GREEN'S FUNCTION-STOCHASTIC APPROACH TO SOLVING LINEAR, NONLINEAR AND NONHOMOGENEOUS EVOLUTION TRANSPORT PROBLEMS

by

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ABSTRACT

THOMAS DOWUONA NORTEY. Green's function-stochastic approach to solving linear, nonlinear and nonhomogeneous evolution transport problems.
(Under the direction of DR. RUSSELL KEANINI)

A general, analytical Green's function-stochastic approach for solving linear, nonlinear and non-homogeneous evolution and transport problems is presented.

Analysis of practical and natural engineering situations reveal the basic elements from which an understanding of such systems is best put together. Partial differential equations that evolve from such analysis sometimes become difficult to solve by classical approaches. While numerical solutions are often sought in some studies, Green's function methods often offer a powerful alternative.

Green's function in the case of linear models, provide integral solutions that depend strictly on known boundary conditions, initial conditions, and when necessary, space and/or time varying system forcing functions. The approach of Green's function best reveals the responses to the problems effected by Dirac delta functions. Green's function methods are especially useful when the system is subject to random boundary and/or initial conditions, and/or random forcing. Under these conditions, Green's function based solution describes exactly the random response of the modeled system. In addition, the Green's -based solution allows explicit calculation of the space-and possibly time-dependent mean response, as well as the space and time-dependent system variance about the mean evolves in space and time is, in turn typically

Problems of physics and engineering that reveal probabilistic characteristics or involve boundary conditions that are random in nature as well as possibly stochastic in-system forcing can be analysed by Green's function methods. For such problems, Green's function methods can be usefully combined with methods from theory of stochastic differential equations. Here, the time and space-dependent evo-

lution of the variables describing the system response is often modeled as a random walk/stochastic process, the evolution of which is governed by advection-diffusion stochastic differential equation. The probabilstic decription of how the stochastic process, evolves in space and time is, in typically embedded in the transition probability density, p, which gives the (conditional) probability of observing the stochastic process/random system response, at some specified position and time, given its position at an earlier time. Importantly and as detailed and exploited in this dissertation, the equation governing the evolution of the transition density, p - the Chapmann-Kolmogorov equation - corresponds exactly, under fairly general conditions, to the Green's function describing the system's response. Thus, the important problem of modeling the random response of linear systems subject to random forcing can be powerfully tackled by initially determining the system's Green's function, and explicitly identifying the system variable describing system response as stochastic process /random walker. Using this recipe provides scientists and engineers with a near-complete, rigorous, physics-based, probabilistic picture of how the system evolves under random forcing and /or random boundary and initial conditions.

This dissertation first analysed the flow problem from continuum approach employing conservation principles of mass and momentum. Such analysis led to a pure diffusion problem. The diffusion problem is then solved for a semi-infinite and finite medium with moving boundary. The Green's function method is then employed to solve the diffusion equation but this time with time - dependent boundary motions. Some useful Green's function results were obtained. The Chapmann-Kolmogorov equation is then derived and simplified to the Fokker- Planck equation for application to randomly-forced incompressible flow problems. Finally, two simple example flow problems, the random response of a semi-infinite fluid layer, and the random response of a finite layer, both driven by the random boundary motion are considered.

DEDICATION

This dissertation is dedicated to my wife Julienne and my children, Gabrielle, Joel and Jeremy.

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CHAPTER 1: INTRODUCTION

The analysis of flow fields initiated by moving boundaries have long been of interest to physicists and engineers. Many experimental and numerical techniques have been used to investigate the flow field initiated by moving boundaries. A variety of solutions are known for laminar flow with moving boundaries. The moving boundary may be part of the domain for which the flow field is being solved. It may be semi-infinite with a moving boundary or a finite domain between two long infinite plates with one of the plates given an initial velocity. Stokes was one of the earlier scientists who investigated flow initiated by a moving boundary in a semi-infinite medium. Couette also investigated wall-driven flow in which the domain is finite between boundaries, one stationary and the other moving with a velocity. In Couette's flow the domain may be between linear boundaries or two cylindrical boundaries. In Stokes investigations , the boundary velocities were one with uniform motion and the an oscillatory motion. In both cases, the domain describes a semi-infinite domain with fluid initially at rest and bounded below by a solid plane.

The flow field developed varies according to the function describing the motion of the boundary that initiated the flow. The boundary motion could be uniform or vary with time. Solutions already exist in literature for diffusion equations with uniform boundary motion condition. Classical methods of solving partial differential equations could be used to solve the resulting partial differential equations deduced from the continuity and momentum Navier Stokes equations applied to the flow filed with uniform boundary motion and initial conditions. When the partial differential diffusion equation governing the flow field has a time-dependent source term and time-dependent boundary conditions, the classical approach could not be used to solve the flow field. The Duhamel theory or Green's function method may have to be adopted.

Duhamel's principle relates the construction of the solution of inhomogeneous equations to one involving a homogeneous equation. The method is valid for initial and boundary value problems for hyperbolic and parabolic equations. We consider the parabolic equation of form:

$$\rho(x)\ U_x(x,t) + L[U(x,t)] = g(x,t) \qquad (1-1)$$

where

$$L[U] = -\frac{\partial}{\partial x}(\rho(x)\frac{\partial u}{\partial x}) + q(x)u$$

$\rho(x) > 0$ and $g(x,t)$ a given forcing or source term. We assume $U(x,t)$ satisfies homogeneous initial conditions at t =0. For the initial and boundary value problem of $(1-1)$ in a boundary region C, we again assume that homogeneous boundary condition $U(x,t) = 0$. Duhamel's principle proceeds as follows. Consider a homogeneous version of $(1-1)$, that is :

$$\rho(x)V_t(x,t) + L[v(x,t)] = 0 \qquad (1-2)$$

for the function $V(x,t)$, which is assumed to satisfy the same boundary conditions (if any are given) as $U(x,t)$. It is assumed that the problem above for $V(x,t)$ can be solved by the separation of variables for the initial and boundary value problem. The solution depends on the parameter τ (ie initial initial time), so we write it as $v(x,t;\tau)$.

Duhamel's principle states that the solution $v(x,t)$ of the given inhomogeneous problem is :

$$U(x,t) \ = \int_0^t v(x,t;\tau)d\tau \qquad (1-3)$$

A motivation for the method is obtained by noting that the effect of the term $g(x,t)$ can be characterized as resulting from a superposition of impulses at times, $t=\tau$ over the time span $0 \leq \tau \leq t$

This section of the chapter describes to a short extent what Green's function about. Green's function can be considered as an integral kernel which can be used to solve boundary value problems. It can be used to solve both ordinary and partial differential equations of physics and engineering. Green's function provides means to describe the response to an arbitrary differential equation with or without a source term. The equation may have boundary conditions but may or may not have initial condition. Partial differential equations of physics and engineering namely, Poisson's equation, the diffusion equation and the wave equation with boundary conditions that make it difficult to solve by classical approach could be solved by Green's function approach. Data prescribed at one point contribute to the solution at other spatial remote points. Analysis reveals the basic elements from which an understanding of such systems is best put together, namely point sources described by Dirac delta functions, and the Green's function which describes their effects. Most partial differential equations of physics and engineering can be tackled by Green's function techniques which bring out their similarities and at the same time highlight their significant differences. Green's function encourages a spontaneously efficient common approach to propagation from prescribed initial conditions, volume distributed sources, surface distributed sources and line distributed sources or boundary data. Even with time-independent equations, data prescribed at one point contribute to the solution at other remote points. The usefulness of Green's function lies in the fact that the solution of the original problem can be represented only in terms of Green's function. Once the Green's function is known, the response distribution in the medium is readily computed. Green's function has close ties to the transition probability distribution function of a stochastic field. Derived representations for the generating function of Green's function of stochastic problems are stated either with the use of derived diffusion equation in different forms. Solutions of a second-order stochastic differential equation in the framework of the stochastic field theory is constructed. The combined Green's function-stochastic approach leads to determination of explicit integral formulas for computing probability density functions and constituents expectations. Once these are obtained, then the nonlinear and non-homogeneous problems are tackled by limiting integrals to small time steps. The probability density functions become descriptive solutions to stochastic flow fields and can be used to obtain the moment generating functions which are the properties of each flow. The Green's function-stochastic method is applied to flows with a moving boundaries. The resulted flow field is due to the type of domain, the initial condition and the function governing the motion of each boundary.

1.1 Dirac Delta Function:

Dirac delta function is a means by which, point-sources, its application and the responses generated are represented. Intuitively, the function $\delta(x)$ is defined to be zero, when $x \neq 0$ and is infinite at $x = 0$ in such a way that the area under it is unity. To express this we write:

$$\delta(x) = 0, \; if x \neq 0 \quad and \int_{-\eta_1}^{\eta_2} dx \; \delta(x) = 1 \qquad (1-4)$$

The integral in (1-11) could be written as :

$$\int_{-\infty}^{\infty} dx \; \delta(x) = 1 \qquad (1-5)$$

If we reverse the limits on the integral :

$$\int_{\eta_2}^{-\eta_1} dx\ \delta(x) = -\int_{-\eta_1}^{\eta_2} dx\ \delta(x) = -1 \quad (1-6)$$

If f(x) is a well behaved function, continuous and differentiable as often as required, then

$$\int_{-\eta_1}^{\eta_2} dx\ \delta(x) f(x) = f(0) \int_{-\eta_1}^{\eta_2} dx\ \delta(x) = f(0) \quad (1-7)$$

f(x) is called a test function. The equality in equation (1-14) holds because $\delta(x \neq 0) = 0$ entails that there is no contribution to the integral from anywhere where $f(x) \neq f(0)$. Consequently f(x) may be replaced by f(0). The delta-function and its derivatives make sense only if multiplied by a sufficiently well- behaved test function and then integrated over some finitely wide range of x. Infact despite its name, $\delta(x)$ is not really a function at all, it is more properly described as a "distribution" or a generalized function. Often it is helpful to think of it as a "functional". While an ordinary function f maps numbers x unto number f(x), a functional maps any ordinary (test) function f unto the numbers f(0). The delta -function can be related to convolutions. A convolution C is a special kind of operator mapping functions unto other functions of the same variable say g onto G written symbolically as Cg = G. the convolution C is represented by a function c(x-y), and one defines

$$G(x) = \int_{-\infty}^{\infty} dy c(x-y) g(y) = \int_{-\infty}^{\infty} dy\ c(y) g(x-y) \qquad (1-8)$$

For general c, G is of course a function different from g. In the special case where $c(x-y) = \delta(x-y)$, the equation

$$\int_{\eta_2}^{-\eta_1} dx \delta(x) = -\int_{-\eta_1}^{\eta_2} dx \delta(x) = -1$$

entails that

$$G(x) = \int_{-\infty}^{\infty} dy\ \delta(x - y)g(y) = g(x) \qquad (1 - 9)$$

Thus $\delta(x - y)$ regarded as a convolution represents the identity (unit) operator, 1, which maps any test function unto the same function.

1.2 The Diffusion Equation:

The flow of fluid in a domain is analysed from two perspectives. First from the perspective of a macroscopic view and from the perspective of a near microscopic view. In the macrscopic view, the application of conservation principles of mass and momentum(Navier Stokes equations) led to the diffusion equation. The diffusion equation was derived in the continuum limit employing, forces, fluxes, concentrations of particles, potentials and transient quantities within the flow. Reynolds number which helps to categorize flows as laminar or turbulent is a ratio of inertial forces to viscous forces in a fluid. Therefore we can say that at very low Reynolds numbers, the viscous effects outweigh the inertial effects. If our initial analysis of the flow field has led us to a pure diffusion situation, it brings to our attention that Reynolds number must be very low and viscous forces must be very dominant. The quantities or properties of the fluid that are undergoing diffusion must be of interest to us . These may include mass, momentum, concentration and vorticity. The solution method to be adopted to solve the flow field from a continuum mechanics view is dictated by the boundary and initial conditions. The diffusion equation is generally stated as:

$$\frac{\partial u(x,t)}{\partial t} = D\nabla^2 u(x,t)$$

Unlike the mechanics of solids, the mechanics of fluids is more considered as particles in motion. The motion of a small particle is dominated by fast timescales, short distances and collisions with neighbouring particles, that yield highly irregular and rapidly changing motion. It can be said that what highlights the difference between macro and microscopic life is an issue of scale. To approximate the mechanics of collision between molecules which occur on time scale orders of about $10^{12}s$ and distances of order of about $10^{-9}m$, the idea of the stochastic behaviour and random walk is adopted. The ideas of vorticity and random walk are in place only to allow us to ignore the massive complexity of such a small fast system to yield a more tractable problem.[ref] Since the diffusion concept alerts us to look beyond the slow observations of particle motion into the world of molecular dynamics, it can be said the diffusion equation allows us to talk about the statistics of randomly moving particles. This yields us to the temptation of abandoning Newtonian mechanics and notion of inertia in favor of a system that directly responds to fluctuations in the surrounding environment. The Brownian motion of a particle can be described as stochastic process. Another closely related concept to stochastic process is the random walk. In distinction to the Brownian motion where the randomness appears as continuous Wiener process, the random walk proceeds by discrete steps. Random walks provide a basis for understanding a wide range of phenomena and require the use of many mathematical techniques to solve the related problems. Random walk look the same on all scales and the general features of the statistical behavior are independent of the microscopic details. The concept of random walk is described as follows:

A man starts from point O and walks l yards in a straight line; He then turns through any angle whatever and walks another l yards in a straight line. He repeats this process n times. The probability is required that after these n stretches he is at a distance between r and $r + \delta r$ from the starting point O. the drunkard takes a series of steps of equal length away from the last point but each at a random angle. The random walk on a line is much simpler. The positions are spaced regularly along a line. The walker has two possibilities; either one step to

the right(+1) with probability P or one step to the left(-1) with probability q = 1- p. For a symmetric case like a pure diffusion situation $q = p = 1/2$.
After n steps, the position of the random walker is given by:

$$s(n) = \sum_{i=1}^{n} l_i;\ with\ l_i = 1 \qquad (1-10)$$

A series of fluctuating values s(n) is obtained when the random walk is repeated many times. Interesting quantities are the averages over an ensemble of m different realizations. For an ifinitely large ensemble, $m \to \infty$, the probability theory allows us to caculate the probability distribution s(n), as well as the mean values. For the symmetric case $p = q = 1/2$, we have zero mean value. $< s(n) >= 0$. Since s(n) and s(-n) are equally likely, the root-mean-square

$$\sigma(n) = \sqrt{< s(n)^2 > - < s(n) >^2} = \sqrt{n} \qquad (1-11)$$

characterizes the average deviation amplitude from the mean value. The theoritical results given by (1-10) and (1-11) as well as the probability distribution over the position of the random-walker follow easily from the properties of the Markov chain describing the process, namely the probability P(m, n+1) that the walker is at position m after n+1 steps is given by the set of probability P(m,n) after n steps in accordance with the master equation.

$$P(m, n+1) = pP(m-1, n) + qP(m+1, n)$$

In 1996, J. T. C. Liu investigated the problem of flow induced by the impulsive motion of an infinite flat plate in a dusty gas. This kind of problem was first considered by Stokes. Liu's work considered the corresponding Rayleigh problem for a viscous incompressible dusty gas. Two elementary limiting situations were confirmed. For 'small' times, the viscous diffusion layer grows parabolically;- $(\nu t)^{\frac{1}{2}}$ as if the particle phase were absent. For 'large" times, the viscous diffusion layer also grows parabolically, but as $(\nu t)^{\frac{1}{2}}$ where ν is the kinematic viscousity based upon the viscosity of the gas but the total density of the combined gas and solid phases.

Hassan Aref and Eric D. Siggia, in 1980 studied the initial value problem defined by two parallel vortex sheets of opposite signs. They numerically simulated the roll-up of the sheets into vortex street. The breakdown of the metastable street into a two-dimensional, turbulent shear flow was also studied. Using dimensional arguments, they derived the relevant scaling theory, and showed that it applies to a flow started from two random vortex sheets. For the turbulent flow that follows from the breakdown of a regular vortex street two length scales with different power-law growth in time appear to be necessary. The important differences in the asymptotic structures of the flows initialized from random and regular sheets were revealed. The initial condition considered by them consisted of two vortex sheets of opposit sign discretized into point vortices. When the same perturbation was applied to both sheets, they rolled up to produce a staggered array of finite area vortices. The breakdown of the vortex street led to a two-dimensional turbulent shear flow. They argued that when a street of point vortices breaks down, the momentum thickness increases as $t^{\frac{1}{2}}$

In 2004, H Basirzadeh and A. V. Kamyad introduced an approach for solving a wide range of moving boundary problems by using calculus of variations and measure theory. They transformed the problems equivalently into an optimal control problem. They then modified the new problem into one consisting of the minimization of a linear functional over a set aof Radon measures. By using the solution of finite linear programming the suboptimal measures were obtained. Finally the approximate optimal control as constructed, and then an approximate solution for moving boundary function on specific time. They obtained solution for the position of the moving boundary at various times. Also obtain solutions that showed the variations of the piecewise constant control functions and trajectory functions with time.

T. C. Illingworth and I. O Golosnoy in 2005 developed a numerical scheme to find transient solutions to diffusion problems in two distinct phases that were separated by a moving boundary. In order to model the differential equations governing the diffusion process, they assume that the diffusion coefficients were functions of composition only, and the equilibrium concentration were constant. They included the interface position as a continuous variable in the model and solved a finite-difference form of the differential differential equation. Difficulties arose in tracking the motion of the interface because of the discontinuity in the concentration profile there. Since the chemical activity of each species varies continuously across the sample, describing the way in which diffusion affects activity(rather than concentration) could potentially overcome these problems. Another way they dealt with the discontinuity at the interface was to use discretization of space which took into account the motion of the interface.. from their scheme, they were able to to predict how the interface position varies as a function of time for a particular planar system. the numerical scheme used conserved solute(to within rounding accuracy) in every calculation. A concentration profile implied that the position of the interface moved in a parabolic manner.

$$s(t) = k\sqrt{4D_B t}$$

where k is a constant that depended on the geometry of the system; as well as the concentration. The interface position varied as with the square root of time.

