LIMIT THEOREMS FOR REACTION DIFFUSION MODELS

by

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ABSTRACT

YAQIN FENG. Limit theorems for reaction diffusion models. (Under the direction
of DR. STANISLAV A. MOLCHANOV)

We introduce two different reaction diffusion models: evolution of one-cell popula-
tions in the presence of mitosis and continuous contact model.

In the first model we consider the time evolution of the supercritical reaction
diffusion-equation on the lattice $\mathbb{Z}^d$ when each particle together with it spatial coor-
dinate has an extra parameter (mass). In the moment of the division the mass of
the particle which is growing linearly after the birth is divided in random propor-
tion between two offspring (mitosis). Using the technique of moment equations we
study asymptotically the mass-space distribution of the particles. For each site in the
bulk of the population mass distribution of the particles is the solution of the special
differential-functional equation with linearly transformed argument. We prove several
limit theorems for such population and study in detail the statistics of the masses of
the particles.

The continuous contact model describes the space and time stationary behavior
of the particles. The central result here is the existence of limit distributions for
continues time critical homogeneous-in-space branching processes with heavy tails
spatial dynamics in dimension $d = 2$. In dimension $d \geq 3$, the same results are
true without any special assumptions on the underlying (non-degenerated) stochastic
dynamics.

ACKNOWLEDGMENTS

INTRODUCTION

We present here two different models: mathematical model of plankton and continuous contact model.

One of our central goals is explanation with appropriate models of the following empirical facts: Firstly, biopopulations, at least those which are not strongly suppressed by civilization, may exhibit stationary in space and time. Secondly, the spatial distribution of many species has a pattern of patches, which is strongly deviated from the Poissonian point field. Thirdly, the distribution of parameters of some species, such as mass, size etc is non-Gaussian.

The main ideas are based on the Fisher-Kolmogorov-Petrovsky-Piscunov model of the evolution of a new gene after mutation. Classical FKPP equation which was introduced by Fisher [Fisher, 1937] and studied in details by Kolmogorov, Petrovsky and Piscunov[Kolmogorov et al., 1937] had appeared in the biological context. It describes the early stages of growth of the population of a new species (appeared as result of mutation somewhere in the space) due to their random motion (diffusion). Our central object will be, as in FKPP theory, the branching process with random dynamics in the space. We'll exclude the direct interaction between particles, but the birth-death processes will create, however, some type of attractive mean field interaction.

Our plankton model generalizes of the classical FKPP scheme. Cell growth in plankton is characterized by cell division [Round et al., 1990]. Cells grow and segregate a full complement of components to each offspring cell. Some related works on Plankton can be found in [Hall and Wake, 1989],[Begg, 2007] and [Begg et al., 2008]. However, their work has in general not included the space dynamics of population. Thus, we will consider the particles with masses as the model for the population of plankton. In the process of the division of such particles (mitosis), their masses are

randomly distributed between offspring. As a result, the FKPP equations have now not a differential but a functional-differential nature. We are interested in the limit theorems for such population as well as the statistics of masses of the particles.

The continuous contact model can be used to describe the spatial plant ecology like the model of a forest. One of the aims of this model is to study existence of the evolution states for the contact process. A key idea to approach the problem is the study of corresponding correlation functions. We describe the collection of positions of n particles $x_1, \cdots, x_n$ as configuration of particles. They can be interpreted as a cloud of particles and there is at most one element of particles occupying a single position x_i. We will focus on the existence of limit distribution through the study of the correlation function.

In Chapter 1, we introduce the FKPP model and present some classical results. We begin with moment generating function for Galton-Watson process and list the limit distribution for supercritical, critical and subcritical case. We then discuss the FKPP equation and the moment equations for the correlation function. In order to study the behavior the population, we consider large deviation for the random walk based on Crammer's approach. The definition of the population Front is then given and we describe the shape of the front in different cases.

In Chapter 2, we introduce the reaction diffusion model of one-cell populations in presence of mitosis. It is different from the classical FKPP model due to the presence of extra parameter mass. This population can be described by non-linear FKPP-type functional differential equations for the generating function of the number of particles in the fixed domain D_1 and Laplace transform of the total mass in the domain D. From these equations one can derive the linear moment equations. The operator of the moment equations implies two independent Markov processes: underlying symmetric random walk and Mass process. The Mass process has very good property and we prove the limit distribution as time $t \to \infty$. Asymptotic of mass distribution around

0 and ∞ is also studied. Furthermore, we present the limit distribution of number of population at each site in the lattice, we prove that it convergences to the same limit distribution as in classic Galton-Watson process. Finally, a special solvable model is also given. With the assumption of the constant growth rate of the mass, the joint limit distribution has a simple form. Also, central limit theorem is obtained for the conditional mass process.

In the last chapter, we describe the continuous contact process in $\mathbb{R}^d$. For critical case, it was shown by [Kondratiev et al., 2008] the existence of limit distribution in the dimension $d \geq 3$ for the underlying random walk with finite second moment. However, for $d = 1, 2$,the second moment of the underlying process diverges. To avoid the divergence, we add long jumps to the underlying process. Under some regularity conditions, we prove the existence of the ergodic limiting states. We also prove the estimation for the variance of the number of particles in a region which can be used for future investigation of central limit theorems.

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CHAPTER 1: THE CLASSIC FKPP MODEL

1.1 FKPP Model

In the classical FKPP theory we start from the single particle at the point $x \in \mathbb{Z}^d$ (one can consider also the case $\mathbb{R}^d$) instead of $\mathbb{Z}^d$. The particle performs the symmetric random walk on $\mathbb{Z}^d$ (time is continuous) with generator $\kappa\Delta$, where $\kappa > 0, \Delta\Phi(x) = \sum_{x':\|x'-x\|=1} (\Phi(x') - \Phi(x))$. The particle can die at the time interval $(t, t + dt)$ with probability μdt (μ is the mortality rate) or can be splitted into two particles (at the site $x \in \mathbb{Z}^d$) with probability βdt (β is the birth rate). For two offspring we have the same independent stochastic dynamics. Let $n(t, x)$=numbers of particles at the site $x \in \mathbb{Z}^d$ and moment t, $N(t) = \sum_{x \in \mathbb{Z}^d} n(t, x)$ is the total number of the particles. Since μ, β are constant, the process $N(t)$ is the classical Galton-Watson process with generating function $\phi(t, z) = Ez^{N(t)}$, which is the solution of the backward Kolmogonov equation:

$$\frac{\partial \phi}{\partial t} = \beta\phi^2 - (\beta + \mu)\phi + \mu \tag{1.1}$$

$$\phi(0, z) = z$$

The distribution of $N(t)$ due to particular simplicity of the equation (1.1) can be calculated explicitly, see [Gikhman and Skorokhod, 1974].

$$\phi(t, z) = \frac{(z - \alpha) - (z - 1)\alpha e^{t(\beta-\mu)}}{(z - \alpha) - (z - 1)e^{t(\beta-\mu)}}, \ \ \alpha = \frac{\mu}{\beta}(\mu \neq \beta) \tag{1.2}$$

$$P(N(t) = 0) = \frac{1 - e^{t(\beta-\mu)}}{1 - \frac{e^{t(\beta-\mu)}}{\alpha}} \xrightarrow{t \longrightarrow \infty} \alpha \tag{1.3}$$

$$P(N(t) = k) = (1 - \alpha)^2 \frac{\left(1 - e^{t(\beta-\mu)}\right)^{k-1} e^{t(\beta-\mu)}}{\left(\alpha - e^{t(\beta-\mu)}\right)^{k+1}}, k \geq 1 \qquad (1.4)$$

Then

$$EN(t) = \phi'(t, z)|_{z=1} = e^{(\beta-\mu)t}$$

Let's consider three cases:

- $\beta > \mu$, i.e $\alpha > 1$, then $P(N(t) = 0) \to 1$ as $t \to \infty$, the branching process is degenerated for $t \to \infty$.

- $\beta < \mu$, i.e $\alpha < 1$, then $P(N(t) = 0) \to \alpha < 1$ as $t \to \infty$, the process is degenerated with some probability $\alpha < 1$. However for any fixed $P(N(t) = k) \to 0, k \to \infty$, it means that in the case of non-degeneracy, the process $N(t)$ will tend to for ∞.

- If $\alpha = 1$, which corresponds to the critical case, the calculations are a bit different:

$$\frac{d\phi}{dt} = \beta(1 - \phi)^2, \ with \ \phi(0, z) = z$$

Then

$$\phi(t, z) = 1 - \frac{1 - z}{1 + \beta t(1 - z)} \qquad (1.5)$$

We then have

$$P(N(t) = 0) = \frac{\beta t}{1 + \beta t} \to 1, t \to \infty \qquad (1.6)$$

$$P(N(t) = k) = \frac{(\beta t)^{k-1}}{(1 + \beta t)^{k+1}}, k \geq 1 \qquad (1.7)$$

For large t, one can find approximations for the distribution of $N(t)$:

Lemma 1.1. *Assume that $\alpha < 1$ (supercritical case), then*

$$P(\frac{N(t)}{e^{(\beta-\mu)t}} < a) \longrightarrow G(a) \tag{1.8}$$

The limit distribution $G(a)$ has an atom $\alpha = \frac{\mu}{\beta}$ at 0 and exponential density for $a > 0$. Formally

$$\frac{dG}{da} = \alpha\delta_0(a) + (1-\alpha)\lambda e^{-\lambda a}, \quad a \geq 0 \tag{1.9}$$

where $EN(t) = e^{(\beta-\mu)t}, \alpha = \frac{\mu}{\beta}, \lambda = 1 - \alpha$. The atom $\alpha\delta_0(a)$ corresponds to the possibility of degeneration of the population on the initial stages of evolution.

Proof. Clearly,

$$P(\frac{N(t)}{e^{(\beta-\mu)t}} = 0) = P(N(t) = 0)$$

$$= \frac{1 - e^{t(\beta-\mu)}}{1 - \frac{e^{t(\beta-\mu)}}{\alpha}} \xrightarrow{t \to \infty} \alpha$$

Meanwhile,

$$P(\frac{N(t)}{e^{(\beta-\mu)t}} > a) = P(N(t) > ae^{(\beta-\mu)t})$$

$$= \frac{(1-\alpha)^2 \frac{\left(1-e^{t(\beta-\mu)}\right)^{(ae^{(\beta-\mu)t})-1} e^{t(\beta-\mu)}}{\left(1-\alpha e^{t(\beta-\mu)}\right)^{(ae^{(\beta-\mu)t})+1}}}{1 - \frac{1-e^{t(\beta-\mu)}}{1-\alpha e^{t(\beta-\mu)}}}$$

$$= (1-\alpha)\frac{\left(1 - e^{t(\beta-\mu)}\right)^{(ae^{(\beta-\mu)t})-1}}{\left(1 - \alpha e^{t(\beta-\mu)}\right)^{(ae^{(\beta-\mu)t})}}$$

$$= (1-\alpha^2)\frac{1}{1-\alpha}\left[\frac{1}{1 + (1-\alpha)e^{t(\beta-\mu)}}\right]^{ae^{(\beta-\mu)t}}$$

$$\longrightarrow (1-\alpha^2)\frac{1}{1-\alpha}e^{-\frac{a}{(1-\alpha)}}$$

Therefore, the density of $G(a)$ satisfies

$$\frac{dG}{da} = \alpha\delta_0(a) + (1-\alpha)\lambda e^{-\lambda a}, \quad a \geq 0$$

$\square$

Lemma 1.2. *Let* $\alpha = 1$ *(* $\beta = \mu$ *, critical case), then* $E\left[N(t)|N(t) > 0\right] = 1 + \beta t$ *and*

$P\left(\frac{N(t)}{1+\beta t} > x | N(t) > 0\right) \xrightarrow[t \to \infty]{} e^{-x}, x \geq 0.$

Proof. First

$$P(N(t) = k | N(t) > 0) = \frac{P(N(t) = k)}{P(N(t) > 0)} = \frac{P(N(t) = k)}{\frac{1}{1+\beta t}}$$

It gives

$$P(N(t) = k | N(t) > 0) = \frac{(\beta t)^{k-1}}{(1+\beta t)^k}, k \geq 1$$

and

$$E(Z^{N(t)}|N(t) > 0) = \sum_{k=1}^{\infty} z^k \frac{(\beta t)^{k-1}}{(1+\beta t)^k} = \frac{z}{1+\beta t(1-z)}$$

Differentiation this conditional generating functions provides the conditional expectation

$$E(N(t)|N(t) > 0) = \left[\frac{z}{1+\beta t(1-z)}\right]'|_{z=1} = 1 + \beta t$$

$$E[e^{-\frac{\lambda N(t)}{1+\beta t}}|N(t) > 0] = \frac{e^{-\frac{\lambda}{1+\beta t}}}{1+\beta t(1-e^{-\frac{\lambda}{1+\beta t}})} \xrightarrow[t \to \infty]{} \frac{1}{1+\lambda}$$

It is the Laplace transform of the exponential distribution with parameter 1. $\qquad\square$

1.2 FKPP equation

Let's consider now the random variable $n(t, x_1), \cdots\cdots, n(t, x_k), N(t)$ and their joint generating function

$$u(t, x; z_1, \cdots, z_k, z) = E_x z_1^{n(t,x_1)} \cdots z_k^{n(t,x_k)} z^{N(t)}$$

elementary and well-known calculations provides the FKPP-type equation:

$$\frac{\partial u}{\partial t} = \kappa \Delta u + \beta u^2 - (\beta + \mu)u + \mu \tag{1.10}$$

$$u(0, x) = \begin{cases} z_1 z, x = x_1 \\ \ldots \ldots \\ z_k z, x = x_k \\ z, x \notin x_i, i = 1, \cdots, k \end{cases}$$

Of course, we can assume that all x_i are different.

