DYNAMICS OF THE SHIFT ACTION ON LINEAR SEQUENCE SPACES OVER GROUPS BEYOND $\mathbb{Z}$

by

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ABSTRACT

SERGEI L. MILES. Dynamics of the shift action on linear sequence spaces over groups beyond $\mathbb{Z}$. (Under the direction of DR. KEVIN MCGOFF & DR. WILLIAM BRIAN)

In linear dynamics, bounded linear operators over infinite-dimensional Banach spaces have been shown to be able to exhibit interesting characteristics including topological transitivity, topological mixing, and even chaos in the sense of Devaney. This dissertation will examine weighted ℓ^p sequence spaces together with the shift action as the operator. In the case the shift action is over the semi-group $\mathbb{N}$, the above topological properties have been previously characterized by conditions on the weight sequence associated with a given weighted ℓ^p space. This work will present recent results for new characterizations of these properties when the group action over a countable group is instead considered. Additionally, an example choice of the weight sequence in this setting will be presented which yields points which are periodic while having an infinite orbit.

Lastly, new implications for infinite and 0 topological entropy for the weighted ℓ^p space with the shift action over $\mathbb{N}$ will be given. In particular, when the weight sequence is summable over a subset of $\mathbb{N}$ with positive upper density then infinite entropy may be achieved. Furthermore, when an arbitrary ratio of the weights is bounded above then 0 entropy is guaranteed.

DEDICATION

Thank you to my parents for advising, guiding, and financing me along the way through my entire academic career. Also a thank you to Madison Melton for encouraging and supporting me through the graduate school experience; your kindness has made the journey more manageable.

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CHAPTER 1: INTRODUCTION

1.1 Document Structure

A dynamical system is a pair (X, T) where X is a compact metric space and T is a continuous operator on X. Chapter 1 will explain the setting for what class of dynamical systems will be studied (see section 1.3) and will present the questions which motivate this work. Furthermore, a brief discussion of other related works at the end of chapter 1 will be provided which will motivate other open questions.

Chapter 2 will then showcase the new results for shift group actions over an arbitrary countable group when paired with the weighted ℓ^p space. The results will highlight characterizations for common topological dynamics properties such as topological transitivity, weakly mixing, mixing, and chaos in the sense of Devaney. Definitions of which in this setting are provided in section 1.4.1. Additionally, counterexamples will be given showing where traditional characterizations in the setting of the shift action over the semi-group $\mathbb{N}$ no longer hold true.

The open question of topological entropy in this setting is given separate special attention in chapter 3. New implications for infinite and zero topological entropy will be presented. However, the question is not fully answered and so a discussion of open questions in this direction is also provided at the end of chapter 3.

Lastly, chapter 4 will briefly summarize the results and showcase other open questions for additional areas of research.

1.2 Problem Statement

When studying various classes of dynamical systems one of the first goals is to characterize under what conditions do topological transitivity, weakly mixing, and mixing occur. A linear dynamical system is a pair (X, T) where X is a Banach space and T is a linear continuous operator on X. Notably, X is not compact here where in traditional topological dynamics X is compact. Additionally, when X is a finite dimensional Banach space then it has been shown to be impossible for the dynamics to produce transitivity, weakly mixing, or mixing [1]. So it would seem linear dynamics would not be very interesting; however, when the Banach space is infinite dimensional then even chaos in the sense of Devaney may occur. In this way, linear dynamics can be just as chaotic as nonlinear dynamics.

There are many examples of topologically transitive linear dynamics such as the translation map on holomorphic functions (Birkhoff, 1929), the derivative map on the set of holomorphic functions (MacLane, 1952), and the scaled backward shift on the ℓ^p space (Rolewicz, 1969) just to name a few. In symbolic dynamics there is an active interest in classifying group actions. The goal of this work is to then consider group actions in the setting of linear dynamics. In particular, in the context of the weighted ℓ^p space together with the backward shift operator (discussed in section 1.3). Characterizations for common dynamics properties in the setting of the backward shift over $\mathbb{N}$ have already been found, and so this dissertation will work to find new characterizations in the setting of a group action over a countable group and compare these conditions with the traditional characterizations.

Lastly, there is often an interest in comparing chaos with topological entropy. In some settings there can be chaos while having 0 topological entropy; and conversely, infinite entropy with the absence of chaos may also occur. However, complete characterizations for infinite, zero, or positive finite entropy in the setting of the weighted ℓ^p space with the backward shift operator have not been found and is an active open area of research. The last part of this work will aim to give new implications in this question.

1.3 Class of Model Systems: ℓ^p

Traditional ℓ^p spaces, with $1 \leq p < \infty$, are defined in the following manner

$$\ell^p = \{x = (x_i)_{i=1}^{\infty} \in \mathbb{K}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^p < \infty\}$$

which when taken together with the shift action over the semi-group $\mathbb{N}$, call it σ, becomes a linear dynamical system pair (ℓ^p, σ). The shift action σ is the backward shift operator where $\forall x \in \ell^p$, all entries are shifted to the left and the first entry is dropped with no memory retention:

$$\sigma(x) = \sigma(x_1, x_2, x_3, ...) = (x_2, x_3, ...)$$

Similarly, $\forall n \in \mathbb{N}$, the n^{th} shift of x may be determined as

$$\sigma^n(x) = (x_{n+1}, x_{n+2}, ...)$$

This space with the p-norm then forms a metric space, and σ is both linear and continuous. Continuity in this setting will be discussed in greater detail in section 1.4.1. But, not even topological transitivity can be achieved and so this is a rather dull linear dynamical system.

However, let $(v_i)_{i=1}^{\infty}$ be a sequence such that $\forall i \geq 1, \;\; v_i > 0$, and consider instead the weighted ℓ^p space:

$$\ell^p(v) = \{x = (x_i)_{i=1}^{\infty} \in \mathbb{K}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^p v_i < \infty\}$$

Here interesting properties can now occur, even a property as extreme as Chaos in the sense of Devaney. Note the p-norm will also be suitably modified

$$||x|| = (\sum_{i=1}^{\infty} |x_i|^p v_i)^{1/p}$$

and so this metric will then induce the topology on $\ell^p(v)$.

Additionally, observe $\ell^p(v)$ has the countable basis $\mathcal{B} = \{e_n\}$ where $\forall n \geq 1$, e_n is the sequence

$$(e_i)_{i=1}^{\infty} \text{ where } e_i = \begin{cases} 1, & \text{if } i = n \\ 0, & \text{otherwise} \end{cases}$$

For example, $e_1 = (1, 0, 0, ...)$. Then by linearity and properties of vector spaces, $\forall x \in \ell^p(v)$, x may be written as a linear combination of the basis vectors, $x = \sum_{i=1}^{\infty} x_i e_i$. Thus, by linearity of the shift action, $\sigma(x)$ may be rewritten

$$\sigma(x) = x_1\sigma(e_1) + x_2\sigma(e_2) + ... = \sum_{i=2}^{\infty} x_i e_{i-1}$$

Note: $\forall n \geq 2$, $\sigma(e_n) = e_{n-1}$ and $\forall m < n$, $\sigma^m(e_n) = e_{n-m}$.

These definitions and properties will be used throughout the document. Lastly, for sake of ease, hence forward in the introduction let $X = \ell^p(v)$.

1.4 Background

This section has been broken into two subsections since much of the most common properties analyzed in topological dynamics have already been characterized in previous works in the case of the space $\ell^p(v)$. Section 1.4.1 will highlight these characterizations. However, special attention should be given to the question of topological entropy since this is an active open area of research even today. Additionally, the definitions provided will be given in the context of $(\ell^p(v), \sigma)$, and sources to the original definitions will be cited.

1.4.1 Properties of Topological Dynamics

The first requirement on the operator of a dynamical system pair (X, T) is T must be continuous. With the topology induced by the metric on X, the typical ε, δ definition will suffice. However, in the case of an operator on a Banach space, recall T is continuous if and only if T is bounded. When $X = \ell^p(v)$, it is often easier to check for boundedness of σ rather than continuity.

Definition 1 (Bounded Operator)**.** *Let σ be the shift action on X, then $\forall n \in \mathbb{N}$, σ^n is called* ***bounded*** *if $\exists M > 0$ such that $\forall x \in X$*

$$||\sigma^n(x)|| \leq M||x||$$

In Grosse-Erdmann and Manguillot's textbook [1], they put forward the characterization σ is bounded if and only if $\sup\limits_{n\in\mathbb{N}} \frac{v_n}{v_{n+1}} < \infty$.

In dynamics, the goal is often to determine to what extent can points travel around the space via the operator. Can arbitrary open sets land inside and hit each other after transformation, do the sets stay together, or do they only hit each other for a brief moment? Varying degrees for a system's ability to mix are highlighted by the following definitions, the weakest form of mixing being topological transitivity.

Definition 2 (Topological Transitivity)**.** *Let σ be a continuous shift action on X, then σ is called* ***topologically transitive*** *if $\forall$ nonempty open $U, V \subseteq X$, $\exists n \geq 0$ such that $\sigma^n(U) \cap V \neq \emptyset$.*

For $n = 0$, σ^n is the identity map. Notably, the following theorem will be helpful in this setting:

Theorem (Birkhoff Transitivity Theorem)**.** *Let T be a continuous map on a separable complete metric space X without isolated points. Then the followinging assertions are equivalent:*

(i) T is topologically transitive;

(ii) $\exists x \in X$ such that the orbit of x is dense in X.

If one of these conditions holds, then the set of points in X with dense orbit is a dense G_δ-set.

A proof of which is discussed in [1]. Notably, this gives rise to an equivalent interpretation for topological transitivity in the setting of linear dynamics with the following definition.

Definition 3 (Hypercyclicity)**.** *Let σ be a continuous shift action on X, then σ is called* ***hypercyclic*** *if $\exists x \in X$ such that the orbit of x is dense in X.*

Thus, σ is topologically transitive if and only if σ is hypercyclic. Hypercyclicity is a property in linear dynamics which at times is easier to show than topological transitivity. Furthermore, this will then lead to the characterization σ is topologically transitive if and only if $\inf_{n\in\mathbb{N}} v_n = 0$, discussed in [1].

A stronger notion for an operator's ability to mix the space is the property weakly mixing.

Definition 4 (Weakly Mixing)**.** *Let σ be a continuous shift action on X, then σ is called* ***weakly mixing*** *if $\forall$ nonempty open $U, V \subseteq X$*

$$N(U,U) \cap N(U,V) \neq \emptyset$$

where $N(U,V) = \{n \in \mathbb{N} : \sigma^n(U) \cap V \neq \emptyset\}$.

In other words, there exists shifts of U which lands in U and V. Weakly mixing may also be equivalently determined by the next definition. This will be used later in the paper.

