is focused on Urban Math Education in Charlotte North Carolina. The title of my study is *Multi-representation of Linear Functions for pre-College Students in a STEM perspective*. It is important to note that multi-representations of linear functions are widely used in medicine, engineering and industry. The students with a thorough understanding of multi-representations of linear function are expected to be more successful in acquiring jobs and meeting on job expectations in Science, Technology, Engineering, and Math (STEM) related careers and more. The purpose of this study is to illustrate the importance of multi-representation of linear functions involving the understanding and concept development for Pre-College students in a STEM perspective. I would like request your permission for your child to participate in my study. The details for my research study are as follows.

I will use case study methodology for my research. It will involve your child's Pre-College Algebra1teacher and three students from his/her class. The students will be randomly selected for participation upon receiving parent permission/consent for being in the study and confirming a 3.0 or higher grade point average (GPA).

The study will consist of two parts: a pilot study and the main study. For the pilot study the three selected participants will be given several math problems (equations, story-problems, graphing, etc.) in order to learn how familiar they are with these types of math problems. Then, participants will work on six problems related to multi-representations of linear functions. After completion of the pilot study, the main study will begin. The main study will include classroom observations and assessment of artifacts (test, quiz, class work, homework etc) collected from classroom to explore student teacher interaction. A 40-60 minutes interview with participating students will be conducted and video recorded to help me to explore the children's understanding of linear functions using multiple representations in class.

I will also interview the teacher separately once at the beginning and again after his/her completion of unit on linear functions. The teacher interviews will be used to understand his/her teaching perspective and strategies for concept building in linear functions. The interview recordings will NOT be accessible to any other person except me and my advisor. All recordings will be destroyed after the study is completed. Only the researcher and will have access to information from your child. At the conclusion of the study, children's responses will be reported as group results using pseudo names only.

Participation in this study is voluntary. Your decision whether or not to allow your child to participate will not affect the services normally provided to your child as part of the Pre-College program. Your child's participation in this study will not lead to the loss of any benefits to which he or she is otherwise entitled. Even if you give your permission for your child to participate, your child is free to refuse to participate. If your child agrees to participate, he or she is free to end participation at any time. You and your child are not waiving any legal claims, rights, or remedies because of your child's participation in this research study.

All information about student and teacher participation in study including their identity will be kept confidential. The following measures will be taken to ensure confidentiality: 1.

Pseudonym will be used instead of real names of participants. 2. All recordings will be labeled with the pseudonym. 3. Only researcher and research advisor will have access to the recorded materials. 4. All recorded materials will be saved in a locked cabinet on a single computer not connected to internet the UNC Charlotte office for STEM Education and will be destroyed at the end of study.

There are no known risks to participate in the study. However there may be risks which are currently unforeseeable. There may benefits from student participation in this study because it helps the students to be more successful in school and in their future careers. It also helps the students to reflect on their own learning goals and perspectives.

Should you have any questions or desire further information, please call me or email You may also contact my advisor,. If you have questions about your or your child's rights as participant in a research study, you may contact the UNC Charlotte Office of Research Compliance .

Keep this letter completing the bottom portion and fax it to my attention or send it with the student in a sealed envelope. Please remember to keep a copy for your records and send the original to me. Thank you in anticipation.

Sincerely,

PhD student
Department of Curriculum and Instructions
College of Education

Please indicate whether or not you wish to allow your child to participate in the research study mentioned above (Multi- representations of Linear Functions for Pre-College Students in a STEM perspective) by checking one of the statements below, signing your name and fax or send it with the student in a sealed envelope. Sign both copies and keep one for your records.

_____ I grant permission for my child to participate in the study *Multi- representations of Linear Functions for Pre-College Students in a STEM perspective*.

_____ I do not grant permission for my child to participate in the study *Multi- representations of Linear Functions for Pre-College Students in a STEM perspective*.

______________________________ ______________________________
Signature of Parent/Guardian Printed Parent/Guardian Name

______________________________ ____________ ____________________
Printed Name of Child Date Parent phone/ email contact

# APPENDIX F: TEACHER CONSENT

College of Education Department of Middle, Secondar, and K-12 Education.
(Parental permission form to participate in the study)

Dear Teacher, Date:______________

I am a doctoral student in the department of Curriculum and Instructions at University of North Carolina Charlotte. My research is focused on Urban Math Education Charlotte North Carolina. I am conducting a research project on "Multi- representations of Linear Functions for Pre-College Students in a STEM perspective". I request your participation in my research study.

The study consists of two parts, a pilot study and a main study. The pilot study will involve only students in your call who agree to participate and have parental permission/consent to participate. I will ask students' to complete a task assessment to learn how familiar they are with the type of problems related to multi-representation linear functions. The student participants will also complete additional multi-representation linear functions problems after the task assessment.

Your participation will begin with the main study. The main study includes classroom observations and assessment of artifacts (test, quiz, class work, homework etc) collected from classroom to explore student teacher interaction. There will be two classroom observations for three weeks or until the unit for linear function is completed. During the observations I will have no interaction with you or the students in class. Two teacher interviews, duration 40-60 minutes will be held separately to understand the teaching perspective and strategies used for building student concept for linear functions. The first interview will focus on your previous experience in teaching linear functions using multi- representations and your expectations from students to be able to learn at the end of your teaching the unit on linear functions. The second interview will be at the end of the unit on linear functions. This will be geared towards knowing about how much and to what extent your expectations were met and what is your idea about the use of multi-representation of linear functions towards concept building that may be helpful in STEM careers. The interviews will be recorded and will NOT be accessible to any other person except me and my faculty advisor. All recordings will be destroyed after the study is completed. Only the advisor and myself will have access to information from you. At the conclusion of the study, your responses will be reported as group results with students' and teacher pseudo names.