From (1.10), one can get the moment equations or equations for the correlation functions. The simplest one has a form: If $m_1(t, x) = E_0 n(t, x)$, then

$$\frac{\partial m_1(t, x)}{\partial t} = \kappa \Delta m_1 + (\beta - \mu)m_1 \tag{1.11}$$

$$m_1(0, x) = \delta_0(x)$$

i.e

$$m_1(t, x) = P(t, x, 0)e^{(\beta - \mu)t}$$

Here $P(t, x, 0)$ is the transition probability of the random walk with generator $\kappa \Delta$. It is well known that $P(t, x, y)$ satisfy the standard heat equation:

$$\frac{\partial P(t, x, y)}{\partial t} = \kappa \Delta_x P(t, x, y) = \kappa \Delta_y P(t, x, y) \tag{1.12}$$

$$P(0, x, y) = \delta_y(x)$$

Let's solve (1.12) by using the Fourier transform. For any function $\Phi(.) \in l^1(\mathbb{Z}^d)$, $\|\Phi\|_1 = \sum_{y \in \mathbb{Z}^d} |\Phi(y)|$, one can define the Fourier transform $\hat{\Phi}(\phi), \phi = (\phi_1, \cdots, \phi_d) \in T^d = [-\pi, \pi]^d$ as a continuous function on the dual group (Euclidian forms) T^d. Applying this transform to equation (1.12) with respect to, say the first variable x, and $y = 0$,

we'll get

$$\frac{\partial \hat{P}(t, \phi)}{\partial t} = \kappa \hat{\Delta}(\phi)\hat{P} \tag{1.13}$$

$$\hat{P}(0, \phi) = 1$$

Here $-\Phi(\phi) = \hat{\Delta}(\phi) = 2 \sum_{j=1}^{d} (\cos \phi_j - 1)$ is the Fourier transform or the discrete Laplacian Δ. It follows from (1.13) that

$$\hat{P}(t, \phi) = \sum_{x \in \mathbb{Z}^d} P(t, x, 0)e^{i(\phi, x)} = \sum_{x \in \mathbb{Z}^d} P(t, 0, x)e^{i(\phi, x)}$$

$$= E_0[e^{i(\phi, X(t))}] = e^{2\kappa t [\sum_{j=1}^{d} (\cos \phi_j - 1)]} = e^{\kappa t \hat{\Delta}(\phi)}$$

i.e. We know the characteristic function of the random variable $X(t)$. Due to the inverse Fourier transform formula we get

$$P(t, x, 0) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{-t\kappa \Phi(\phi) - i(\phi, x)} \, d\phi$$

$\Phi(\phi) = 2 \sum_{j=1}^{d} (1 - \cos \phi_j)$ is the Fourier symbol of discrete Laplacian Δ.

1.3 Large deviations for the random walk

We'll need large deviations asymptotic for $|y| = O(t)$ in the study of the front of the population for FKPP model. Corresponding estimations will be proven using Crammer's approach ([Molchanov, 2009])(which in our case is related to the Doob transformation of the transition probabilities). Let's start from equation (1.12) but apply not Fourier but Laplace transform:

$$\hat{P}(t, \lambda) = \sum_{y \in \mathbb{Z}^d} P(t, 0, y)e^{(\lambda, y)}, \quad \lambda \in \mathbb{R}^d \tag{1.14}$$

The same calculation as before shows that:

$$\hat{P}(t, \lambda) = e^{\kappa t H(\lambda)}, \quad H(\lambda) = 2 \sum_{j=1}^{d} (\cosh \lambda_j - 1) \tag{1.15}$$

Lemma 1.3. *For fixed $\lambda \in \mathbb{R}^d$, the kernal*

$$Q_\lambda(t, x, y) = e^{-\kappa t H(\lambda)} [e^{-(\lambda, x)} P(t, x, y) e^{(\lambda, y)}] \tag{1.16}$$

is the transition probability for the Markov chain $X^\lambda(t)$ on $\mathbb{Z}^d$ with continuous time and generator

$$L_\lambda f(x) = k \sum_{j=1}^{d} \left((f(x + e_j) - f(x)) e^{\lambda_j} + (f(x - e_j) - f(x)) e^{-\lambda_j} \right) \tag{1.17}$$

Proof. The Chapman-Kolmogonov equation

$$\sum_{z \in \mathbb{Z}^d} Q_\lambda(t, x, z) Q_\lambda(s, z, y) = Q_\lambda(t + s, x, y)$$

follows directly from (1.15),(1.16) and the fact that $Q_\lambda(t, x, y) = Q_\lambda(t, x - y, 0)$. Using the definition

$$
\begin{aligned}
L_\lambda f(x) &= \lim_{\triangle t \to 0} \frac{E_x f(x^\lambda(\triangle t)) - f(x)}{\triangle t} \\
&= \lim_{\triangle t \to 0} \frac{\sum_{y \in \mathbb{Z}^d} Q_\lambda(\triangle t, x, y)[f(y) - f(x)]}{\triangle t}
\end{aligned}
$$

and explicit formula for $Q_\lambda(\triangle t, x, y)$, we will get (1.17). $\qquad \square$

Consider the process $X^\lambda(t), s \geq 0$ with the generator L_λ, it has(like $X(s)$ with the generator $\kappa \Delta = L_0$) independent increments. Let's note that

$$EX_i^\lambda(dt) = 2\kappa \sinh \lambda_i dt, \quad i = 1, 2, \cdots, d,$$

$$cov(X_i^\lambda(dt), X_j^\lambda(dt)) = EX_i^\lambda(dt) \otimes X_j^\lambda(dt) = 2\kappa \cosh \lambda_i dt \delta_{ij}$$

$$(one \ can \ neglect \ EX_i^\lambda(dt) EX_j^\lambda(dt))$$

i.e.

$$\begin{aligned}
\vec{a}^\lambda(t) &= EX^\lambda(t) = (2\kappa t \sinh \lambda_i) \quad i = 1, 2, \cdots, d, \\
B^\lambda(t) &= cov X^\lambda(t) = (2\kappa t \cosh \lambda_i \delta_{ij}) \quad i = 1, 2, \cdots, d.
\end{aligned} \tag{1.18}$$

If $\vec{\lambda} \leq A$ and $t \to \infty$, then uniformly over λ

$$\begin{aligned}
P\{X^\lambda(t) &= [EX^\lambda(t)]\} \sim \frac{1}{(2\pi)^{d/2}\sqrt{det B^\lambda(t)}} \\
&= \frac{1}{(4\pi\kappa t)^{d/2}\sqrt{\displaystyle\prod_{j=1}^{d}\cosh \lambda_j}}
\end{aligned} \tag{1.19}$$

From (1.17) and (1.19) we'll derive asymptotic formula for $P(t, 0, y)$ acting uniformly on $y = (y_1, \cdots, y_d) : \|\frac{y}{t}\| \leq A$ For fixed y, find $\vec{\lambda}$ from the equation:

$$\frac{E_0|X^\lambda(t)|_j}{t} = \frac{y_j}{t} = 2\kappa \sinh \lambda_j$$

so

$$\lambda_j = arcsinh\left(\frac{y_j}{2\kappa t}\right)$$

Then from (1.16)

$$\begin{aligned}
P(t, 0, y) &= e^{\kappa t H(\lambda)} e^{-(\lambda, y)} Q_\lambda(t, 0, y) \\
&\sim \frac{e^{\kappa t H(\lambda) - (\lambda, y)}}{(4\pi\kappa t)^{d/2}\sqrt{\displaystyle\prod_{j=1}^{d}(1 + \frac{y_j^2}{4\kappa^2 t^2})}}
\end{aligned}$$

Plug the expression of λ_j,

$$\sim \frac{e^{-\displaystyle\sum_{j=1}^{d} y_j arcsinh\frac{y_j}{2\kappa t} + 2\kappa t \sum_{j=1}^{d}(\sqrt{1 + \frac{y_j^2}{4\kappa^2 t^2}} - 1)}}{(4\pi\kappa t)^{\frac{d}{2}}\sqrt{\displaystyle\prod_{j=1}^{d}(1 + \frac{y_j^2}{4\kappa^2 t^2})}}$$

If $|y| = O(t)$,then using the Taylor expansion, we'll find

$$P(t, 0, y) \sim \frac{1}{(4\pi kt)^{\frac{d}{2}}} e^{\sum_{j=1}^{d}(-(\frac{1}{4\kappa t} + \frac{1}{8\kappa^2 t^2})y_j^2 + \frac{1}{192\kappa^3 t^3}y_j^4 + O(\frac{|y|^6}{|t|^5}))}$$

It means that the Gaussian approximation acts up to $|y| = O(t^{\frac{3}{4}})$, in Cramer case it is true in general for $|y| = O(t^{\frac{2}{3}})$. Higher degree $\frac{3}{4} > \frac{2}{3}$ related in our situation to the vanishing of the third moments of $X(t)$.

For $|y| \sim ct$

$$P(t, 0, y) \sim \frac{1}{(4\pi\kappa t)^{\frac{d}{2}}} \exp(-2\kappa t(\mathcal{H}\frac{y}{2\kappa t}))$$

Here $\mathcal{H}(\vec{z}) = \sum_{j=1}^{d} z_j \ arcsinh z_j - \sum_{j=1}^{d}(\sqrt{1 + z_j^2} - 1)$

From the above discussion, we can summarize the asymptotic of $P(t, 0, x)$ for different relations between t and x by the following large deviation theorem.

Theorem 1.1.

a) If $|y| = o(t^{\frac{3}{4}})$,then

$$P(t, 0, y) \xrightarrow{t \to \infty} \frac{\exp\left(-\frac{|x|^2}{4\kappa t}\right)}{(4\pi\kappa t)^{\frac{d}{2}}}$$

b) For fixed $y = O(t)$, as $t \to \infty$,

$$P(t, 0, y) \sim \frac{1}{(4\pi\kappa t)^{\frac{d}{2}}} \exp(-2\kappa t\mathcal{H}(\frac{y}{2\kappa t}))$$

where

$$\mathcal{H}(\vec{a}) = \sup_{\vec{h}} \left[(\vec{a}, \vec{h}) - 2\kappa \sum_{j=1}^{d}(\cosh h_j - 1)\right]$$

i.e

$$\mathcal{H}(\vec{a}) = \sum_{j=1}^{d} \left[a_j arcsin(\frac{a_j}{2\kappa}) - 2\kappa\sqrt{1 + \frac{a_j^2}{4\kappa^2}}\right]$$

1.4   The front of population

The behavior of the spreading region is of great interest. Each path along the genealogical tree of the population on the time interval $[0, t]$ is a random walk with the typical range $O(\sqrt{t})$. The number of the particles is growing exponentially like $\exp\{\beta t\}$ and due to small large deviation probabilities the "radius" of the population has order $O(t)$. Kolmogorov [Kolmogorov et al., 1937] described the traveling wave front is the solution of the following Reaction-diffusion equation:

$$\frac{\partial v}{\partial t} = \kappa \Delta v + \beta v(1 - v) \tag{1.20}$$

(1.20) was first suggested by Fisher [Fisher, 1937] as a model for spread of a favored gene in a population. We can view v as the population density with one representing the saturation density, then the first term describes the diffusion in space and the second term describes the birth and death of the population[Kadanoff, 2006]. It is (1.10) after substitution $v = 1 - u$. In one dimension case, see [Kolmogorov et al., 1937], the particle solution like solution (1.20) is given by

$$v(t, x) = \phi(x - ct)$$

As $z \to -\infty$, $\phi(z) \to 1$ and $z \to \infty$, $\phi(z) \to 0$. The function $\phi(.)$ presents the parametrization of the separatrix connecting two critical points of the ODE

$$\kappa \phi'' + c\phi' + \beta\phi(1 - \phi) = 0$$

Such definition of the "'front"' is not the only interesting one. In the classical work of FKPP, the definition of the front is a bit of different. The front is defined by

$$F(t) = \{x : u(t, z, x) = E_0 z^{N(t,x)} \leq \frac{1}{2}\}$$

From the point of view of the population dynamics, another definition are also possible. Let $m_1(t, x, \Gamma) = E_x n(t, \Gamma)$,

$$\frac{\partial m_1(t, x, y)}{\partial t} = \kappa \Delta_x m_1(t, x, y) + \beta m_1(t, x, y)$$

$$m_1(0, x, y) = \delta_x(y)$$

i.e.

$$m_1(t, 0, y) = \frac{\exp\left(-\frac{y^2}{4\kappa t} + \beta t\right)}{(4\kappa \pi t)^{\frac{d}{2}}}$$

Define the "'density front"' by the relation $m_1(t, 0, y) = 1$ will give $|y| \approx 2\sqrt{\kappa\beta}t$. This is not a Kolmogorov's definition of the front, however, it also propagates linearly in time-space. It is convenient for the future moment's calculations. We will use this definition in the future. From this definition of the front $F(t) = \{x : m_1(t, x) \approx 1\}$, i.e. $\ln P(t, x, 0) + t(\beta - \mu) = 0$, we can formulate our front equation as follows:

- For $|x| = o(t^{\frac{3}{4}})$,

$$P(t, x, 0) \sim \frac{1}{(4\pi\kappa t)^{\frac{d}{2}}} e^{\sum_{j=1}^{d}\left(-\left(\frac{1}{4\kappa t} + \frac{1}{8\kappa^2 t^2}\right)x_j^2 + \frac{1}{192\kappa^3 t^3}x_j^4 + O\left(\frac{|x|^6}{|t|^5}\right)\right)}$$

  which gives the Front equation

$$\sum_{j=1}^{d}\left(\frac{1}{4\kappa t} + \frac{1}{8\kappa^2 t^2}\right)x_j^2 \approx (\beta - \mu)t$$

  Therefore,

$$|x| \approx 2t\sqrt{\kappa(\beta - \mu)}$$

- For $x = O(t)$,

$$P(t, x, 0) \sim \exp\{-t\mathcal{H}(\frac{x}{t})\}$$

  since

$$-t\mathcal{H}(\frac{x}{t}) \sim -t\sum_{i=1}^{d}\frac{|x_i|}{t}\ln\frac{x_i}{\kappa t}$$
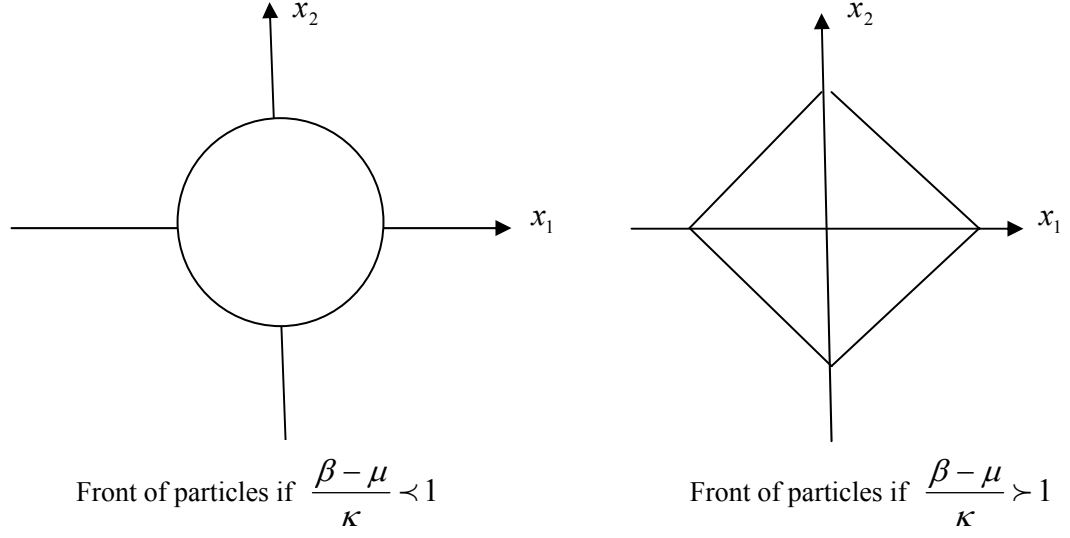
Figure 1.1: Front of the particles

the front F_t of the population now is given by

$$\sum_{i=1}^{d} \frac{|x_i|}{\kappa t} \ln \frac{x_i}{\kappa t} \approx \frac{\beta - \mu}{\kappa}$$

and it means $\sum_{i=1}^{d} |x_i| \approx (\beta - \mu)t.$

CHAPTER 2: MATHEMATICAL MODEL OF PLANKTON

2.1   Introduction

In this part, we study a mathematical model of evolution of plankton based on the FKPP equation. In order to identify further reasonable types of plankton model, we may look into some existing literatures. Hall A. J. and Wake G.C. [Hall and Wake, 1989] had studied a functional differential equation for the steady size distribution of the plankton population. Basse B., Wake G.C., Wall D.J.N. and van Brunt B. [Basse et al., 2004]studied a stochastic model for cell growth in plankton based on a modified Fokker-Planck equation. More work can be found in [Begg et al., 2008].