Pablo Suarez and Abhinandan Chowdhury numerically studied the stochastic Burger's equation with moving boundaries in 2014. Their aim was to investigate the effect of random noise on the Burger's equation describing a field between boundaries moving with a prescribed manner. A time-dependent noise term was chosen. Two types of moving boundary functions were considered, a constant velocity moving boundary and a boundary moving with a rapidly oscillating function. Different intensities of the noise term were tested by their solution. They showed that the solution profile for Burgers equation without the noise term matches the solution profile obtained for the Burger's equation with similar moving boundary. The solution profile between the boundaries does not appear to be perturbed by the presence of the random term. For the case of the rapidly oscillating boundaries the solution profiles at the boundaries, do not appear to be perturbed to a significant extent. But the solution profiles between the boundaries experience very high frequency oscillation with very small magnitude which is predominantly dependent on the value of ϵ the time-dependent noise created substantially amount of perturbation in the boundary, but cannot influence the solution profile between the boundaries in a significant manner.

Christopher J. Vogl, Michael J. Miksis and Stephen H Davis in 2012 investigated a class of one-dimensional moving boundary problems that involves one or more regions governed by anomalous diffusion. A novel numerical method was developed to handle the moving interface. Two moving boundary problems were solved; the first involves a subdivision region to the one side of an interface and a classical diffusion region to the other. Anomalous diffusion describes a process where then mean-squared behaviour of $< x^2(t) >\sim t^\alpha$ where $\alpha \neq 1$. The processes were modelled by an integro-differential equation known as the fractional diffusion equation. the numerical method they developed was based on the method of Diethelm et al. The moving boundary problem was solved on an infinite domain with Heaviside initial data. The boundary condition at the interface were formulated from aa consideration of a vitrification front that was postulated to occur during the anhydrobiosis of certain organisms, by Aksan et al(2000). With vitrification, continuity of water concentration and flux were expected at the formation front. With these conditions, two problems were considered. the first problem had an interface bounded by an anomalous region to the one side and a classical diffusion region to the other. Instead of moving monotonically the interface was found to reverse direction after a given time, which depended on material parameter. The anomalous-classical problem was then generalized to involve subdiffusion on both sides of the interface. This new problem was denoted as the anomalous-anomalous problem. This interface could also reverse direction after a given time, but only if the orders of subdivision are distinct.

In 2008, Dmitri Volfson and Jorge Vinals studied the induced by random vibration of a solid boundary in otherwise quiescent fluid. Their work examined the formation and separation of viscous layers in a fluid which in contact with a solid boundary that is vibrated randomly. Their analysis was motivated by the low level and random acceleration field that affects a number of microgravity experiments. They first studied the case of a planar boundary to generalize the classical result of Stokes. Next they considered a slightly curved boundary and show that steady streaming appears in the ensemble average at first order in the perturbed flow variables. Finally, they addressed the case of a modulated boundary that is vibrated randomly. The study of Volfson and Vinals was motivated by the significant levels of residual accelerations (g-jitter) that have been detected during space missions in which microgravity experiments have been conducted. Direct measurement of these residual accelerations has shown, they have a wide frequency spectrum, ranging approximately from 10^{-4} Hz to 10^{2} Hz.

Despite the efforts of a number of researches over the last decade, there remain areas of uncertainty about the potential effect of such a residual acceleration field on typical micigravity fluid experiments, especially in quantitative terms. the formation of viscous layer around solid boundaries when the flow amplitude has a random component has not been addressed yet despite its potential relevance for a number of microgravity experiments. The study of Volfson and Vinals included the dynamics of colloidal transport, coarsening studies of solid-liquid mixtures in which purely diffusive controlled transport is desired, or the interaction between the viscous layer produced by bulk flow of random amplitude and morphological instability of a crystal-melt interface.

Zhang et al (1993) and thompson et al (1997) adopted a statiscal description of the residual acceleration field on board spacecraft and modeled the acceleration time series as a stochastic process in time. Progress was achieved through the consideration of specific stochastic model according to which each Cartesian-component of the residual accelerationfield g(t0 is modeled as a narrow band boise. This noise is a Gaussian process defined by three independent parameters. Each realization of narrow band noise can be viewed as a temporal sequence of periodic function of angular frequency Ω with amplitude and phase that remain constant only for a finite amount of time.

In the paper of Volfson and Vinals, the flow induced in an otherwise quiescent fluid by the random vibration of a solid boundary, the velocity of the boundary $U_0(t)$ is assumed presribed, and modeled as a narrow band stochastic process. In the monochromatic limit, the variance of the velocity field decays exponentially away from the wall, with a characteristic decay length given by the Stokes layer thickness $\delta_s = (2\nu/\Omega)^{\frac{1}{2}}$, where ν is the kinematic viscosity of the fluid and Ω is the angular frequency of vibration of the boundary. They showed that for any finite correlation time the stationary variance of the tangential velocity asymptotically decays as the inverse squared distance from the wall, in contrast with the exponential decay in the deterministic case. This asymptotic behavior originates from the low frequency range of the power spectrum of the boundary velocity. Volfson and

Vinals next investigated two additional geometries in which the equationsgoverning fluid flow are not linear, and showed that several of the generic features obtained for the case of a planar boundary still hold. they found that if the thickness of Stokes layer, δ_s , and the amplitude of oscillation 'a' are small compared with a characteristic length scale of the boundary L ($\delta_s << L, a << L$) then the generation of secondary steady streaming may be described as follows. Vibration of the rigid boundary gives rise to an oscillatory and nonuniform motion of the fluid. The flow is potential in the bulk, and rotational in the boundary layer because of no slip conditions on the boundary. The bulk flow applies pressure at the outer edge of the boundary layer, which does not vary across the layer. The non-uniformity of the flow leads to vorticity convection in the boundary layer. Both convection and the applied pressure drive vorticity, diffusion and thus induce secondary steady motion which does not vanish outside of the boundary layer.
Volfson and Vinal also investigated the flow field induced by a wavy wall. Wavy wall geometry induced flow had earlier been studied by Lyne(1971) to address the interaction between the flow above the sea bed and ripple pattern on it. Lyne(1971) deduced the existence of steady streaming in the limit in which the amplitude of the wall deviation from planarity is small compared with the thickness of the Stokes layer. Lyne introduced a conformal transformation and obtained an explicit solution in the limit of small kRe, where k is the wavenumber of the wall profile scaled by the thickness of the Stokes layer, and Re is the Reynolds number. The detailed structure of the secondary flow depends on the ratio between the wavelength of the boundary profile and the thickness of the Sokes layer.
Volfson and Vinals addressed the flow created by a gently curved solid boundary that is being vibrated randomly. The perturbatiion parameter they used was the ratio between the amplitude of vibrations and the characteristic inverse curvature of the wall. they found that the average velocity diverges logarithmically away from

the boundary because of the low frequency range of the power spectrum. They also studied the formation of a boundary layer around a wavy boundary that is vibrated randomly. Positive and negative vorticity production in adjacent regions of the boundary introduces a natural decay length in the solution, thus leading toexponential decay of the flow away from the boundary, even in the absence of a low frequency cut-off in the power spectrum of the boundary velocity. Steady streaming is found at second order comprising two or four recirculating cells per period of the boundary profile. Volfson and Vinals examined the case of a planar boundaryvibrated along its own plane. The Navier-Stokes equation reduced to a linear equation, a fact that allows a complete solution of the flow. The simple solution still exhibited several of the qualitative features that are present in the case of random forcing by a curved boundary namely asymptotic power law decay of the velocity field away from the boundary, and sensitive dependence on the low frequency range of the power spectrum of the boundary velocity.

Two types of moving boundary initiated flow fields are considered. These are the Stoke's flow and the Couette flow. In the Stoke's flow, the domain is a semi-infinite medium with the flow initiated by the finite side of the boundary. In the couette flow, the domain is a finite medium between two long plates, considered infinite in length. One of the plates is given a motion while the other is maintained stationary. The moving plate initiates the flow. A variety of solutions are known for laminar flow with moving boundaries. Stokes two problems were some of the very first problems in which the Navier Stokes equations were solved. In both cases, the domain describes a sem-infinite fluid at rest initially and bounded below by a solid plane at $y = 0$ In the first case, the plane is accelerated to a constant velocity Uo. In the second case, the plane is given a steady oscillation at velocity of $Ucoswt$. Numerical approaches have been used to simulate this problem. Some domain methods such as the finite difference method(FDM), finite element method(FEM) and finite volume method(FVM) are difficult to simulate in an infinite or semi-infinite domain with unrestricted boundary conditions. The boundary element method(BEM), another numerical scheme that involves mesh reduction has been used by Zhu et al and Bulgakov et al to solve diffusion equations. S.P. Hu, C.M. Fan, C. W. Chen and D.L Young employed the methods of Fundamental solutions(MFS) a numerical scheme to solve the Stoke's problem. Before Young et al , Chen et al and Barakrishnan and Ramachandran applied the MFS for diffusion equations by using the modified Helmholtz fundamental solution. An important issue of the MFS is the locations of source points, which was circumvented with the consideration of the source positions as unknown variables by Fairwether et al. Young et al applied the unsteady MFS successfully for multi-dimensional diffusion equations. The Laplace transform or the finite difference scheme is used to deal with the time derivative of the governing equations. This is due to the fact that the MFS is treated always in the spatial domain with respect to the location of the source points and the field points. Young et al used the fundamental solution of Stokes problem with the unsteady MFS without the need for Laplace transform or finite difference method to take care of the time derivative term.

CHAPTER 2: GOVERNING EQUATIONS AND BOUNDARY WITH UNIFORM MOTION

2.1 Governing Equation

Fluid flow in a continuum medium can be analyzed by the application of conservation principles. These are the principles of conservation of mass, conservation of momentum and conservation of energy. For most incompressible flows, the principles conservation of mass and conservation of momentum may be all that are needed to solve the flow field. The conservation of energy princilple may not be necessarily needed in the analysis of some fluid problems and the derivation of the governing equations.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (2-1)$$

We assume a two dimensional flow field. There is therefore no variation of any quantity in the z-direction. Since the boundary is of infinite length no velocity changes in x, ie $\frac{\partial u}{\partial x} = 0$. The continuity equation reduces to:

$$\frac{\partial v}{\partial y} = 0 \qquad (2-2)$$

Upon partial integration, it is found that v is a function of x only. The flow is induced by a moving boundary. The boundary has no velocity component in the y-direction. Therefore for no-slip condition v = 0 at the boundary. Since there is no variation of y-component velocity with respect to y, v = 0 everywhere within the flow. Lets consider the x-momentum equation of the Navier Sokes equations;

$$(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) - \frac{1}{\rho}\frac{\partial p}{\partial x} + gx \qquad (2-3)$$

The flow is fully developed in x-direction, v = 0 and flow is assumed two dimensional. We shall take the pressure to be constant throughout the fluid. There is no gravitation effect in x direction. The x-momentum equation reduces to:

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (2-4)$$

The equation obtained is commonly called the diffusion equation. It is a diffusion equation with no source term. Many formal methods are available for the solution of this equation. Since there is no typical physical length in the formulated problem, it is logical to suspect that the solution, u will be a function of some combination of y and t. For the diffusion differential equation to be solvable, we need boundary and initial conditions for it. The boundary and initial conditions directs the solution approach adopted for the problem. This reasearch is also involved in solving flow field induced by moving boundaries. In this research, two types of medium are considered, a semi-infinite medium and a finite medium between two long infinite plates. The solution to the flow field developed is influenced by the function of the motion given to boundary. A uniform boundary motion is first considered and when the boundary motion is time-dependent. Of interesting consideration is when the flow is initiated by a random boundary motion. The analysis of this type of flow is handled in chapter five. Presently in this chapter we will consider when the boundary is given a uniform velocity. We will solve uniform boundary situation for both the semi-infinite medium and the finite medium. Let us first consider when the boundary motion is uniform, also known as Stoke's first problem.

2.2 Flow In A Semi-Infinite Medium

In this problem the boundary is given a constant velocity to initiate the flow. The problem equation with its boundary and initial conditions is set up as shown.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad (2-5a)$$

$$u(y,t) = U_0 \ at \ y = 0 \qquad (2-5b)$$

$$u(y,t) = 0 \ for \ t = 0 \qquad (2-5c)$$

The diffusion equation in a semi-infinite medium with zero initial velocity and constant velocity of the boundary can be solved by normal classical method for solving such problems. A new dependent variable is Q(y,t) is defined as

$$Q(y,t) = u(y,t) - U_0 \qquad (2-6)$$

The problem takes the form

$$\frac{\partial Q}{\partial t} = \nu \frac{\partial^2 Q}{\partial y^2} \qquad (2-7a)$$

Boundary Condition and initial condition:

$$Q(y,t) = 0 \ at \ y = 0 \qquad (2-7b)$$

$$Q(y,t) = -U_0 \ for \ t = 0 \qquad (2-7c)$$

Problem $(2-7)$ can now be solved employing the method of separation of variables. Assume solution of form

$$Q(y,t) = Y(y)T(t) \qquad (2-8)$$

substitute $(2-8)$ into $(2-7a)$, we obtain

$$YT' = kY''T \qquad (2-9)$$

where "k" is substituted for the diffusion

$$\frac{Y''}{Y} = \frac{T'}{\nu T} = -\alpha^2 \qquad (2-10)$$

where α is a positive constant

$$Y'' + \alpha^2 Y = 0 \qquad (2-11)$$

$$T' + \alpha^2 \nu T = 0 \qquad (2-12)$$

Solving equation $(2-12)$

$$T' + \alpha^2 \nu T = 0 \qquad (2-13)$$

$$T(t) = Ce^{-\nu\alpha^2 t} \qquad (2-14)$$

Solving equation $(2-11)$

$$Y'' + \alpha^2 Y = 0$$

From the boundary conditions of equation $(2-7b)$

$$Y(0) = 0$$

General solution for $(2-11)$

$$Y(y) = A\cos\alpha y + B\sin\alpha y \qquad (2-15)$$

where A nad B are constants

$$Y(0) = 0, A = 0$$

$$Y(y) = B \sin \alpha y \qquad (2-16)$$

Since equation $(2-15)$ has eigenvalues , we can express it in terms of eigenfunction as such,

$$Y_n(y) = B_n \sin \alpha_n y \qquad (2-17)$$

Similarly equation (2-14) can be written in terms of eigenfunction as

$$T_n(t) = C_n e^{-\nu \alpha_n^2 t} \qquad (2-18)$$

Substitute $(2-17) and (2-18)$ into $(2-8)$

$$Q_n(y,t) = (B_n \sin \alpha_n y)(C_n e^{-\nu \alpha_n^2 t}) \qquad (2-19)$$

solution for $(2-7a)$ takes the form

$$Q(y,t) = \sum_{n=1}^{\infty} Q_n(y,t) \qquad (2-20)$$

Combining coefficients, equation (2-20) can be written as :

$$Q(y,t) = \sum_{n=1}^{\infty} (D_n \sin \alpha_n y)(e^{-\nu \alpha_n^2 t}) \qquad (2-21)$$

Taking the limit of the sum as n gets larger and larger, the general solution for $(2-7a)$ can be expressed as

$$Q(y,t) = \int_0^{\infty} D e^{-\ \nu \alpha^2 t} \sin \alpha y \ d\alpha \qquad (2-22)$$

Appying initial condition

$$Q(y,0) = \int_0^\infty D \sin \alpha y \; d\alpha \qquad (2-23)$$

Suppose the initial condition were a function of y, equation $(2-23)$ can be written as

$$F(y) = \int_0^\infty D(\alpha) \sin \alpha y \; d\alpha \qquad (2-24)$$

Representing F(y) by a Fourier series, Fourier coefficients take the form:

$$D(\alpha) = \frac{2}{\pi} \int_0^\infty F(y') \sin \alpha y' \; dy' \qquad (2-25)$$

$$Q(y,t) = \frac{2}{\pi} \int_0^\infty D(\alpha) e^{- \; \nu\alpha^2 t} \sin \alpha y \; d\alpha \qquad (2-26)$$

$$Q(y,t) = \int_0^\infty \frac{2}{\pi} \int_0^\infty F(y') \sin \alpha y' e^{-\nu\alpha^2 t} \sin \alpha y \; d\alpha dy' \qquad (2-27)$$

$$Q(y,t) = \frac{2}{\pi} \int_0^\infty F(y') \int_0^\infty e^{-\nu\alpha^2 t} \sin \alpha y' \sin \alpha y \; d\alpha dy' \qquad (2-28)$$

The integration with respect to α is evaluated by making use of the following.

$$2 \sin \alpha y' \sin \alpha y = \cos \alpha(y-y') - \cos \alpha(y-y') \qquad (2-29)$$

$$\int_0^\infty e^{-\nu\alpha^2 t} \cos \alpha(y-y') d\alpha = \sqrt{\frac{\pi}{4\nu t}} \; exp[-\frac{(y-y')^2}{4\nu t}] \qquad (2-30)$$

$$\int_0^\infty e^{-\nu\alpha^2 t} \cos \alpha(y+y') d\alpha = \sqrt{\frac{\pi}{4\nu t}} \; exp[-\frac{(y+y')^2}{4\nu t}] \qquad (2-31)$$

$$\frac{2}{\pi} \int_0^\infty e^{-\nu\alpha^2 t} \sin \alpha y' \sin \alpha y \; d\alpha = \frac{1}{(4\pi\nu t)^{\frac{1}{2}}} [exp(-\frac{(y-y')^2}{4\nu t} - exp(-\frac{(y+y')^2}{4\nu t}] \qquad (2–32)$$

The solution $Q(y,t)$ becomes

$$Q(y,t) = \frac{1}{(4\pi\nu t)^{\frac{1}{2}}} \int_0^\infty F(y)[exp(-\frac{(y-y')^2}{4\nu t}) - exp(-\frac{(y+y')^2}{4\nu t})] \qquad (2-33)$$

In the case when the function F(y) takes on a constant value, ie

$$F(y) = -U_0$$

Equation $(2-33)$ can be written as:

$$\frac{Q(y,t)}{-U} = \frac{1}{(4\pi\nu t)^{\frac{1}{2}}}[\int_0^\infty exp(-\frac{(y-y')^2}{4\nu t}) - \int_0^\infty exp(-\frac{(y+y')^2}{4\nu t})] \qquad (2-34)$$