Definition 5 (Weakly Mixing (Alternate)). *Let σ be a continuous shift action on X, then σ is called **weakly mixing** if $\sigma \times \sigma$ is topologically transitive.*

Where $\sigma \times \sigma$ is the product shift action on the product space $X \times X$. Notably, in general dynamics weakly mixing implies topological transitivity. But the reverse implication is not true in general. However, a different scenario is presented via the following theorem given in [1].

Theorem 1 (Grosse-Erdmann and Manguillot). *Let X be a Banach space in which $\{e_n\}_n$ is a basis, and let σ be the backward shift on X. Then the following are equivalent:*

i) σ *is hypercyclic;*

ii) σ *is weakly mixing;*

iii) $\exists$ *an increasing sequence* $\{n_k\}_{k=1}^{\infty}$ *of positive integers such that* $e_{n_k} \to 0$ *in* X *as* $k \to \infty$.

Therefore, using the previous characterization, σ is weakly mixing if and only if $\inf\limits_{n\in\mathbb{N}} v_n = 0$. The next tier for an operator's ability to mix the space is the property mixing.

Definition 6 (Mixing). *Let σ be a continuous shift action on X, then σ is called **mixing** if $\forall$ nonempty open $U, V \subseteq X$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$*

$$\sigma^n(U) \cap \sigma(V) \neq \emptyset \ .$$

In general, mixing $\Rightarrow$ weakly mixing $\Rightarrow$ topological transitivity. Via a similar notion as to theorem 1, consider the following theorem.

Theorem 2 (Grosse-Erdmann and Manguillot). *Let X be a Banach space in which $\{e_n\}_n$ is a basis, and let σ be the backward shift on X. Then the following are equivalent:*

i) σ is mixing;

ii) $e_n \to 0$ in X as $n \to \infty$.

A proof of which may be found in [1]. Thus σ is mixing if and only if $\lim_n v_n = 0$. Note, theorems 1 and 2 are explicitly stated here since they will give motivation for the new theorems presented later in chapter 2.

Some of the most extreme examples of dynamical systems showcase some of these properties, and there is active debate and research into what the exact definition should be for more chaotic systems. Some suggest infinite topological entropy, discussed in the next section, while others suggest chaos in the sense of Devaney. Other definitions of chaos are also presented in other works, but this dissertation will utilize Devaney's definition. However, another notion is first needed.

Definition 7 (Periodic Point). *Let σ be a continuous shift action on X, and let $x \in X$. Then x is called a **periodic point** of σ if $\exists n \in \mathbb{N}$ such that $\sigma^n(x) = x$.*

Notably, even the existence of a nontrivial periodic point implies the weight sequence $\{v_n\}_{n=1}^{\infty}$ must be summable [1].

Definition 8 (Chaos in the sense of Devaney). *Let σ be a continuous shift action on X, then σ is called **chaotic in the sense of Devaney** if σ satisfies the following conditions:*

i) σ is hypercyclic;

ii) $\exists$ a set of periodic points which is dense in X.

In general, the existence of a nontrivial periodic point does not alone imply Devaney chaos. But in the setting of the backward shift on $\ell^p(v)$ the existence is enough to get chaos [1]. Thus, σ is chaotic in the sense of Devaney if and only if $\sum_n v_n < \infty$.

Now, with all of the properties of interest defined, the following hierarchy in this setting may be observed.

Table 1.1: Property Characterizations for shift semi-group action over $\mathbb{N}$.

Property	Characterization
Continuity $\Leftrightarrow$ Boundedness	$\sup_{n \in \mathbb{N}} \frac{v_n}{v_{n+1}} < \infty$
Topological Transitivity $\Leftrightarrow$ Hypercyclicity	$\inf_{n \in \mathbb{N}} v_n = 0$
Weakly Mixing	$\inf_{n \in \mathbb{N}} v_n = 0$
Mixing	$\lim_{n \to \infty} v_n = 0$
Devaney Chaos	$\sum_{n \in \mathbb{N}} v_n < \infty$

1.4.2 Topological Entropy

In his work [2], Bowen presents the definition for topological entropy which is most commonly and widely used in the field. Two equivalent definitions are provided in which one uses the concept of separating sets and the other uses the concept of spanning sets.

Definition 9 (Separating Set)**.** *Let $K \subseteq X$ be compact, σ be a continuous shift action on X, $E \subseteq K$, $n \in \mathbb{N}$, and let $\varepsilon > 0$. Then E is called* ***(n, ε)-separated*** *if $\forall$ distinct $x, y \in E$, $\exists j \in \mathbb{N}$ with $0 \leq j < n$ such that $||\sigma^j(x) - \sigma^j(y)|| > \varepsilon$. Furthermore, let $s_n(\varepsilon, K)$ denote the maximal cardinality of any (n, ε)-separated set $E \subseteq K$.*

Definition 10 (Spanning Set)**.** *Let $K \subseteq X$ be compact, σ be a continuous shift action on X, $F \subseteq K$, $n \in \mathbb{N}$, and let $\varepsilon > 0$. Then F* ***(n, ε)-spans*** *the set K if $\forall x \in K$, $\exists y \in F$ such that $||\sigma^j(x) - \sigma^j(y)|| \leq \varepsilon$ for all $0 \leq j < n$.*

Furthermore, let $r_n(\varepsilon, K)$ denote the minimal cardinality of any set F which (n, ε)-spans K.

So, conceptually, a set is separating if its points can eventually be a fixed distance apart, and a set spans another set if points of the second set can be closely followed by points of the first set. Now Bowen's definition may be considered.

Definition 11 (Topological Entropy)**.** *Let σ be a continuous shift action on X, and let $K \subseteq X$ be compact. Then*

$$h(\sigma, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\varepsilon, K)$$

and the ***topological entropy of $\boldsymbol{\sigma}$*** *is determined by*

$$h(\sigma) = \sup_{K compact} h(\sigma, K)$$

The definitions listed here will be used extensively in chapter 3. Bowen also discusses in greater detail topological entropy in the setting of noncompact spaces in other works [3].

1.5 Main Results

This work will first aim to offer a new table and hierarchy of the topological dynamics properties, shown in table 1.1, in the case when the shift group action over an arbitrary countable group is instead considered rather than the semi-group $\mathbb{N}$. Notably, memory of entries is no longer lost via a single shift as in the case of the semi-group action. This will be highlighted in greater detail in chapter 2. Also in the case the group is $\mathbb{Z}^2$ there is now a notion of direction when scanning through a given point. Whereas, in the case of $\mathbb{N}$, there is only a single direction to consider. While ideas like theorem 1 and theorem 2 will be used to construct new characterizations, special concern will have to be applied to inverse shifts since there is no longer memory loss.

For this reason, the infimum condition in table 1.1 will not suffice. A counterexample will be discussed in section 2.3. But new theorems similar to theorems 1 and 2 will be given and proven. So in this way topological transitivity, hypercyclicity, and weakly mixing will still be equivalent properties when the inverse direction of the shift is also considered.

In the case of mixing, the same characterization will still hold when a limit of the weight sequence is taken over an enumeration of the group. However, the original proof strategy of using the definition of hypercyclicity will not work. And so a different strategy using topological transitivity will instead be presented.

In the case of the semi-group action, it should be noted that a point $x \in X$ being periodic implies the point also has finite orbit. However, in the case of a group action, it will become possible to have a periodic point which still has an infinite orbit. Furthermore, it will be shown in section 2.5 the system could then possess nontrivial periodic points without the action being mixing. For this reason, density of periodic points may no longer be an appropriate requirement for Devaney Chaos. Density of points with finite orbit may be preferred instead. Notably, summability of the weight sequence over all of the group will not be necessary.

Lastly, chapter 3 will showcase new implications and tests for infinite and zero topological entropy in the case of the semi-group action on X. It will be shown that a finite bound on an arbitrary ratio of the weight sequence will result in zero topological entropy in section 3.2. Additionally, summability of the weight sequence over a subset of $\mathbb{N}$ with positive upper density will be shown to imply infinite topological entropy in section 3.1. However, neither proof will be an equivalence theorem. And so section 3.3 will highlight the open questions in this regard.

1.6 Related Works

For a broader consideration of linear dynamics in general, Menet's work [4] provides a hierarchy of properties for arbitrary linear dynamics. Additionally, Menet examines other properties of linear dynamics such as reiteratively hypercyclic, $\mathcal{U}$-frequently hypercyclic, and frequently hypercyclic. Other investigations of hypercyclic operators, and linear operators in general, may also be found in Bayart and Matheron's text [5].

Competing definitions for chaos may also be considered via Li-Yorke chaos presented by Li and Yorke [6]. Also Schweizer and Smítal's definition, distributional chaos [7], may be examined. Additionally, in Bartoll, Martínez-Giménez, and Peris' paper [8] they show for the backward shift on the weighted ℓ^p space Devaney chaos is equivalent to the strong specification property.

Linear operators and infinite topological entropy are discussed in the setting of L^p by Brian and Kelly in [9]. Also Brian, Kelly, and Tennant give an example of a shift operator on $\ell^p(v)$ which possesses infinite topological entropy while not being Devaney chaotic in [10]. The theorem for infinite topological entropy in chapter 3 will similarly follow the argument made in their paper. For an alternate consideration for the definition of topological entropy see Liu, Wang, and Wei's work [11] where they present a definition which is metric-independent while Bowen's is metric-dependent.

CHAPTER 2: PROPERTIES OF THE SHIFT ACTION OVER A GROUP

2.1 The Setting

Let G be a countable group, let $p \in \mathbb{N}$ such that $1 \leq p < \infty$, and let $(v_g)_{g \in G}$ be a set such that $\forall g \in G,\ \ v_g > 0$. Then define

$$\ell^p_G(v) = \{x = (x_g)_{g\in G} \in \mathbb{K}^G : \sum_{g\in G} |x_g|^p v_g < \infty\}$$

together with the corresponding p-norm such that $\forall x \in \ell^p_G(v)$

$$||x|| = (\sum_{g\in G} |x_g|^p v_g)^{1/p}$$

which forms a normed linear sequence space. Throughout chapter 2, let $X = \ell^p_G(v)$. Note: G countable implies G may be enumerated, $(g_i)_{i=1}^{\infty}$, and so X has the basis $\mathcal{B} = \{e_h\}_{h\in G}$ where $\forall h \in G$, e_h is the sequence

$$(e_{g_i})_{i=1}^{\infty} \text{ where } e_{g_i} = \begin{cases} 1, & \text{if } g_i = h \\ 0, & \text{otherwise} \end{cases}$$

For example, if $G = \mathbb{Z}$ then $e_0 = (..., 0, 0, 1, 0, 0, ...)$. thus, by definition of a basis, $\forall x \in X,\ \ x = \sum\limits_{g\in G} x_g e_g$. Furthermore, Let σ be the shift action on X then $\forall h \in G$

$$\sigma^h(x) = \sum_{g\in G} x_g e_{gh^{-1}}$$

But, in order to be a true dynamical system, the first requirement will be continuity of σ.