We extended the above model by considering the space dynamics of plankton population. Cell growth in plankton is characterized by cell division [Round et al., 1990]. Cells grow and segregate a full complement of components to each offspring cell. To describe the characteristic of plankton, we equip our FKPP model with the introduction of the extra parameter mass. In the moment of the division the mass of the particle which is growing linearly after the birth is divided in random proportion between two offspring. We study asymptotically the mass-space distribution of the particles and prove several limit theorems. In cases where all the parameter are constants and mortality rate is neglectable, the joint distribution can be found explicitly and thus provides a benchmark for future's investigation for more complex situations.

2.2   Main model

Here $\beta > \mu$ (supercritical case). As in the original FKPP model, put $n(0, x) = \delta_0(x), x \in \mathbb{Z}^d$ a single initial particle. The main difference from the classical FKPP

situation is the presence of an extra parameter for each particle, called its mass. The initial particle has mass m at $t = 0$ and position $x \in \mathbb{Z}^d$. For $t > 0$, the particle performs a random walk with generator $\mathcal{L} = \kappa\Delta$, its mass increases linearly: $m \rightarrow m(t) = m + vt, v > 0$, i.e., the underlying Markow process $(x(t), m(t))$ in $\mathbb{Z}^d \times \mathbb{R}^1_+$ has the generator $\mathcal{L} = \kappa\Delta + v\frac{\partial}{\partial m}$. At the moment of the first transformation τ_1 (exponentially distributed with parameter $\beta + \mu$), the particle either dies or splits into two particles at the point $x(\tau_1)$. The mass $m_1 = m + v\tau_1$ of the particle at the moment of splitting is distributed between two offspring in random proportion:

$$m'_2 = m_1\theta, \quad m''_2 = m_1(1 - \theta),$$

θ has distribution $q(d\theta)$. The random variable θ is independent on the prehistory and has the symmetric distribution $q(d\theta)$ on $[0, 1]$:

$$\theta \overset{law}{=} 1 - \theta$$

(to preserve the symmetry between offspring). The new particles perform independent evolutions with the same parameters $\mu, \beta, k, q(d\theta)$ (starting from $x(\tau_1)$ and masses $m_1 = \theta m(\tau_1), m_2 = (1 - \theta)m(\tau_1)$). They can be considered as single point

$$x_1 = x(\tau_1), m'_2, x_2 = x(\tau_1), m''_2 \in (\mathbb{Z}^d \times \mathbb{R}^1_+)^2$$

Like in the standard theory of the reaction-diffusion equations, we can present the evolution of the particles field as a Markov Process in the Fock space

$$X = \varnothing \cup \underbrace{(\mathbb{Z}^d \times \mathbb{R}^1_+)}_{F_1} \cup \cdots \underbrace{(\mathbb{Z}^d \times \mathbb{R}^1_+)^n}_{F_n} \cup \cdots$$

Remark 2.1. *This model gives the realistic description of the one-cell species populations (like plankton) which demonstrate the long time exponential growth (the Malthusian behavior). Of course, even in this case, some limitations (Oxygen, light etc.) will stop sooner or later the further growing. It is very important that our model includes*
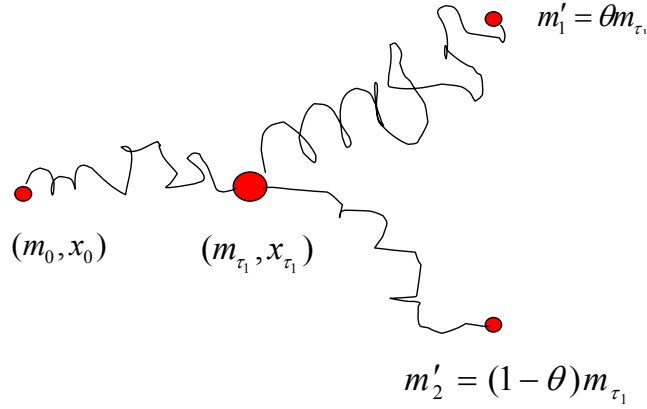
$$m_1' = \theta m_{\tau_1}$$

$$(m_0, x_0) \quad (m_{\tau_1}, x_{\tau_1})$$

$$m_2' = (1 - \theta) m_{\tau_1}$$

Figure 2.1: Evolution of the particles

the mass of particles and the phenomenon of the mitosis. The only essential deviation on the exponential distribution of the "life time" τ of the particles (between the birth and the transformation or the reaction). It is better but harder to study the semi-Markovian models with the general distribution of τ. This topic will be discussed later in the construction of semi-Markovian "mass" process.

For each open set $\Gamma \subset \mathbb{Z}^d \times \mathbb{R}_+^1$, in particular for $D_1 \times D$, where $D_1 \in \mathbb{Z}^d, D \in (0, \infty)$, Let's define the following notations:

$$n(t, x) = numbers\ of\ particles\ at\ the\ site\ x \in \mathbb{Z}^d\ and\ moment\ t;$$

$$N(t) = \sum_{x \in \mathbb{Z}^d} n(t, x); \quad n_D(t, x) = \sum_{i=1}^{n(t,x)} I_{m_i \in D}; \quad N_D(t) = \sum_{x \in \mathbb{Z}^d} n_D(t, x);$$

$$m(t, x, D) = \sum_{i=1}^{n_D(t,x)} m_i; \quad M(t) = total\ masses\ at\ moment\ t.$$

To find the finite dimensional distributions for $n_D(t)$, we calculate the corresponding generating functions and Laplace transforms. In the simplest case, it is the function:

$$u_z(t, x, y, m; D) = E_{x,m} z^{n_D(t,y)}, |z| \leq 1$$

We also introduce the generating function of the form:

$$u_{z,k}(t, x, y, m; D) = E_{x,m} z^{n_D(t,y)} e^{-km(t,y,D)}, |z| \leq 1, k \geq 0$$

The following results are the basis for the further analysis:

Theorem 2.1. *Let* $u_z(t, x, y, m; D) = E_{x,m} z^{n_D(t,y)}$, *then* $u_z(t, x, y, m; D)$ *satisfy the following functional differential equation:*

$$\begin{aligned}
\frac{\partial u(t,x,y,m;D)}{\partial t} &= \kappa \Delta u(t, x, y, m; D) + \frac{\partial u(t,x,y,m;D)}{\partial m} \cdot v \\
&\quad + \beta \int_0^1 u(t, x, y, \theta m; D) \cdot u(t, x, y, (1 - \theta)m; D) q(d\theta) \\
&\quad - (\beta + \mu) u(t, x, y, m; D) + \mu
\end{aligned} \tag{2.1}$$
$$u_z(0, x, y, m; \Gamma) = z^{\delta_x(y)} I_m(D)$$

Proof. The proof is similar to the proof of Theorem 2.2 in the following. □

Theorem 2.2. *Let* $u_{z,k}(t, x, y, m; D) = E_{x,m} z^{n_D(t,y)} e^{-km(t,y,D)}$, *then* $u_{z,k}(t, x, y, m; D)$ *satisfy the following functional differential equation:*

$$\begin{aligned}
\frac{\partial u_{z,k}(t,x,y,m;D)}{\partial t} &= \kappa \Delta u_{z,k}(t, x, y, m; D) + \frac{\partial u_{z,k}(t,x,y,m;D)}{\partial m} v \\
&\quad + \beta \int_0^1 u_{z,k}(t, x, y, \theta m; D) \cdot u_{z,k}(t, x, y, (1 - \theta)m; D) q(d\theta) \\
&\quad - (\beta + \mu + kv) u_{z,k}(t, x, y, m; D) + \mu
\end{aligned} \tag{2.2}$$
$$u_{z,k}(0, x, y, m; \Gamma) = z^{\delta_x(y)} e^{-km} I_m(D).$$

The proofs of both theorems are practically identical. Let's give the sketch of the calculations in Theorem 2.2.

Proof. The formal derivation of this equation based on the standard technique: balance of the probabilities in the infinitesimal initial time interval $(0, dt)$. Namely, let's consider

$$u_{z,k}(t + dt, x, y, m; D) = E_{x,m} z^{n_D(t+dt,y)} e^{-km(t+dt,y,D)}$$

and then let's split the interval $[0, t + dt]$ into two parts $[0, dt] \cup [dt, t + dt]$. At the moment $t = 0$, we have one particle in the point x with mass m and during $[0, dt]$,

we observe one of the following hypothesis:

- H_0: nothing happen, the particle remains in x, no annihilation, no splitting:

$$P(H'_x) = 1 - \beta dt - \mu dt - 2d\kappa dt$$

- H'_x: transition of the initial particle in one of the neighboring points x', $\|x - x'\| = 1$

$$P(H'_x) = \kappa dt$$

- H_+: splitting of the initial particle into two particles:

$$P(H_+) = \beta dt$$

- H_-: annihilation of the initial particles:

$$P(H_+) = \mu dt$$

After annihilation, $n_D(t, y) = 0$, for $t > dt$. Now one can apply the full expectation formula:

$$u_{z,k}(t + dt, x, y, m; D) = u(t, x, y, m + vdt; D)(1 - \beta dt - \mu dt - 2d\kappa dt)e^{-kv(x,m)dt} + \mu dt$$

$$+\beta dt \int_0^1 u_{z,k}(t + dt, x, y, \theta(m + vdt); D) \cdot u_{z,k}(t + dt, x, y, (1 - \theta)(m + vdt); D)q(d\theta)$$

Theorem 2.2 is obtained by letting $dt \to 0$ $\qquad \square$

Due to the nonlinearity, moment generating function is not the best source of the information about the particle field, it is better to work with the statistical moments. The factorial moments of $n_D(t, y)$ can be calculated by partial derivative of moment generating function with respect to z at $z = 1$. The moments of mass can be obtained by differentiated with respect to k at $k = 0$.

a) If $l_1(t, x, y, m; D) = E_{x,m} n_D(t, y) = \frac{\partial u_{z,k=0}(t,x,y,m;D)}{\partial z}\big|_{z=1}$

$$\begin{cases} \frac{\partial l_1(t,x,y,m;D)}{\partial t} = \kappa \Delta l_1(t, x, y, m; D) + \frac{\partial l_1(t,x,y,m;D)}{\partial m} v \\ \qquad - (\beta + \mu) l_1(t, x, y, m; D) + 2\beta \int_0^1 l_1(t, x, y, \theta m; D) q(d\theta) \\ l_1(0, x, y, m; \Gamma) = \delta_x(y) I_m(D) \end{cases}$$

If $L_1(t, x, y, m; D) = E_{x,m} m(t, y, D) = -\frac{\partial u_{z=1,k}(t,x,y,m;D)}{\partial k}\big|_{k=0}$

$$\begin{cases} \frac{\partial L_1(t,x,y,m;D)}{\partial t} = \kappa \Delta L_1(t, x, y, m; D) + \frac{\partial L_1(t,x,y,m;D)}{\partial m} \cdot v \\ \qquad + v - (\beta + \mu) L_1(t, x, y, m; D) + 2\beta \int_0^1 L_1(t, x, y, \theta m; D) q(d\theta) \\ L_1(0, x, y, m; D) = \delta_x(y) m I_m(D) \end{cases}$$

If we repeat the similar calculations with the second derivatives, we will get

b) If $l_2(t, x, y, m, D) = E_{x,m} n_D(t, y)(n_D(t, y) - 1) = \frac{\partial^2 u_{z,k=0}(t,x,y,m;D)}{\partial z^2}\big|_{z=1}$

$$\begin{cases} \frac{\partial l_2(t,x,y,m;D)}{\partial t} = \kappa \Delta l_2(t, x, y, m; D) + \frac{\partial l_2(t,x,y,m;D)}{\partial m} \cdot v \\ \qquad - (\beta + \mu) l_2(t, x, y, m; D) + 2\beta \int_0^1 l_2(t, x, y, \theta m; D) q(d\theta) \\ \qquad + 2\beta \int_0^1 l_1(t, x, y, \theta m; D) l_1(t, x, y, (1 - \theta) m; D) q(d\theta) \\ l_2(0, x, y, m; D) = 0 \end{cases}$$