Introducing a new variable:

$$-\eta = \frac{(y-y')}{\sqrt{(4\nu t)}} \; dy' = \sqrt{(4\nu t)}\; d\eta \; for\ first\ integral$$

$$-\eta = \frac{(y+y')}{\sqrt{(4\nu t)}} \; dy' = \sqrt{(4\nu t)}\; d\eta \; for\ second\ integral$$

$$\frac{Q(y,t)}{-U} = \frac{1}{\sqrt{\pi}}\int_{-\frac{y}{\sqrt{4\nu t}}}^\infty e^{-\eta^2} d\eta - \int_{\frac{y}{\sqrt{4\nu t}}}^\infty e^{-\eta^2} d\eta \qquad (2-35)$$

since $e^{-\eta^2}$ is symmetrical about $\eta = 0$

$$\frac{Q(y,t)}{-U} = \frac{2}{\sqrt{\pi}}\int_0^{\frac{y}{\sqrt{4\nu t}}} e^{-\eta^2} d\eta \qquad (2-36)$$

$$\frac{Q(y,t)}{-U} = erf(\frac{y}{\sqrt{4\nu t}}) \qquad (2-37)$$

$$Q(y,t) = -U_0 \; erf(\frac{y}{\sqrt{4\nu t}}) \qquad (2-38)$$

but $u(y,t) = Q(y,t) + U_0$

$$u(y,t) = U_0 - U_0 \; erf(\frac{y}{\sqrt{4\nu t}}) \qquad (2-39)$$

$$u(y,t) = U_0[1 - erf(\frac{y}{\sqrt{4\nu t}})] \qquad (2-40)$$

$$u(y,t) = U_0 \; erfc \; (\frac{y}{\sqrt{4\nu t}}) \qquad (2-41)$$

Solution equation $(2-40)$ expresses $u(y,t)$ in terms of U_0 and an error function.The influence of the moving boundary extends to infinity soon after after the boundary starts moving. At large distances ie as $y \to \infty$ the $erfc(\infty) \to 0$, the viscous effect is very minimal. There is still a minute viscous influence throughout the flow. From the tables of error function, $erf(1.99) = 0.99511$. It is seen that when $\frac{y}{\sqrt{4\nu t}} = 1.99$, u(y,t) is reduced to one percent of U_0 that is when $y = 4\sqrt{\nu t}$. This result shows that viscosity diffuses from the solid boundary into the fluid and that the distance at which a given effect occurs varies with $\sqrt{\nu t}$. The elapsed time t may be expressed in terms of the distance x through which the plate travels ie $t = \frac{x}{U_0}$ so that we can write

$$y \sim x\sqrt{\frac{\nu}{U_0 x}}$$

. If we assume the viscosity to be constant throughout, then it can be said that the distance at which the viscosity diffuses from the solid boundary is proportional to $\sqrt{t}$. The diffusion of viscous effect from the boundary into the flow is a recurring feature of real fluid flows.

2.3 Flow In a Finite Medium.

The governing diffusion equation is now solved for a finite medium between two long infinite plates a distance h apart.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad (2-42a)$$

$$u(y,t) = 0 \; at \; y = 0 \qquad (2-42b)$$

$$u(y,t) = U_0 \; at \; y = h \qquad (2-42c)$$

$$u(y,t) = 0 \; for \; t = 0 \qquad (2-42d)$$

$$u(y,t) = v(y,t) + U_o\frac{y}{h} \qquad (2-43)$$

The problem stated in terms of the variable $v(y,t)$ becomes

$$\frac{\partial v}{\partial t} = \nu\frac{\partial^2 v}{\partial y^2} \qquad (2-44a)$$

$$v(y,t) = 0 \; at \; y = 0 \qquad (2-44b)$$

$$v(y,t) = 0 \; at \; y = h \qquad (2-44c)$$

$$v(y,t) = -U_o\frac{y}{h} \; for \; t = 0 \qquad (2-44d)$$

The method of separation of variables is adopted in solving problem $(2-44)$. Assume solution

$$v(y,t) = Y(y)T(t) \qquad (2-45)$$

$$YT' = \nu Y''T \qquad (2-46)$$

$$\frac{Y''}{Y} = \frac{T'}{\nu T} = -\alpha^2 \qquad (2-47)$$

where α is a constant. Two separate equations of the y and t respectively can be written as:

$$Y'' + \alpha^2 Y = 0 \qquad (2-48)$$

$$T' + \alpha^2\nu T = 0 \qquad (2-49)$$

From the boundary conditions of $(2-44b)$ *and* $(2-44c)$

$$v(0,t) = Y(0)T(t) = 0$$

$$v(h,t) = Y(h)T(t) = 0$$

Thus $Y(0) = 0,\ and\ Y(h) = 0$ The solution to equation $(2-48)$ takes the form

$$Y(y) = A\cos\alpha y + B\sin\alpha y \qquad (2-50)$$

Since $Y(0) = 0,\ A = 0$. To satisfy the second condition,

$$Y(h) = B\sin\alpha h = 0 \qquad (2-51)$$

The constant B cannot be zero. Therefore $\sin\alpha h = 0$, and $\alpha = \frac{n\pi}{h}$ Substituting these eigenvalues, we have

$$Y_n(y) = B_n \sin\frac{n\pi}{h} \qquad (2-52)$$

Equation $(2-49)$ is stated as

$$T' + \alpha^2\nu T = 0 \qquad (2-53)$$

The solution of which is :

$$T(t) = Ce^{-\nu\alpha^2 t} \qquad (2-54)$$

In terms of eigen funtion and eigen values equation $(2-54-18)$ is written as:

$$T_n(t) = C_n e^{-\nu\alpha^2 t} \qquad (2-55)$$

Substituting $(2-52)$ and $(2-55)$ into $(2-45)$,

$$V_n(y,t) = Y_n(y)T_n(t) \qquad (2-56)$$

$$V_n(y,t) = a_n e^{-\nu\alpha^2 t}\sin\alpha y \qquad (2-57)$$

where $a_n = B_n C_n$ The series form for V_n becomes

$$V_n(y,t) = \sum_{n=1} a_n e^{-\nu \alpha_n^2 t} \sin \alpha y \qquad (2-58)$$

The initial condition equation of $(2-44d)$ can be rewritten as:

$$V_n(y,t) = F(y') \qquad for \quad t = 0 \qquad (2-59)$$

substituting $(2-59)$ into $(2-58)$, the initial condition becomes

$$F(y') = \sum_{n=1} a_n \sin \alpha_n y \qquad (2-60)$$

Fourier coefficient a_n is given by

$$a_n = \frac{2}{h} \int_0^h F(y') \sin \alpha y' \ dy' \qquad (2-61)$$

Substituting $(2-61)$ into $(2-58)$ gives:

$$V(y,t) = \sum_{n=1} \Big[\frac{2}{h} \int_0^h F(y') \sin(\alpha_n \ y') dy'\Big] e^{-\nu \alpha_n^2 t} \sin \alpha y \qquad (2-62)$$

$$F(y') = -\frac{U_o}{h} y \qquad (2-63)$$

Now

$$\int_0^h F(y') \sin \alpha_n y' \ dy' = \frac{U_o}{\alpha_n} (-1)^n \qquad (2-64)$$

The solution equation $V(y,t)$ becomes:

$$V(y,t) = \sum_{n=1} \frac{2}{h} \frac{U_o}{\alpha_n} (-1)^n e^{-\nu \alpha_n^2 t} \sin \alpha_n y \qquad (2-65)$$

From equation $(2-43)$ the solution of $u(y,t)$ is given as:

$$u(y,t) = v(y,t) + \frac{U_o}{h}y \qquad (2-66)$$

Substituting $(2-65)$ into $(2-66)$) gives:

$$u(y,t) = \frac{U_o}{h}y + \frac{2U_o}{h}\sum_{n=1}\frac{1}{\alpha_n}(-1)^n e^{-\nu\alpha_n^2 t}\sin\alpha_n y \qquad (2-67)$$

$$u(y,t) = \frac{U_o}{h}y + \frac{2U_o}{h}\sum_{n=1}\frac{h}{n\pi}(-1)^n e^{-\nu(\frac{n\pi}{h})^2 t}\sin\frac{n\pi}{h}y \qquad (2-68)$$

where $\alpha_n = \frac{n\pi}{h}$

In the finite medium, the viscosity diffuses from the moving boundary into the fluid. For a fixed time, and very close to the moving boundary, the diffusive effect is a higher percentage of U_0. As time become longer and longer and also very close to the boundaries the diffusion of viscosity reduces rapidly and the diffusion is more contolled by the relation $u)y,t) \sim \frac{U_0 u}{h}$ and independent of time. The diffusion of the viscosity is a recurring feature as indicated by the eigen value $\frac{n\pi}{h}$

CHAPTER 3: FLOW FIELD WITH TIME-DEPENDENT MOVING BOUNDARY: DUHAMEL APPROACH

3.1 Flow In A Semi-Infinite Medium

When the diffusion equation has a time-dependent boundary condition and or a source term which varies with time, the problem cannot be solved by the classical method. A different approach has to be adopted. One of the methods by which it can be solved is by Duhamel's theory, which is based on Duhamel's superposition integral.

If we consider the diffusion equation over a semi-infinite strip ($0 \leq y \leq 1$) in a distance-time plane, the non-homogeneous condition along the lines $y = 0$ and $y = 1$ prescribe the solution variable ϕ as a constant along each of these boundaries. The problem was solved by determining a particular solution ϕ_s of the governing differential equation which satisfy those conditions, and then using conventional separation - of- variables methods to determine the "correction" $\phi_T = \phi - \phi_s$ which then is to satisfy homogeneous conditions along the two boundaries $y = constant$. Restating the general diffusion equation and its boundary and initial conditions:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad (3-1a)$$

$$u(y,t) = F(t) \; at \; y = 0, \qquad (3-1b)$$

$$u(y,t) = f(y) = 0 \; for \; t = 0, \qquad (3-1c)$$

As a first step we solve the problem in the special case when F(t) is unity. Denote this solution by $u = A(y,t)$, we obtain with $f(y) = 0, u_1 = 1$, the result;

$$A(y,t) = [1 - erf\ (\frac{y}{\sqrt{4\nu t}})] \qquad (3-2)$$

$$A(y,t) = erfc\ (\frac{y}{\sqrt{4\nu t}}) \qquad (3-3)$$

From the linearity of the problem, it is seen that, at the instant following $t = \tau_n$, the solution distribution corresponding to the step-function approximation F(t) is given by the sum:

$$u = F(0)A(y,t) + [F(\tau_1) - F(0)]A(y,t-\tau_1) + [F(\tau_2) - F(\tau_1)]A(y,t-\tau_2)$$

$$+ ...... + [F(\tau_n) - F(\tau_{n-1})]A(y,t-\tau_n) \qquad (3-4)$$

$$F(\tau_{k+1}) - F(\tau_k) = \Delta F_k \qquad (\tau_{k+1}) - (\tau_k) = \Delta\tau_k \qquad (3-5)$$

$$u = F(0)A(y,t) + \sum_{k=0}^{n-1} A(y,t-\tau_{k+1})(\frac{\Delta F}{\Delta\tau})\Delta\tau_k \qquad (3-6)$$

$$u = F(0)A(y,t) + \int_0^t A(y,t-\tau)F(\tau)d\tau \qquad (3-7)$$

Assuming F(t) is differentiable. This is a version of Duhamel's principle, and it gives the desired solution in terms of the basic function $A(y,t)$. An alternative form is obtained by integration by parts,

$$u(y,t) = F(0)A(y,t) + [A(y,t-\tau)F(\tau)]_0^{\tau=t} - \int_0^t F(\tau)\frac{\partial}{\partial\tau}A(y,t-\tau)d\tau \qquad (3-8)$$

$$\frac{\partial}{\partial\tau}A(y,t-\tau) = -\frac{\partial}{\partial t}A(y,t-\tau) \qquad (3-9)$$

$$u(y,t) = F(0)A(y,0) + \int_0^t F(\tau)\frac{\partial}{\partial\tau}A(y,t-\tau)d\tau \qquad (3-10)$$

$A(y.0)$ vanishes when $y \geq 0$

$$u(y,t) = \int_0^t F(\tau)\frac{\partial}{\partial t}A(y,t-\tau)d\tau \qquad (3-11)$$

$$u(y,t) = \int_0^t -F(\tau)\frac{\partial}{\partial \tau}A(y,t-\tau)d\tau \qquad (3-12)$$

When (3-11) is used, where;

$$A(y,t) = [1 - erf\ (\frac{y}{\sqrt{4\nu t}})]$$

$$\frac{\partial A(y,t-\tau)}{\partial t} = \frac{y}{\sqrt{(4\pi\nu)}(t-\tau)^{\frac{3}{2}}} \exp\Big[(-\frac{y^2}{4\nu(t-\tau)})\Big] \qquad (3-13)$$

The solution to problem (3-1), the diffusion equation with no source term and a moving boundary of function F(t) in a semi-infinite medium can be expressed generally as

$$u(y,t) = \frac{y}{\sqrt{4\pi\nu}} \int_0^t \frac{F(\tau)}{(t-\tau)^{\frac{3}{2}}}\ exp\Big[-\frac{y^2}{4\nu(t-\tau)}\Big]d\tau \qquad (3-14)$$

If we introduce the independent variable: $\eta = \frac{y}{\sqrt{4\nu(t-\tau)}}$

$$t - \tau = \frac{y^2}{4\nu\eta^2} \qquad and \qquad d\tau = \frac{2}{\eta}(t-\tau)d\eta$$

$$u(y,t) = \frac{2}{\sqrt{\pi}}\int_{\frac{y}{\sqrt{4\nu t}}}^{\infty} e^{-\eta^2} f(t - \frac{y^2}{4\nu\eta^2})d\eta \qquad (3-15)$$

Suppose the boundary condition at $y = 0$ is given as;

$$F(t) = U_0 t \qquad (3-16)$$

Substituting into (3-15), the solution gives

$$u(y,t) = \frac{2U_0}{\sqrt{\pi}} \int_{\frac{y}{\sqrt{4\nu t}}}^{\infty} e^{-\eta^2}(t - \frac{y^2}{4\nu\eta^2})d\eta \qquad (3-17)$$

If the function of the boundary motion were to be sinusoidal;

$$F(t) = U_0 \; Cos \; \omega t \qquad (3-18)$$

the solution equation takes the form

$$u(y,t) = \frac{2U_0}{\sqrt{\pi}} \int_{\frac{y}{\sqrt{4\nu t}}}^{\infty} e^{-\eta^2} Cos[\omega(t - \frac{y^2}{4\nu\eta^2})]d\eta$$

$$\frac{u(y,t)}{U_0} = \frac{2}{\sqrt{\pi}} \int_{\frac{y}{\sqrt{4\nu t}}}^{\infty} e^{-\eta^2} Cos[\omega(t - \frac{y^2}{4\nu\eta^2})]d\eta \qquad (3-19)$$

The obtained equation above can be split into two term equation with change in integral limits as shown.

$$\frac{u(y,t)}{U_0} = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-\eta^2} Cos[\omega(t - \frac{y^2}{4\nu\eta^2})]d\eta$$

$$-\frac{2}{\sqrt{\pi}} \int_{0}^{\frac{y}{\sqrt{4\nu t}}} e^{-\eta^2} Cos[\omega(t - \frac{y^2}{4\nu\eta^2})]d\eta \qquad (3-20)$$

The first integral can be evaluated. The solution equation becomes:

$$\frac{u(y,t)}{U_0} = exp[-y(\frac{\omega}{2\nu})^{\frac{1}{2}}]Cos[\omega t - y(\frac{\omega}{2\nu})^{\frac{1}{2}}]$$

$$-\frac{2}{\sqrt{\pi}} \int_{0}^{\frac{y}{\sqrt{4\nu t}}} e^{-\eta^2} Cos[\omega(t - \frac{y^2}{4\nu\eta^2})]d\eta \qquad (3-21)$$

3.2: Flow in A Finite Medium:

Duhamel's theory is applied to flow in a finite medium between long infinite plates distance h apart. Flow field is governed by pure diffusive equation. The boundary condition here is different from the semi-infinite medium. Problem is solved for the case when the boundary moves with a linearly increasing velocity and also when it moves with a sinusoidal motion. As in previous section solution obtained is expressed in terms of an auxilliary one which is the solution when the time dependent boundary is replaced by unity. The governing equation is stated with its boundary and initial conditions.