2.2 Continuity

Recall for a linear operator on a normed linear space, the operator is continuous if and only if the operator is bounded. So in order to check for continuity, boundedness is used in the following theorem.

Theorem 3 (Bounded Shift Group Action)**.** *Let σ be the shift group action on X, and let $g \in G$. Then σ^g is* ***bounded*** *if and only if*

$$\sup_{k \in G} \frac{v_k}{v_{kg}} < \infty$$

Proof. Let $g \in G$ and suppose σ^g is bounded. This implies $\exists M > 0$ such that $\forall x \in X$

$$||\sigma^g(x)|| \leq M||x||$$

$$\sum_{i \in G} |x_{ig}|^p v_i \leq M^p \sum_{i \in G} |x_i|^p v_i$$

Let $k \in G$ and let $x = e_{kg}$. Then

$$|x_{kg}|^p v_k \leq M^p |x_{kg}|^p v_{kg}$$

$$\frac{v_k}{v_{kg}} \leq M^p$$

But g and k were arbitrary. Thus, $\forall g, k \in G$

$$\frac{v_k}{v_{kg}} \leq M^p$$

which implies $\forall g \in G$

$$\sup_{k \in G} \frac{v_k}{v_{kg}} \leq M^p < \infty$$

Now let $g \in G$ and suppose the supremum condition holds. Let $k \in G$ and by assumption $\exists M > 0$ such that

$$\frac{v_k}{v_{kg}} \leq M$$

Next let $x \in X$ and observe

$$v_k \leq M v_{kg}$$

$$|x_{kg}|^p v_k \leq M |x_{kg}|^p v_{kg}$$

But k was arbitrary. Therefore,

$$||\sigma^g(x)|| \leq M^{1/p} ||x||$$

$\square$

In the case of the semi-group action over $\mathbb{N}$, an arbitrary shift is a multiple of shifting backwards once. So it sufficed to obtain boundedness via $\sup\limits_{n \in \mathbb{N}} \frac{v_n}{v_{n+1}} < \infty$. But in the case of the group action, it must be verified any arbitrary shift will obey the supremum condition. Otherwise, continuity is determined the same way in either setting.

2.3 Topological Transitivity and Weakly Mixing

Topological transitivity is the first property where the characterization in the case of the semi-group action will fail to hold true. Consider this example when $G = \mathbb{Z}$, and the weight sequence is determined $\forall i \in G$ by

$$v_i = \begin{cases} \frac{1}{i^p}, & \text{if } i > 0 \\ 1, & \text{otherwise} \end{cases}$$

So $(v_i)_{i=1}^{\infty} = (..., 1, 1, 1, 1, \frac{1}{1^p}, \frac{1}{2^p}, \frac{1}{3^p}, ...)$, σ is bounded by theorem 3, and $\inf\limits_{i \in G} v_i = 0$. However, with this choice of the weights, σ is not topologically transitive. Let

$U = \{x \in X : ||x - (e_0 + e_1)|| < \frac{1}{2}\}$ and let $V = \{x \in X : ||x - (e_{-3} + e_{-1})|| < \frac{1}{2}\}$. Then U and V are nonempty open sets. Note the following $\forall x \in U$

$$\text{if } i = 0 \text{ or } 1,\ |x_i - 1| < \frac{1}{2} \Rightarrow 0.5 < x_i < 1.5$$

$$\forall i < 0,\ |x_i| < \frac{1}{2}$$

$$\forall i > 1,\ |x_i| < \frac{i}{2}$$

Similarly, $\forall x \in V$

$$\text{if } i = -3 \text{ or } -1,\ |x_i - 1| < \frac{1}{2} \Rightarrow 0.5 < x_i < 1.5$$

$$\forall i < -3,\ i = -2,\ \text{or } i = 0,\ |x_i| < \frac{1}{2}$$

$$\forall i > 0,\ |x_i| < \frac{i}{2}$$

This may be viewed visually via the figure below.

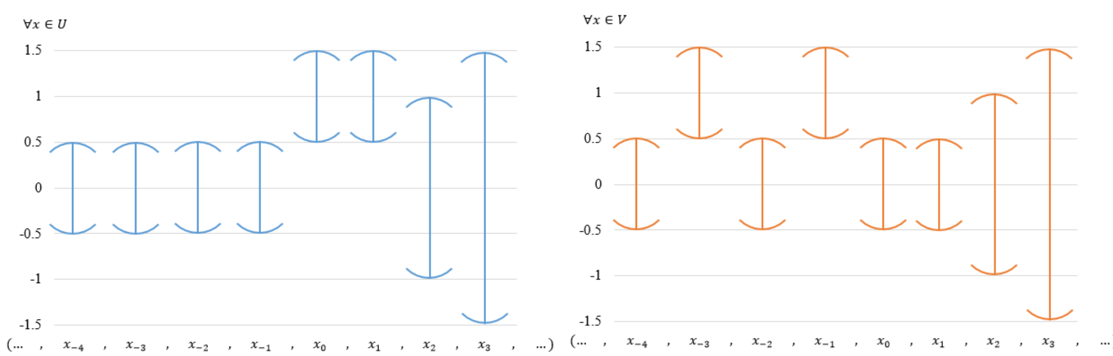

Figure 2.1: Example open sets which break Transitivity.

So no shift of U can land inside of V; therefore, σ is not topologically transitive. In the case of the semi-group action over $\mathbb{N}$, having a single direction in which a subsequence of the weights approaches 0 is sufficient since each shift to the left drops an entry of x with no memory retention. But in the case of the group action, entries are not lost after shifting. For this reason, it will become necessary for the inverse direction of a subsequence of the weights to also approach 0.

Recall, by the Birkhoff transitivity theorem, σ is topologically transitive if and only if σ is hypercyclic. In order to prove theorem 1, Grosse-Erdmann and Manguillot used the hypercyclicity criterion [1].

Theorem (Hypercyclicity Criterion)**.** *Let $X = \ell^p(v)$. Let σ be the backward shift. Now if $\exists$ dense X_0, $Y_0 \subseteq X$, $\exists$ a sequence $(n_i)_{i=1}^{\infty}$ of positive integers, and $\exists$ maps $S_n : Y_0 \to X$ $\forall n \geq 1$ such that $\forall x \in X_0$ $\forall y \in Y_0$ the following holds:*

i) $\sigma^{n_i}(x) \to 0$

ii) $S_n(y) \to 0$

iii) $\sigma^{n_i}(S_n(y)) \to y$

*Then σ is **weakly mixing**; moreover, σ is also **hypercyclic**.*

For a proof see [1]. In order to construct the new theorem similar to theorem 1, a new criterion in the setting of the group action will be needed first. The following definition will also be used.

Definition 12. *A sequence of group elements $(g_n)_{n=1}^{\infty}$ **tends to infinity** if $\forall$ finite $K \subseteq G$, $\exists N$ large enough such that $\forall n > N$, $g_n \notin K$.*

Theorem 4 (Group Action Hypercyclicity Criterion)**.** *Let $X = \ell^p_G(v)$ with G a countable group, and let σ be a bounded shift action. Now if $\exists$ dense X_0, $Y_0 \subseteq X$, $\exists$ a sequence $(g_n)_{n=1}^{\infty}$ of elements of G which tends to infinity, and $\exists$ maps $S_n : Y_0 \to X$ $\forall n \geq 1$ such that $\forall x \in X_0$ $\forall y \in Y_0$ the following holds:*

i) $\sigma^{g_n}(x) \to 0$

ii) $S_n(y) \to 0$

iii) $\sigma^{g_n}(S_n(y)) \to y$

*Then σ is **weakly mixing**; moreover, σ is also **hypercyclic**.*

Proof. (This argument also follows closely the argument made in [1] for the hypercyclicity criterion.) Note by X separable $\exists$ a sequence $(y_i)_{i=1}^{\infty}$ of elements of Y_0 such that the sequence is dense in X, and by assumption Y_0 is dense in X.

Claim: $\exists\ x_j \in X$ and $\exists\ n_j \in \mathbb{N}$ with $j \geq 1$ such that

$$x = x_1 + S_{n_1}(y_1) + x_2 + S_{n_2}(y_2) + x_3 + ...$$

is well-defined and hypercyclic in X.

Proof: $\forall j \geq 1$ recursively construct x_j and n_j such that $\forall l \leq j-1$

1) $||x_j|| < \frac{1}{2^j}$ and $||\sigma^{g_{n_l}}(x_j)|| < \frac{1}{2^j}$,

2) $||S_{n_j}(y_j)|| < \frac{1}{2^j}$ and $||\sigma^{g_{n_l}}(S_{n_l}(y_j))|| < \frac{1}{2^j}$,

3) $||\sigma^{g_{n_j}}(S_{n_j}(y_j)) - y_j|| < \frac{1}{2^j}$ and $||\sigma^{g_{n_j}}(\sum_{l=1}^{j-1}(x_l + S_{n_l}(y_l)) + x_j)|| < \frac{1}{2^j}$.

For $j = 1$, $x_1 = 0$ satisfies 1) by continuity of σ. By assumption, *ii*) and *iii*) implies $\exists n_1 \in \mathbb{N}$ large enough such that 2) and 3) are satisfied.

Let $j \geq 2$, and suppose $x_1, ..., x_{j-1}$ and $n_1, ..., n_{j-1}$ have been chosen to satisfy 1), 2), and 3). Now by the density of X_0, $\exists x_j \in X$ such that 1) holds and

$$\sum_{l=1}^{j-1}(x_l + S_{n_l}(y_l)) + x_j \in X_0$$

Again, by *ii*) and *iii*) pick n_j large enough such that 2) and 3) are satisfied. Also note, by 1) and 2)

$$x = \sum_{j=1}^{\infty}(x_j + S_{n_j}(y_j)) \text{ converges in } X$$

Finally, $\forall j \geq 1$ observe:

$$||\sigma^{g_{n_j}}(x) - y_j|| =$$

$$||\sigma^{g_{n_j}}(\sum_{l=1}^{j-1}(x_l + S_{n_l}(y_l)) + x_j) + \sigma^{g_{n_j}}(S_{n_j}(y_j)) - y_j + \sum_{l=j+1}^{\infty} \sigma^{g_{n_j}}(x_l) + \sum_{l=j+1}^{\infty} \sigma^{g_{n_j}}(S_{n_l}(y_l))||$$

$$\leq \frac{1}{2^j} + \frac{1}{2^j} + \sum_{l=j+1}^{\infty} \frac{1}{2^l} + \sum_{l=j+1}^{\infty} \frac{1}{2^l} = \frac{4}{2^j}$$

Therefore, σ is hypercyclic. Lastly, with this same construction it may be shown $\sigma \times \sigma$ can satisfy $1), 2),$ and $3)$ with $X_0 \times X_0$ and $Y_0 \times Y_0$ as dense subsets of $X \times X$ and using the maps $S_n \times S_n$. In so doing, $\sigma \times \sigma$ is shown to be hypercyclic and thus σ is weakly mixing by definition 5. $\square$

Now with theorem 4, the following theorem may be proven.