If $L_2(t, x, y, m; \Gamma) = E_{x,m} m(t, x, D)^2 = \frac{\partial^2 u_{z=1,k}(t,x,y,m;D)}{\partial k^2}\big|_{k=0}$

$$\begin{cases} \frac{\partial L_2(t,x,y,m;D)}{\partial t} = \kappa \Delta L_2(t, x, y, m; D) + \frac{\partial L_2(t,x,y,m;D)}{\partial m} \cdot v(x, m) \\ \qquad + 2v L_1(t, x, y, m; D) - (\beta + \mu) L_2(t, x, y, m; D) \\ \qquad + 2\beta \int_0^1 L_2(t, x, y, \theta m; D) q(d\theta) \\ \qquad + 2\beta \int_0^1 L_1(t, x, y, \theta m; D) L_1(t, x, y, (1 - \theta) m; D) q(d\theta) \\ L_2(0, x, y, m, D) = \delta_x(y) m^2 I_m(D) \end{cases}$$

c)   If $c_2(t, x, y, m; D) = E_{x,m} n_D(t, y) \mu(t, y, D) = -\frac{\partial^2 u_{z,k}(t,x,y,m;D)}{\partial z \partial k}\big|_{z=1,k=0}$

$$\begin{cases} \frac{\partial c_2(t,x,y,m;D)}{\partial t} = \kappa \Delta c_2(t, x, y, m; D) + \frac{\partial c_2(t,x,y,m;D)}{\partial m} \cdot v + v l_1(t, x, y, m; D) \\ \qquad -(\beta + \mu)c_2(t, x, y, m; D) + 2\beta \int_0^1 c_2(t, x, y, \theta m; D) q(d\theta) \\ \qquad +2\beta \int_0^1 l_1(t, x, y, \theta m; D) L_1(t, x, y, (1-\theta)m; D) q(d\theta) \\ c_2(0, x, y, m; D) = m\delta_x(y) \end{cases}$$

Similar equation can be presented for moments of any order. Using the Leibniz formula, the higher order factorial moments has similar structure with the following recurrent relation:

d)   If $l_p(t, x, y, m; D) = E_{x,m}[n_D(t, y)(n_D(t, y) - 1) \cdots (n_D(t, y) - p - 1)]$

$$= \frac{\partial^p u_{z,k=0}(t, x, y, m; D)}{\partial z^p}\big|_{z=1}$$

$$\begin{cases} \frac{\partial l_p(t,x,y,m,D)}{\partial t} = 2\beta[\int_0^1 (l_p(t, x, y, \theta m, D) - l_p(t, x, y, m, D)) q(d\theta)] \\ \qquad +\frac{\partial l_p(t,x,y,m,D)}{\partial m} v + \kappa \Delta l_p(t, x, y, m, D) + (\beta - \mu)l_p(t, x, y, m, D) \\ \qquad +\beta \int_0^1 \sum_{i=1}^{p-1} \binom{p}{i} l_i(t, x, y, \theta m, D) l_{p-i}(t, x, y, (1-\theta)m, D) q(d\theta) \\ m_l(0, x, y, m, D) = 0 \end{cases}$$

Theorem 2.3. *Put $E_{x,m} n_D(t, y) = \int_D \rho(t, (x, m), (y, w)) dw$, then $\rho(t, (x, m), (y, w))$ satisfy the following functional differential equation:*

$$\begin{aligned} \frac{\partial \rho(t,(x,m),(y,w))}{\partial t} &= 2\beta \int_0^1 [\rho(t, (x, \theta m), (y, w)) - \rho(t, (x, m), (y, w))] q(d\theta) \\ &+ \frac{\partial \rho(t,(x,m),(y,w))}{\partial m} \cdot v + \kappa \Delta \rho(t, (x, m), (y, w)) + (\beta - \mu)\rho(t, (x, m), (y, w)) \end{aligned} \qquad (2.3)$$

From (2.3), we observe that the operator without $(\beta - \mu)$ describe two independent Markov processes:

$$\mathcal{L}_x f = \kappa \Delta f$$

It is the symmetric random walk with diffusive coefficient $\kappa > 0$.

$$\mathcal{L}_m f = v\frac{\partial f}{\partial m} + 2\beta \left(\int_0^1 [f(\theta m) - f(m)] q(d\theta) \right)$$
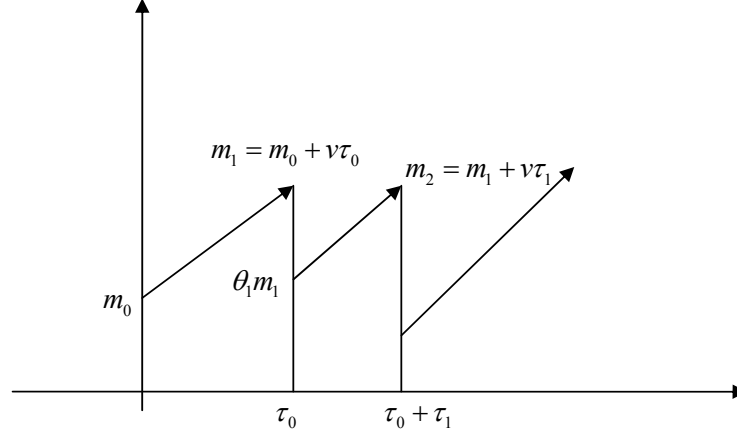
Figure 2.2: Mass process

It is the mass process on half axis $m > 0$.

2.3   Mass process

Let's consider the mass process with the generator

$$\mathcal{L}_m f = v\frac{\partial f}{\partial m} + 2\beta\left(\int_0^1 [f(\theta m) - f(m)]q(d\theta)\right)$$

Process $m(t)$ is the Markov chain with continuous time and Poissonian jumps. Mass process with "Poissonian jump" linearly depending on mass the generator of the mass has the following meaning: one particle starts at $t = 0$ with initial mass m, it grows with the linear rate v, after exponential time τ_1 with parameter 2β, one particles with mass m split into two particles with corresponding masses

$$m_1 = (m + v\tau_1)\theta$$

$$m_2 = (m + v\tau_1)(1 - \theta)$$

Where θ and $1 - \theta$ has the same law $q(d\theta)$. See figure 2.2.

The transition density of the mass process $\rho(t, m, m')$, i.e. the fundamental solution

of

$$
\begin{cases}
\frac{\partial \rho(t,m,m')}{\partial t} = \mathcal{L}_m \rho(t,m,m') \\
\rho(0,m,m') = \delta_{m'}(m)
\end{cases}
$$

have a limit $\Pi(m') = \lim_{t \to \infty} \rho(t,m,m')$.

According to the definition of the conjugate operator, applying method of change of variable and integration by parts, it is easy to find the conjugate operator.

Lemma 2.1. *The conjugate operator*

$$
\mathcal{L}_m^\star g = -v \frac{\partial g}{\partial m} + 2\beta \left(\int_0^1 [g(\frac{m}{\theta}) - g(m)] q(d\theta) \right)
$$

Theorem 2.4. *Process $m(t)$ has the invariant density $\Pi(m)$ such that $\mathcal{L}^\star \Pi = 0$. This density has the same law as the distribution density of the random geometric series*

$$
\xi = v\tau_0 + v\tau_1\theta_1 + \cdots + v\tau_n\theta_1\cdots\theta_n + \cdots \tag{2.4}
$$

Where $\tau_i, i \geq 0$ are i.i.d $\exp(2\beta)$ random variable and $\theta_i, i \geq 0$ are i.i.d random variable with the law $q(d\theta)$, τ_i and θ_i are independent.

Proof. First, let's find the limiting distribution of mass at the moment of the jumps $(\tau_i + 0)$, At the initial moment

$$
m_0 = m
$$

After the moment of the first splitting,

$$
m(\tau_1+) = \theta_1(m + v\tau_1)
$$

Similarly, at the moment of the second splitting,

$$
\begin{aligned}
m((\tau_1 + \tau_2)+) &= \theta_2(m_1 + v\tau_2) \\
&= \theta_1\theta_2 m + v\tau_2\theta_2 + v\tau_1\theta_1\theta_2 \\
&\overset{law}{=} \theta_1\theta_2 m + v\tau_1\theta_1 + v\tau_2\theta_1\theta_2
\end{aligned}
$$

In general,

$$m((\tau_1 + \cdots + \tau_n)+) \overset{law}{=} \theta_1 \cdots \theta_n m + v\tau_1\theta_1 + \cdots + v\tau_n\theta_1 \cdots \theta_n$$

so as $t \to \infty$, the limit will have the the form

$$m((\tau_1 + \cdots + \tau_n)+) \xrightarrow[n\to\infty]{law} v\tau_1\theta_1 + \cdots + v\tau_n\theta_1 \cdots \theta_n + \cdots$$

Second, for fixed $t >> 1$, the previous splitting had taken place at the time $t - \tau_0$, where $\tau_0 \sim \exp(2\beta)$, it gives

$$m(t) \xrightarrow[t\to\infty]{law} v\tau_0 + v\tau_1\theta_1 + \cdots + v\tau_n\theta_1 \cdots \theta_n + \cdots$$

$\square$

Now we'll calculate the moment for the invariant limiting distribution $\Pi(m)$, the calculation will be based on the following fact:

$$\xi = v\tau_0 + \theta_1\tilde{\xi}$$

Here $\tilde{\xi} \overset{law}{=} \xi$ and τ_{-1}, θ_1 are independent on $\tilde{\xi}$. Since τ is exponential random variable with parameter 2β, so

$$E\tau = \frac{1}{2\beta}, E\tau^2 = \frac{1}{2\beta^2}$$

As a result,

$$E\xi = Ev\tau_0 + E\theta_1\tilde{\xi}$$

so

$$E\xi = \frac{v}{2\beta} + \frac{1}{2}E\xi$$

thus,

$$E\xi = \frac{v}{\beta}$$

The second moment

$$E\xi^2 = v^2 E(\tau^2) + 2v E(\tau\theta_1\tilde{\xi}) + E(\theta_1^2\tilde{\xi})^2$$

from the independence, then

$$E\xi^2 = \frac{v^2}{\beta^2(1 - E\theta_1^2)}$$

so

$$var\xi = \frac{v^2}{\beta^2}\left(\frac{1}{1 - E\theta_1^2} - 1\right).$$

Similarly, the third moments

$$E\xi^3 = \frac{3v^3}{2\beta^3(1 - E\theta_1^2)(1 - E\theta_1^3)}.$$

Let's find the asymptotic of $\Pi(m)$ for large m and small m. Since

$$\xi = v\tau_0 + v\tau_1\theta_1 + \cdots + v\tau_n\theta_1\cdots\theta_n + \cdots$$

Therefore,

$$
\begin{aligned}
E_\theta[e^{-\lambda\xi}] &= E_\theta[e^{-\lambda(v\tau_0 + v\tau_1\theta_1 + v\tau_2\theta_1\theta_2\cdots)}]\\[2mm]
&= E_\theta[e^{-\lambda v\tau_0}]E_\theta[e^{-\lambda v\tau_1\theta_1}]\cdots[e^{-\lambda v\tau_2\theta_1\cdots\theta_n}]\cdots\\[2mm]
&= \frac{1}{(1 + \frac{\lambda v}{2\beta})(1 + \frac{\lambda v\theta_1}{2\beta})\cdots(1 + \frac{\lambda v\theta_1\cdots\theta_n}{2\beta})\cdots}\\[2mm]
&= \frac{c_0}{1 + \frac{\lambda v}{2\beta}} + \frac{c_1}{1 + \frac{\lambda v\theta_1}{2\beta}} + \cdots + \frac{c_n}{1 + \frac{\lambda v\theta_1\cdots\theta_n}{2\beta}} + \cdots\\[2mm]
&= \frac{c_0}{1 + \frac{\lambda}{\frac{2\beta}{v}}} + \frac{c_1}{1 + \frac{\lambda}{\frac{2\beta}{v\theta_1}}} + \cdots + \frac{c_n}{1 + \frac{\lambda}{\frac{2\beta}{v\theta_1\cdots\theta_n}}} + \cdots
\end{aligned}
$$

Here,

$$c_0 = \frac{1}{(1 - \theta_1)(1 - \theta_1\theta_2)\cdots}$$

$$c_1 = \frac{1}{(1 - \frac{1}{\theta_1})(1 - \theta_2)(1 - \theta_2\theta_3)\cdots}$$

$$c_n = \frac{1}{(1 - \frac{1}{\theta_1 \cdots \theta_n})(1 - \frac{1}{\theta_2 \cdots \theta_n}) \cdots (1 - \frac{1}{\theta_n})(1 - \theta_{n+1})(1 - \theta_{n+1}\theta_{n+2}) \cdots}$$

Now one can find conditional density $p_\xi(m)$ of random variable if $\vec{\theta} = (\theta_1, \theta_2, \cdots)$ are known,

$$
\begin{aligned}
p_\xi(m) &= E_{\vec{\theta}}[\frac{2\beta}{v} e^{-\frac{2\beta m}{v}} c_0] + E_{\vec{\theta}}[\frac{2\beta}{v\theta_1} e^{-\frac{2\beta m}{v\theta_1}} c_1] + \cdots \\
&= E_{\vec{\theta}}(\frac{2\beta}{v} e^{-\frac{2\beta m}{v}} \frac{1}{(1 - \theta_1)(1 - \theta_1\theta_2) \cdots}) \\
&+ E_{\vec{\theta}}(\frac{2\beta}{v\theta_1} e^{-\frac{2\beta m}{v\theta_1}} \frac{1}{(1 - \frac{1}{\theta_1})(1 - \theta_2)(1 - \theta_2\theta_3) \cdots}) + \cdots \\
&= \frac{2\alpha\beta}{v} e^{-\frac{2\beta m}{v}} - E_{\theta_1} \frac{2\alpha\beta}{1 - \theta_1} e^{-\frac{2\beta m}{v\theta_1}} + \cdots
\end{aligned}
\tag{2.5}
$$

where

$$\alpha = E \frac{1}{(1 - \theta_1)(1 - \theta_1\theta_2) \cdots} \tag{2.6}$$

Let's formulate several analytic results about the invariant density .

Theorem 2.5. *Assume that $Supp\theta = [a, 1 - a]$, $0 < a \leq \frac{1}{2}$, then for large m,*

$$\Pi(m) \xrightarrow[m \to \infty]{} \frac{2\alpha\beta}{v} e^{-\frac{2\beta m}{v}} + R(m)$$

The remainder term with the maximum on the boundary has order

$$R(m) \sim \frac{2\alpha\beta}{a} e^{-\frac{2\beta m}{v(1-a)}} L(m)$$

Where $L(m) \xrightarrow[m \to \infty]{} 0$ and $L(m)$ depends on the structure of the distribution $q(d\theta)$ near the maximum point $\theta_{critical} = 1 - a$.