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\nu}\frac{\partial u}{\partial t} \qquad (3-22a)$$

$$u(0,t) = 0\ at\ y = 0 \qquad (3-22b)$$

$$u(h,t) = F(t)\ at\ y = h \qquad (3-22c)$$

$$u(y,0) = f(y) \equiv 0\ for\ t = 0 \qquad (3-22d)$$

An auxilliary equation is formed with F(t) in (3-22c) being replaced by unity.

$$\frac{\partial^2 A}{\partial y^2} = \frac{1}{\nu}\frac{\partial A}{\partial t} \qquad (3-23a)$$

$$A(0,t) = 0\ at\ y = 0 \qquad (3-23b)$$

$$A(h,t) = 1\ at\ y = h \qquad (3-23c)$$

$$A(y,0) = 0\ for\ t = 0 \qquad (3-23d)$$

As a first step toward the solution of this problem, an auxilliary equation is formed with F(t) in (3-23 c) being replaced by unity. The solution of $(3-23)$ is

$$A(y,t) = \frac{y}{l} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi y}{l}e^{-\frac{n^2 t}{\lambda}} \qquad (3-24)$$

The realationship between solution to the pure diffusion equation of $(3-22)$ and its auxilliary solution is given by:

$$u(y,t) = F(0)A(y,t) + \int_0^t A(y,t-\tau)F'(\tau)d\tau \qquad (3-25)$$

$$u(y,t) = \int_0^t F(\tau)\frac{\partial}{\partial t}A(y,t-\tau)d\tau \qquad (3-26)$$

The integral appearing in (3-25) or (3-26) is often known as superposition integral. When (3-26) is used, the solution of the given problem takes the form

$$u(y,t) = \frac{2}{\pi\lambda}\sum_{n=1}^{\infty}(-1)^{n+1)}n\Big[\int_0^t F(\tau)e^{\frac{n^2\tau}{\lambda}}d\tau\Big]e^{-\frac{n^2t}{\lambda}}\sin\frac{n\pi y}{h} \qquad (3-27)$$

wheras (3-25) leads to the equivalent form

$$u(y,t) = \frac{y}{h}F(t)+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\Big[F(0)+\int_0^t F'(\tau)e^{\frac{n^2\tau}{\lambda}}d\tau\Big]e^{-\frac{n^2t}{\lambda}}\sin\frac{n\pi y}{h} \qquad (3-28)$$

Suppose the boundary motion function at $y = h$ is given by

$$F(t) = U_0 t \qquad (3-29)$$

Substituting this into (3-27) and performing the integration gives:

$$u(y,t) = \frac{2\lambda U_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1)}}{n^3}\Big[\frac{n^2t}{\lambda} - (1 - e^{-\frac{n^2t}{\lambda}})\Big]\sin\frac{n\pi y}{h} \qquad (3-30)$$

and substituting (3-29) into (3-28) gives:

$$u(y,t) = U_0\frac{y}{h}t + \frac{2\lambda U_0}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{1 - e^{-\frac{n^2t}{\lambda}}}{n^3})\sin\frac{n\pi y}{h} \qquad (3-31)$$

The equivalence of (3-30) and (3-31) is verified by noticing the validity of the expansion

$$\frac{y}{h} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi y}{h} \qquad (0 \le y < h) \qquad (3-32)$$

Comparing the integrals of $(2-36)$ and $(3-17)$, there is similarity to some extent. There is a factor of $(t - \frac{y^2}{4\nu\eta^2})$ in the integrand of $(3-17)$ due to the time t influencing the velocity U_0 of the boundary. The diffusion of viscosity when the boundary moves with linearly increasing velocity has some similarity to diffusion effect of viscosity when the boundary moves with uniform velocity. The integral of $(3-17)$ has not been explicitly solved. It is therefore difficult to comment deeply on the results. In the case of the finite medium, at fixed times and close to the boundaries, the diffusion effect is more controlled by the first term of equation $(3-31)$ due to the eigen function $\sin\frac{n\pi y}{h}$ in the second term. As time gets longer and longer, the second term still contributes to the diffusive effect but effect is majorly controlled by the first term of $U_0\frac{y}{h}t$. The diffusion effect is recurring in the finite medium.

If the motion of the boundary is now sinusoidal, and represented by

$$F(t) = U_o \cos(\omega t - \beta) \qquad (3-33)$$

From equation $(3-26)$, the solution $u(y,t)$ becomes

$$u(y,t) = -\frac{2D}{h}\int_{\tau=0}^{t} U_o \cos(\omega t - \beta) \sum_{n=1} e^{-D\alpha_n^2(t-\tau)}(-1)^n \alpha_n \sin \alpha_n y \qquad (3-34)$$

CHAPTER 4: FLOW FIELD WITH TIME-DEPENDENT MOVING BOUNDARY; GREEN'S FUNCTION APPROACH

4.1 Greens Function In An Infinite Medium

Fluid flow in a semi-infinite medium with time-dependent moving boundary can also be solved by employing Greens Function method. The general diffusion equation in a semi-infinite medium with initial and boundary conditions as stated is considered.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad (4-1a)$$

$$u(0,t) = g(t)\ at\ y = 0 \qquad (4-1b)$$

$$u(y,0) = f(y)\ for\ t = 0 \qquad (4-1c)$$

To solve the semi-infinite medium diffusion problem by using Green's function, it seem proper to solve the diffusion problem in an unbounded space and then employ the method of images to obtain the solution for the semi-infinite medium. Rewriting equation $(4-1)$ in the unbounded space :

$$\psi(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial y^2}) = 0 \qquad (4-2a)$$

$$\psi(y,t_0) = F(y)\ for\ t = t_0 \qquad (4-2b)$$

where D is the diffusion coefficient, t_0 is the initial time, instead of zero in (4-1c) and ψ is the solution variable. Let the differential operator L be defined as:

$$L = (\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial y^2}) \qquad (4-3)$$

The intention is to determine $\psi(y,t)$ for all $t > t_0$ given:

$$L\psi = 0 \ and \ \psi(y,t_0) = F(y) \ for \ t = t_0 \qquad (4-4)$$

Define the propagator K_0 by

$$LK_0(y,t|y',t') = 0, \ for \ t > t_0 \qquad (4-5a)$$

$$K_0(y,t'|y',t') = \delta(y-y'), \ (at \ equal \ times) \qquad (4-5b)$$

$$K_0 = 0 \ as \ y \to \infty, \qquad (4-5c)$$

K_0 satisfies the homogeneous equation without a delta function on the right, Therefore it is not a Green's function. K_0 is subject to the boundary conditions $(4-5c)$. Once K_0 is known, then the problem is solved by the following expressions.

$$\psi(y,t) = \int_{-\infty}^{\infty} dy' \ (K_0(y,t|y',t_0)F(y') \qquad (4-6)$$

T o obtain an expression for K_0 we proceed by the following:
Equation $(4-6)$ is substituted into $(4-4)$ and written as

$$L\psi = L\int dy' K_0\psi(y') = \int dy' \ (LK_0)\psi(y') = 0 \qquad (4-7)$$

Satisfying the initial condition of equation $(4-5b)$:

$$\psi(y,t_0) = \int_{-\infty}^{\infty} dy' \ (K_0(y,t_0|y',t_0)F(y') = \int dy'\delta(y-y')F(y') = F(y) \qquad (4\text{--}8)$$

If K_0 is expressed as a Fourier integral with respect to its y-independence,

$$K_0(y,t|y',t') = \int_{-\infty}^{\infty} dk \ A(k,t,y',t')exp(iky) \qquad (4-9)$$

Substitute the Fourier integral of K_0 into $4-5a$

$$LK_0 = \int dk L(A\ exp(iky)) = \int dk\ [\frac{\partial A}{\partial t} + \nu k^2 A] exp(iky) = 0 \qquad (4-10)$$

$$\frac{\partial A}{\partial t} = -\nu k^2 A, \qquad A = B(k, y', t) exp(-\nu k^2 t) \qquad (4-11a, b)$$

Substitute $(3-32b)$ into $(3-31)$, set $t = t'$ and appeal to $(4-5b)$.

$$\int dk\ exp(-\nu k^2 t') B(k, y', t') exp(iky)\ = \delta(y - y') \qquad (4-12)$$

Using the standard Fourier representation of $\delta(y-y')$ $(4-12)$ becomes

$$\int dk\ exp(-\nu k^2 t') B(k, y', t') exp(iky) = \frac{1}{2\pi}\int dk\ exp(ik(y - y')) \qquad (4-13)$$

Equating coefficients of exp(iky),

$$B(k, y', t') = \frac{1}{2\pi} exp(\nu k^2 t') exp(iky')$$

whence

$$A(k, t, y', t') = \frac{1}{2\pi} exp(-\nu(t - t')k^2) exp(iky) \qquad (4-14)$$

$$K_0(y, t|y', t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ exp(-\nu(t - t')k^2) exp(ik(y - y')) \qquad (4-15)$$

Let $\xi = y - y'$ and $\tau = t - t'$ Equation $(4-15)$ is rewritten as

$$K_0(\xi, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ exp(-\nu\tau k^2 + ik\xi) \qquad (4-16)$$

$K_0(\xi, \tau)$ is evaluated from first principles by completing the square in the exponent in $(4-16)$

$$(-\nu\tau k^2 + ik\xi) = -\nu\tau(k^2 - \frac{ik\xi}{\nu\tau})$$

$$-\nu\tau(k^2 - \frac{ik\xi}{\nu\tau}) = -\nu\tau[(k - \frac{i\xi}{2\nu\tau})^2 + \frac{\xi^2}{4\nu^2\tau^2}] \qquad (4-17)$$

substituting $(4-17)$ into$(4-16), K_0$ is written as:

$$K_0(\xi,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ exp\Big[-\nu\tau[(k - \frac{i\xi}{2\nu\tau})^2 + \frac{\xi^2}{4\nu^2\tau^2}]\Big] \qquad (4-18)$$

$$K_0(\xi,\tau) = \frac{1}{2\pi} exp(-\frac{\xi^2}{4\nu\tau})\int_{-\infty}^{\infty} dk\ exp[-\nu\tau(k - \frac{i\xi}{2\nu\tau})^2] \qquad (4-19)$$

Changing integration variables to $\sigma = (k - \frac{i\xi}{2\nu\tau})$ Then the integration becomes $\int_c d\sigma\ exp(-\nu\tau\sigma^2)$ This form of integration conforms to the Gaussian integral, the solution of which is:

$$\int_c = \int_{-\infty}^{\infty} d\sigma\ exp(-\nu\tau\sigma^2) = (\frac{\pi}{\nu\tau})^{\frac{1}{2}} \qquad (4-20)$$

Substituting $(4-20)$ into $(4-19)$, the expression for K_0 can now be expressed as:

$$K_0(\xi,\tau) = (\frac{\pi}{\nu\tau})^{\frac{1}{2}} \frac{1}{2\pi} exp(-\frac{\xi^2}{4\nu\tau})$$

$$K_0(\xi,\tau) = \frac{1}{(4\pi\nu\tau)^{\frac{1}{2}}} exp(-\frac{\xi^2}{4\nu\tau}) \qquad (4-21)$$

Equation $(4-16)$ can now be written as:

$$\psi(y,t) = \int_{-\infty}^{\infty} dy'[\frac{1}{[4\pi D(t-t_0)]^{\frac{1}{2}}} exp(-\frac{(y-y')^2}{4D(t-t_0)})]\psi(y',t_0) \qquad (4-22)$$

Equation $(4-22)$ expresses the solution for the diffusion equation in an unbounded space but with an initial condition given.

To expand the diffusion problem of equation $(4-2)$ and to make it more general, we include a source term. Restating problem $(4-2)$ but with a source term and initial condition:

$$\psi(\frac{\partial}{\partial t} - \nu\frac{\partial^2}{\partial y^2}) = \rho(y,t) \qquad (4-23a)$$

$$\psi(y, t_0) = F(y) \ for \ t = t_0 \qquad (4-23b)$$

where $\rho(y, t)$ is the source term. To solve problem $(4-23)$, an auxiliary problem for the same region is considered. If G_0 is the Green's function in unbounded space, then introducing G_0 into problem $(4-23)$, we obtain:

$$LG_0 = (\frac{\partial}{\partial t} - \nu\nabla^2)G_0(r, t \mid r', t') = \delta(t-t')\delta(r-r'), \qquad (4-24a)$$

$$G_0 = 0 \ for \ t < t') \qquad (4-24b)$$

$$G_0 = 0 \ as \ r \to \infty \qquad (4-24c)$$

In the auxiliary problem of $(4-24)$, the source is a unit impulsive source for a three-dimensional problem, the $\delta(r-r')$ represents a point source located at r', while the delta function $\delta(t-t')$ indicates that it is an instantaneous source releasing its energy spontaneously at time t' In the case of two-dimensional problems, $\delta(r-r')$ is a two dimensional delta function that characterizes a line energy source located at r', while for the one-dimensional problems $\delta(r-r')$ is a one-dimensional delta function which represents a plane surface energy source located at r'. The physical significance of the Green's function $G(r, t \mid r', t')$ for the three-dimensional problems is as follows: It represents the diffusive quantity at the location r, at time t, due to an instantaneous point source of unit strength, located at the point r', releasing its energy spontaneously at time t = t'. The auxiliary problem satisfied by Green's function is valid over the same region as the original physical problem $(4-23)$, but the boundary conditions $(4-24b)$ is the homogeneous version of the boundary conditions $(4-23b)$ and the initial condition is zero. On the basis of this definition, the physical significance of Green's function may be interpreted as;

$$G(r, t \mid r', t') \equiv G(effect \mid impulse) \qquad (4-25)$$

The first part of the argument, $(r,\ t)$ represents the effect, that is the diffusive quantity in the medium at the location r at time t, while the second part, $(r',\ t')$ represents the impulse, that is, the impulsive (instantaneous) point source located at r', releasing its energy spontaneously at time t'.

The usefulness of Green's function lies in the fact that the solution of the original problem $(4-2)$, can be represented only in terms of Green's function. Therefore, once the Green's function is known, the diffusive quantity distribution $\psi(r,t)$ in the medium is readily computed.

$$(\frac{\partial}{\partial t} - \nu\nabla^2)G_0(R,\tau) = \delta(\tau)\delta(R), \qquad (4-26a)$$

$$G_0 = 0\ for\ \tau < 0) \qquad (4-26b)$$

$$G_0 = 0\ as\ R \to \infty \qquad (4-26c)$$

$$K_0\ fully\ determines\ G_0$$

$$G_0(R,\tau) = \theta(\tau)K_0(R,\tau) \qquad (4-27)$$

When $\tau > 0,\ G_0\ and\ K_0\ coincides.$

This leads to a more explicit form of the initial condition on G_0, obtained by taking the limit of equation $(4-27)$ as $\tau \to 0$;

$$\lim_{\tau\to 0+} G_0(R,\tau) = \lim_{\tau\to 0+} K_0(R,\tau) = K_0(R,0) = \delta(R) \qquad (4-28)$$

The Green's function in unbounded space G_0, therefore can be equated to the expression for K_0, in $(4-21)$.

$$G_0 = \frac{1}{[4\pi\nu(t-t')]^{\frac{1}{2}}} exp(-\frac{(r-r')^2}{4\nu(t-t')}) \qquad (4-29)$$

Alternative Method For Determining Green's Function For Unbounded Space

Restating the diffusion equation with boundary and initial conditions :

$$\frac{\partial^2 u}{\partial y^2} - \frac{1}{\nu}\frac{\partial u}{\partial t} = \rho(y,t) \qquad (4-30)$$

The Green's function for the one dimensional diffusion equation is governed by

$$\frac{\partial G}{\partial t} - \nu\frac{\partial^2 G}{\partial y^2} = \delta(y-y')\delta(t-t') \qquad (4-31a)$$

$$\lim_{|y|\to\infty} |G(y,t|y',t')| < \infty \qquad (4-31b)$$

$$G(y,0 \mid y',t') = 0 \qquad (4-31c)$$

To find the Green's function $G(y,t \mid y',t')$, we begin by taking the Laplace transform of (4-31)

$$g(y,t \mid y',t') = \int_0^\infty e^{-st}G(y,t \mid y',t')dt \qquad (4-32)$$

$$\frac{\partial^2 g}{\partial y^2} - \frac{s}{\nu}g = -\frac{\delta(y-y')}{\nu}e^{-st'} \qquad (4-33)$$

Next we take the Fourier transform of (4-33) :

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ikt}dt \qquad (4--34)$$

$$\widehat{F}(\frac{d^2 g}{dy^2}) = (-ik)^2\widehat{F}(g) = -k^2\bar{G}$$

$$\widehat{F}(\frac{s}{\nu}g) = \frac{s}{\nu}\bar{G}, \qquad \widehat{F}[\delta(y-y')] = \frac{e^{iky'}}{\sqrt{2\pi}}$$

Fourier transform of (4-33) therefore becomes:

$$-(k^2\bar{G} + \frac{s}{\nu}\bar{G}) = -\frac{e^{iky'}.e^{-st'}}{\sqrt{2\pi}\nu} \qquad (4-35)$$

$$\bar{G}(k^2+\frac{s}{\nu})=\frac{1}{\sqrt{2\pi}}\frac{e^{iky'}e^{-st'}}{\nu} \qquad (4-36)$$

$$\bar{G}=\frac{1}{\sqrt{2\pi}}\frac{1}{(k^2+\frac{s}{\nu})}\frac{e^{iky'}e^{-st'}}{\nu} \qquad (4-37)$$

Fourier inverse transform is given by :

$$F(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(k)e^{-iky}dk \qquad (4-38)$$

Taking the inverse Fourier transform of (4-37), we obtain:

$$g=\frac{e^{-st'}}{2\pi\nu}\int_{-\infty}^{\infty}\frac{e^{-i(y-y')k}}{(k^2+\frac{s}{\nu})}dk \qquad (4-39)$$

Equation (4-39) is transformed into a closed contour and evaluated by the residue theorem. The residue theorem is given by :

$$\int_{-\infty}^{\infty}f(y)dy=2\pi i\sum Resf(z) \qquad (4-40)$$

In equation (4-39) $f(z)=\frac{1}{k^2+\frac{s}{D}}$

$$Resf(z)=[\frac{1}{2k}]_{z=z_1}\ where\ z_1=k_1=\frac{i\sqrt{s}}{\sqrt{\nu}} \qquad (4-41)$$

$$Resf(z)=Res[\frac{1}{2k}]_{z=z_1}=\frac{i\sqrt{\nu}}{2\sqrt{s}} \qquad (4-42)$$

$$\int_{-\infty}^{\infty}\frac{e^{-i(y-y')k}}{(k^2+\frac{s}{D})}dk=2\pi i[e^{-i(y-y')\frac{i\sqrt{s}}{\sqrt{\nu}}}]\frac{i\sqrt{\nu}}{2\sqrt{s}} \qquad (4-43)$$

$$g=\frac{e^{-st'}}{2\pi\nu}\int_{-\infty}^{\infty}\frac{e^{-i(y-y')k}}{(k^2+\frac{s}{\nu})}dk=\frac{e^{-st'}}{2\pi\nu}2\pi i[e^{-i(y-y')\frac{i\sqrt{s}}{\sqrt{\nu}}}]\frac{i\sqrt{\nu}}{2\sqrt{s}} \qquad (4-44)$$

$$g=\frac{e^{-st'}[e^{-i(y-y')\frac{\sqrt{s}}{\sqrt{\nu}}}]}{\sqrt{\nu}\sqrt{s}} \qquad (4-45)$$

$$g = \frac{e^{-|y-y'|\frac{\sqrt{s}}{\sqrt{\nu}} - st'}}{2\sqrt{\nu}\sqrt{s}} \qquad (4-46)$$

Taking the inverse of the Laplace transform of (4-46) we obtain:

$$G(y,t|y',t') = \frac{1}{\sqrt{4\pi\nu(t-t')}} exp\Big(-\frac{(y-y')^2}{4\nu(t-t')}\Big) \qquad (4-47)$$

This expression for $G(y,t|y',t')$ compares well with equation (4-29). While $(4-29)$ is for generalized coordinates expression $(4-47)$ is for one dimensional coordinate. the generalized solution in terms of Green's function can be expressed as:

$$\psi(y,t) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dV \; G_0(r,t|r',t)\rho(r',t') \qquad (4-48)$$

In other words $\psi(r,t)$ is just the linear combinations of the contributions from all the elementary sources $\rho(r',t')dV'dt'$ that have ever acted.

$$L\psi = \int_{-\infty}^{t} dt'dV'[LG_0]\rho(r',t') = \int_{-\infty}^{t} dt' \int_{-\infty}^{t} dV \; \delta(t-t')\delta(r-r')\rho(r',t') \qquad (4-49)$$

Equation $(4-48)$ may be written as:

$$\psi(y,t) = \int_{t_0}^{t} dt' \int dV' \; G_0(r,t|r',t)\rho(r',t') = \int_{t_0}^{t} dt' \int dV' \; K_0(r,t|r',t)\rho(r',t') = f_0(r,t) \qquad (4-50)$$

where f_0 stands for the integrals regarded as explicit constructs from the data ρ. The general inhomogeneous problem can now be solved by combining the solutions $(4-48)$ of the homogeneous equation for given $\psi(r,t_0)$ with the solution $(4-50)$ of the inhomogeneous equation with given ρ.

$$\psi(r,t) = f_0(r,t) + h_0(r,t)$$

$$\psi(r,t) = \int_{t_0}^{t} dt' \int dV' \; G_0(r,t|r',t)\rho(r',t') + \int dV' \; (G_0(r,t|r',t_0)\psi(r',t_0) \qquad (4-51)$$

$$\psi(r,t) = \int_{t_0}^{t} dt' \int dV' \ \frac{1}{[4\pi\nu(t-t')]^{\frac{n}{2}}} exp(-\frac{\mid r-r' \mid^2}{4\nu(t-t')})\rho(y',t_0) \ +$$

$$\int dV' \frac{1}{[4\pi = \nu(t-t_0)]^{\frac{n}{2}}} exp(-\frac{\mid r-r' \mid^2}{4\nu(t-t_0)})\psi(r',t_0) \qquad (4-52)$$

The first term on the right hand side of equation $(4-52)$ is for the contribution of the source term $\rho(y', t_0)$, to the solution $\psi(r, t)$. The second term on the right is for the contribution of the initial condition $\psi(r', t_0)$ to the solution $\psi(r, t)$.