There are no known risks to participate in the study. However there may be risks which are currently unforeseeable. There are no direct benefits to you but this research may provide insight into the challenges algebra 1 students in our schools have related to multi-representations and End of Course testing, college entrance exams. This study may help the students to be more successful in school and in their future careers. It also helps the students to reflect on their own learning goals and perspectives.

Participation in this study is voluntary. Any information that is obtained in connection with this study and that can be identified with you will remain confidential and will be disclosed only with your permission or as required by law. Confidentiality will be maintained by means of securing data as anonymous pseudonym under lock and key in the password protected single computer at the office of UNC Charlotte College of Education.

Should you have any questions or desire further information, please call me or email me. You may also contact my advisor about my research study. If you have questions about your or your child's rights as participant in a research study, you may contact the UNC Charlotte Office of Research Compliance.

Keep this letter, completing the bottom portion and fax it to my attention fax or you may return it to me in a sealed envelope to the UNC Charlotte Center for STEM Education. Please remember to keep a copy for your records and send the original to me. Thank you in anticipation.

Sincerely,

PhD student
Department of Curriculum and Instructions
College of Education

Please indicate whether or not you wish to participate in the research study mentioned above (Multi-representations of Linear Functions for Pre-College Students in a STEM perspective) by checking one of the statements below, signing your name and fax it or send it in a sealed envelope. Sign both copies and keep one for your records.

_____ I will voluntarily participate in the study *Multi- representations of Linear Functions for Pre-College Students in a STEM perspective* mentioned above

_____ I will not voluntarily participate in the study *Multi- representations of Linear Functions for Pre-College Students in a STEM perspective* mentioned above.

______________________________ Date ___________________________________

Signature

_______________________________ Contact email / Phone # ____________________

Printed Name

# APPENDIX G: EMAIL INVITATION

UNC CHARLOTTE
College of Education
Center for Science, Technology, Engineering, and Mathematics [STEM] Education

Dear ________________________ Date: _____________

I am a doctoral student at UNC Charlotte, preparing to collect data for my dissertation research study. I am contacting you on the recommendation of the Assistant Director Center for STEM Education at UNC Charlotte College of Education. ,who spoke to you about my topic and told me that you had shown some interest in being a participant in my research. I would like to invite you to join me in my study. The focus of my study is to explore the student learning of linear functions using multiple representations.
As you are aware teachers have a major impact on student's concept learning and development, I would like to observe your Algebra 1 pre-college students and talk to you about teaching strategies used to teach linear functions to pre-college students in mathematics.

If you are interested in participating, I would like to observe your class twice a week until the unit on linear functions is completed. I will also request to interview you twice, once at the beginning of unit and the second interview when you have completed the unit. The anticipated time frame is summer semester 2013, starting in June. Each interview will last between 40 minutes to an hour. We can schedule these interviews at a time and location that is convenient for you at the beginning and end of your unit on linear functions. You may benefit from participation in this study because it will provide a space for you to reflect on your practice as a teacher. A pseudonym for you will be used in my writings.
If you would like more information, or if you feel you would be interested in participating in my study, please contact me
Thank you for your time and have a great day,

Sincerely

Doctoral Candidate in Curriculum and Instruction Urban Mathematics Education

# APPENDIX H: STUDENT ASSESSMENT

Dear Student,

I am a doctoral student at UNC Charlotte working on my research project titled as “Multi-representations of linear functions for Pre-College students in a STEM perspective.” For this research case study I would like to invite you as a volunteer student participant.

My study consists of two parts: a pilot study in which you will be asked to show all your work and solve sample linear functions problems to the best you can. These sample problems will be similar to what you will be asked to discuss during the interview process of the second part of the study (main study). In the second part of study called the main study, I will visit your classroom as an observer twice a week until the unit on linear functions is completed. I will not interact with you or the teacher during the observations. I will just sit in the back of the room and take notes about the learning strategies and student-teacher interactions during the lesson. There will be no audio or video recording involved for observations. After the unit on linear functions is completed in class, I would like to have a one-on one interview with you and two more of your class mates. The purpose of interview is to find out about your impressions and learning of linear functions in class. These interviews will be video recorded for listening to your comments correctly and precisely. We will follow the university required guidelines for collecting audio video data and it will be kept confidential as mentioned below.

I would also like you to be aware of the fact that you will not receive any direct benefits (monetary or otherwise) from participating in this research. However, your participation may help me to understand some of the difficulties faced by you and your fellow Algebra 1 students when working with and transitioning between different representations. It may also help the education system to understand the need of better training and professional development for Algebra 1 teachers to help student understand the benefit of knowing how to interpret multi-representations of the same problem in different perspective.

<u>Confidentiality</u>: Only the principal researcher and research advisor will have access to research results associated with your identity. In the event of publication of this research, no personally identifying information will be disclosed.

Questions about this research study should be directed to the primary investigator and person in charge or to her supervisor Questions about your rights as a research participant should be directed to the UNC Charlotte Institutional Review Board Office

If you agree to take part in this research, please sign the bottom page or send it electronically.
Thank you in anticipation.

I certify that I have read this form and volunteer to participate in this research study: “Multi-representations of linear functions for Pre-College students in a STEM perspective”

__________________________________ Grade: __________________
(Print) Student Name

__________________________________ Date: __________________
Student Signature