Proof. From (2.5), due to the Laplace method, it is trivial to get the result. □

The behavior of $p_\xi(m)$ as $m \to 0$ is much more interesting. Here we will use the exponential Chebyshev's inequality, more detailed analysis in the case when $q(d\theta)$ contains finitely many atoms, see Derfel [Derfel, 1989] and [Derfel, van Brunt, and Wake, Derfel et al.].

Theorem 2.6. *Assume that $Supp\theta = [a, 1-a]$, $0 < a \leq \frac{1}{2}$, then if $m \to 0$, then*

$$P\{\xi \leq m\} \sim e^{c_1 \ln^2(\frac{1}{m})}$$

Proof. Let's start from the standard calculations, for $\lambda > 0$ and fix $a \leq \theta_i \leq 1 - a$, $i = 1, 2, \cdots$

$$
\begin{aligned}
P\left\{\xi \leq m | \vec{\theta}\right\} &= P\{e^{-\lambda\xi} > e^{-\lambda m} | \vec{\theta}\} \leq \min_{\lambda > 0} \frac{Ee^{-\lambda\xi}}{e^{-\lambda m}} \\
&= \min_{\lambda > 0} e^{\lambda m - \ln(1 + \frac{\lambda v}{2\beta}) - \ln(1 + \frac{\lambda v\theta_1}{2\beta}) - \ln(1 + \frac{\lambda v\theta_1\theta_2}{2\beta}) - \cdots}
\end{aligned}
\tag{2.7}
$$

Equation for the critical point $\lambda_0 = \lambda_0(m)$ has a form:

$$m = \frac{\frac{v}{2\beta}}{1 + \frac{\lambda v}{2\beta}} + \frac{\frac{v\theta_1}{2\beta}}{1 + \frac{\lambda v\theta_1}{2\beta}} + \cdots + \frac{\frac{v\theta_1\cdots\theta_k}{2\beta}}{1 + \frac{\lambda v\theta_1\theta_2}{2\beta}} + \cdots$$

i.e.

$$m = \frac{1}{\frac{2\beta}{v} + \lambda} + \frac{1}{\frac{2\beta}{v\theta_1} + \lambda} + \cdots + \frac{1}{\frac{2\beta}{v\theta_1\cdots\theta_k} + \lambda} + \cdots$$
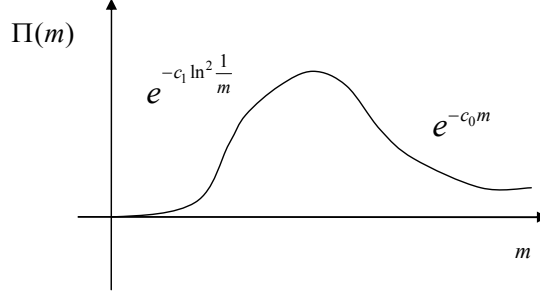
Define $k(\lambda) = \min\{k : \frac{2\beta}{v\theta_1\cdots\theta_k} \sim \lambda\}$, then $m \sim \frac{k(\lambda)}{\lambda}$. From $\frac{2\beta}{v\theta_1\cdots\theta_k} \sim k$, we then have $k(\lambda) \sim \frac{\ln\lambda}{E\ln(\frac{1}{\theta})}$. Hence, the critical point

$$\lambda \sim \frac{\ln(\frac{1}{m})}{mE\ln(\frac{1}{\theta})} \tag{2.8}$$

Substitute (2.8) into Chebyshev's inequality (2.7) gives

$$
\begin{aligned}
P\left\{\xi \leq m | \vec{\theta}\right\} &\leq e^{\frac{\ln(\frac{1}{m})}{E\ln(\frac{1}{\theta})} - \ln\left(1 + \frac{v\ln(\frac{1}{m})}{2m\beta E\ln(\frac{1}{\theta})}\right) - \cdots - \ln\left(1 + \frac{v\ln(\frac{1}{m})\theta_1\cdots\theta_k}{2m\beta E\ln(\frac{1}{\theta})}\right)} \\
&\leq e^{\frac{\ln(\frac{1}{m})}{E\ln(\frac{1}{\theta})} - \ln\left(1 + \frac{v\ln(\frac{1}{m})}{2m\beta E\ln(\frac{1}{\theta})}\right) - \cdots - \ln\left(1 + \frac{v\ln(\frac{1}{m})a^k}{2m\beta E\ln(\frac{1}{\theta})}\right)} \\
&\leq e^{\frac{\ln(\frac{1}{m})}{E\ln(\frac{1}{\theta})} - \sum_{i=0}^{k} \ln\left(\frac{v\ln(\frac{1}{m})a^i}{2m\beta E\ln(\frac{1}{\theta})}\right)} \\
&\leq e^{-c_1 \ln^2(\frac{1}{m})}
\end{aligned}
$$

$\square$

Figure 2.3: Asymptotic behavior of $\Pi(m)$

2.4 Moment equations and limit theorems

In this section, we study the distribution of $n(t,0)$. The basic technique is the calculation of the moments. Joint distribution of $N(t)$ and $n(t,0)$ and conditional distribution are also of our interest. Although we will not discuss in details here, our aim is to lay a solid ground for future investigation of other limit theorems.

Consider $u(t,x;z) = E_x z^{n(t,0)}$, similar to FKPP equation (1.10)

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \kappa\Delta u + \beta u^2 - (\beta + \mu)u + \mu \\
u(0,x) &= z\delta_0(x)
\end{aligned}
\tag{2.9}
$$

Differentiation over z with substitution $z = 1$ gives the equation for the first moment $m_1(t,x) = E_x(n(t,0))$

$$
\begin{aligned}
\frac{\partial m_1(t,x)}{\partial t} &= \kappa\Delta m_1 + (\beta - \mu)m_1 \\
m_1(0,x) &= \delta_0(x)
\end{aligned}
\tag{2.10}
$$

i.e

$$
m_1(t,x) = P(t,x,0)e^{(\beta-\mu)t}
$$

where

$$
P(t,x,y) \sim \frac{e^{-\frac{|x-y|^2}{4\kappa t}}}{(4\pi\kappa t)^{\frac{d}{2}}}, \quad |x-y| \sim O(t^{\frac{3}{4}})
\tag{2.11}
$$

Differentiation of (2.9) over z l times and substitution $z = 1$ (gives us the equation for the l-th moment $m_l(t, x) = E_x[n(t, 0) \cdots (n(t, 0) - l + 1)], l \geq 2$. $m_l(t, x)$ satisfies the non-homogeous equation

$$\frac{\partial m_l(t, x)}{\partial t} = \kappa \Delta m_l + (\beta - \mu) m_l + \beta \sum_{p=1}^{l-1} \binom{l}{p} m_p(t, x) m_{l-p}(t, x) \qquad (2.12)$$

$$m_l(0, x) = 0$$

Lemma 2.2. *(Carleman C.) If $\int_{\mathbb{R}} x^l d\mu_n(x) \xrightarrow{n \rightarrow \infty} \nu_l$ and $|\nu_l| \leq l! c_0^l$, then ν_l are moments of some measures μ and $\mu_n \xrightarrow{w} \mu$.*

Lemma 2.3. *(Duhamel's Principle)*

$$\begin{cases} \frac{\partial m(t,x)}{\partial t} = \kappa \Delta m + v(x) m + g(t, x) \\ m(0, x) = m_0(x) \end{cases} \qquad (2.13)$$

then the solution is given by:

$$m(t, x) = E_x \left[e^{\int_0^t v(x_s) \, ds} \right] m_0(x_t) + E_x \left[\int_0^t e^{\int_0^s v(x_u) \, du} g(t - s, x_s) \right]$$

Our goal is to prove

Theorem 2.7. *If $\beta > \mu \geq 0$ and $|x| \sim o(t^{\frac{3}{4}})$, then*

$$P_x \left(\frac{n(t, 0)}{m_1(t, x)} < a \right) \longrightarrow G(a)$$

where G(a) is

$$\frac{dG}{da} = \alpha \delta_0(a) + (1 - \alpha) \lambda e^{-\lambda a}, \quad a \geq 0$$

which is the same limiting distribution as for $\frac{N(t)}{E(N(t))}$ in Lemma(1.1).

Proof. The proof of this theorem is based on th direct calculation of the moments. It is sufficient to check

$$E \left[\frac{n(t, 0)}{m_1(t, x)} \right]^l \xrightarrow{l \rightarrow \infty} \mu_l = l! \left(\frac{\beta}{\beta - \mu} \right)^{l-1}, \quad l = 1, 2, \cdots$$

Note that the moments are increasing not to fast and we can use Carleman's moment theorem. It is equivalent to prove the similar relation for the factorial moments:

$$E\left[\frac{n(t,0)(n(t,0)-1)\cdots(n(t,0)-l+1)}{m_1^l(t,x)}\right]^l \xrightarrow[l\to\infty]{} \mu_l = l!\left(\frac{\beta}{\beta-\mu}\right)^{l-1}, \quad l=1,2,\cdots$$

i.e.

$$E\left[n(t,0)(n(t,0)-1)\cdots(n(t,0)-l+1)\right]^l \xrightarrow[l\to\infty]{} \mu_l = l!e^{l(\beta-\mu)t}p^l(t,x,0)(\frac{\beta}{\beta-\mu})^{l-1},$$

$$l=1,2,\cdots$$

Let's start inductive proof from the second moment, $l=2$. The differential equation and intial condition for the second moment are:

$$\frac{\partial m_2(t,x)}{\partial t} = k\Delta m_2 + (\beta-\mu)m_2 + 2\beta m_1^2(t,x)$$

$$m_2(0,x) = 0$$

Due to Duhamel's formula,

$$m_2(t,x) = \int_0^t 2\beta \sum_{y\in\mathbb{Z}^d} m_1^2(t-s,y)e^{(\beta-\mu)s}p(s,y,x)\,ds$$

$$\sim \int_0^t 2\beta \sum_{y\in\mathbb{Z}^d} \frac{e^{2(\beta-\mu)(t-s)}e^{-\frac{2|y|^2}{4\kappa(t-s)}}}{(4\pi\kappa s)^d}e^{(\beta-\mu)s}p(s,y,x)\,ds$$

The main contribution give the results of s close to 0 since $m_1^2(t-s,y)$ changes slowly, therefore,

$$m_2(t,x) \sim \int_0^t 2\beta \sum_{y\in\mathbb{Z}^d} \frac{e^{-\frac{2|x|^2}{4\kappa t}}}{(4\pi\kappa t)^d}e^{(\beta-\mu)(2t-s)}\,ds$$

$$\sim 2\beta\frac{e^{-\frac{2|x|^2}{4\kappa t}}}{(4\pi\kappa t)^d}\int_0^t e^{(\beta-\mu)(2t-s)}\,ds$$

$$\sim 2!\frac{e^{-\frac{2|x|^2}{4\kappa t}}}{(4\pi\kappa t)^d}e^{2(\beta-\mu)t}\frac{\beta}{\beta-\mu}$$

Assume by induction that

$$m_k(t, x) \sim k! e^{k(\beta-\mu)t}(p(t, x, 0))^k (\frac{b}{\beta - \mu})^{k-1}, \quad k \le l - 1$$

and use the general equation for m_l in terms of $m_k m_{l-k}, k = 1, 2 \cdots, l - 1$. It gives

$$m_l(t, x) = \beta \sum_{k=1}^{l-1} \binom{l}{k} \int_0^t \sum_{y \in \mathbb{Z}^d} m_k(t - s, y) m_{l-k}(t - s, y) e^{(\beta-\mu)s} p(s, y, x) \, ds$$

$$\sim \beta \sum_{k=1}^{l-1} \binom{l}{k} \int_0^t \sum_{y \in \mathbb{Z}^d} e^{l(\beta-\mu)(t-s)} p^l(t - s, y, 0) k!(n - k)!(\frac{\beta}{\beta - \mu})^{l-2} e^{(\beta-\mu)s} p(s, y, x) \, ds$$

$$\sim \beta \sum_{k=1}^{l-1} \binom{l}{k} \int_0^t \sum_{y \in \mathbb{Z}^d} e^{l(\beta-\mu)(t-s)} p^l(t - s, y, 0) k!(n - k)!(\frac{\beta}{\beta - \mu})^{l-2} e^{(\beta-\mu)s} p(s, y, x) \, ds$$

$$\sim \beta l!(\frac{\beta}{\beta - \mu})^{l-2} p^l(t, x, 0) \int_0^t e^{l(\beta-\mu)t - (l-1)(\beta-\mu)s} \, ds$$

$$\sim \beta l!(\frac{\beta}{\beta - \mu})^{l-2} p^l(t, x, 0) \frac{e^{(\beta-\mu)lt}}{(l - 1)(\beta - \mu)}$$

$$\sim l! e^{l(\beta-\mu)t} p^l(t, x, 0)(\frac{\beta}{\beta - \mu})^{l-1}$$

which gives the desired result. $\qquad\square$

Remark 2.2. *Theorem 2.7 has the following interpretation: In the Malthusian population, the exponential growth of the total population proportionally effect population in each site.*

<h2 style="text-align:center">2.5   Solvable model with constant coefficients</h2>

Throughout the subsection we restrict ourselves to the special case. Our interest concentrate on the limiting distribution of the mass process under the condition the total number of particles $N(t)$ is fixed. In particular, we assume that the entire coefficients are constant, i.e.

$$\beta > 0, \mu = 0, v(x, m) = v, q = 0.5, m = 0, \Delta = [0, \infty),$$

then $(N(t), M(t))$ is a Markov process. To study the behavior of the process $(N(t), M(t))$, consider the moment generating function of $\phi(t, z, k) = Ez^{N(t)}e^{-kM(t)}$, then $\phi(z, k)$ satisfies

$$\phi(t + dt, z, k) = \phi^2(t, z, k)\beta dt + (1 - \beta dt)e^{-kvdt}\phi(t, z, k)$$

Hence

$$\frac{\partial \phi}{\partial t} = \beta\phi^2 - (\beta + kv)\phi$$

$$\phi(0, z, k) = z$$

Solving the euqations, we have

$$\phi(t, z, k) = \frac{e^{-\beta t - kvt}}{\frac{1}{z} - \frac{1 - e^{-\beta t - kvt}}{1 + \frac{vk}{\beta}}} \tag{2.14}$$

From the above derivation, we now state the limit distribution of $N(t)$ and $M(t)$ after normalization.