4.2 The Method Of Images

The Green's function may be required in a semi-infinite medium. Considering a three dimension medium in a semi-infinite medium. The domain can be considered to be a halfspace V, that is bounded by the x-y plane plus say an infinite hemisphere on which G is likewise to vanish. When we place an x-y plane at $z = 0$ and mirror a point $(x', y', +z)$ on the positive z-axis in the x-y plane, the mirrored image of the point will be at $x', y', -z)$. We are looking for the Green's function, when a unit negative source is situated at $r' \equiv (x', y', -z)$. By symmetry along the z-axis, Green's function vanishes on the median plane between r', a location above the plane whose z-coordinate is positive, and r'' which is at location that a reflection of the point r' in the x-y plane, ie Green's function vanishes on the x-y plane. Similar to equation $(4-29)$ and employing the method of images, we can express the Green's function referencing the mirrored image point as the excitation point as

$$G_0(r, t|r'', t'') = \frac{1}{[4\pi\nu(t-t')]^{\frac{1}{2}}} exp(-\frac{(r-r')^2}{4\nu(t-t')}) \qquad (4-53)$$

Defining $R'' = r - r''$ and $R = r - r'$, the Green's function for the halfspace semi-infinite medium can expressed as

$$G_D(r, t \mid r', t') = G_0(r, t \mid r', t') - G_0(r, t \mid r'', t'') \qquad (4-54)$$

Combining equations $(4-47)$ and $(4-54)$, we obtain the Green's function for diffusion equation in semi-infinte medium as:

$$G_D(r, t \mid r', t') = \frac{1}{[4\pi\nu(t-t')]^{\frac{1}{2}}} \left(exp\Big(-\frac{(r-r')^2}{4\nu(t-t')}\Big) - exp\Big(-\frac{(r+r')^2}{4\nu(t-t')}\Big) \right) \qquad (4\text{--}55)$$

We want to determine the flow field for a semi-infinite medium with a moving boundary. The top boundary is given a constant velocity 'v' The boundary moves as $y = vt$. The equation to be solved is stated as:

$$\frac{\partial^2 u}{\partial y^2} - \frac{1}{\nu}\frac{\partial u}{\partial t} = \rho(y,t) \qquad (4-56)$$

$$u(0,t) = v \ at \ y = 0$$

$$u(y,0) = 0$$

The Green's function for the one dimentional diffusion equation is governed by

$$\frac{\partial G}{\partial t} - \nu\frac{\partial^2 G}{\partial y^2} = \delta(y-y')\delta(t-t') \qquad (4-57a)$$

$$G(y,t|y',t')|_{y=vt} = 0, \qquad t' < t \qquad (4-57b)$$

$$lim_{y\to\infty}|G(y,t|y't'| < \infty$$

$$G(y,0 \mid y',t') = 0 \qquad 0 < y \qquad (4-57c)$$

We begin by writing Green's function as linear combination of the free -space Green's function plus a presently unknown function $\Gamma(y,t)$. The purpose of the free-space Green's function is to eliminate the delta functions in (4-56a) while $\Gamma(y,t)$ is a homogeneous solution that was introduced so that $g(y,t|y',t')$ satisfies the boundary conditions. Therefore,

$$G(y,t|y',t') = \frac{1}{\sqrt{4\pi\nu(t-t')}}exp\Big(-\frac{(y-y')^2}{4\nu(t-t')}\Big) + \Gamma(y,t) \qquad (4-58)$$

Substitute (4-58) into (4-57a), we find that

$$\frac{\partial \Gamma}{\partial t} - \nu\frac{\partial^2 \Gamma}{\partial y^2} = \delta(y-y')\delta(t-t') \qquad (4-59a)$$

with the boundary conditions that

$$\Gamma(y,t)|_{y=vt} = -\frac{1}{\sqrt{4\pi\nu(t-t')}} exp\Big(-\frac{(vt-y')^2}{4\nu(t-t')}\Big) \qquad (4-59b)$$

and

$$lim_{y\to\infty}|\Gamma(y,t)| < \infty \qquad t' < t \qquad (4-59c)$$

and the initial condition that

$$\Gamma(y,0) = 0 \qquad 0 < y \qquad (4-59d)$$

To eliminate the moving boundary, we introduce the new independent variable $z = y - vt$ and $\tau = t - t'$ and the dependent variable

$$\Gamma(y,t) = exp\Big[-\frac{vz}{2\nu} - \frac{v^2\tau}{4\nu}\Big]\ \omega(z,\tau) \qquad (4-60)$$

substitute (4-60) into (4-59), we obtain

$$\frac{\partial\omega}{\partial\tau} = \nu\frac{\partial^2\omega}{\partial z^2}, \quad 0 < z,\tau \qquad (4-61a)$$

with the boundary conditions

$$\omega(0,\tau) = -\frac{1}{2\sqrt{\pi\nu\tau}}\ exp\Big[\frac{v\epsilon_0}{2\nu} - \frac{{\epsilon_0}^2\tau}{4\nu\tau}\Big] \qquad (4-61b)$$

and

$$lim_{z\to\infty}|\omega(z,\tau)| < \infty \qquad 0 < \tau \qquad (4-61c)$$

and the initial condition

$$\omega(z,0) = 0 \qquad 0 < z \qquad (4-61d)$$

where $\epsilon_0 = \epsilon - vt'$ To solve (4-61), we take their Laplace transform

$$\nu \frac{d^2W}{dz^2} - sW = 0, \qquad (4-62a)$$

with

$$W(0,s) = -\frac{1}{2\sqrt{\nu s}}\ exp\Big[-\frac{\epsilon_0\sqrt{s}}{\sqrt{\nu}} + \frac{v\epsilon_0}{2\nu}\Big] \qquad (4-62b)$$

and

$$lim_{z\to\infty}|W(z,s)| < \infty \qquad (4-62c)$$

The solution to (4-62) is

$$W(z,s) = -\frac{1}{2\sqrt{\nu s}}\ exp\Big[-\frac{(z+\epsilon_0)\sqrt{s}}{\sqrt{\nu}} + \frac{v\epsilon_0}{2\nu}\Big] \qquad (4-63)$$

Taking the inverse of (4-63) we obtain,

$$\omega(z,\tau) = -\frac{H(\tau)}{2\sqrt{\pi\nu\tau}}\ exp\Big[-\frac{(z+\epsilon_0)^2}{4\nu\tau} + \frac{v\epsilon_0}{2\nu}\Big] \qquad (4-64)$$

so that

$$\Gamma(y,t) = -\frac{H(t-t')}{2\sqrt{\pi\nu(t-t')}}\ exp\Big[-\frac{(y+\epsilon-2vt')^2}{4\nu(t-t')} + \frac{v(\epsilon-vt')}{\nu}\Big] \qquad (4-65)$$

substitute (4-65) into (4-58) gives the Green's function:

$$G(y,t|y',t') = \frac{H(t-t')}{\sqrt{4\pi\nu(t-t')}} exp\Big(-\frac{(y-y')^2}{4\nu(t-t')}\Big)$$

$$-\frac{H(t-t')}{\sqrt{4\nu\pi(t-t')}}\ exp\Big[-\frac{(y+\epsilon-2vt')^2}{4\nu(t-t')} + \nu\frac{v(\epsilon-vt')}{\nu}\Big] \qquad (4-66)$$

$$G(y,t|y',t') = \frac{H(t-t')}{2\sqrt{\pi\nu(t-t')}}\Big[exp(-\frac{(y-y')^2}{4\nu(t-t')}) - exp(-\frac{(y+\epsilon-2vt')^2}{4\nu(t-t')} + \frac{v(\epsilon-vt')}{\nu})\Big] \qquad (4\text{--}67)$$

4.3 Green's Function Method For Flow In A Finite Medium.

The fluid flow field bounded by a finite medium of two infinite plates and governed by a pure diffusion equation with the top boundary moving with a time dependent motion can also be solved by the Greens' function method. Restating the governing equation with its boundary and initial conditions:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad (4-68a)$$

$$u(y,t) = 0 \ at \ y = 0 \qquad (4-68b)$$

$$u(y,t) = F(t) \ at \ y = h \qquad (4-68c)$$

$$u(y,t) = 0 \ for \ t = 0 \qquad (4-68d)$$

In order to be able to establish an expression of solution to the diffusion problem with boundary and initial values employing the Green's function, one need to explore the characteristics of time-invariance and symmetry of the Greens funtion. The popagator K, is similar to the Green's function but not exactly the same as the Green's function. It satisfies the homogeneous version of the diffusion equation.

$$LK = (\frac{\partial}{\partial t} - \nu \nabla^2) K(r,t \mid r',t) = 0 \ for \ t \geq t' \qquad (4-69a)$$

where L is the operator $(\frac{\partial}{\partial t} - \nu \nabla^2)$
At equal times,

$$K(r,t' \mid r',t') = \delta(r - r') \qquad (4-69b)$$

$$K(r,t \mid r',t') = 0 \ for \ r \ on \ S \qquad (4-69c)$$

where the boundary surface is designated by "S": The Greens function is defined by :

$$LG(r,t \mid r',t') = \delta(t-t')\delta(r-r') \qquad (4-70a)$$

$$G(r,t \mid r',t') = 0 \ for \ t < t' \qquad (4-70b)$$

$$G(r,t \mid r',t') = 0 \ for \ r \ on \ S \qquad (4-70c)$$

Green's function can be related to the propagator K as:

$$G(r,t \mid r',t') = \theta(t-t')K(r,t \mid r',t')' \qquad (4-71)$$

K and G are functions of t and t' only through $\tau = t - t'$ hence they can be written as $K(r,r';\tau) \ and \ G(r,r';\tau)$ Since $\tau = (t-t')$ is unaffected by the replacements $t \to -t', \ \ t' \to -t,$ we have:

$$K(r,t \mid r',t') = K(r,-t' \mid r',-t) \qquad (4-72)$$

Rewriting equations $(4-69a) \ and (4-69b)$

$$(\frac{\partial}{\partial t} - \nu\nabla^2)K(r,t \mid r',t) = 0, \quad K(r,t' \mid r',t') = \delta(r-r') \qquad (4-73a,b)$$

In $(4-73a)$ if (r',t') is replaced by (r'',t'') and rearranged by using $(4-72)$, we obtain:

$$(\frac{\partial}{\partial t} - \nu\nabla^2)K(r,t \mid r'',t'') = (\frac{\partial}{\partial t} - \nu\nabla^2)K(r,-t'' \mid r'',-t) = 0 \qquad (4-74)$$

Changing the names of the two variables as $t \to -t$, hence:

$$\frac{\partial}{\partial t} \to \frac{-\partial}{\partial t}, \quad t'' \to -t'' \quad this \ yields:$$

$$(-\frac{\partial}{\partial t}-\nu\nabla^2)K(r,t''\mid r'',t)=0,\quad K(r,-t''\mid r'',t'')=\delta(r-r")\qquad(4-75a,b)$$

Multiply $(4-73a)$ by $K(r,t''\mid r'',t)$ and $(4-75a)$ by $K(r,t\mid r',t')$ Take the difference, and integrate it:

$$\int_{t_1}^{t_2}dt\int_V dV\{[K(r,t''\mid r'',t)\frac{\partial}{\partial t}K(r,t\mid r',t')+\frac{\partial K(r,t''\mid r'',t)}{\partial t}K(r,t\mid r',t')]$$

$$-\nu[K(r,t''\mid r'',t)\nabla^2K(r,t\mid r',t')-(\nabla^2K(r,t''\mid r'',t))K(r,t\mid r',t')]\}=0\quad(4-76)$$

The contents of the second pair of square brackets are integrated with respect to volume, using Green's theorem, which yields:

$$\int_{t_1}^{t_2}dt\int_S dS[K(r,t''\mid r'',t)\partial_nK(r,t\mid r',t')-\partial_nK(r,t''\mid r'',t)K(r,t\mid r',t')]=0$$

Accordingly, the contents of the first pair square bracket in $(5-9)$ must also integrate to zero. Inspection reveals that these contents are just the total time-derivative of the products of the two propagators. In shorthand form,

$$\int_{t_1}^{t_2}dt\int_V dV\frac{\partial}{\partial t}(KK)=\int_{t_1}^{t_2}dt\frac{d}{dt}\int_V dV\frac{\partial}{\partial t}(KK)\quad(4-77)$$

$$=\int_V dV(KK)\Big|_{t=t_2}-\int_V dV(KK)\Big|_{t=t_1}=0\quad(4-78)$$

In other words the volume integral $\int_V dVK(r,t"\mid r",t)K(r,t\mid r',t')$ is independent of t.

Finally we equate the values of this integral at $t=t"$ and $t=t'$, determining them by exploiting the delta-functions on the right of $(4-70b)$ and $(4-68b)$. Finally we equate the values of this integral at $t=t''$ and $t=t'$, determining them by exploiting the delta-functions on the right of $(4-75b)$ and $(4-73b)$.

$$\int_V dVK(r,t''\mid r'','')K(r,t''\mid r',t')=\int_V dV\delta(r-r'')K(r,t''\mid r',t'')=K(r'',t''\mid r',t')$$

$$= \int_V dV K(r, t'' \mid r'', t') K(r, t' \mid r', t') = \int_V dV K(r, t'' \mid r'', t') \delta(r - r')$$

$$= K(r', t'' \mid r'', t') \quad (4-79)$$

$$K(r'', t'' \mid r', t') = K(r', t'' \mid r'', t') \quad (4-80)$$

If the double primes are dropped, equation $(4-80)$ can be written as:

$$K(r, t \mid r', t') = K(r', t \mid r, t') \quad (4-81)$$

The time translation invariance of $(4-72)$ and the symmetry of $(4-81)$ together imply reciprocity relation.

$$K(r, t \mid r', t') = K(r', -t' \mid r, -t) \quad (4-82)$$

By virtue of $G(r, r'\tau) = \theta(\tau)K(r, r' : \tau)$', the Greens function entails the same properties of time translation, symmetry and reciprocity, namely:

$$G(r, t \mid r', t') = G(r, -t' \mid r', -t) = G(r', t \mid r, t') = G(r', -t' \mid r, -t) \quad (4-83)$$

The symmetry relations of $(4-83)$ show the K and G obey the same boundary conditions as functions of r' as of r. The diffusion equation can be written employing the Green's function and the reciprocity relation as:

$$(\frac{\partial}{\partial t} - \nu\nabla^2) G(r, t \mid r', t') = (\frac{\partial}{\partial t} - \nu\nabla^2) G(r', -t' \mid r, -t) = \delta(t-t')\delta(r-r') \qquad (4\text{–}84)$$

Since $t \to -t'$, hence $\frac{\partial}{\partial t} = -\frac{\partial}{\partial t}$ Also $r \to -r'$, hence $\nabla'^2 = \nabla^2$ This yields:

$$(-\frac{\partial}{\partial t} - \nu\nabla'^2) G(r, t \mid r', t') = \delta(-t + t')\delta(r - r') = \delta(t - t')\delta(r - r') \qquad (4-85)$$

Equation $(4-85)$ ia called the reciprocal equation or often called the adjoint equation. In order to solve the general diffusion problem, an expression involving Green's function is derived that can be used to solve it. Rewrite the inhomogeneous diffusion problem using primed variables.

$$(\frac{\partial}{\partial t'} - \nu \nabla'^2)\psi(r', t') = \rho(r', t') \qquad (4-86)$$

under initial conditions $\psi(r', t_0)$ and inhomogeneous boundary conditions on the boundary S.

$$(-\frac{\partial}{\partial t'} - \nu \nabla'^2)G(r, t \mid r', t') = \delta(t - t')\delta(r - r') \qquad (4-87)$$

Multiply $(4-86)$ by $G(r, t \mid r', t')$ and $(4-87)$ by $\psi(r', t')$

$$G(r, t \mid r', t')(\frac{\partial}{\partial t'} - \nu \nabla'^2)\psi(r', t') = \rho(r', t')G(r, t \mid r', t') \qquad (4-88)$$

$$\psi(r', t')(-\frac{\partial}{\partial t'} - \nu \nabla'^2)G(r, t \mid r', t') = \delta(t - t')\delta(r - r')\psi(r', t' \qquad (4-89)$$