Theorem 5. *Let $X = \ell^p_G(v)$ with G a countable group. Suppose σ is a bounded shift action on X. Then the following are equivalent:*

*i) σ is **hypercyclic***

*ii) σ is **weakly mixing***

iii) $\exists$ a sequence $(g_n)_{n=1}^{\infty}$ of elements of G which tends to infinity such that $\forall$ finite $F \subseteq G$, $\forall f \in F$, $e_{fg_n} \to 0$ and $e_{fg_n^{-1}} \to 0$ in X as $n \to \infty$.

Proof. (i $\to$ iii) Suppose σ is hypercyclic. Note: G is countable and so may be enumerated $(g_n)_{n=1}^{\infty}$. Next without loss of generality let $p = 1$, let 1 denote the identity element of G, let $m > 2$, and consider the open sets:

$$U_m = \{x : ||x - (me_{g_1} + me_{g_2} + ... + me_{g_m})|| < \frac{B_m}{m}\}$$

$$V_m = \{x : ||x - ((m+1)e_{g_1} + (m+1)e_{g_2} + ... + (m+1)e_{g_m})|| < \frac{B_m}{(m+1)}\}$$

where $B_m = min\{v_{g_1}, v_{g_2}, ..., v_{g_m}\}$. For some $m > 2$, let $U = U_m$, $V = V_m$, and recall σ hypercyclic implies the action is topologically transitive. Thus by assumption, $\exists x \in X$ and $\exists g \in G$ such that

$$x \in U \cap \sigma^g(V)$$

So $x \in U$ gives $\forall i \leq m,\ \ |x_{g_i} - m|v_{g_i} < \frac{B_m}{m}$ and $\forall t > m$ if $h = g_t$, then $|x_h|v_h < \frac{B_m}{m}$

which further implies

$$x_{g_i} > m - \frac{B_m}{m v_{g_i}} \tag{2.1}$$

And $x \in \sigma^g(V)$ gives $\sigma^{g^{-1}}(x) \in V$. So $\forall i \leq m,\ \ |x_{g_i g^{-1}} - (m+1)| v_{g_i} < \frac{B_m}{m+1}$ and $\forall t > m$ if $h = g_t$, then $|x_{hg^{-1}}| v_h < \frac{B_m}{(m+1)}$ which further implies

$$x_{g_i g^{-1}} > (m+1) - \frac{B_m}{(m+1) v_{g_i}} \tag{2.2}$$

Now let $i \leq m$, pick $h = g_i g$. Note, $\forall j \leq m,\ \ h \neq g_j$ by the choice of U and V and the fact $m + \frac{1}{m} < m + 1 - \frac{1}{m+1}$. Then $|x_{g_i}| v_{g_i g} < \frac{B_m}{(m+1)}$ and by (equation 2.1):

$$\begin{aligned}
(m - \frac{B_m}{m v_{g_i}}) v_{g_i g} &< \frac{B_m}{m+1} \\
v_{g_i g} &< \frac{B_m}{m+1} \cdot \frac{1}{m - \frac{B_m}{m v_{g_i}}} = \frac{B_m}{(m+1)m - \frac{(m+1)B_m}{m v_{g_i}}} \\
||e_{g_i g}|| = v_{g_i g} &< \frac{m v_{g_i} B_m}{m v_{g_i}(m+1)m - (m+1)B_m} = \frac{m v_{g_i} B_m}{m^3 v_{g_i} + m^2 v_{g_i} - m B_m - B_m}
\end{aligned}$$

Note: i was arbitrary, and so as m increases construct a sequence of elements of G, $(g_k)_{k=1}^{\infty}$, where $g_k = g$ given by the topological transitivity for U_m and V_m. This gives

$$\lim_{k \to \infty} ||e_{g_i g_k}|| < \lim_{2 < m \to \infty} \frac{m v_{g_i} B_m}{m^3 v_{g_i} + m^2 v_{g_i} - m B_m - B_m}$$

Then as $m \to \infty$ and since B_m is a non-increasing sequence the right hand limit goes to 0. This gives $||e_{g_i g_k}|| \to 0$ as $k \to \infty$.

Similarly, let $i \leq m,\ \ h = g_i g^{-1}$ gives $|x_{g_i g^{-1}}| v_{g_i g^{-1}} < \frac{B_m}{m}$. Now apply (equation

2.2) to get:

$$
\begin{aligned}
((m+1)-\frac{B_m}{(m+1)v_{g_i}})v_{g_ig^{-1}} &< \frac{B_m}{m} \\
v_{g_ig^{-1}} &< \frac{B_m}{m}\cdot\frac{1}{(m+1)-\frac{B_m}{(m+1)v_{g_i}}} = \frac{B_m}{m(m+1)-\frac{mB_m}{(m+1)v_{g_i}}} \\
||e_{g_ig^{-1}}|| = v_{g_ig^{-1}} &< \frac{(m+1)v_{g_i}B_m}{(m+1)v_{g_i}m(m+1)-mB_m} \\
&= \frac{mv_{g_i}B_m + v_{g_i}B_m}{m^3v_{g_i}+2m^2v_{g_i}+mv_{g_i}-mB_m}
\end{aligned}
$$

Again i was arbitrary, and by using the same sequence of elements of G constructed above, as $k\to\infty$, $||e_{g_ig_k^{-1}}||\to 0$. Now let $F\subseteq G$ with F finite and let $f\in F$. Then $\exists m>2$ such that g_m is listed in the enumeration after f with a corresponding g_k from the constructed sequence. So $f=g_i$ for some $i\leq m$ which gives

$$||e_{fg_k}||\to 0 \text{ and } ||e_{fg_k^{-1}}||\to 0 \text{ as } k\to\infty$$

But f was arbitrary, so $\forall f\in F$ these norms converge to 0. For $p>1$ pick suitably large enough m to achieve the same result.

(iii $\to$ ii) Aiming to use theorem 4, let

$X_0 = Y_0 = \{x\in X : x$ has nonzero entries for only finitely many $g\in G\}$. Note, X_0 and Y_0 are dense in X. Next take the sequence given by assumption $(g_k)_{k=1}^{\infty}$ and define $S_k : Y_0\to X$ where $S_k(y)=\sigma^{{g_k}^{-1}}(y)$, $\forall y\in Y_0$ $k\geq 1$. Also by assumption, σ is a bounded shift action and so S_k is continuous $\forall k\geq 1$.

Now let $x\in X_0$ then x is a finite linear combination of the basis vectors. So x has non-zero entries at finitely many group element locations, call them $f_1, f_2, ..., f_\alpha$ for some $\alpha<\infty$. Next observe by linearity:

$$\sigma^{g_k}(x) = \sigma^{g_k}(x_{f_1}e_{f_1}+x_{f_2}e_{f_2}+...+x_{f_\alpha}e_{f_\alpha}) = x_{f_1}e_{f_1g_k^{-1}}+x_{f_2}e_{f_2g_k^{-1}}+...+x_{f_\alpha}e_{f_\alpha g_k^{-1}}$$

$$\text{So } \sigma^{g_k}(x) \to 0 \text{ as } k \to \infty \text{ by iii).}$$

Similarly, $\forall y \in Y_0$

$$S_k(y) \to 0 \text{ as } k \to \infty.$$

Also, by definition of S_k, $\sigma^{g_k}(S_k(y)) = y$, $\forall y \in Y_0$. Thus:

$$\sigma^{g_k}(S_k(y)) \to y, \forall y \in Y_0 \text{ as } k \to \infty.$$

Therefore, by theorem 4, σ is weakly mixing.

(ii $\to$ i) Lastly, weakly mixing implies hypercyclicity in general. $\square$

Now by theorem 5, σ is topologically transitive $\iff$ hypercyclic $\iff$ weakly mixing $\iff$ $\exists$ a sequence $(g_n)_{n=1}^{\infty}$ of elements of G which tends to infinity such that $\forall$ finite $F \subseteq G$, $\forall f \in F$

$$\lim_n v_{fg_n} = 0 \quad \textit{and} \quad \lim_n v_{fg_n^{-1}} = 0.$$

However, if G is abelian, then this characterization may be simplified some.

Lemma 1. *Let $X = \ell^p_G(v)$ with G a countable abelian group. Suppose σ is a bounded shift action on X. Then the following are equivalent:*

i) $\exists$ a sequence $(g_n)_{n=1}^{\infty}$ of elements of G which tends to infinity such that $\forall$ finite $F \subseteq G$, $\forall f \in F$, $e_{fg_n} \to 0$ and $e_{fg_n^{-1}} \to 0$ in X as $n \to \infty$.

ii) $\exists$ a sequence $(g_n)_{n=1}^{\infty}$ of elements of G which tends to infinity such that $e_{g_n} \to 0$ and $e_{g_n^{-1}} \to 0$ in X as $n \to \infty$.

Proof. (i $\to$ ii) This direction follows automatically when $F = \{1\}$ where 1 is the identity element of G.

(ii $\to$ i) Aiming to use theorem 4, let

$X_0 = Y_0 = \{x \in X : x$ has nonzero entries for only finitely many $g \in G\}$. Note, X_0

and Y_0 are dense in X. Let $(g_n)_{n=1}^{\infty}$ be an enumeration of G. Next define $S_n : Y_0 \to X$ where $S_n(y) = \sigma^{{g_n}^{-1}}(y),\ \ \forall y \in Y_0\ \ n \geq 1$. By assumption, σ is a bounded shift action and so S_n is continuous $\forall n \geq 1$.