Theorem 2.8. *The joint distribuiton of* $(N(t)e^{-\beta t}, M(t)e^{-\beta t}) \xrightarrow[t\to\infty]{} (\xi, \frac{v}{\beta}\xi)$, *where* ξ *is an exponential random variable with parameter 1.*

Proof. let's consider the Laplace transform

$$E\left[e^{-k_1 N(t)e^{-\beta t} - k_2 M(t)e^{-\beta t}}\right] = E[(e^{-k_1 e^{-\beta t}})^{N(t)}e^{-(k_2 e^{-\beta t})M(t)}]$$

$$= \phi(t, e^{-k_1 e^{-\beta t}}, k_2 e^{-\beta t})$$

$$= \frac{e^{-\beta t - k_2 vte^{-\beta t}} + \frac{vk_2}{\beta}e^{-2\beta t - k_2 vte^{-\beta t}}}{e^{k_1 e^{-\beta t}} + \frac{vk_2}{\beta}e^{-\beta t + k_1 e^{-\beta t}} - 1 + e^{-\beta t - k_2 vte^{-\beta t}}}$$

$$\xrightarrow[t\to\infty]{} \frac{-\lambda}{-k_1\beta - k_2 v - \beta}$$

$$= \frac{1}{1 + k_1 + k_2\frac{v}{\beta}}$$

At the same time, the Laplace transform of $(\xi, \frac{v}{\beta}\xi)$ is

$$E\left[e^{-k_1\xi - k_2\frac{v}{\beta}\xi}\right] = E\left[e^{-(k_1 + k_2\frac{v}{\beta})\xi}\right] = \frac{1}{1 + k_1 + k_2\frac{v}{\beta}}$$

This gives the theorem. □

If we fixed total number of particle, What is the behavior of $M(t)$? The following two theorems answer the above question.

Theorem 2.9.

$$E\left[e^{-kM(t)}|N(t)=n\right] = e^{-kvt}\left(\frac{1-e^{-\beta t-kvt}}{(1-e^{-\beta t})\left(1+\frac{vk}{\beta}\right)}\right)^{n-1}$$

Proof. From (2.14)

$$\phi(t,z,k) = \frac{e^{-\beta t-kvt}}{\frac{1}{z}-\frac{1-e^{-\beta t-kvt}}{1+\frac{vk}{\beta}}} = \sum_{n=1}^{\infty}e^{-\beta t-kvt}\left(\frac{1-e^{-\beta t-kvt}}{1+\frac{vk}{\beta}}\right)^{n-1}z^n$$

From the total probability formula:

$$\phi(t,z,k) = Ez^{N(t)}e^{-kM(t)} = \sum_{n=1}^{\infty}P(N(t)=n)E\left[e^{-kM(t)}|N(t)=n\right]z^n$$

together with the fact

$$P(N(t)=n) = e^{-\beta t}(1-e^{-\beta t})^{n-1}$$

we have

$$E[e^{-kM(t)}|N(t)=n] = e^{-kvt}\left(\frac{1-e^{-\beta t-kvt}}{(1-e^{-\beta t})\left(1+\frac{vk}{\beta}\right)}\right)^{n-1} \qquad (2.15)$$

□

Theorem 2.10. *Given that $N(t)=n$,*

$$\frac{M(t)-\frac{v}{\beta}n}{\frac{v}{\beta}\sqrt{n}}|N(t)=n \to N(0,1)$$

Proof. since $n(t)=\xi e^{\beta t}$

$$E\left[e^{-k\left(\frac{M(t)-\frac{v}{\beta}n}{\frac{v}{\beta}\sqrt{n}}\right)}\right]\Big|_{N(t)=\xi e^{\beta t}} = e^{k\sqrt{\xi e^{\beta t}}}e^{\frac{-k\beta t}{\sqrt{\xi e^{\beta t}}}}\left[\frac{1-e^{-\beta t-\frac{k\beta t}{\sqrt{\xi e^{\beta t}}}}}{(1+\frac{k}{\sqrt{\xi e^{\beta t}}})(1-e^{-\beta t})}\right]^{\xi e^{\beta t}}$$

$$= e^{k\sqrt{\xi e^{\beta t}}-\frac{k\beta t}{\sqrt{\xi e^{\beta t}}}+\xi e^{\beta t}\left[\ln(1-e^{-\beta t-\frac{k\beta t}{\sqrt{\xi e^{\beta t}}}})-\ln(1-e^{-\beta t})-\ln(1+\frac{k}{\sqrt{\xi e^{\beta t}}})\right]}$$

Using Taylor expansion $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ and $\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$

gives $e^{\frac{k^2}{2}}$ as $t \to \infty$. $\square$

CHAPTER 3: CONTINOUS CONTACT PROCESS

3.1   Introduction

The contact model is one of the simplest ones in the theory of interacting Particle systems. The contact model on the lattice was first constructed and studied by T. E. Harris [Harris, 1974] and its name is due to the interpretation as a model for infection spreading. Lattice contact models have very reach properties and some essential applications. See, e.g. T. M. Liggett [Liggett, 1985], [Liggett, 1999].

In 2006, The continuous contact model was first proposed by Yu. Kondratiev and A. Skorokhod [Kondratiev and Skorokhod, 2006]. Under certain general assumptions on the infection spreading characteristics in the Euclidean space $\mathbb{R}^d$, they introduce the continuous contact process as a spatial birth-and-death process in the configuration space of the system. Series of works have been done by Yu. Kondratiev and his group, see especially [Kondratiev et al., 2008]. For the continuous contact process, we are especially interested in the critical case because the existence of the invariant measure. For other cases the limiting invariant state does not exist[Kondratiev et al., 2008]. Meanwhile, the dimensionality also plays an important role. For $d \geq 3$, we will prove that the limiting distribution exists if the second moment of the underlying random walk is finite. However, for $d = 1$ or $d = 2$, the second moment diverges. To avoid the divergence, we can assume the underlying random walk to allow for long jumps. Such assumption leads to the main models here. Detailed analysis of the discrete and continuous models can be found in [Feng et al., 2010].

From a biological viewpoint it is a natural model of a "forest": there is no motion of particles in space, but each "tree" can produce a new "seed," and the seeds are

$$p\{\xi \in (z, z+dz)\} = a(z)dz$$
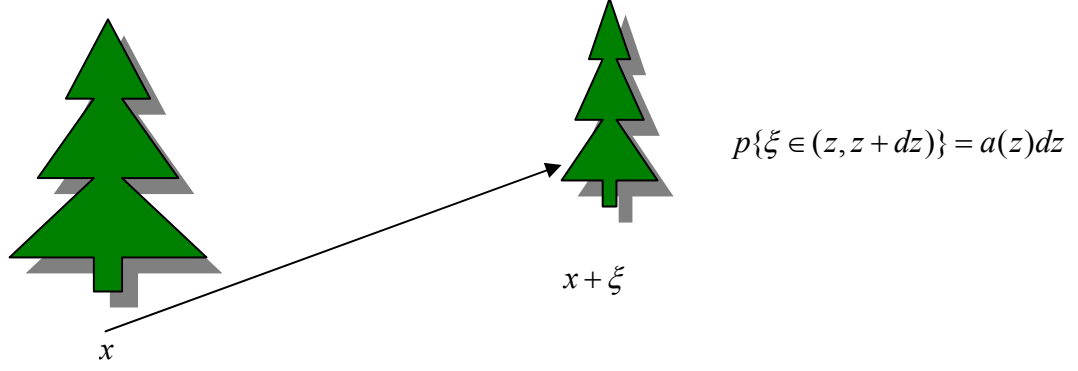
$$x + \xi$$

$$x$$

Figure 3.1: Tree

randomly distributed around the tree.

3.2   Contact processes in $\mathbb{R}^d$, $d \geq 1$

Assume that the initial field of trees has a Poissonian structure with density ρ_0, i.e., $\forall (\Gamma \subset \mathcal{B}(\mathbb{R}^d), m(\Gamma) := |\Gamma| < \infty$ we have

$$P\{n(0, \Gamma) = k\} = e^{-\lambda(\Gamma)} \frac{(\lambda(\Gamma))^k}{k!}, k \geq 0,$$

$$\lambda(\Gamma) = |\Gamma|\rho_0.$$

During the time dt each tree can either die with probability μdt or produce a single seed with probability βdt. If $x \in \mathbb{R}^d$ was the location of this tree the seed jumps to the point $x + z \in \mathbb{R}^d$ with density $a(z)$. We will assume that $a(z) = a(-z)$ (symmetry), $a(\cdot) \in C(\mathbb{R}^d) \cap L^1(\mathbb{R}^2)$, and $\int_{\mathbb{R}^d} a(x)\, dx = 1$. Let $\mu = \beta$ (criticality), the rate of the transformation (either death or birth of a seed) be equal to 2β and at the moment of transformation τ $(P\{\tau > s\} = e^{-2\beta s})$ each of two possibilities (death or the birth of a seed) have probability $\frac{1}{2}$.

Let us note (in contrast to the $\mathbb{Z}^d$ case) that the field $n(t, x)$ has multiplicity one

and the correlation function $k_t^{(n)}(x_1, \cdots, x_n)$ has the sense of the densities:

$$k_t^{(n)}(x_1, \cdots, x_n)dx_1 \cdots dx_n = P\{to\ find\ at\ the\ moment\ t \geq 0\ single\ particles$$

$$inside\ the\ sets\ x_1 + dx_1, \cdots, x_n + dx\}, n \geq 1$$

The first two moments have the following form:

$$\frac{\partial k_t^{(1)}(x)}{\partial t} = -\beta k_t^{(1)}(x) + \beta \int_{\mathbb{R}^d} a(x - z)k_t^{(1)}(z)\, dz$$

$$k_0^{(1)}(x) = \rho_0 \tag{3.1}$$

and

$$\begin{aligned}
\frac{\partial k_t^{(2)}(x_1, x_2)}{\partial t} &= -2\beta k_t^{(2)}(x_1, x_2) + \beta k_t^{(1)}(x_1)a(x_1 - x_2) + \beta k_t^{(1)}(x_2)a(x_2 - x_1) \\
&+ \beta \int_{\mathbb{R}^d} a(x_1 - z)k_t^{(2)}(x_2, z)\, dz + \beta \int_{\mathbb{R}^d} a(x_2 - z)k_t^{(2)}(x_1, z)\, dz \\
k_0^{(2)}(x_1, x_2) &= \rho_0^2 \tag{3.2}
\end{aligned}$$

Section 3.3 will contain the asymptotic analysis of the moments $k_t^{(l)}(\cdot)$, $l \geq 1$, $t \to \infty$. We will prove that in dimension $d \geq 3$ there exists a limiting distribution for $n(t, \cdot)$, $t \to \infty$ without any conditions on the moments of the density $a(z)$ (in contrast to [Kondratiev et al., 2008] where the limiting distribution was justified under the conditions $\int_{\mathbb{R}^d}|z|^2 a(z)dz < \infty$, $d \geq 3$). In dimensions $d = 1, 2$ ($d = 2$ is the most important case for biological applications), we will prove the existence of a limiting distribution for $n(t, \cdot)$, $t \to \infty$ under some regularity conditions on $a(z)$, $|z| \to \infty$.

Roughly speaking, we assume that for $d = 2$ the density $a(z)$ belongs to the domain of attraction of a symmetric stable distribution with parameter $0 < \alpha < 2$ (let us stress that symmetric for $d \geq 2$ does not mean isotropic). For $d = 1$ the density $a(\cdot)$ must be from the domain of attraction of a symmetric stable law with parameter $0 < \alpha < 1$.

As for the discrete case, we also derive the asymptotic for the variance of trees in a

region, with an eye toward establishing a central limit theorem, although we do not do that here.

We will not present, the analysis of the moments (correlation functions) of order $n \geq 3$, but will concentrate only on the second moment (using Fourier analysis, which will be the main tool for the highter moments as well). It will be proven that for $d \geq 3$ and *any spatial dynamics* or for $d = 2$ and *heavy tails spatial dynamics* the density of the second correlation function $k^{(2)}(t, x_1, x_2)$ has a nontrivial limit $k^{(2)}(\infty, x_1, x_2)$, $t \to \infty$. Together with the conservation of the first moment (density), $k^{(1)}(t, x) \equiv \rho_0$, it will provide the fundamental fact of *tightness* for the finite dimensional distributions of the point field $n(t, \cdot)$. In any limit theorem about the ergodicity (existence of the limit distribution) for the Markov process (in our case these processes have infinite dimensional phase space: the space of the locally finite configurations $\{x_i, i = 1, 2, ...\}$ of the particles) the proof of tightness is the first and most important step.

The complete proof of the existence of the limiting distribution (the convergence of the field $n(t, \cdot)$ to a statistical equilibrium) will be published later.

3.3   Correlation function and Fourier analysis

This model was introduced by Yu. Kondratiev and their work also contains existence theorems for the limiting distribution in dimensions $d \geq 3$.