Subtracting $(4-89)$ from $(4-88)$

$$G(r, t \mid r', t')(\frac{\partial}{\partial t'}\psi(r', t') + \psi(r', t')(\frac{\partial}{\partial t'}G(r, t \mid r', t')$$

$$-\nu[G(r, t \mid r', t')\nabla'^2\psi(r', t') - \psi(r', t)\nabla'^2 G(r, t \mid r', t')]$$

$$= G(r, t \mid r', t')\rho(r', t') - \psi(r', t)\delta(t - t')\delta(r - r') \qquad (4-90)$$

Integrate with respect to dV' over V and with respect to t' from t_0 to t^+, where $t^+ = t + \epsilon$,

$$\int_{t_0}^{t^+} dt \int_V dV \frac{\partial}{\partial t'}[G(r, t \mid r', t')\psi(r', t')] - \nu[G(r, t \mid r', t')\nabla'^2\psi(r', t') - \psi(r', t)\nabla'^2 G(r, t \mid r', t')]$$

$$= \int_{t_0}^{t^+} dt \int_V dV G(r,t \mid r',t')\rho(r',t') - \int_{t_0}^{t^+} \int_V dV \psi(r',t)\delta(t-t')\delta(r-r') \qquad (4-91)$$

Applying Green's theorem to the second term on the left and noting that:

$$\int_{t_0}^{t^+} \int_V dV \psi(r',t)\delta(t-t')\delta(r-r') = \psi(r,t) \qquad (4-92)$$

$$\int_V dV'[G(r,t \mid r',t')\psi(r',t')]|_{t'=t_0}^{t'=t^+} - \nu[\int_{t_0}^{t^+} dt' \int_S dS[G(r,t \mid r',t')\partial_n'\psi(r',t') - \psi(r',t)\partial_n' G(r,t \mid r',t')]$$

$$= \int_{t_0}^{t^+} dt \int_V dV G(r,t \mid r',t')\rho(r',t') - \psi(r',t) \qquad (4-93)$$

In the first term on the left, the upper limit vanishes because $t' = t^+ > t$, so that G vanishes. In the second term on the left, we replace t^+ by t.

$$-\int_V dV' G(r,t \mid r',t_0)\psi(r',t_0) - \nu[\int_{t_0}^{t^+} dt' \int_S dS[G(r,t \mid r',t')\partial_n'\psi(r',t') - \psi(r',t)\partial_n' G(r,t \mid r',t')]$$

$$= \int_{t_0}^{t^+} dt \int_V dV G(r,t \mid r',t')\rho(r',t') - \psi(r,t) \qquad (4-94)$$

Solving for $\psi(r,t)$

$$\psi(r,t) = \int_{t_0}^{t} dt' \int_V dV' G(r,t \mid r',t')\rho(r',t')$$

$$+\nu \int_{t_0}^{t} dt' \int_S dS'[G(r,t \mid r',t')\partial_n'\psi_s(r',t') - (\partial_n' G(r,t \mid r',t')\psi_s(r',t')]$$

$$+ \int_V dV' G(r,t \mid r',t_0)\psi(r,t_0) \qquad (4-95)$$

To determine the Green's function $G(y,t/y',t')$, for problem $(4-68)$ a homogeneous version with an initial condition of 'F(y)' is stated as :

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad (4-96a)$$

$$u(y,t) = 0 \ at \ y = 0 \qquad (4-96b)$$

$$u(y,t) = 0 \ at \ y = h \qquad (4-96c)$$

$$u(y,t) = F(y) \ for \ t = 0 \qquad (4-96d)$$

By employing the method of separation of variables similar to that used in chapter three, the solution to $(4-96)$ can be obtained as :

$$u(y,t) = [\sum_{n=1} \frac{2}{h} \int_{y'=0}^{h} e^{-D\alpha_n^2 t} \sin(\alpha_n y) \sin \alpha y'] F(y') dy' \qquad (4-97)$$

The homogeneous solution in terms of homogeneous Green's function for bounded diffusion equation with initial condition at time equal zero can be written as;

$$u(y,t) = \int_R G(y, t/y', 0) F(y') dy' \qquad (4-98)$$

Comparing equations $(4-97)$ and $(4-98)$, the homogeneous Green's function is given as:

$$G(y, t/y', 0) = \frac{2}{h} \sum_{n=1} e^{-D\alpha_n^2 t} \sin(\alpha_n y) \sin \alpha y' \qquad (4-99)$$

If the initial excitation where at a time " t' " other than zero, the Green's Function will be stated as:

$$G(y, t/y', t') = \frac{2}{h} \sum_{n=1} e^{-D(\frac{n\pi}{h})^2 (t-t')} \sin(\frac{n\pi}{h}) y) \sin \frac{n\pi}{h}) y' \qquad (4-100)$$

where $\alpha_n = \frac{n\pi}{h}$ There are other means of obtaining the Green's function of the diffusion equation we are dealing with. One other way is by use of Laplace transform and eigen function expansion. This technique is illustrated next.

Consider one dimensional diffusion equation over the interval $0 < y < h$

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial y^2} = f(y,t) \qquad 0 < y < h, \qquad (4-101)$$

To find the Green's function for this problem, we consider the following problem with Green's function " G " substituted into the governing equation:

$$\frac{\partial G}{\partial t} - \nu \frac{\partial^2 G}{\partial y^2} = \delta(y, y')\delta(t - t') \qquad 0 < y, y' < h \; 0 < t, t' \qquad (4-102a)$$

with the boundary conditions:

$$\alpha_1 G(0, t|y', t') + \beta_1 G_y(0, t|y', t') = 0, \quad 0 < t \qquad (4-102b)$$

and

$$\alpha_2 G(h, t|y', t') + \beta_2 G_y(h, t|y', t') = 0, \quad 0 < t \qquad (4-102c)$$

$$G(y, 0|y', t') = 0, \quad 0 < y < h \qquad (4-102d)$$

Taking the Laplace transform of (4-102) we have

$$\frac{\partial^2 g}{\partial y^2} - \frac{s}{D} g = -\frac{\delta(y, y')}{\nu} e^{-st'} \qquad 0 < y < h \qquad (4-103a)$$

with

$$\alpha_1 g(0, s|y', t') + \beta_1 g'(0, s|y', t') = 0, \quad 0 < t \qquad (4-103b)$$

and

$$\alpha_2 g(h, s|y', t') + \beta_2 g'(h, t|y', t') = 0, \quad 0 < t \qquad (4-103c)$$

Applying the technique of eigen function expansion, we have

$$g(y, s|y, t') = e^{-st'} \sum_{n=1}^{\infty} \frac{\varphi_n(y')\varphi_n(y)}{s + Dk_n^2} \qquad (4-104)$$

where $\varphi_n(y)$ is the nth orthonormal eigenfunction to the regular Sturm-Liouville problem

$$\varphi''(y) + k^2 \varphi(y) = 0, \qquad 0 < y < h, \qquad (4-105a)$$

subject to the boundary conditions

$$\alpha_1\varphi(0) + \beta_1\varphi'(0) = 0 \qquad (4-105b)$$

and

$$\alpha_2\varphi(h) + \beta_2\varphi'(h) = 0 \qquad (4-105c)$$

Taking the inverse of the Laplace transform (4-104) we have the Green's function

$$G(y,t|y',t') = \Big[\sum_{n=1}^{\infty}\varphi_n(y')\varphi_n(y)e^{-k_n^2D(t-t')}\Big]H(t-t') \qquad (4-106)$$

Restating equation (4-104)

$$\frac{\partial G}{\partial t} - \nu\frac{\partial^2 G}{\partial y^2} = \delta(y,y')\delta(t-t') \quad 0 < y,\ y' < h \ 0 < t,t' \qquad (4-107a)$$

with boundary and initial conditions

$$G(0,t|y',t') = G(h,t|y',t') = 0, \qquad (4-107b)$$

$$G(y,0|y',t') = 0, \qquad 0 < y < h \qquad (4-107c)$$

The Sturm-Liouville problem is

$$\varphi''(y) + k^2\varphi(y) = 0, \qquad 0 < y < h, \qquad (4-108)$$

with boundary conditions $\varphi(0) = \varphi(L) = 0$. The nth orthonormal eigenfunction to (4-108) is

$$\varphi_n(y) = \sqrt{\frac{2}{h}}\sin\Big(\frac{n\pi y}{h}\Big) \qquad (4-109)$$

Substitute (4-109) into (4-106) the Green's function for equation (4-107) is given as:

$$G(y,t|y',t') = \frac{2}{h}\Big\{\sum_{n=1}^{\infty} \sin\Big(\frac{n\pi y'}{h}\Big) \sin\Big(\frac{n\pi y}{h}\Big) e^{-\nu(\frac{n\pi}{h})^2(t-t')}\Big\} H(t-t') \qquad (4\text{—}110)$$

In shorthand representation , the solution equation of $(4-95)$ can be written as

$$\psi(r,t) = f(r,t) + g(r,t) + h(r,t) \qquad (4-111)$$

where

$$f(r,t) = \int_{t_0}^{t} dt' \int_{v} dv' G(r,t/r',t')\rho(r',t')$$

$$g(r,t) = D\int_{t_0}^{t} dt' \int_{v} ds'[G(r,t/r',t')\partial_n'\psi_s(r',t') - \partial_n' G(r,t/r',t')\psi_s(r',t')]$$

$$h(r,t) = \int_{v} dv' G(r,t/r',t_o)\psi((r',t_o)$$

In equation $(4-95)$, the term f(r,t) represents the contribution from the source term $\rho(r,t)$, $g(r,t)$ represents the contribution from the boundary condition to the general solution while $h(r,t)$ represents the contribution from the initial conditions to the solution. In solving the Couette flow problem of equation $(4-68)$, there is no source term in the diffusion problem, thus $\rho(r,t) = 0$. Therefore $f(r,t) = 0$. The initial condition of $(4-68d)$ is also zero. There is no contribution from the initial condition. That is $h(r,t) = 0$. The solution $\psi(r,t)$ is solely in response to the boundary condition, that is the flow initiated by the motion of the boundary at h. Since equation $(4-68)$ is being solvde in one dimension, the solution $u(y,t)$ can be expressed as:

$$u(y,t) = \nu\int_{t_0}^{t} dt' \int_{y} dy'[G(y,t/y',t')\partial_n' u_s(y',t') - \partial_n' G(y,t/y',t')u_h(y',t')] \qquad (4\text{—}112)$$

In the Couette problem of equation $(4-46)$, the boundary condition of $(4-46c)$ is specified. This makes the Green's Function on the boundary at $y = h$ zero. That is :

$$G(y,t/y',t') = 0 \quad at\ y = h \qquad (4-113)$$

Equation $(4-112)$ reduces to:

$$u(y,t) = \nu \int_{t'=0}^{t} dt' \int_{y'=0}^{h} dy' [-\frac{\partial G(y,t/y',t')}{\partial y'} F(t')] \qquad (4-114)$$

Rewriting equation $(4-110)$;

$$G(y,t/y',t') = \frac{2}{h} \sum_{n=1} e^{-\nu(\frac{n\pi}{h})^2(t-t')} \sin(\frac{n\pi}{h})y) \sin \frac{n\pi}{h})y' \qquad (4-115)$$

$$\frac{\partial G(y,t/y',t')}{\partial y'} = \frac{2}{h} \sum_{n=1} e^{-\nu(\frac{n\pi}{h})^2(t-t')} (\frac{n\pi}{h}) \sin \frac{n\pi}{h} y \cos \frac{n\pi}{h} y' \qquad (4-116)$$

Performing the spatial integration in equation $(4-114)$, it reduces to;

$$u(y,t) = -\nu \int_{t'=0}^{t} \frac{\partial G(y,t/y',t')}{\partial y'} |_{y'=h} F(t') dt' \qquad (4-117)$$

$$u(y,t) = -\frac{2\nu}{h} \sum_{n=1} e^{-\nu(\frac{n\pi}{h})^2(t)} (\frac{n\pi}{h}) \sin \frac{n\pi}{h} y \cos \frac{n\pi}{h} h \int_{t'=0}^{t} e^{-\nu(\frac{n\pi}{h})^2(t')} F(t') dt' \qquad (4-118)$$

$$u(y,t) = -\frac{2\nu}{h} \sum_{n=1} (-1)^n e^{-\nu(\frac{n\pi}{h})^2(t)} (\frac{n\pi}{h}) \sin \frac{n\pi}{h} y \int_{t'=0}^{t} e^{-\nu(\frac{n\pi}{h})^2(t')} F(t') dt' \qquad (4-119)$$

CHAPTER 5 : FLOW WITH RANDOM MOVING BOUNDARY

5.1 Stochastic Processes

In this chapter, the flow fields within the domain of both semi-infinite and finite are considered but with the boundary moving with a random motion. The flow field therefore has random characteristics and the solution methods are stochastic in nature. We begin first by looking at a general stochastic analysis of a flow field. Systems in which a certain time-dependent random variable $\chi(t)$ exists, can be described as a stochastic process, in which systems evolve probabilistic in time. The probability events can be assigned both joint and conditional probabilities, which we define as follows:

If U and V are two discrete random variables, the probability distribution for their simultaneous occurrence can be represented by a function with values f(u,v) for any pair of values (u,v) within the range of random variables U and V. It is customary to refer to this function as the joint probability function of U and V. The probability of an event B occurring when it is known that some event A has occurred is called a conditional probability and is denoted by P(B/A) which is read "the probability of B given A".

A set of values, $x_1, x_2, x_3, ...$ etc. of the random variable $\chi(t)$ can be measured at times $t_1, t_2, t_3, ...$ A set of joint probability densities exist $p(x_1, t_1; x_2, t_2; x_3, t_3; ....)$ which describe the system completely. In terms of these joint probability density functions, a conditional probability density function can be defined:

$$p(x_1, t_1; x_2, t_2; .... \mid y_1, \tau_1; y_2, \tau_2; ...)$$

$$= p(x_1, t_1; x_2, t_2; ....; y_1, \tau_1; y_2, \tau_2; ....)/p(y_1, \tau_1; y_2, \tau_2; ....) \qquad (5-1)$$

If the future can be determined by the knowledge of the most recent condition, the process is known as a Markov process, if the times are such that:

$$t_1 \geq t_2 \geq t_3 \geq ....\tau_1 \geq \tau_2 \geq ..... \qquad (5-2)$$

The concept of an evolution equation leads us to consider the conditional probabilities as predictions of the future values of $\chi(t)$ $(ie, x_1, x_2, , ...at\ times\ t_1, t_2, ...)$ given the knowledge of the past values $(y_1, y_2, ...at\ times\ \tau_1, \tau_2, ...)$. If the times satisfy $(5-2)$, the conditional probability is determined entirely by the knowledge of the most recent condition, as stated in the following equation.

$$p(x_1, t_1; x_2, t_2; .... \mid y_1, \tau_1; y_2, \tau_2; ...) = p(x_1, t_1; x_2, t_2; ....; \mid y_1, \tau_1) \qquad (5-3)$$

By definition of the conditional probability density:

$$p(x_1, t_1; x_2, t_2; \mid y_1, \tau_1;) = p(x_1, t_1 \mid x_2, t_2; y_1, \tau_1)p(x_2, t_2; \mid y_1, \tau_1) \qquad (5-4)$$

and using Markov assumption, we find:

$$p(x_1, t_1; x_2, t_2; y_1, \tau_1;) = p(x_1, t_1 \mid x_2, t_2;)p(x_2, t_2; \mid y_1, \tau_1) \qquad (5-5)$$

An arbitrary joint probability can be expressed simply as:

$$p(x_1, t_1; x_2, t_2; x_3, t_3; ....x_n, t_n;) =$$

$$p(x_1, t_1 \mid x_2, t_2;)p(x_2, t_2; \mid x_3, t_3)p(x_3, t_3; \mid x_4, t_4)......p(x_n-1, t_n-1; \mid x_n, t_n)p(x_n, t_n) \qquad (5\text{–}6)$$

provided

$$t_1 \geq t_2 \geq t_3 \geq .....t_{n-1} \geq t_n$$

5.2 Chapman-Kolmogorov Equations

It is required that summing over all mutually exclusive events of one kind in a joint probability eliminates that variable, ie

$$\sum_i P(A \cap B_i) = P(A) = \sum_i P(A/B_i)P(B_i)$$

$$\sum_i P(A_i \cap B_j \cap C_k....) = P(B_j \cap C_k \cap ...); \qquad (5-7)$$

When this is applied to stochastic process we get:

$$p(x_1, t_1) = \int dx_2\ p(x_1, t_1; x_2, t_2) = \int dx_2\ p(x_1, t_1; \mid x_2, t_2)p(x_2, t_2) \qquad (5-8)$$

This equation is an identity valid for all stochastic processes. It can also be written that

$$p(x_1, t_1 \mid x_3, t_3)) = \int dx_2\ p(x_1, t_1; x_2, t_2 \mid x_3, t_3))$$

$$= \int dx_2\ p(x_1, t_1; \mid x_2, t_2; x_3, t_3)p(x_2, t_2 \mid x_3, t_3) \qquad (5-9)$$

If $t_1 \geq t_2 \geq t_3$ the t_3 dependence in the doubly conditioned probability can be dropped and we can write:

$$p(x_1, t_1 \mid x_3, t_3)) = \int dx_2\ p(x_1, t_1 \mid x_2, t_2)p(x_2, t_2 \mid x_3, t_3) \qquad (5-10)$$

Equation $(5-10)$ is the Chapman-Kolmogorov equation.

Differential Chapman Kolmogorov Equation.