By (ii), $\exists$ a sequence of positive integers $(n_k)_{k=1}^{\infty}$ such that $e_{g_{n_k}} \to 0$ and $e_{g_{n_k}^{-1}} \to 0$ in X as $k \to \infty$. Now let $y \in Y_0$ then y is a finite linear combination of the basis vectors. Without loss of generality consider $y = e_{g_i}$ where $i \geq 1$. Observe by G abelian:

$$S_{n_k}(y) = \sigma^{{g_{n_k}}^{-1}}(e_{g_i}) = e_{g_i g_{n_k}} = e_{g_{n_k} g_i} = \sigma^{g_i^{-1}}(e_{g_{n_k}})$$

And so by assumption and continuity,

$$S_{n_k}(y) = \sigma^{g_i^{-1}}(e_{g_{n_k}}) \to 0 \text{ as } k \to \infty.$$

Similarly, let $x \in X_0$ and without loss of generality $x = e_{g_j}$ for some $j \geq 1$ and observe:

$$\sigma^{g_{n_k}}(x) = \sigma^{g_{n_k}}(e_{g_j}) = e_{g_j {g_{n_k}}^{-1}} = e_{{g_{n_k}}^{-1} g_j} = \sigma^{{g_j}^{-1}}(e_{{g_{n_k}}^{-1}}) \to 0 \text{ as } k \to \infty.$$

Lastly, note:

$$\sigma^{g_{n_k}}(S_{n_k}(y)) = \sigma^{g_{n_k}}(\sigma^{{g_{n_k}}^{-1}}(y)) = y, \forall k \geq 1.$$

Therefore, by theorem 4, σ is weakly mixing. And by theorem 5, this implies i).

$\square$

With lemma 1 in hand, the following corollary is achieved together with theorem 5.

Corollary 1. *Let $X = \ell^p_G(v)$ with G a countable abelian group. Suppose σ is a bounded shift action on X. Then the following are equivalent:*

i) σ is ***hypercyclic***

ii) σ is ***weakly mixing***

iii) $\exists$ a sequence $(g_n)_{n=1}^{\infty}$ of elements of G which tends to infinity such that $e_{g_n} \to 0$ and $e_{g_n^{-1}} \to 0$ in X as $n \to \infty$

And so by corollary 1, if G is abelian, σ is topologically transitive $\iff$ hypercyclic $\iff$ weakly mixing $\iff$ $\exists$ a sequence $(g_n)_{n=1}^{\infty}$ of elements of G which tends to infinity such that

$$\lim_n v_{g_n} = 0 \quad \textit{and} \quad \lim_n v_{g_n^{-1}} = 0.$$

2.4 Mixing

First, for clarity in the setting of the shift group action, mixing may be defined in the following manner.

Definition 13 (Mixing Group Action)**.** *Let $X = \ell^p_G(v)$ with G a countable group, let $(g_n)_{n=1}^{\infty}$ be an enumeration of G, and let σ be a bounded shift action on X. σ is called* ***mixing*** *if $\forall$ nonempty open $U, V \subseteq X$ $\exists N \in \mathbb{N}$ large enough such that $\forall n > N$, $\sigma^{g_n}(U) \cap V \neq \emptyset$.*

In order to prove theorem 2, Grosse-Erdmann and Manguillot used the Kitai criterion [1].

Theorem (Kitai's Criterion)**.** *Let $X = \ell^p(v)$. Let σ be the backward shift. Now if $\forall$ sequence $(n_i)_{i=1}^{\infty}$ of positive integers, $\exists$ dense X_0, $Y_0 \subseteq X$, and $\exists$ maps $S_n : Y_0 \to X$ $\forall n \geq 1$ such that $\forall x \in X_0$ $\forall y \in Y_0$ the following holds:*

i) $\sigma^{n_i}(x) \to 0$

ii) $S_n(y) \to 0$

iii) $\sigma^{n_i}(S_n(y)) \to y$

Then σ is ***mixing****.*

For a proof see [1]. In order to construct the new theorem similar to theorem 2, a new criterion in the setting of the group action will be needed first.

Theorem 6 (Group Action Kitai Criterion)**.** *Let $X = \ell^p_G(v)$ with G a countable group. Suppose σ is a bounded shift action on X. If for any sequence $(g_n)_{n=1}^{\infty}$ of elements of G which tends to infinity, $\exists$ dense X_0, $Y_0 \subseteq X$, maps $S_n : Y_0 \to Y_0$ such that $\forall x \in X_0$ and $\forall y \in Y_0$ the following holds:*

i) $\sigma^{g_n}(x) \to 0$

ii) $S_n(y) \to 0$

iii) $\sigma^{g_n}(S_n(y)) = y, \ \ \forall n \geq 1$

Then σ is ***mixing****.*

Proof. Let $(g_n)_{n=1}^{\infty}$ be a sequence of elements of G which tends to infinity. Suppose $\exists$ dense X_0, $Y_0 \subseteq X$ and maps $S_n : Y_0 \to Y_0$ such that $\forall x \in X_0$ and $\forall y \in Y_0$ $i), ii)$, and $iii)$ hold true. Let U, $V \subseteq X$ be nonempty open sets. By the density assumption:

$$\exists x \in X_0 \cap U \text{ and } \exists y \in Y_0 \cap V$$

This further gives by $i), ii)$, and $iii)$

$$\sigma^{g_n}(x) \to 0,\ S_n(y) \to 0, \text{ and } \sigma^{g_n}(S_n(y)) = y,\ \forall n \geq 1.$$

Now let $z_n = S_n(y)$. As $n \to \infty$, $\exists N \in \mathbb{N}$ large enough such that $\forall n \geq N$

$$x + z_n \in U \text{ and } \sigma^{g_n}(x + z_n) = \sigma^{g_n}(x) + \sigma^{g_n}(z_n) = \sigma^{g_n}(x) + y \in V$$

So $\forall n \geq N$

$$\sigma^{g_n}(U) \cap V \neq \emptyset$$

But $(g_n)_{n=1}^{\infty}$, U, and V were arbitrary. Therefore, σ is mixing. $\square$

Notably this criterion highlights for a mixing group action, travelling in any one direction (any sequence of G tending to infinity) results in a point approaching 0 in the norm. A similar notion is arrived at via the definition for homoclinic points.

Definition 14 (Homoclinic to 0)**.** *Let $x \in \ell^p_G(v)$, x is called **homoclinic to 0** if $\forall$ sequence $(g_k)_{k=1}^{\infty}$ of elements of G which tends to infinity*

$$\sigma^{g_k}(x) \to 0 \textit{ in } \ell^p_G(v)$$

So a natural equivalency between mixing and the existence of points homoclinic to 0 will be shown in the next theorem. Then with theorem 6, the following theorem may be proven.

Theorem 7. *Let $X = \ell^p_G(v)$ with G a countable group. Suppose σ is a bounded shift action on X. Then the following are equivalent:*

*i) σ is **mixing***

ii) $\forall$ sequence $(g_k)_{k=1}^{\infty}$ of elements of G which tends to infinity, $e_{g_k} \to 0$ and $e_{g_k^{-1}} \to 0$ in X as $k \to \infty$

*iii) $\exists$ a set of points **homoclinic to 0** dense in X*

*iv) $\exists$ a nontrivial point **homoclinic to 0**.*

Proof. $(i \to ii)$ Note G countable implies G may be enumerated $(g_n)_{n=1}^{\infty}$. Let 1 denote the identity element of G, let $m > 2$, and consider the open sets:

$$U_m = \{x : ||x - (me_{g_1} + me_{g_2} + ... + me_{g_m})|| < \frac{B_m}{m}\}$$

$$V_m = \{x : ||x - ((m+1)e_{g_1} + (m+1)e_{g_2} + ... + (m+1)e_{g_m})|| < \frac{B_m}{(m+1)}\}$$

where $B_m = min\{v_{g_1}, v_{g_2}, ..., v_{g_m}\}$. Suppose σ is mixing and let $(g_k)_{k=1}^{\infty}$ be an arbitrary sequence of elements of G which tends to infinity. Next pick m large enough such that g_m is listed in the enumeration after $1 \in G$. Let $U = U_m$ and $V = V_m$. By assumption $\exists K \in \mathbb{N}$ large enough such that $\forall k \geq K$

$$\exists x \in U \cap \sigma^{g_k}(V)$$

So $x \in U$ gives $\forall i \leq m,\ \ |x_{g_i} - m|v_{g_i} < \frac{B_m}{m}$ and $\forall t > m$ if $h = g_t$, then $|x_h|v_h < \frac{B_m}{m}$ which further implies

$$x_{g_i} > m - \frac{B_m}{mv_{g_i}} \tag{2.3}$$

And $x \in \sigma^{g_k}(V)$ gives $\sigma^{g_k^{-1}}(x) \in V$. So $\forall i \leq m,\ \ |x_{g_i g_k^{-1}} - (m+1)|v_{g_i} < \frac{B_m}{m+1}$ and $\forall t > m$ if $h = g_t$, then $|x_{hg_k^{-1}}|v_h < \frac{B_m}{(m+1)}$ which further implies

$$x_{g_i g_k^{-1}} > (m+1) - \frac{B_m}{(m+1)v_{g_i}} \tag{2.4}$$

Next let $h = 1g_k^{-1}$. Note $\forall j \leq m,\ \ 1g_k^{-1} \neq g_j$ by the choice of U and V and the fact

$\frac{1}{m} + m < (m+1) - \frac{1}{m+1}, \ \ \forall m > 2$. Then $|x_{1g_k^{-1}}|v_{1g_k^{-1}} < \frac{B_m}{m}$ and by (equation 2.4):

$$\begin{aligned}
((m+1) - \frac{B_m}{(m+1)v_1})v_{1g_k^{-1}} &< \frac{B_m}{m} \\
v_{1g_k^{-1}} &< \frac{B_m}{m} \cdot \frac{1}{(m+1) - \frac{B_m}{(m+1)v_1}} = \frac{B_m}{m(m+1) - \frac{mB_m}{(m+1)v_1}} \\
||e_{1g_k^{-1}}|| = v_{1g_k^{-1}} &< \frac{(m+1)v_1B_m}{(m+1)v_1m(m+1) - mB_m} \\
&= \frac{mv_1B_m + v_1B_m}{m^3v_1 + 2m^2v_1 + mv_1 - mB_m}
\end{aligned}$$

But as m increases, a large enough k may always be found such that the above inequality holds. This gives:

$$\lim_{k\to\infty} ||e_{1g_k^{-1}}|| < \lim_{2<m\to\infty} \frac{mv_1B_m + v_1B_m}{m^3v_1 + 2m^2v_1 + mv_1 - mB_m}$$

So as $m \to \infty$, and since B_m is a non-increasing sequence, the right hand limit goes to 0. This gives $||e_{1g_k^{-1}}|| = ||e_{g_k^{-1}}|| \to 0$ as $k \to \infty$.