Consider the initial Poissonian field in $\mathbb{R}^d$. The death rate of the particles is equal to the birth rate β. At the moment τ: $p\{\tau > s\} = e^{-2\beta s}$, a particle either dies with probability $\frac{1}{2}$ or produces a new seed, which is randomly distributed with density $a(z), z \in \mathbb{R}^d$. Assume that $a(z) = a(-z)$ and

$$\int_{\mathbb{R}^d} a(z) \, dz = 1.$$

This is a simplified model of a forest. After a birth at the point x, the tree remains in x but the seed produces a new tree at $x + \xi \in \mathbb{R}^d$. The central object in this model

is the set of correlation functions:

$$k_t^{(n)}(x_1, \cdots, x_n)dx_1 \cdots dx_n = P\{to\ find\ at\ the\ moment\ t \geq 0\ single\ particles$$

$$inside\ the\ sets\ x_1 + dx_1, \cdots, x_n + dx\}, n \geq 1$$

The derivation of the differential equations for $k_t^{(n)}(x_1, \cdots, x_n)$ based on the standard technique:balance of the probabilities in the infinitesimal initial time interval $(0, t+dt)$. Namely, let's split the interval $[0, t+dt]$ into two parts $[0, t] \cup [t, t+dt]$. During $[t, t+dt]$, we observe one of the following hypothesis:

- nothing happen with probability $1 - \beta n dt$

- birth of a particle from inside cloud with probability $\beta a(x_i - x_j)dt$

- birth of a particle from outside cloud with probability $\beta a(x_i - z)dt$

Now one can apply the full expectation formula:

$$
\begin{aligned}
k_{t+dt}^{(n)}(x_1, \cdots, x_n) &= (1 - \beta n dt)k_t^{(n)}(x_1, \cdots, x_n) \\
&+ \sum_{i=1}^{n} k_t^{(n-1)}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \sum_{j:j \neq i} \beta a(x_i - x_j) \\
&+ \sum_{i=1}^{n} k_t^{(n)}(x_1, \cdots, x_{i-1}, z, x_{i+1}, \cdots, x_n) \int_{\mathbb{R}^n} \beta a(x_i - z)
\end{aligned}
$$

Rewrite the above expression, we have

$$
\begin{aligned}
\frac{\partial k_t^{(n)}(x_1, \cdots, x_n)}{\partial t} &= -\beta n k_t^{(n)}(x_1, \cdots, x_n) \\
&+ \beta \sum_{i=1}^{n} \sum_{j:j \neq i} a(x_i - x_j)k_t^{(n-1)}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \\
&+ \beta \sum_{i=1}^{n} \int_{\mathbb{R}^n} a(x_i - z)k_t^{(n)}(x_1, \cdots, x_{i-1}, z, x_{i+1}, \cdots, x_n)\ (3.3)
\end{aligned}
$$

To analyze the correlation equation, let us apply a Fourier transform to the correlation function and define $\hat{k}^{(n)}(\varphi_1, \varphi_2, \cdots, \varphi_n) := \lim_{t \to \infty} \hat{k}_t^{(n)}(\varphi_1, \varphi_2, \cdots, \varphi_n)$.

When $n = 1$, it is trival that

$$k^{(1)}(x) = k_t^{(1)}(x) = \rho_0.$$

When $n = 2$, apply a Fourier transform to equation (3.2). Then:

$$\frac{\partial \hat{k}_t^{(2)}(\varphi_1, \varphi_2)}{\partial t} = -(2 - \hat{a}(\varphi_1) - \hat{a}(\varphi_2))\beta \hat{k}_t^{(2)}(\varphi_1, \varphi_2)$$

$$+(2\pi)^d \rho_0(\hat{a}(\varphi_1) + \hat{a}(\varphi_2))\delta_0(\varphi_1 + \varphi_2)$$

$$\hat{k}_0^{(2)}(\varphi_1, \varphi_2) = (2\pi)^{2d} \rho_0^2 \delta_0(\varphi_1)\delta_0(\varphi_2) \tag{3.4}$$

The solution of this equation is:

$$\hat{k}_t^{(2)}(\varphi_1, \varphi_2) = (2\pi)^{2d} \rho_0^2 \delta_0(\varphi_1)\delta_0(\varphi_2) e^{-\beta(2 - \hat{a}(\varphi_1) - \hat{a}(\varphi_2))t}$$

$$+(2\pi)^d \rho_0 \frac{(\hat{a}(\varphi_1) + \hat{a}(\varphi_2))\delta_0(\varphi_1 + \varphi_2)}{2 - \hat{a}(\varphi_1) - \hat{a}(\varphi_2)}(1 - e^{-\beta(2 - \hat{a}(\varphi_1) - \hat{a}(\varphi_2))t})$$

Letting $t \to \infty$, we obtain:

$$\hat{k}^{(2)}(\varphi_1, \varphi_2) = \lim_{t \to \infty} \hat{k}_t^{(2)}(\varphi_1, \varphi_2) = (2\pi)^{2d} \rho_0^2 \delta_0(\varphi_1)\delta_0(\varphi_2) + (2\pi)^d \rho_0 \frac{\hat{a}(\varphi_1)\delta_0(\varphi_1 + \varphi_2)}{1 - \hat{a}(\varphi_1)}.$$

Thus, if the inverse Fourier transform $k^{(2)}(x_1, x_2)$ exists, it should take the form:

$$k^{(2)}(x_1, x_2) = \lim_{t \to \infty} k_t^{(2)}(x_1, x_2) = \rho_0^2 + \frac{\rho_0}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\varphi_1(x_2 - x_1)} \frac{\hat{a}(\varphi_1)}{1 - \hat{a}(\varphi_1)} \, d\varphi_1.$$

When $n = 3$, similarly, the limiting correlation function has the form:

$$\hat{k}^{(3)}(\varphi_1, \varphi_2, \varphi_3) = (2\pi)^{3d} \rho_0^3 \delta_0(\varphi_1)\delta_0(\varphi_2)\delta_0(\varphi_3)$$

$$+(2\pi)^{2d} \rho_0^2 (\underbrace{\frac{(\hat{a}(\varphi_1) + \hat{a}(\varphi_2))}{2 - \hat{a}(\varphi_1) - \hat{a}(\varphi_2)}} \delta_0(\varphi_1 + \varphi_2)\delta_0(\varphi_3) + \textit{two similar terms})$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\textit{cluster containing } (x_1, x_2), \ x_3 \textit{ is separated}}$$

$$+(2\pi)^d \rho_0(\frac{(\hat{a}(\varphi_1) + \hat{a}(\varphi_2))}{3 - \hat{a}(\varphi_1) - \hat{a}(\varphi_2) - \hat{a}(\varphi_3)} \frac{\hat{a}(\varphi_1 + \varphi_2)\delta_0(\varphi_1 + \varphi_2 + \varphi_3)}{2 - \hat{a}(\varphi_1 + \varphi_2) - \hat{a}(\varphi_3)}$$

$$+\textit{two similar terms}).$$

The Fourier transform of the limiting correlation function has a cluster structure.

The general formula, although a bit complicated, is clear from the presentation of the first three moments.

Theorem 3.1. *The Fourier transform of the limiting correlation function is a solution to the recurrent system of equations:*

$$\hat{k}^{(n)}(\varphi_1, \cdots, \varphi_n) = (2\pi)^{nd} \rho_0^n \delta_0(\varphi_1) \cdots \delta_0(\varphi_n)$$
$$+ \sum_{j=1}^{n} \sum_{\substack{i \neq j \\ i_1 < i_2 < \cdots < i_{n-3}, \neq i, j}} \frac{(\hat{a}(\varphi_i) + \hat{a}(\varphi_j)) \hat{k}^{(n-1)}(\varphi_i + \varphi_j, \varphi_{i_1} \cdots, \varphi_{i_{n-3}})}{n - \hat{a}(\varphi_1) - \cdots - \hat{a}(\varphi_n)}$$

$$(3.5)$$

Proof. Apply a Fourier transform to the correlation function $k_t^{(n)}(x_1, \cdots, x_n)$. Then, from equation (3.3), we have:

$$\frac{\partial \hat{k}_t^{(n)}(\varphi_1, \cdots, \varphi_n)}{\partial t} = -(n - \hat{a}(\varphi_1) - \cdots - \hat{a}(\varphi_n)) \beta \hat{k}_t^{(n)}(\varphi_1, \cdots, \varphi_n)$$
$$+ \beta \sum_{j=1}^{n} \sum_{\substack{i \neq j \\ i_1 < i_2 < \cdots < i_{n-3}, \neq i, j}} (\hat{a}(\varphi_i) + \hat{a}(\varphi_j)) \hat{k}_t^{(n-1)}(\varphi_i + \varphi_j, \varphi_{i_1} \cdots, \varphi_{i_{n-3}})$$

By solving this equation with the initial conditions and letting $t \to \infty$, we obtain the recurrent formula. □

Remark 3.1. *The Fourier transform of the limiting correlation function $\hat{k}^{(n)}(\varphi_1, \cdots, \varphi_n)$ has a cluster structure, similar to the expansion of the resolvent of the multiparticle Schrödinger operator. It contains:*

- *n individual 1-clusters, given by the term:*

$$(2\pi)^{nd} \rho_0^n \delta_0(\varphi_1) \cdots \delta_0(\varphi_n)$$

- *one 2 cluster and $(n - 2)$ 1-clusters, with the typical term:*

$$(2\pi)^{(n-1)d} \rho_0^{n-1} \frac{\hat{a}(\varphi_1) + \hat{a}(\varphi_2)}{2 - \hat{a}(\varphi_1) - \hat{a}(\varphi_2)} \delta_0(\varphi_1 + \varphi_2) \delta_0(\varphi_3) \cdots \delta_0(\varphi_n)$$

- *one 3 cluster and $(n-3)$ 1-clusters with the typical term:*

$$(2\pi)^{(n-2)d}\rho_0^{n-2}\frac{\hat{a}(\varphi_1)+\hat{a}(\varphi_2)}{3-\hat{a}(\varphi_1)-\hat{a}(\varphi_2)-\hat{a}(\varphi_3)}\frac{\hat{a}(\varphi_1+\varphi_2)+\hat{a}(\varphi_3)}{2-\hat{a}(\varphi_1+\varphi_2)-\hat{a}(\varphi_3)}\cdot$$

$$\delta_0(\varphi_1+\varphi_2+\varphi_3)\delta_0(\varphi_4)\cdots\delta_0(\varphi_n)$$

$$+(2\pi)^{(n-2)d}\rho_0^{n-2}\frac{\hat{a}(\varphi_1)+\hat{a}(\varphi_3)}{3-\hat{a}(\varphi_1)-\hat{a}(\varphi_2)-\hat{a}(\varphi_3)}\frac{\hat{a}(\varphi_1+\varphi_3)+\hat{a}(\varphi_2)}{2-\hat{a}(\varphi_1+\varphi_3)-\hat{a}(\varphi_2)}\cdot$$

$$\delta_0(\varphi_1+\varphi_2+\varphi_3)\delta_0(\varphi_4)\cdots\delta_0(\varphi_n)$$

$$+(2\pi)^{(n-2)d}\rho_0^{n-2}\frac{\hat{a}(\varphi_2)+\hat{a}(\varphi_3)}{3-\hat{a}(\varphi_1)-\hat{a}(\varphi_2)-\hat{a}(\varphi_3)}\frac{\hat{a}(\varphi_2+\varphi_3)+\hat{a}(\varphi_1)}{2-\hat{a}(\varphi_2+\varphi_3)-\hat{a}(\varphi_1)}\cdot$$

$$\delta_0(\varphi_1+\varphi_2+\varphi_3)\delta_0(\varphi_4)\cdots\delta_0(\varphi_n)$$

and so forth.

Example 3.1. *Assume that there are n particles grouped as $(x_1, x_2, x_3), (x_4, x_5),$ $x_6, \cdots, x_n$, then the typical expression will be:*

$$(2\pi)^{(n-3)d}\rho_0^{n-3}\frac{\hat{a}(\varphi_1)+\hat{a}(\varphi_2)}{5-\hat{a}(\varphi_1)-\hat{a}(\varphi_2)-\hat{a}(\varphi_3)-\hat{a}(\varphi_4)-\hat{a}(\varphi_5)}\cdot$$

$$\frac{\hat{a}(\varphi_1+\varphi_2)+\hat{a}(\varphi_3)}{4-\hat{a}(\varphi_1+\varphi_2)-\hat{a}(\varphi_3)-\hat{a}(\varphi_4)-\hat{a}(\varphi_5)}\cdot\frac{\hat{a}(\varphi_4)+\hat{a}(\varphi_5)}{3-\hat{a}(\varphi_1+\varphi_2+\varphi_3)-\hat{a}(\varphi_4)-\hat{a}(\varphi_5)}\cdot$$

$$\delta_0(\varphi_1+\varphi_2+\varphi_3)\delta_0(\varphi_4+\varphi_5)\delta_0(\varphi_6)\cdots\delta_0(\varphi_n)+\ similar\ terms.$$

Lemma 3.1. *For $d \geq 3$, let $0 \leq a(z) \in L^1(\mathbb{R}^d)$ be an arbitrary even function such that $\int_{\mathbb{R}^d} a(x)\,dx = 1$. Then, the Fourier transform of $a(x)$, which is $\hat{a}(\varphi) = \int_{\mathbb{R}^d} e^{-i(\varphi,x)}a(z)\,dz$, satisfies the following estimate:*

$$|1 - \hat{a}(\varphi)| = O(\varphi^2),\ as\ |\varphi| \to 0. \tag{3.6}$$

Lemma 3.2. *For $d=2$, suppose for the contact process that $a(z)$ has the spatial distribution*

$$a(z) = \frac{h(\theta)}{|z|^{2+\alpha}}\left(1 + O\left(\frac{1}{|z|^2}\right)\right),\ z \neq 0$$

with $0 < \alpha < 2$, $\theta = \arg\frac{z}{|z|} \in [-\pi, \pi) = T^1$, $h_1, h_2 \in C^2(T^1)$, $h > 0$ and so satisfies

the heavy tails assumption. Then, as $|\varphi| \to 0$, $1 - \hat{a}(\varphi) = O(|\varphi|^\alpha)$.