The assumptions made are closely connected with the continuity properties of the process under consideration. Because of the form of the continuity condition,

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x-z|>\varepsilon} dx\ p(x, t + \Delta t \mid z, t) = 0 \qquad (5-11)$$

uniform in z, t and Δt For all $\varepsilon > 0$, the following conditions are required:

$$i) \lim_{\Delta t \to 0} p(x, t + \Delta t \mid z, t)/\Delta t = W(x \mid z, t) \qquad (5-12)$$

$$ii) \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x-z|>\varepsilon} dx(x_i - z_i) p(x, t + \Delta t \mid z, t) = A_i(z, t) + O(\varepsilon) \qquad (5-13)$$

$$iii) \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|x-z|>\varepsilon} dx(x_i - z_i)(x_j - z_j) p(x, t + \Delta t \mid z, t) = B_i j(z, t) + O(\varepsilon) \qquad (5-14)$$

All higher-order coefficients of the form $(5-13)$ *and* $(5-14)$ must vanish. Consider the time evolution of the expectation of a function, f(z) which is twice continuously differentiable.

$$\partial_t \int dx f(x) p(x, t \mid y, t') = \lim_{\Delta t \to 0} \{ \int dx f(x) [p(x, t + \Delta t \mid y, t') - p(x, t \mid y, t')] \} / \Delta t =$$

$$\lim_{\Delta t \to 0} \{ \int dx \int dz f(x) p(x, t + \Delta t \mid z, t') p(z, t \mid y, t') - \int dz f(z) p(z, t \mid y, t') \} / \Delta t \qquad (5-15)$$

The integral over x is divided into two regions, $|x - z| \geq \varepsilon$ *and* $|x - z| < \varepsilon$. When $|x - z| < \varepsilon$, since $f(z)$ is by assumption twice differentiable, we may write

$$f(x) = f(z) + \sum_i \frac{\partial f(z)}{\partial z_i}(x_i - z_i) + \sum_{ij} \frac{1}{2} \frac{\partial^2 f(z)}{\partial z_i \partial z_j}(x_i - z_i)(x_j - z_j)$$

$$+ |x - z|^2 R(x, z) \qquad (5-16)$$

but $|R(x, z)| \to 0$ as $|x - z| \to 0$

Substitute equation $(5-16)$ into $(5-15)$,

$$\partial_t \int dx f(x) p(x,t \mid y,t') = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\{\int \int_{|x-z|<\varepsilon} dxdz \Big[\sum_i (x_i - z_i) \frac{\partial f}{\partial z_i} + \sum_{ij} \frac{1}{2}(x_i - z_i)(x_j - z_j) \frac{\partial^2 f}{\partial z_i \partial z_j} \Big]$$

$$\times\, p(x, t+\Delta t \mid z,t) p(z,t \mid y,t')$$

$$+ \int \int_{|x-z|<\varepsilon} dxdz \, |x-z|^2 \, R(x,z) p(x,t+\Delta t \mid z,t) p(z,t \mid y,t')$$

$$+ \int \int_{|x-z|\geq\varepsilon} dxdz f(x) p(x,t+\Delta t \mid z,t) p(z,t \mid y,t')$$

$$+ \int \int_{|x-z|<\varepsilon} dxdz f(z) p(x,t+\Delta t \mid z,t) p(z,t \mid y,t')$$

$$- \int \int dxdz f(z) p(x,t+\Delta t \mid z,t) p(z,t \mid y,t')\} \qquad (5-17)$$

For lines 1 and 2 of equation $(5-17)$, we take the limits inside the integral to obtain

$$\int dz \Big[\sum_i A_i \frac{\partial f}{\partial z_i} + \sum_{ij} \frac{1}{2} B_{ij} \frac{\partial^2 f}{\partial z_i \partial z_j} \Big] p(z,t \mid y,t') + \ O(\varepsilon) \qquad (5-18)$$

Line 3 is a remainder term and vanishes as $\varepsilon \to 0$ Lines 4 through 6 can all be put together to obtain

$$\int \int_{|x-z|\geq\varepsilon} dx\ dz f(z) [W(z, \mid x,t) p(x,t \mid y,t') - W(x, \mid z,t) p(z,t \mid y,t')] \qquad (5-19)$$

Taking the limit of equation $(5-17)$ *as* $\varepsilon \to 0$, its found,

$$\partial_t \int dz f(z) p(z,t \mid y,t') = \int dz \Big[\sum_i A_i(z,t) \frac{\partial f(z)}{\partial z_i} + \sum_{ij} \frac{1}{2} B_{ij}(z) \frac{\partial^2 f(z)}{\partial z_i \partial z_j} \Big] p(z,t \mid y,t')$$

$$+ \int dz f(z) \{ \int dx [W(x, \mid z,t) p(x,t \mid y,t') - W(x, \mid z,t) p(z,t \mid y,t')]\} \qquad (5-20)$$

Integrating equation $(5-20)$ by parts we obtain

$$\int dz f(z)\partial_t p(z,t \mid y,t') = \int dz f(z)\{-\sum_i \frac{\partial}{\partial z_i} A_i(z,t)p(z,t \mid y,t') + \sum_{ij} \frac{1}{2}\frac{\partial^2}{\partial z_i \partial z_j} B_{ij}(z,t)p(z,t \mid y,t')$$

$$+ \int dx[W(z, \mid x,t)p(x,t \mid y,t') - W(x, \mid z,t)p(z,t \mid y,t')]\} + surface\, terms. \qquad (5\text{--}21)$$

If we choose $f(z)$ to be arbitrary but nonvanishing only in an arbitrary region R' entirely in R, it can be deduced that for all z in the interior of R,

$$\partial_t p(z,t \mid y,t') = \{-\sum_i \frac{\partial}{\partial z_i} A_i(z,t)p(z,t \mid y,t') + \sum_{ij} \frac{1}{2}\frac{\partial^2}{\partial z_i \partial z_j} B_{ij}(z,t)p(z,t \mid y,t')$$

$$+ \int dx[W(z, \mid x,t)p(x,t \mid y,t') - W(x, \mid z,t)p(z,t \mid y,t')]\} \qquad (5-22)$$

Equation $(5-22)$ is called the differential Chapman-Kolmogorov Equation. It can be shown that, under certain conditions, if $A(x,t), B(x,t)$ *and* $W(x \mid y,t)$, that a non-negative solution to the differential Chapman-Kolmogorov equation exists, and this solution also satisfies the Chapman-Kolmogorov equation. The condition to be satisfied is the initial condition,

$$p(z,t \mid y,t) = \delta(y-z) \qquad (5-23)$$

which follows from the definition of conditional probability density, and any appropriate boundary conditions.

5.3 Diffusion Process: The Fokker-Planck Equation

If we assume $W(z, \mid x,t)$ to be zero, the differential Chapman Kolmogorov equation reduces to the Fokker- Planck equation:

$$\frac{\partial p(z,t \mid y,t')}{\partial t} = -\sum_i \frac{\partial}{\partial z_i}[A_i(z,t)p(z,t \mid y,t')] + \frac{1}{2}\sum_{ij} \frac{\partial^2}{\partial z_i \partial z_j}[B_{ij}(z,t)p(z,t \mid y,t')] \quad (5\text{--}24)$$

Equation $(5-24)$ is known as the diffusion process. $A(z,t)$ is known as the drift vector and the matrix $B(z,t)$ as the diffusion matrix. If the drift term is zero, a pure diffusion equation is obtained and the term $B(z,t)$ becomes the diffusion coefficient.

$$\frac{\partial p(z,t \mid y,t')}{\partial t} = \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial z_i \partial z_j}[B_{ij}(z,t)p(z,t \mid y,t')] \qquad (5-25)$$

If equation $(5-25)$ is combined with equation $(5-23)$ as its initial condition, it can be realized that the conditional probability $p(z,t \mid y,t)$ is equivalent to the Green's function discussed in chapter four. The requirement for continuity of the sample paths is satisfied if $W(x \mid z,t)$ is zero. Hence, the Fokker Planck equation describes a process in which $\chi(t)$ has continuous sample paths. For small time changes, Δt, the solution of the Fokker Planck will still be on the whole sharply peaked, and hence derivatives of $A_i(z,t)$ and $B_{ij}(z,t)$ will be negligible compared to those of p. We are thus reduced to solving approximately

$$\frac{\partial p(z,t \mid y,t')}{\partial t} = -\sum_i A_i(y,t)\frac{\partial}{\partial z_i}p(z,t \mid y,t') + \frac{1}{2}\sum_{ij} B_{ij}(y,t)\frac{\partial^2}{\partial z_i \partial z_j}p(z,t \mid y,t') \qquad (5\text{–}26)$$

where we have neglected the time dependence of A_i and B_{ij} for small $(t-t')$. If equation $(5-26)$ is subject to the initial condition:

$$p(z,t \mid y,t) = \delta(z-y) \qquad (5-27)$$

we get:

$$p(z,t+\Delta t \mid y,t) = (2\pi)^{-\frac{N}{2}}\{det[B(y,t)]\}^{\frac{1}{2}}[\Delta t]^{-\frac{1}{2}}$$

$$\times\ exp\{-\frac{1}{2}\frac{[z-y-A(y,t)\Delta t][B(y,t)]^{-1}[z-y-A(y,t)\Delta t]}{\Delta t}\} \qquad (5-28)$$

Equation $(5-28)$ is a Gaussian distribution with variance matrix B(y,t) and mean

$y + A(y,t)\Delta t$. It also depicts the picture of a system moving with systematic drift, whose velocity is $A(y,t)$ on which is superimposed a Gaussian fluctuation with covariance matrix $B(y,t)\Delta t$.

5.4 Stochastic Differential Equation

Equation $(5-25)$ is a pure diffusion process equation with no drift part to it. Momentum and vorticity equations describe time-dependent diffusion. Diffusion of physical entities, typically diffusing particles, can also be described in stochastic terms via a stochastic differential equation:

$$d\chi(t) = \sqrt{2\nu}\; dw(t) \qquad (5-29)$$

where $\chi(t)$ is the position of the physical entity, ν is the diffusion coefficient, and $w(t)$ is a Wiener process. We proposed that individual vortex sheets, produced during short time increments, Δt, at the fluid layer's moving boundary, can be viewed as a swarm of thinner elemental vortex sheets. The swarm of elemental vortex sheets(constituting the thicker vortex sheets formed over Δt), in turn, is viewed as a stochastic process, ie of a swarm of random walkers. The evolution/random motion of each elemental vortex sheet is described by the above stochastic differential equation.

The flow field is randomly generated, therefore a rapidly and irregularly fluctuating random function of time $\chi(t)$ is considered. For a differential equation involving such a random variable, we turn to Langevin's equation which describes such a stochastic process. Langevin equation can be described as an ordinary differential equation in which a rapidly and irregularly fluctuating random function of time occurs. Langevin equation can be written in the form :

$$\frac{d\chi}{dt} = a(x,t) + b(x,t)\epsilon(t) \qquad (5-30)$$

where x is a variable of interest, $a(x,t)$ and $b(x,t)$ are certain known functions and $\epsilon(t)$ is the rapidly fluctuating random term. An idealised mathematical formulation of the concept of a "rapidly" varying highly irregular function is that for $t \neq t'$, $\epsilon(t)$ *and* $\epsilon(t')$ are statistically independent. It is required that the mean of $\epsilon(t)$ must be zero, ie $<\epsilon(t)>= 0$. Any nonzero mean can be absorbed into the definition of $a(x,t)$ and thus require that :

$$<\epsilon(t)\epsilon(t')>= \delta(t-t') \qquad (5-31)$$

The differential equation $(5-30)$ is expected to be integrable and hence, it can be written:

$$u(t) = \int_0^t dt'\epsilon(t') \qquad (5-32)$$

It can thus be written:

$$\int_0^t \epsilon(t')dt' = u(t) = W(t) \qquad (5-33)$$

The integral of $\epsilon(t)$ is $W(t)$ which is itself not differentiable. However the corresponding equation :

$$\chi(t) - \chi(0) = \int_0^t a[x(s),s]ds + \int_0^t b[x(s),s]\epsilon(s)ds \qquad (5-34)$$

the integral of $\epsilon(t)$ is interpreted as the Wiener process $W(t)$, that

$$dW(t) = W(t+dt) - W(t) = \epsilon(t)dt \qquad (5-35)$$

Equation $(5-34)$ can therefore be written as :

$$\chi(t) - \chi(0) = \int_0^t a[x(s),s]ds + \int_0^t b[x(s),s]dW(s) \qquad (5-36)$$

What is the probability of the random variable $\chi(t)$ at (x, t) given that it ws at (x', t') ? We focus on the transition density, $P(x, t/x', t')$. We want to find a governing differential equation that describe the space-time evolution of $P(x, t/x', t')$. Einstein first made the connection between the random motion of a diffusing particle and the equation corresponding to a Fokker-Planck equation that describes the space-and-time dependent evolution of the particle's transition density, p. For an introduction of a random walker and the diffusion process, we consider a man moving randomly along a line. The steps are of length l so that his position can take on only the value Nl, where N is integral. Let p be the probability that he moves to the right and q the probability that he moves to the left. Let $P(x, N)$ be the probability that the random walker is at site x at the N^{th} time step. The probability $P(x, N)$ satisfies the stochastic difference equation:

$$P(x, N+1) = pP(x-1, N) + qP(x+1, N) \qquad (5-37)$$

where, $P(x, N+1)$ = probability that the random walker will be at x after $(N+1)$ time steps, $P(x-1, N)$ = probability that the random walker will be at $x-1$ position after N time step, $P(x+1, N)$ = probability that the random walker will be at $x+1$ position after N time steps. If we consider the special case, $p = q = \frac{1}{2}$, equation $(5-37)$ can be written as:

$$P(x, N+1) - P(x, N) = \frac{P(x-1, N) + P(x+1, N) - 2P(x, N)}{2} \qquad (5-38)$$

$$\frac{\Delta t[P(x, N+1) - P(x, N)]}{\Delta t} = \frac{[P(x-1, N) + P(x+1, N) - 2P(x, N)](\Delta x)^2}{2(\Delta x)^2} \qquad (5\text{–}39)$$

In the limit of large N, the differences become differentials such that:

$$\Delta t \frac{\partial P}{\partial t} = \frac{(\Delta x)^2}{2} \frac{\partial^2 P}{\partial x^2} \qquad (5-40)$$

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} \qquad where \qquad D = \frac{(\Delta x)^2}{2\Delta t} = \frac{l^2}{2\tau} \qquad (5-41)$$

5.5 Non-Anticipating Functions:

A function G(t) is called a non-anticipating function of t if for all s and t, such that $t < s$. $G(t)$ is statistically independent of $(W(s) - W(t))$. This means that $G(t)$ is independent of the behavior of the Wiener process in the future of t. For a non-anticipating function f of W(t), a general differentiation can be expressed as :

$$df[W(t),t] = \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial f}{\partial w}dW(t) + \frac{1}{2}\frac{\partial^2 f}{\partial W^2}[dW(t)]^2 + \frac{\partial^2 f}{\partial W \partial t}dtdW(t) + ... \quad (5-42)$$

Using the following:

$$(dt)^2 \to 0,\ dtdW(t) \to 0,\ [dW(t)]^2 = dt \qquad (5-43$$

and all higher power greater than 2 vanishing we arrive at

$$df[W(t),t] = (\frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial W^2})dt + \frac{\partial f}{\partial w}dW(t) \qquad (5-44)$$

A stochastic quantity $\chi(t)$ obeys an Ito SDE written as:

$$d\chi(t) = a[x(t),t]dt + b[x(t),t]dW(t) \qquad (5-45)$$

If for all t and t_o,

$$\chi(t) = \chi(0) + \int_{t_0}^{t} a[x(t'),t']dt' + \int_{t_0}^{t} b[x(t'),t']dW(t') \qquad (5-46)$$

$\chi(t)$, the solution to the stochastic differential equation $(5-46)$ is a Markov Process. Given an initial condition $\chi(t_0)$, the future time development is uniquely determined, that is $\chi(t)$ for $t > t_0$ is determined only by i) the particular path of

$W(t)$ for $t > t_0$; and ii) the value of $X(t_0)$. Since $\chi(t)$ is a nonanticipating function of t, $W(t)$ for $t > t_0$ is independent of $\chi(t)$ for $t < t_0$. Thus the time development of $\chi(t)$ for $t > t_0$ is independent of $\chi(t)$ for $t < t_0$ provided $\chi(t_0)$ is known. Hence $\chi(t)$ is a Markov process. Consider an arbitrary function of $x(t)$ f[x(t)]. If the differential of $f[x(t)]$ is expanded to second order in $dW(t)$

$$df[x(t)] = f[x(t) + dx(t)] - f[x(t)] = f'[x(t)]dx(t) + \frac{1}{2}f''[x(t)]dx(t)^2 + ...$$

$$= f'[x(t)]\{a[x(t), t]dt + b[x(t), t]dW(t)\} + \frac{1}{2}f''[x(t)]b[x(t), t]^2[dW(t)]^2 + ... \quad (5{-}47)$$

where all other terms have been discarded since they are of higher order. Using $[dW(t)]^2 = dt$, we obtain

$$df[x(t)] = \{a[x(t), t]f'[x(t)] + \frac{1}{2}b[x(t), t]^2 f''[x(t)]\}d(t) + .b[x(t), t]f'[x(t)]dW(t) \quad (5{-}48)$$

Equation (5−47) is Ito's formula. Connection between Fokker Planck Equation and Stochastic Differential Equation. Consider the time development of an arbitrary function$f(x, t)$. Using Ito's formula,

$$\frac{< df[x(t)] >}{dt} =< \frac{df[x(t)]}{dt} >= \frac{d}{dt} < f[x(t)] >$$

$$=< a[x(t), t]\frac{df}{dx} + \frac{1}{2}b[x(t), t]^2\frac{d^2 f}{dx^2} \quad (5-49)$$

However $\chi(t)$ has a conditional probability density $p(x, t \mid x_0, t_0)$ and

$$\frac{d}{dt} < f[x(t)] >= \int dx f(x)\partial_t p(x, t|x_0, t_0)$$

$$= \int dx[a(x, t)\partial_x f + \frac{1}{2}b(x, t)^2\partial_x^2]p(x, t \mid x_0, t_0) \quad (5-50)$$

Integrating by parts and discarding surface terms, we obtain:

$$\int dx f(x)\partial_t p = \int dx f(x)\{-\partial_x[a(x,t)p + \frac{1}{2}\partial_x^2 b(x,t)^2 p]\} \qquad (5-51)$$

Since $f(x)$ is arbitrary,

$$\partial_t p(x,t \mid x_0,t_0) = -\partial_x[a(x,t)p(x,t \mid x_0,t_0)] + \frac{1}{2}\partial_x^2[b(x,t)^2 p(x,t \mid x_0,t_0)] \qquad (5-52)$$

This is a complete equivalence to a diffusion process defined by a drift coefficient $a(x,t)$ and a diffusion coefficient $b(x,t)^2$. For small time changes, Δt, the solution of the Fokker Planck equation will still be on the whole sharply peaked, and hence the derivatives of $A_i(z,t)$ and $B_ij(z,t)$ will be negligible compared to those of the probability density. In one dimension the Fokker Planck equation takes the form:

$$\frac{\partial f(x,t)}{\partial t} = -\frac{\partial}{\partial x}[A(x,t)f(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[B(x,t)f(x,t)] \qquad (5-53a)$$

The Fokker Planck equation is valid for the conditional probability, that is

$$f(x,t) = p(x,t|x_0,t_0)$$

for any initial condition (x_0,t_0)

$$p(x,t_0|x_0,t_0) = \delta(x-x_0) \qquad (5-53b)$$

For one time probability :

$$p(x,t) = \int dx_0 \; p(x,t;x_0,t_0) = \int dx_0 \; p(x,t|x_0,t_0)p(x_0,t_0) \qquad (5-54)$$

One time probability is also valid for $p(x,t)$ with initial condition

$$p(x,t)\Big|_{t=t_0} = p(x,t_0)$$

5.6 Boundary Conditions

We consider the forward equation of Chapman Kolmogorov.

$$\frac{\partial p(z,t)}{\partial t} = -\sum_i \frac{\partial}{\partial z_i}[A_i(z,t)p(z,t)] + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial z_i \partial z_j}[B_{ij}(z,t)p(z,t)] \qquad (5-55)$$

The above equation can be written as :

$$\frac{\partial p(z,t)}{\partial t} = -\sum_i \frac{\partial}{\partial z_i} J_i(z,t)) = 0 \qquad (5-56)$$

where

$$J_i(z,t) = A_i(z,t)p(z,t) - \frac{1}{2}\sum_j \frac{\partial}{\partial z_j} B_{ij}(z,t)p(z,t) \qquad (5-57)$$

$J_i(z,t)$ is called the probability current. Equation $(5-56)$ has the form of a local conservation equation and can be written in an integral form as follows:
Consider some region R with a boundary S and define

$$P(R,t) = \int dz\ p(z,t)$$

Then (5-56) is equivalent to

$$\frac{\partial P(R,t)}{\partial t} = -\int_s ds.n.J(z,t) \qquad (5-58)$$

where n is the outward normal to S. Thus equation (5-58) indicates that the total loss of probability is given by the surface integral of J over the boundary of R.