Similarly, let $i \leq m, \ \ h = 1g_k$ gives $|x_1|v_{1g_k} < \frac{B_m}{m+1}$. Now apply (equation 2.3) to get:

$$\begin{aligned}
(m - \frac{B_m}{mv_1})v_{1g_k} &< \frac{B_m}{m+1} \\
v_{1g_k} &< \frac{B_m}{m+1} \cdot \frac{1}{m - \frac{B_m}{mv_1}} = \frac{B_m}{(m+1)m - \frac{(m+1)B_m}{mv_1}} \\
||e_{1g_k}|| = v_{1g_k} &< \frac{mv_1B_m}{mv_1(m+1)m - (m+1)B_m} = \frac{mv_1B_m}{m^3v_1 + m^2v_1 - mB_m - B_m}
\end{aligned}$$

But as m increases, a large enough k may always be found such that the above inequality holds. This gives:

$$\lim_{k\to\infty} ||e_{1g_k}|| < \lim_{2<m\to\infty} \frac{mv_1B_m}{m^3v_1 + m^2v_1 - mB_m - B_m}$$

So as $m \to \infty$, and since B_m is a non-increasing sequence, the right hand limit goes to 0. This gives $||e_{1g_k}|| = ||e_{g_k}|| \to 0$ as $k \to \infty$.
But $(g_k)_{k=1}^{\infty}$ was arbitrary. Therefore, $ii)$ holds.

$(ii \to i)$ Aiming to use theorem 6, let $(g_n)_{n=1}^{\infty}$ be an arbitrary sequence of elements of G which tends to infinity and let
$X_0 = Y_0 = \{x \in X : x$ has nonzero entries for only finitely many $g \in G\}$. Note, X_0 and Y_0 are dense in X. Next define $S_n : Y_0 \to Y_0$ such that $\forall y \in Y_0,\ S_n(y) = \sigma^{g_n^{-1}}(y)$ $\forall n \geq 1$. Let $x \in X_0$ and $y \in Y_0$, then by this definition:

$$\sigma^{g_n}(S_n(y)) = \sigma^{g_n}(\sigma^{g_n^{-1}}(y)) = y, \forall n \geq 1.$$

Lastly, note x is a finite linear combination of the basis vectors. So x has non-zero entries at finitely many group element locations, call them $f_1, f_2, ..., f_\alpha$ for some $\alpha < \infty$. Then by linearity observe

$$\sigma^{g_n}(x) = \sigma^{g_n}(x_{f_1}e_{f_1} + x_{f_2}e_{f_2} + ... + x_{f_\alpha}e_{f_\alpha}) = x_{f_1}e_{f_1g_n^{-1}} + x_{f_2}e_{f_2g_n^{-1}} + ... + x_{f_\alpha}e_{f_\alpha g_n^{-1}}$$

Note: $\forall j \leq \alpha$ with the sequence $(h_n = f_j g_n^{-1})_{n=1}^{\infty},\ e_{h_n} \to 0$ by $ii)$. So by assuming $ii)$, this gives $\sigma^{g_n}(x) \to 0$. Similarly, $S_n(y) \to 0$. Therefore, by theorem 6, σ is mixing.

$(ii \to iii)$ Assume property $ii)$ holds true. Let
$X_0 = \{x \in X : x$ has nonzero entries for only finitely many $g \in G\}$ which is dense in X.

Claim: X_0 is a set of points homoclinic to 0.

Proof: Let $x \in X_0$, then $\exists\ F = \{f_1, ..., f_j\} \subseteq G$ such that $x_{f_i} \neq 0\ \ \forall i \leq j$ and $\forall g \in G \backslash F\ \ x_g = 0$. Let $(g_k)_{k=1}^{\infty}$ be an arbitrary sequence of elements of G which tends to infinity. Then observe

$$\sigma^{g_k}(x) = x_{f_1}e_{f_1g_k^{-1}} + ... + x_{f_j}e_{f_jg_k^{-1}}$$

But $\forall i \leq j$, $(h_k = f_i g_k^{-1})_{k=1}^{\infty}$ is also a sequence of elements of G which tends to infinity. And so $ii)$ implies $\forall i \leq j$, $x_{f_i} e_{f_i g_k^{-1}} \to 0$ as $k \to \infty$. Therefore, $\sigma^{g_k}(x) \to 0$ in X. But $(g_k)_{k=1}^{\infty}$ was arbitrary, thus x is homoclinic to 0. Also x was arbitrary, so X_0 is a set of points homoclinic to 0.

$(iii \to iv)$ Trivially true.

$(iv \to ii)$ Suppose $\exists$ a nontrivial point homoclinic to 0, call it x. Then $\exists f \in G$ such that $x_f \neq 0$. Next let $(g_k)_{k=1}^{\infty}$ be an arbitrary sequence of elements of G which tends to infinity. Since $\sigma^{g_k}(x) \to 0$ this implies $x_f e_{f g_k^{-1}} \to 0$ and so $e_{f g_k^{-1}} \to 0$.

Now consider the sequence $(h_k = g_k^{-1} f)_{k=1}^{\infty}$ of elements of G which tends to infinity. By definition of homoclinic to 0 this gives

$$\sigma^{h_k}(x) \to 0$$

which implies

$$e_{f h_k^{-1}} = e_{g_k} \to 0$$

Similarly, when $(h_k = g_k f)_{k=1}^{\infty}$ this shows $e_{g_k^{-1}} \to 0$. □

And so by theorem 7, σ is mixing $\iff$ $\exists$ a nontrivial point homoclinic to 0 $\iff$ $\exists$ a set of points homoclinic to 0 dense in X $\iff$ for an enumeration $(g_n)_{n=1}^{\infty}$ of G

$$\lim_{n\to\infty} v_{g_n} = 0$$

2.5 Ambiguity for Chaos

Definition 7 and definition 8 provide the notion of periodic points and its relationship to Devaney chaos. In the setting of the semi-group action over $\mathbb{N}$, a point being periodic implies the same point has a finite orbit. This is because $\sigma^n(x) = x$ for some $n \in \mathbb{N}$ implies there are only finitely many distinct shifts of x up to the n^{th} shift. However, in the setting of the group action, it is possible to be periodic in a particular direction of the group while still having an infinite orbit.

Consider this example when $G = \mathbb{Z}^2$ with the weight sequence $(v_{(i,j)})_{(i,j)\in G}$ where $\forall (i,j) \in G$ define

$$v_{(i,j)} = (1/2)^{|j|}$$

i.e. the lattice of G with these associated weights looks like

$$\begin{array}{ccccc}
 & & \vdots & & \\
\ddots & & \frac{1}{4} & & \iddots \\
 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\
\cdots & 1 & \boxed{1} & 1 & \cdots \\
 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \\
\iddots & & \frac{1}{4} & & \ddots \\
 & & \vdots & &
\end{array}$$

Figure 2.2: Example weight sequence which yields periodic points with infinite orbit.

The box in the figure indicates the identity location $(0,0)$ in the lattice. Notably, $x = \sum_{j\in\mathbb{Z}} e_{(0,j)} \in \ell^p_G(v)$ since $||x|| < \infty$. And yet x has an infinite orbit as each shift to the left is a new unique point. The supremum condition for boundedness is also satisfied, and so this is an example of a linear continuous operator on a Banach space which possesses periodic points with an infinite orbit.

Also observe with this choice for the weight sequence, σ is not mixing since there are subsequences of the weights which are constant. This also diverges from the traditional setting where, in the case of the semi-group action, the existence of a nontrivial periodic point $\Rightarrow$ summability of the weight sequence $\Rightarrow$ mixing. All though topological transitivity is still implied by the existence of nontrivial periodic points.

In order to better highlight this distinction, section 2.5.1 will discuss necessary conditions for density of periodic points. While section 2.5.2 will consider density of points with finite orbit.

2.5.1 Density of Periodic Points

For an arbitrary countable group G, Let 1 denote the identity element. Also note any periodic point has an associated nontrivial periodicity. If $x \in X$ is periodic, then $\exists g \in G \backslash \{1\}$ such that $\sigma^g(x) = x$. Additionally, $\forall k \in \mathbb{Z}, \;\; \sigma^{g^k}(x) = x$. So copies of x entries are stored at multiples of g locations, but in order for $x \in X$ the weight sequence along multiples of a periodicity must be summable. Or rather, the weight sequence in the tail of these sums must be small. This then leads to the next theorem. For clarity, $\forall g \in G$, let $< g >$ denote the cyclic subgroup $\{g^k \in G : k \in \mathbb{Z}\}$.

Theorem 8 (Group Action Density of Periodic Points)**.** *Let $X = \ell^p_G(v)$ with G a countable group. Suppose σ is a bounded shift action on X. Then $\exists$ a **set of periodic points dense in X** if and only if $\forall$ finite $F \subseteq G\backslash\{1\}$, $\forall \varepsilon > 0$, $\exists g \in G\backslash\{1\}$ such that*

$$\sum_{h \in F(<g>\backslash\{1\})} v_h < \varepsilon$$

Proof. Suppose $\exists$ a set of periodic points which is dense in X. So $\forall x \in X$ which is a nontrivial periodic point $\exists\ g \in G\backslash\{1\}$ such that $\sigma^g(x) = x$. Furthermore, note $\forall f \in G$

$$x_f = x_{fg^k}\ ,\ \forall k \in \mathbb{Z}.$$

Now let $F \subseteq G\backslash\{1\}$ be a finite set and let $v = min\{v_f : f \in F\}$. Let $\varepsilon > 0$ and without loss of generality suppose $\varepsilon < v$. Next define $x^* = \sum\limits_{f \in F} 2e_f$. By assumption $\exists$ periodic $x \in X$ such that $x \in B_{\varepsilon^{1/p}}(x^*)$. (This is the $\varepsilon^{1/p}$ ball around x^*). Note: since $\varepsilon < v$ this implies $x \neq 0$ and $\forall f \in F, \;\; x_f > 1$. So by definition $\exists g \in G\backslash\{1\}$

such that $\sigma^g(x) = x$. Observe x may be written as:

$$x = \sum_{h \in F<g>} x_h e_h + z$$

Where z is some error term with negligible impact to the norm of x.

Lastly, by construction

$$\begin{aligned}
\varepsilon^{1/p} &> ||x - x^*|| \\
&\geq ||\sum_{f \in F}(x_f - 2)e_f + \sum_{h \in F(<g>\backslash\{1\})} x_h e_h|| \\
&\geq ||\sum_{h \in F(<g>\backslash\{1\})} x_h e_h|| \\
&\geq ||\sum_{h \in F(<g>\backslash\{1\})} e_h|| \\
&= (\sum_{h \in F(<g>\backslash\{1\})} v_h)^{1/p} \\
\varepsilon &> \sum_{h \in F(<g>\backslash\{1\})} v_h
\end{aligned}$$

Next suppose the right hand side. Let $y \in X$ and let $\varepsilon > 0$. By density of X_0, $\exists x^* \in X_0$ such that $x^* \in B_{\varepsilon/2}(y)$.

Claim: $\exists$ periodic $x \in X$ such that $x \in B_{\varepsilon/2}(x^*) \subseteq B_\varepsilon(y)$.