Proof.

$$1 - \hat{a}(\varphi) = 1 - \int_{\mathbb{R}^2} e^{-i(\varphi, x)} a(x)\, dx = \int_{\mathbb{R}^2} (1 - \cos(\varphi, x)) a(x)\, dx.$$

If $\vec{x} = (x_1, x_2) = r(\cos\theta, \sin\theta)$, $\varphi = |\varphi|(\cos\gamma, \sin\gamma)$, then

$$
\begin{aligned}
1 - \hat{a}(\varphi) &= \int_{|\vec{x}| \geq 0}^{\infty} \frac{h(\theta)}{|\vec{x}|^{2+\alpha}} (1 - \cos(\varphi, \vec{x})) d\vec{x} + O(|\varphi|^2) \\
&= \int_0^{\infty} \frac{dr \cdot r}{r^{2+\alpha}} \int_{-\pi}^{\pi} h(\theta)(1 - \cos(r|\varphi| \cdot |\cos(\theta - \gamma)|)) d\theta + O(|\varphi|^2) \\
&= \int_{-\pi}^{\pi} h(\theta) \int_0^{\infty} \frac{dr}{r^{1+\alpha}} (1 - \cos(r|\varphi| \cdot |\cos(\theta - \gamma)|)) d\theta + O(|\varphi|^2).
\end{aligned}
$$

Using the substitution $t = r|\varphi| \cdot |\cos(\theta - \gamma)|$ we obtain

$$1 - \hat{a}(\varphi) = O(|\varphi|^2) + |\varphi|^\alpha \int_{-\pi}^{\pi} h(\theta)|\cos(\theta - \gamma)|^\alpha d\theta \cdot \int_0^{\infty} \frac{1 - \cos t}{t^{1+\alpha}} dt.$$

Set $c_\alpha := \int_0^{\infty} \frac{1-\cos t}{t^{1+\alpha}} dt$, $\mathcal{H}(\gamma) := \int_{-\pi}^{\pi} h(\theta)|\cos(\theta - \gamma)|^\alpha d\theta$, $\gamma = \arg\varphi$, $\mathcal{H}(\gamma) \in C(T^1)$, $\mathcal{H}(\gamma) > 0$. Then:

$$1 - \hat{a}(\varphi) = c_\alpha |\varphi|^\alpha \mathcal{H}(\gamma) + O(|\varphi|^2), \quad |\varphi| \to 0.$$

$\square$

Theorem 3.2. *Let $a(z) > 0$ be an arbitrary even continous function such that $\int_{\mathbb{R}^d} a(z)dz = 1$ and $\hat{a}(\varphi) = \int_{\mathbb{R}^d} e^{-i\varphi x} a(x)dx \in \mathbb{L}^1(\mathbb{R}^d)$.*

- *For $d \geq 3$, there exists a limiting distribution $k^{(n)}(x_1, \cdots, x_n)$ such that:*

$$k_t^{(n)}(x_1, \cdots, x_n) \xrightarrow[t \to \infty]{} k^{(n)}(x_1, \cdots, x_n).$$

- *For $d = 2$, assume $a(z) \sim \frac{h(\theta)}{\|z\|^{2+\alpha}}$, with $0 < \alpha < 2$, $\theta = \arg \frac{z}{\|z\|} \in [-\pi, \pi) =$*

$T^1, h \in C^2(T^1), h > 0$. *Then, there exists a limiting distribution* $k^{(n)}(x_1, \cdots, x_n)$ *such that:*

$$k_t^{(n)}(x_1, \cdots, x_n) \xrightarrow[t \to \infty]{} k^{(n)}(x_1, \cdots, x_n).$$

- *For* $d = 1$, *assume* $a(z) \sim \frac{h(\theta)}{\|z\|^{2+\alpha}}$, *with* $0 < \alpha < 1$, $\theta = \arg \frac{z}{\|z\|} \in [-\pi, \pi) = T^1, h \in C^2(T^1), h > 0$. *Then, there exists a limiting distribution* $k^{(n)}(x_1, \cdots, x_n)$ *such that*

$$k_t^{(n)}(x_1, \cdots, x_n) \xrightarrow[t \to \infty]{} k^{(n)}(x_1, \cdots, x_n).$$

Proof. In order to clarify the existence of the limiting distribution $k^{(n)}(x_1, \cdots, x_n)$, we should check

$$k^{(n)}(x_1, \cdots, x_n) = \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{i(\varphi_1 x_1 + \cdots + \varphi_n x_n)} \hat{k}^{(n)}(\varphi_1, \cdots, \varphi_n) \, d\varphi_1 \cdots d\varphi_n < \infty.$$

- For $d \geq 3$, Let us consider the case $n = 2$. As calculated earlier, $k^{(2)}(x_1, x_2)$ has the expression

$$k^{(2)}(x_1, x_2) = \rho_0^2 + \frac{\rho_0}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\varphi_1(x_2 - x_1)} \frac{\hat{a}(\varphi_1)}{1 - \hat{a}(\varphi_1)} \, d\varphi_1.$$

By lemma 3.1, $1 - \hat{a}(\varphi) = O(|\varphi|^2)$ as $\varphi \to 0$. If $d \geq 3$, then $\frac{\hat{a}(\varphi)}{1 - \hat{a}(\varphi)} = O(\frac{1}{|\varphi|^2})$ at $\varphi = 0$, i.e., $\frac{\hat{a}(\varphi)}{1 - \hat{a}(\varphi)} \in \mathbb{L}^1(\mathbb{R}^d)$. Therefore,

$$k^{(2)}(x_1, x_2) = \rho_0^2 + \frac{\rho_0}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\varphi_1(x_2 - x_1)} \frac{\hat{a}(\varphi_1)}{1 - \hat{a}(\varphi_1)} \, d\varphi_1 \in \mathbb{L}^\infty(\mathbb{R}^d).$$

For mathematical induction from $(n-1) \to n$, from [Kondratiev et al., 2008], we find that the central issue turns out to be the proof that $\frac{\hat{a}(\varphi)}{1 - \hat{a}(\varphi)} \in \mathbb{L}^1(\mathbb{R}^d)$, which we have just proved for $n = 2$.

- For $d = 2$, $\int_{\mathbb{R}^d} z^2 a(z) \, dz = \infty$, but if we add the extra assumption that $a(z) \sim \frac{h(\theta)}{\|z\|^{2+\alpha}}$, the integrability of $\frac{\hat{a}(\varphi)}{1 - \hat{a}(\varphi)}$ at $\varphi = 0$ still holds. By lemma 3.2, $1 - \hat{a}(\varphi) = O(|\varphi|^\alpha)$ as $\varphi \to 0$.

    If d=2 and $0 < \alpha < 2$, then $\frac{\hat{a}(\varphi)}{1 - \hat{a}(\varphi)} \in \mathbb{L}^1(\mathbb{R}^d)$.

- For $d = 1$, the proof is similar to $d = 2$.

$\square$

Theorem 3.3. *All terms in the cluster expansion of $k^{(l)}(x_1, \cdots, x_l)$ are positive. This is the manifestation of the weak FKG-property of the limiting field.*

Proof. The typical term in the cluster expansion is the product of the following three kinds of term:

- $\delta_0(\varphi_1 + \cdots + \varphi_k), k \leq d$; it is the Fourier transform of a positive constant.

- $\hat{a}(\varphi_1 + \cdots + \varphi_k), k \leq d$; it is the Fourier transform of $a(z)$, which is positive.

- $\frac{1}{k - (\hat{a}(\varphi_1) + \cdots + \hat{a}(\varphi_k))}$; this term can be expressed in the geometric form:

$$
\frac{1}{k - (\hat{a}(\varphi_1) + \cdots + \hat{a}(\varphi_k))} = \frac{1}{k} \frac{1}{1 - \frac{\hat{a}(\varphi_1) + \cdots + \hat{a}(\varphi_k)}{k}}
$$
$$
= \frac{1}{k} \left(1 + \left(\frac{\hat{a}(\varphi_1) + \cdots + \hat{a}(\varphi_k)}{k} \right) + \cdots + \left(\frac{\hat{a}(\varphi_1) + \cdots + \hat{a}(\varphi_k)}{k} \right)^n + \cdots \right)
$$
$$
= \sum_{n=0}^{\infty} \frac{1}{k^{n+1}} \sum_{i_1 + i_2 + \cdots + i_k = n} \binom{n}{i_1, i_2, \cdots, i_k} \hat{a}(\varphi_1)^{i_1} \hat{a}(\varphi_2)^{i_2} \cdots \hat{a}(\varphi_k)^{i_k}
$$

If $i_1 \neq 0$, then $\hat{a}(\varphi_1)^{i_1}$ is the Fourier transform of $\underbrace{a * a * \cdots * a}_{i_1 \, fold \; convolution}(z_1)$.

If $i_1, \cdots, i_k \neq 0, k \leq d$, then $\hat{a}(\varphi_1)^{i_1} \hat{a}(\varphi_2)^{i_2} \cdots \hat{a}(\varphi_k)^{i_k}$ is Fourier transform of the product of those convolutions, i,e.,

$$
(\underbrace{a * a * \cdots * a}_{i_1 \, fold \; convolution}(z_1)) \cdots (\underbrace{a * a * \cdots * a}_{i_k \, fold \; convolution}(z_k)),
$$

which is positive. Therefore, $\frac{1}{k - (\hat{a}(\varphi_1) + \cdots + \hat{a}(\varphi_k))}$ is also the Fourier transform of a positive measure. Hence, for the typical term in the cluster expansion $\frac{\hat{a}(\varphi_1 + \cdots + \varphi_k) \delta_0(\varphi_1 + \cdots + \varphi_l)}{k - (\hat{a}(\varphi_1) + \cdots + \hat{a}(\varphi_k))}$,

together with the locally integrability of the term, we conclude that the inverse Fourier transform of this term is positive. $\qquad\square$

3.4 The variance of the population for the continuous case

Finally, we prove a result for the variance of the number of particles in a region. Our result can be used to establish a central limit theorem for the contact process, although we do not do so in this paper.

Theorem 3.4. *Assume the contact process analyzed above. Let* $1-\hat{a}(\varphi) = \beta(|\varphi|)|\varphi|^{\alpha}$, *assume* $\frac{\hat{a}(\varphi)}{\beta(|\varphi|)}$ *is bounded, and* $\beta(0) = \lim_{|\varphi|\to 0} \dfrac{1-\hat{a}(\varphi)}{|\varphi|^{\alpha}}$. *If* Q_r *denotes a ball of radius* r *with the center at the origin, then as* r *increases, the variance of the number of particles in* Q_r *grows as* $r^{d+\alpha}$.

Proof. Let $n(Q_r)$ denote the number of particles in Q_r as $t \to \infty$, Then, the variance of $n(Q_r)$, which we again call V, is:

$$V := \text{Var}\,(n(Q)) = \int_{Q_r}\int_{Q_r}(k^{(2)}(x,y) - \rho_0^2)dxdy.$$

Because of the spatial invariance of $k^{(2)}$, we set $B(x-y) := k^{(2)}(x,y) - \rho_0^2$.

$$
\begin{aligned}
V \;:\; &= \iint_{Q_rQ_r} B(x-y)dxdy = \iint_{\mathbb{R}^d\mathbb{R}^d} I_{Q_r}(x)I_{Q_r}(y)B(x-y)dxdy \\
&= \iint_{\mathbb{R}^d\mathbb{R}^d} \frac{dxdy}{(2\pi)^{3d}} \iiint_{\mathbb{R}^d\mathbb{R}^d\mathbb{R}^d} \hat{I}_{Q_r}(\varphi_x)e^{-i(\varphi_x,x)}\hat{I}_{Q_r}(\varphi_y)e^{-i(\varphi_y,y)}\hat{B}(\varphi_z)e^{-i(\varphi_z,x-y)}d\varphi_x d\varphi_y d\varphi_z \\
&= \iint_{\mathbb{R}^d\mathbb{R}^d} \frac{dxdy}{(2\pi)^{3d}} \iiint_{\mathbb{R}^d\mathbb{R}^d\mathbb{R}^d} \hat{I}_{Q_r}(\varphi_x)\hat{I}_{Q_r}(\varphi_y)\hat{B}(\varphi_z)e^{-i(\varphi_x+\varphi_z,x)}e^{-i(\varphi_y-\varphi_z,y)}d\varphi_x d\varphi_y d\varphi_z \\
&= \frac{1}{(2\pi)^d} \iint_{\mathbb{R}^d\mathbb{R}^d}\int_{\mathbb{R}^d} \delta_0(\varphi_x+\varphi_z)\delta_0(\varphi_y-\varphi_z)\hat{I}_{Q_r}(\varphi_x)\hat{I}_{Q_r}(\varphi_y)\hat{B}(\varphi_z)d\varphi_x d\varphi_y d\varphi_z \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{I}_{Q_r}(-\varphi_z)\hat{I}_{Q_r}(\varphi_z)\hat{B}(\varphi_z)d\varphi_z.
\end{aligned}
$$

We can write the Fourier transform of the indicator function I as: :

$$\hat{I}_{Q_r}(\varphi) = \int_{Q_r} e^{i(\varphi,x)} dx = \left(\frac{2\pi r}{|\varphi|}\right)^{d/2} J_{d/2}(r|\varphi|),$$

where $J_{d/2}$ is the Bessel function of the first kind of order $\frac{d}{2}$ [Gikhman and Skorokhod, 1974]. Also including the result for $\hat{B}(\varphi) = \frac{\rho_0 \hat{a}(\varphi)}{1-\hat{a}(\varphi)}$ from section 3.3, we obtain:

$$V = \rho_0 r^d \int_{\mathbb{R}^d} \frac{1}{|\varphi|^d} \left(J_{d/2}(r|\varphi|)\right)^2 \frac{\hat{a}(\varphi)}{1-\hat{a}(\varphi)} d\varphi. \tag{3.7}$$

Together with the assumption $1 - \hat{a}(\varphi) = \beta(|\varphi|)|\varphi|^\alpha$, we now can write (3.7) in polar coordinates. This gives:

$$V = \rho_0 r^d \int_0^\infty \frac{c_d}{|\varphi|^{1+\alpha}} \left(J_{d/2}(r|\varphi|)\right)^2 \frac{\hat{a}(\varphi)}{\beta(|\varphi|)} d|\varphi|,$$

where $c_d = \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$.

We set $x := r|\varphi|$ to get:

$$V = \rho_0 c_d r^{d+\alpha} \int_0^\infty \frac{1}{x^{1+\alpha}} \left(J_{d/2}(x)\right)^2 \frac{\hat{a}(\frac{x}{r})}{\beta(\frac{x}{r})} dx.$$

Since $\frac{\hat{a}(\varphi)}{\beta(|\varphi|)}$ is bounded, and as $r \to \infty$ we have $\frac{\hat{a}(\varphi)}{\beta(|\varphi|)} \to \frac{1}{\beta(0)}$. Consequently, as $r \to \infty$,

$$V \to \frac{\rho_0 c_d}{\beta(0)} r^{d+\alpha} \int_0^\infty \frac{1}{x^{1+\alpha}} \left(J_{d/2}(x)\right)^2 dx.$$

Finally, for small x, $\left(J_{d/2}(x)\right)^2 \leq cx^d$ and as x becomes large, $\left(J_{d/2}(x)\right)^2 < \frac{1}{x}$, which means that $\int_0^\infty \frac{1}{x^{1+\alpha}} \left(J_{d/2}(x)\right)^2 dx$ converges, giving that:

$$V \xrightarrow[r \to \infty]{} c_0 r^{d+\alpha}$$

where $c_0 = \frac{\rho_0 c_d}{\beta(0)} \int_0^\infty \frac{1}{x^{1+\alpha}} \left(J_{d/2}(x)\right)^2 dx$, $c_d = \frac{d\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$. $\qquad\square$

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