Proof: note, $x^* \in X_0$ implies $\exists$ finite $F^* \subseteq G$ such that

$$x^* = \sum_{f \in F^*} x_f^* e_f$$

Let $\alpha = max\{|x_f^*| : f \in F\}$, let $F = F^*\backslash\{1\}$, and let $v = min\{v_f : f \in F\}$. Now by assumption $\exists g \in G\backslash\{1\}$ such that

$$\sum_{h \in F(<g>\backslash\{1\})} v_h < min\{v, (\frac{\varepsilon}{2\alpha})^p\} \leq (\frac{\varepsilon}{2\alpha})^p$$

Note, this choice of v implies $F \cap F(< g > \setminus \{1\}) = \emptyset$ since otherwise v would be in the sum. Next define

$$x = \sum_{h \in <g>} \sigma^{h^{-1}}(x^*)$$

By construction $\sigma^g(x) = x$ so x is periodic. Also if $x_1^* \neq 0$, then $(x - x^*)_1 = 0$. Lastly, observe

$$\begin{aligned} ||x - x^*|| = || \sum_{h \in F(<g> \setminus \{1\})} x_h e_h|| &\leq ||\alpha \sum_{h \in F(<g> \setminus \{1\})} e_h|| \\ &\leq \alpha(\sum_{h \in F(<g> \setminus \{1\})} v_h)^{1/p} \\ &< \frac{\varepsilon}{2} \end{aligned}$$

Thus, $x \in B_\varepsilon(y)$. But y was arbitrary; therefore, $\exists$ a dense set of periodic points in X. $\square$

In other words, if from any starting location continually travelling in some $g \in G$ direction results in a tail of the sum of the weights being small, then there is a dense set of periodic points. This does not imply mixing, but it does imply weakly mixing and thus topological transitivity by theorem 5.

2.5.2 Density of Points with Finite Orbit

In order for a nontrivial point $x \in X$ to have a finite orbit, it must be the case that the stabilizer subgroup associated with x, denoted $Stab(x) = \{g \in G : \sigma^g(x) = x\}$, must have finite index. Otherwise, x would have an infinite orbit. The following theorem uses a similar argument to theorem 8, except tails of the weights along these stabilizers must be small.

Theorem 9 (Group Action Density of Points with Finite Orbit)**.** *Let $X = \ell^p_G(v)$ with G a countable group. Suppose σ is a bounded shift action on X. Then $\exists$ a* ***set of points with finite orbit dense in X*** *if and only if $\forall$ finite $F \subseteq G\backslash\{1\}$, $\forall \varepsilon > 0$, $\exists$ subgroup $H \subseteq G$ with finite index such that*

$$\sum_{h \in F(H\backslash\{1\})} v_h < \varepsilon$$

Proof. Suppose $\exists$ a set of points with finite orbit dense in X. So $\forall x \in X$ which is a nontrivial point with finite orbit this implies $\forall g \in H = Stab(x)$, $\sigma^g(x) = x$ and H has finite index. Furthermore, note $\forall f \in G$

$$x_f = x_{fg^k}\ ,\ \forall k \in \mathbb{Z}.$$

Let $F \subseteq G\backslash\{1\}$ be a finite set and let $v = min\{v_f : f \in F\}$. Let $\varepsilon > 0$ and without loss of generality suppose $\varepsilon < v$. Define

$$x^* = \sum_{f \in F} 2e_f$$

By assumption $\exists x \in X$ with finite orbit such that $x \in B_{\varepsilon^{1/p}}(x^*)$. Since $\varepsilon < v$ this implies $\forall f \in F,\ x_f > 1$. Additionally, $H = Stab(x)$ has finite index. Next observe x may be written as:

$$x = \sum_{h \in FH} x_h e_h + z$$

Where z is some error term with negligible impact to the norm of x. Lastly, by construction

$$\begin{aligned}
\varepsilon^{1/p} &> ||x - x^*|| \\
&\geq || \sum_{f \in F} (x_f - 2) e_f + \sum_{h \in F(H \backslash \{1\})} x_h e_h || \\
&\geq || \sum_{h \in F(H \backslash \{1\})} x_h e_h || \\
&> || \sum_{h \in F(H \backslash \{1\})} e_h || \\
&= (\sum_{h \in F(H \backslash \{1\})} v_h)^{1/p} \\
\varepsilon &> \sum_{h \in F(H \backslash \{1\})} v_h
\end{aligned}$$

Now suppose the right hand side. Let $y \in X$ and let $\varepsilon > 0$. By density of X_0, $\exists x^* \in X_0$ such that $x^* \in B_{\varepsilon/2}(y)$.

Claim: $\exists x \in X$ with finite orbit such that $x \in B_{\varepsilon/2}(x^*) \subseteq B_\varepsilon(y)$.

Note, $x^* \in X_0$ implies $\exists$ finite $F^* \subseteq G$ such that

$$x^* = \sum_{f \in F^*} x_f^* e_f$$

Let $\alpha = max\{|x_f| : f \in F^*\}$, let $F = F^* \backslash \{1\}$, and let $v = min\{v_f : f \in F\}$. Now by assumption $\exists$ a subgroup $H \subseteq G$ with finite index such that

$$\sum_{h \in F(H \backslash \{1\})} v_h < min\{v, (\frac{\varepsilon}{2\alpha})^p\} \leq (\frac{\varepsilon}{2\alpha})^p$$

Note, this choice of v implies $F \cap F(H \backslash \{1\}) = \emptyset$ since otherwise v would be in the sum. Next define

$$x = \sum_{h \in H} \sigma^{h^{-1}} x^*$$

By construction, $\forall g \in H, \ \ \sigma^g(x) = x$ and H has finite index so x has finite orbit.

Also if $x_1^* \neq 0$, then $(x - x^*)_1 = 0$. Lastly, observe:

$$\begin{aligned} ||x - x^*|| = || \sum_{h \in F(H \backslash \{1\})} x_h e_h|| &\leq ||\alpha \sum_{h \in F(H \backslash \{1\})} e_h|| \\ &\leq \alpha(\sum_{h \in F(H \backslash \{1\})} v_h)^{1/p} \\ &< \frac{\varepsilon}{2} \end{aligned}$$

Thus, $x \in B_\varepsilon(y)$. But y was arbitrary; therefore, $\exists$ a dense set of points with finite orbit in X. $\square$

2.6 Hierarchy of Properties

Given the theorems proven in this chapter, the following table may now be presented.

Table 2.1: Property Characterizations for shift group action over a countable group.

Property	Characterization
Continuity $\Leftrightarrow$ Boundedness	$\sup_{k \in G} \frac{v_k}{v_{kg}} < \infty$
Topological Transitivity $\Leftrightarrow$ Hypercyclicity	$\exists$ $(g_n)_{n=1}^{\infty} \subseteq G$ tending to infinity s.t. $\forall f \in G$ $\lim_{n\to\infty} v_{fg_n} = 0$ & $\lim_{n\to\infty} v_{fg_n^{-1}} = 0$
Weakly Mixing	$\exists$ $(g_n)_{n=1}^{\infty} \subseteq G$ tending to infinity s.t. $\forall f \in G$ $\lim_{n\to\infty} v_{fg_n} = 0$ & $\lim_{n\to\infty} v_{fg_n^{-1}} = 0$
Mixing $\Leftrightarrow$ $\exists$ a dense set of points homoclinic to 0	Given an enumeration $(g_n)_{n=1}^{\infty}$ of G $\lim_{n\to\infty} v_{g_n} = 0$
$\exists$ a dense set of periodic points	$\forall$ finite $F \subseteq G\backslash\{1\}$, $\forall \varepsilon > 0$, $\exists g \in G\backslash\{1\}$ s.t. $\sum_{h \in F(<g>\backslash\{1\})} v_h < \varepsilon$
$\exists$ a dense set of points with finite orbit	$\forall$ finite $F \subseteq G\backslash\{1\}$, $\forall \varepsilon > 0$, $\exists$ subgroup $H \subseteq G$ with finite index s.t. $\sum_{h \in F(H\backslash\{1\})} v_h < \varepsilon$

When compared with table 1.1, it can be seen characterizations for continuity, boundedness, and mixing translated nearly identically to the group action setting. Topological transitivity, hypercyclicity, and weakly mixing required consideration of inverse directions but otherwise used similar arguments to the ones in the semi-group action setting. However, for Devaney chaos there is a break down from the original characterization since periodic is no longer equivalent to finite orbit.

CHAPTER 3: TOPOLOGICAL ENTROPY OF WEIGHTED ℓ^p SPACES

3.1 Infinite Topological Entropy

In the setting of the shift semi-group action over $\mathbb{N}$, there has been active research and work into determining whether Devaney chaos and infinite topological entropy are equivalent or not. As recently as 2019, Brian, Kelly, and Tennant constructed in their work [10] an example weight sequence which produced infinite topological entropy while failing to possess Devaney chaos. This was done by maintaining summability of the weights over a subset of $\mathbb{N}$ with positive upper density while the sequence as a whole was not summable. As shown in the following theorem, infinite topological entropy is always achieved under these conditions.

Theorem 10 (Infinite Topological Entropy over $S \subseteq \mathbb{N}$ with positive upper density)**.** *Let $X = \ell^p(v)$ and let σ be the bounded shift action on X. Suppose $\exists\ S \subseteq \mathbb{N}$ such that the following holds:*

i) $\overline{d}(S) = \limsup\limits_{n\to\infty} \frac{S\cap[1,n]}{n} > 0$

ii) $\sum\limits_{n\in S} v_n < \infty$

*Then σ has **infinite topological entropy**.*

Proof. Let $S \subseteq \mathbb{N}$ and suppose $i)$ and $ii)$ hold for S. Next define for $m \in \mathbb{N}$,

$$K_m = \{x \in X : \forall n \in S,\ x_n \in \{0, 1, ...m-1\};\ \text{otherwise},\ x_n = 0\}$$

Note $x = \sum\limits_{n\in S}(m-1)e_n \in K_m$ and $||x|| < \infty$ by $ii)$. Thus, K_m is compact. Furthermore, $i)$ implies $\exists\ \{n_j\}_{j=1}^{\infty} \subseteq \mathbb{N}$ such that

$$\lim_{j\to\infty} \frac{S\cap[1,n_j]}{n_j} = B = \overline{d}(S) > 0$$

Now let $0 < \varepsilon < \min\{v_1, 1\}$ and $\forall j \geq 1$, let $r_j = |S \cap [1, n_j]|$. Then consider (utlizing defintions 9 and 11)

$$s_{n_j,\varepsilon}(\sigma, K_m) = \max\{|F| : F \subseteq K_m \text{ and } F \text{ is } (n_j, \varepsilon)\text{-separated}\}$$

When $F \subseteq K_m$ and let $x, y \in F$, if $\exists n \leq n_j$ where $x_n \neq y_n$, then F is (n_j, ε)-separated. This gives

$$s_{n_j,\varepsilon}(\sigma, K_m) \geq m^{r_j}$$

Lastly observe

$$\begin{aligned} h(\sigma, K_m) &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log\, s_{n,\varepsilon}(\sigma, K_m) \\ &\geq \lim_{\varepsilon \to 0} \lim_{j \to \infty} \frac{1}{n_j} \log\, s_{n_j,\varepsilon}(\sigma, K_m) \\ &\geq \lim_{\varepsilon \to 0} \lim_{j \to \infty} \frac{r_j}{n_j} \log\, m \\ &= B \log\, m \end{aligned}$$

So as $m \to \infty$, $h(\sigma, K_m) \to \infty$. Therefore, $h(\sigma) = \infty$. □

However, this theorem is not an equivalency. It is not known whether infinite topological entropy always implies this kind of summability. What this does indicate is so long as the weights are summable fast enough over some subset (with appropriate density) then infinite topological entropy is achieved. This next theorem highlights a weight subsequence which is bounded by an exponential function converging to 0. But it turns out this theorem also implies theorem 10.

Theorem 11. *Let* $X = \ell^p(v)$ *and* σ *be a bounded shift on* X. *If*

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{v_n} > 0 \ ,$$

then $h(\sigma) = \infty$

Proof. Let $X = \ell^p(v)$, let σ be a bounded shift on X, and suppose

$$\limsup_{n} \frac{1}{n} \log \frac{1}{v_n} = \lambda > 0$$

which implies

$$\lim_{n} \ \sup \{\frac{1}{k} \log \frac{1}{v_k} : k \geq n\} = \lambda \ .$$

Pick $\lambda > \varepsilon_1 > 0$ and let $\{\varepsilon_m\}_{m=1}^{\infty}$ be a monotonically decreasing sequence such that $\varepsilon_m \to 0$. Now by definition of limit convergence, for $\varepsilon_1 > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$|\sup \{\frac{1}{k} \log \frac{1}{v_k} : k \geq n\} - \lambda| < \varepsilon_1$$

This gives $\exists k_1 \geq n$ such that

$$\begin{aligned}
\lambda - \varepsilon_1 < \frac{1}{k_1} \log \frac{1}{v_{k_1}} &< \lambda + \varepsilon_1 \\
k_1(\lambda - \varepsilon_1) &< \log \frac{1}{v_{k_1}} \\
e^{k_1(\lambda-\varepsilon_1)} &< \frac{1}{v_{k_1}} \\
v_{k_1} &< e^{-k_1(\lambda-\varepsilon_1)}
\end{aligned}$$

In this manner, construct the sequence $\{k_m\}_{m=1}^{\infty}$. If for some $m \geq 1$ $k_{m+1} = k_m$, then consider $n > k_m$ and reselect k_{m+1}. This guarantees the sequence is strictly increasing. Furthermore, if this sequence has positive upper density, then $h(\sigma) = \infty$ by theorem 10.

So consider the case $\overline{d}(\{k_m\}_{m=1}^{\infty}) = 0$. Note, by σ bounded this implies $\exists B > 0$ such that $\forall n, j \in \mathbb{N}$

$$\frac{v_n}{v_{n+j}} \leq B^j$$
$$v_n \leq B^j v_{n+j}$$

Or, for fixed $j < n$

$$v_{n-j} \leq B^j v_n$$

Now $\forall m \geq 1$, pick $j_m < k_m$ such that

$$B^{j_m} e^{-\lambda k_m} = e^{-\frac{\lambda}{2} k_m}$$
$$B^{j_m} = e^{\frac{\lambda}{2} k_m}$$
$$j = \frac{\lambda}{2 \log B} k_m$$

Finally, let $I_m = [k_m - j_m, k_m]$ and define

$$S = \bigcup_m I_m = \bigcup_m [k_m - j_m, k_m]$$

Since $|S \cap [1, k_m]| \geq |I_m| = \frac{\lambda}{2 \log B} k_m$, then S has positive upper density. Additionally,

$$\begin{aligned}\sum_{n \in S} v_n &\leq \sum_m \sum_{l \in I_m} v_l \\ &\leq \sum_m e^{-\frac{\lambda}{2} k_m} |I_m| \\ &= \sum_m e^{-\frac{\lambda}{2} k_m} \frac{\lambda}{2 \log B} k_m \\ &\leq \frac{\lambda}{2 \log B} \sum_n e^{-\frac{\lambda}{2} n} n < \infty\end{aligned}$$

Therefore, by theorem 10, $h(\sigma) = \infty$. $\square$

This then gives an alternate way to test for infinite topological entropy but relies on theorem 10 in the process.

3.2 Zero Topological Entropy

Alternatively, the question of zero topological entropy in the setting of the shift semi-group action over $\mathbb{N}$ is equally unanswered. The following is a test for zero topological entropy.

Theorem 12 (Bounded Weight Sequence Ratio Test)**.** *Let $X = \ell^p(v)$ and σ be a bounded shift on X. If $\exists\ B > 0$ such that $\forall m, n \in \mathbb{N}$, $\frac{v_m}{v_{m+n}} < B$, then $h(\sigma) = 0$.*

Proof. Let $X = \ell^p(v)$, let σ be a bounded shift on X, and suppose $\exists\ B > 0$ such that $\forall m, n \in \mathbb{N}$, $\frac{v_m}{v_{m+n}} < B$. For sake of ease and without loss of generality, let $p = 1$. Then Let $x \in X$, recall $x = \sum\limits_{n \in \mathbb{N}} x_n e_n$, and define $\{\alpha_n\}_{n=1}^{\infty}$ such that the following is true

$$x_n = \alpha_n \frac{1}{v_n}$$

Additionally, $||x|| = ||\sum\limits_n x_n e_n|| = \sum\limits_n |\alpha_n| < \infty$ since $x \in X$. Next let $n \in \mathbb{N}$, and observe

$$\begin{aligned}
||\sigma^n(x)|| = \sum_{m>n} |x_m| v_{m-n} &= \sum_m |x_{m+n}| v_m \\
&= \sum_m |\alpha_{m+n}| \frac{v_m}{v_{m+n}} \\
&\leq B \sum_m |\alpha_{m+n}| \\
&\leq B||x||
\end{aligned}$$

Thus, for arbitrary $x \in X$ and arbitrary $n \geq 1$, $||\sigma^n|| \leq B$. Furthermore, let $x, y \in X$, then the maximal distance up to any n^{th} shift of x and y may be bounded in the following manner:

$$\begin{aligned}
d_n(x,y) &= \max_{k<n} ||\sigma^k(x) - \sigma^k(y)|| \\
&= \max_{k<n} ||\sigma^k(x-y)|| \\
&\leq \max_{k<n} B||x-y|| \\
&= B||x-y||
\end{aligned}$$

Now for the determining of topological entropy, Bowen showed in [2] topological entropy may be equivalently calculated via separating or spanning subsets of a compact set. Definition 10 will be used here. Let $K \subseteq X$ be compact, let $n \in \mathbb{N}$, and let $\varepsilon > 0$. Let $r_n(\varepsilon, K)$ denote the smallest cardinality of any set F which (n, ε)-spans K. Then topological entropy may be calculated by

$$h(\sigma, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\varepsilon, K)$$

$$h(\sigma) = \sup_{K \text{ compact}} h(\sigma, K)$$

Next let $K \subseteq X$ be compact, let $\varepsilon > 0$, and choose $\delta = \min\{\frac{\varepsilon}{B}, \varepsilon\}$. Let $F \subseteq K$ and suppose F $(1, \delta)$-spans K. Note, by the choice of δ, F also $(1, \varepsilon)$-spans K. So by definition $\forall x \in K$, $\exists y \in F$ such that $||x - y|| \leq \delta$. Then consider for arbitrary $n \in \mathbb{N}$

$$\begin{aligned}
\max_{k<n} ||\sigma^k(x) - \sigma^k(y)|| &= \max_{k<n} ||\sigma^k(x-y)|| \\
&\leq B||x-y|| \\
&\leq \varepsilon
\end{aligned}$$

Therefore, $\forall n \in \mathbb{N}$, F (n, ε)-spans K. Lastly, since K is compact, then for the δ-ball covering of K there exists a finite subcover. Let F be a finite set which takes one representative from each open set in the subcover. Thus, this F (n, ε)-spans K $\forall n \geq 1$. So $h(\sigma, K) = 0$. But K was arbitrary. This gives $h(\sigma) = 0$. □

3.3 Open Questions

As of yet, no example choice of the weight sequence has been produced which results in positive finite entropy. A first additional open question would be, does such an example exist? Or, do all shift actions on $\ell^p(v)$ strictly produce infinite or zero topological entropy? What conditions would adequately characterize infinite or zero topological entropy in this setting? Lastly, once properly characterized, do these same characterizations translate to the shift group action setting? All of which would be good additional questions to explore.

CHAPTER 4: CONCLUSION

4.1 Summary of Results

In the case of $\ell^p(v)$ with the shift semi-group action over $\mathbb{N}$, much of traditional topological dynamics has already been answered via table 1.1 shown in chapter 1. Continuity, boundedness, and mixing translated to the setting $\ell^p_G(v)$ with the shift group action with relatively the same arguments. But for mixing a new group action version of Kitai's criterion first had to be proven. Topological transitivity, hpyercyclicity, and weakly mixing (while still equivalent to each other) required new characterizations in which the inverse direction of the shift group action also had to be considered. This is due to the fact entries are lost and not memory retained in the case of the shift semi-group action. In the setting of $\ell^p_G(v)$ with the shift group action, the definition for Devaney chaos becomes ambiguous since periodicity of a point is no longer equivalent to the point having a finite orbit. Density of either type of point then required different characterizations. All of the new chracterizations in the setting of the shift group action are highlighted in table 2.1.

Lastly, this work examined the open questions regarding topological transitivity in the case of $\ell^p(v)$ with the shift semi-group action over $\mathbb{N}$. While new tests and implications were found and proven for infinite and zero topological entropy, this is still an active open area of research.

4.2 Open Questions

The open questions concerning topological entropy are discussed in section 3.3. Regarding the shift group action, what are characterizations for other topological dynamics properties such as reiteratively hypercyclic, frequently hypercyclic, and

$\mathcal{U}$-frequently hypercyclic? In the setting of the shift semi-group action, it has previously been shown the strong specification property is equivalent to Devaney chaos [8]. But in $\ell^p_G(v)$, what conditions characterize the strong specification property? Is it still equivalent to density of periodic points, is density of points with finite orbit instead required, or are neither equivalent to specification? Additionally, future work could explore the group action over an uncountable group (namely, weighted L^p spaces). How then are the topological dynamics properties characterized when the group is uncountable?

These are all different additional directions the research could conintue, and this disseration has worked to answer preliminary questions which lead to these new ones.

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