A surface integral over any surface S gives the net flow of probability across that surface. We consider the possible scenarios of boundary conditions that may be attended to in stochastic diffusion equation.

Reflecting Barrier:

We consider the situation where the particle cannot leave a region R, hence there is zero net flow of probability across S, the boundary of R. Thus it is required: $n.J(z,t) = 0$ for $z \epsilon S$ and $n \equiv$ normal to S. Since the particle cannot cross S, it must be reflected there and hence the name reflecting barrier for this condition.

Absorbing Barrier:

Here, one assumes that the moment the particle reaches S, it is removed from the system, thus the barrier absorbs. Consequently the probability of being on the boundary is zero ie $p(z,t) = 0$ for $z \epsilon S$.

Periodic Boundary Condition:

We assume that the process takes place on an interval $[a, b]$ in which the two end points are identified with each other.(this occurs for example if the diffusion is on a circle.) Then we impose the boundary condition of discontinuity ie

I. $lim_{x \to b^-} p(x, t) = lim_{x \to a^+} p(x, t)$ and II. $lim_{x \to b^-} J(x, t) = lim_{x \to a^+} J(x, t)$ Most frequently, periodic boundary conditions are imposed when the functions A(x,t) and B(x,t) are periodic on the same interval so that we have

$A(b, t) = A(a, t)$ and $B(b, t) = B(a, t)$

Prescibed Boundaries

If the diffusion coefficient vanishes at a boundary, we have a situation in which the kind of boundary may be automatically prescribed. Supose the motion occurs only for$x > a$, If a Lipschitz condition is obeyed by $A(x, t)$ *and* $\sqrt{B(x, t)}$ at x = a, and B(x,t) is differentiable at x = a, then

$$\partial_x B(a, t) = 0$$

The stochastic differential equation then has solutions, and we may write

$$dx(t) = A(x, t)dt + \sqrt{B(x, t)}dW(t) \qquad (5-59)$$

In this special case, the situation is determined by the sign of A(x,t). Three cases can be considered.

i) Exit Boundary.

In this case, we suppose

$$A(a, t) < 0$$

so that if particle reaches the point a, it will certainly proceed out of region to $x < a$, hence the name "exit boundary" .

ii) Entrance Boundary.

Suppose $A(a, t) > 0$ In this case, if the particle reaches the point a, the sign of A(a,t) is such as to return it to $x > a$; thus a particle placed to the right of a can never leave the region.

iii) Natural Boundary:

A particle introduced at x =a will certainly enter the region. Hence the name " entrance boundary" . Consider

$$A(a, t) = 0$$

The particle, once it reaches x = a, will remain there. However it can be demonstrated that it cannot ever reach this point. This is a boundary from which we can neither absorb nor at which we can introduce any particles.

5.7 Infinite Medium: One Dimensional Diffusion

For a one dimensional random walk (Wiener process , Brownian motion) with no drift, the Fokker Planck equation of $(5-26)$ can be written as:

$$\frac{\partial p(y,t)}{\partial t} = \nu \frac{\partial^2 p(y,t)}{\partial y^2} \qquad (5-60a)$$

where ν is a constant representing the diffusion coefficient. The diffusion equation is completed by an initial and boundary conditions.

$$p(y, t_0) = \delta(y - y') \qquad (5-60b)$$

$$lim_{x\to\pm\infty} p(y,t) = 0 \qquad (5-60c)$$

The solution of the diffusion equation with initial condition as a delta-peak at y' and natural boundary conditions, is the Gaussian distribution:

$$p(y,t) = \frac{1}{\sqrt{4\pi\nu t}} exp\Big(-\frac{(y-y')^2}{4\nu t}\Big) \qquad (5-61)$$

A generalization for an arbitrary initial distribution $p(y, t = 0) = p_0(y)$ is posible due to the superposition of probability densities created by different sources(initial distributions), which is a property of our diffusion equation due to its linearity. Hence, for

$$p_0(y) \equiv \int_{-\infty}^{+\infty} p_0(y')\delta(y - y')dy' \qquad (5-62)$$

the solution is an integral over the normal distributions corresponding to $p(y, t = 0) = \delta(y - y')$ weighted by the source intensity $p_0(y')$, so

$$p(y, t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} p_0(y')exp\Big(- \frac{(y - y')^2}{4\nu t}\Big)dy' \qquad (5-63)$$

The solution of the diffusion equation can be obtained by one-dimensional Fourier transformation to $p^w(k, t)$ (transformation to inverse space by a generating function) which is defined by :

$$p(y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iky} p^w(k, t)dk. \qquad (5-64)$$

where:

$$p^w(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iky} p(y, t)dy. \qquad (5-65)$$

The left hand side of the diffusion equation $(5 - 60a)$ is transformed as :

$$\frac{\partial}{\partial t} p(y, t) = \frac{\partial}{\partial t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, e^{iky} p^w(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, e^{iky} \frac{\partial p^w(k, t)}{\partial t}. \qquad (5\text{--}66)$$

whereas for the right hand side starting with the first differential ;

$$\frac{\partial}{\partial y} p(y, t) = \frac{\partial}{\partial y} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, e^{iky} p^w(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (ik)\, e^{iky} p^w(k, t)dk \qquad (5\text{--}67)$$

becomes

$$\frac{\partial^2}{\partial y^2}p(y,t) = \frac{\partial}{\partial y}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(ik)\ e^{iky}p^w(k,t)dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(ik)^2\ e^{iky}p^w(k,t)dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}k^2\ e^{iky}p^w(k,t)dk \qquad (5-68)$$

Hence the transformed equation reads

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dk\ e^{iky}\frac{\partial p^w(k,t)}{\partial t} = -\frac{\nu}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dk\ k^2\ e^{iky}p^w(k,t) \qquad (5-69)$$

The integrand of $(5-69)$ must equal:

$$\frac{\partial p^w(k,t)}{\partial t} = -k^2\nu\ p^w(k,t) \qquad (5-70)$$

which leads to a local problem in the k-space. An elementary integration yields the solution

$$p^w(k,t) = p^w(k,t=0)e^{-k^2\nu t} \qquad (5-71)$$

in the form of an exponentially decaying kth Fourier mode. The condition transforms as

$$p^w(k,t=0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{iky}p(y,t=0)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-iky}\delta(y-y')dx = \frac{1}{\sqrt{2\pi}}e^{iky'} \qquad (5-72)$$

so that the solution in the Fourier space is

$$p^w(k,t) = \frac{1}{\sqrt{2\pi}}e^{-iky'}e^{-k^2\nu t} \qquad (5-73)$$

Now taking the inverse transformation to the coordinate space;

$$p(y,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dk\ e^{iky}\frac{1}{\sqrt{2\pi}}e^{-iky'}e^{-k^2Dt}$$

$$p(y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \ e^{ik(y-y')-k^2 Dt} \qquad (5-74)$$

$$p(y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \ e^{-\nu t\left[k^2 - \frac{ik(y-y')}{Dt}\right]} \qquad (5-75)$$

$$p(y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \ e^{-\nu t\left[\left(k - \frac{ik(y-y')}{2\nu t}\right)^2 + \left(\frac{(y-y')}{2\nu t}\right)^2\right]} \qquad (5-76)$$

To solve the above problem, we set the variable:

$$z = (k - \frac{ik(y-y')}{2\nu t}), \ with \ dz \ = dk \qquad (5-77)$$

The integration is made in the complex plane along the line which is parallel to the real axis, although shifted by $c \ = (- \frac{ik(y-y')}{2Dt})$. The integral does not depend on c and therefore the integration path can be shifted to the real axis. If we substitute the variable in $(5-77)$ into equation $(5-76)$, equation $(5-76)$, can be written as:

$$p(y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dz \ e^{-\nu t z^2} e^{-Dt\left(\frac{(x-x_0)}{2\nu t}\right)^2} \qquad (5-78)$$

$$p(y,t) = \frac{1}{2\pi} e^{-\left(\frac{(y-y')}{2\nu t}\right)^2} \int_{-\infty}^{+\infty} dz \ e^{-\nu t z^2} \qquad (5-79)$$

By using the formula

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \qquad (5-80)$$

where $\alpha = \nu t$ equation $(5-79)$ becomes

$$p(y,t) = \frac{1}{2\pi} \sqrt{\frac{\pi}{\nu t}} e^{-\frac{(y-y')^2}{4\nu t}} \qquad (5-81)$$

The solution to the one dimensional random walk Fokker Planck diffusion equation with natural boundaries and initial condition can finally be obtained as

$$p(y,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(y-y')^2}{4\nu t}} \qquad (5-82)$$

The Gaussian integral (5-80) can be calculated as follows. In terms of another variable the Gaussian integral can also be written as:

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \qquad (5-83)$$

The square of the integral can therefore be written as :

$$I^2 = \Big(\int_{-\infty}^{\infty} e^{-\alpha x^2} dx\Big)\Big(\int_{-\infty}^{\infty} e^{-\alpha \beta^2} d\beta\Big) \qquad (5-84)$$

$$I^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\beta \ e^{-\alpha(x^2+\beta^2)} \qquad (5-85)$$

We make the transformation to polar coordinates:

$$x = r \ cos \ \theta, \quad \beta = r \ sin \ \theta, \quad r = \sqrt{x^2+\beta^2} \quad dx \ d\beta = rdrd\theta$$

Thus we have

$$I^2 = \int_0^{2\pi} d\theta \int_r^{\infty} rdr \ e^{-\alpha r^2} dr \qquad (5-86)$$

making the substitution :

$$z = r^2 \quad dz = 2r \ dr$$

Then

$$I^2 = \pi \int_0^{\infty} e^{-\alpha z} dz$$

Also by substituting

$$\alpha z = u, \quad du = \alpha dz$$

$$I^2 = \pi \int_0^\infty \frac{1}{\alpha} e^{-u} du = \frac{\pi}{\alpha} \int_0^\infty e^{-u} du = \frac{\pi}{\alpha} \qquad (5-87)$$

Thus

$$I = \sqrt{\frac{\pi}{\alpha}} \qquad (5-88)$$

The moments of the density distribution function is stated as

$$\langle Y^n \rangle(t) = \int_{-\infty}^{\infty} dy\ X^n p(y,t) \qquad (5-89)$$

The moment of the zeroth order has to be one, as it represents the normalization integral.

$$\langle Y^0 \rangle(t) \equiv 1 = \int_{-\infty}^{\infty} p(y,t) dx \qquad (5-90)$$

To check it, we insert the solution:

$$p(y,t) = \frac{1}{\sqrt{4\pi\nu t}} e^{-\frac{(y-y')^2}{4\nu t}} \qquad (5-91)$$

$$\langle Y^0 \rangle = \int_{-\infty}^{\infty} dy \frac{1}{\sqrt{4\pi\nu t}} e^{-\frac{(y-y')^2}{4\nu t}} \qquad (5-92)$$

Let $(y-y') = z$; $dy = dz$ Denoting $a^2 = \frac{1}{4\nu t}$

$$\langle Y^0 \rangle = \int_{-\infty}^{\infty} \frac{a}{\sqrt{\pi}} e^{-a^2 z^2} dz \qquad (5-93)$$

$$\langle Y^0 \rangle = \frac{2a}{\sqrt{\pi}} \int_0^\infty e^{-a^2 z^2} dz \qquad (5-94)$$

Again employing the integral expression $I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$ we obtain

$$\langle Y^0 \rangle = \frac{2a}{\sqrt{\pi}} \frac{1}{2} \sqrt{\frac{\pi}{a^2}} = 1 \qquad (5-95)$$

The first moment of the probability density function represents the mean. We can calculate the first moment as :

$$\langle Y^1\rangle = \int_{-\infty}^{\infty} y\ p(y,t)dx \qquad (5-96)$$

$$\langle Y^1\rangle = \int_{-\infty}^{\infty} y\frac{1}{\sqrt{4\pi\nu t}}e^{-\frac{(y-y')^2}{4\nu t}}\ dy \qquad (5-97)$$

$$\langle Y^1\rangle = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty} ye^{-\frac{(y-y')^2}{4\nu t}}\ dy \qquad (5-98)$$

To solve for the mean we let $y - y' = z$ so $y = y' + z$. Thus

$$\langle Y^1\rangle = \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}(y' + z)e^{-\frac{z^2}{4nut}}\ dz \qquad (5-99)$$

$$\langle Y^1\rangle = y'\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty} e^{-\frac{z^2}{4\nu t}}\ dz + \frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty} ze^{-\frac{z^2}{4\nu t}}\ dz \qquad (5-100)$$

As in the moment of zero order, the product of $\frac{1}{\sqrt{4\pi\nu t}}$ and the integral in the first term is one. The second term of equation $(5-93)$ is zero. The first moment of the probability density function therefore leads to y'.

$$\langle Y^1\rangle = y' + 0 = y'$$

The mean value $\langle y\rangle = y'$ does not change in time, it keeps the initial value.

The second moment is expressed as

$$\langle Y^2\rangle = \int_{-\infty}^{\infty} y^2\ p(y,t)dy \qquad (5-101)$$

The second moment is related to the standard deviation σ via

$$\langle(y - \langle y\rangle)^2\rangle = \langle(y - y')^2\rangle \equiv \sigma^2 \qquad (5-102)$$

Denoting $\alpha = \frac{1}{4\nu t}$, we have

$$\langle Y^2 \rangle = \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{4\pi\nu t}} e^{-\frac{(y-y')^2}{4\nu t}}\ dy \qquad (5-103)$$

Let $x = y - y',\ dx = dy$ Thus

$$\langle Y^2 \rangle = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} (x+y')^2 e^{-\alpha x^2}\ dx \qquad (5-104)$$

$$\langle Y^2 \rangle = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} (x^2 + 2y'x + y'^2) e^{-\alpha x^2}\ dx \qquad (5-105)$$

$$\langle Y^2 \rangle = y'^2 + 0 + \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\ dx \qquad (5-106)$$

To solve the integral term in (5-106) we use the identity

$$\int_{-\infty}^{\infty} q^2 e^{-\alpha q^2}\ dq = -\frac{d}{d\alpha} \int_{-\infty}^{\infty} e^{-\alpha q^2}\ dq \qquad (5-107)$$

Equation (5-99) becomes

$$\langle Y^2 \rangle = y'^2 + \frac{1}{\sqrt{4\pi\nu t}} \Big(-\frac{d}{d\alpha} \int_{-\infty}^{\infty} e^{-\alpha x^2}\ dx \Big) \qquad (5-108)$$

Again using the identity $I = \int_{-\infty}^{\infty} e^{-\alpha q^2} dq = \sqrt{\frac{\pi}{\alpha}}$ and calculating further we have

$$\langle Y^2 \rangle = y'^2 - \frac{1}{\sqrt{4\pi\nu t}} \frac{d}{d\alpha} \Big(\frac{\sqrt{\pi}}{\sqrt{\alpha}} \Big) \qquad (5-109)$$

$$\langle Y^2 \rangle = y'^2 - \frac{\sqrt{\pi}}{\sqrt{4\pi\nu t}} \Big(-\frac{1}{2} \Big) \alpha^{-\frac{3}{2}} \qquad (5-110)$$

$$\langle Y^2 \rangle = y'^2 + \frac{\sqrt{\pi}}{2\sqrt{4\pi\nu t}} \Big(-\frac{1}{4\nu t} \Big)^{-\frac{3}{2}} \qquad (5-111)$$

$$\langle Y^2 \rangle = y'^2 + 2\nu t \qquad (5-112)$$

which finally gives

$$\sigma^2 \equiv \langle (y - \langle y \rangle)^2 \rangle = 2\nu t \qquad (5-113)$$

$$\sigma = \sqrt{2\nu t} \sim \sqrt{t} \qquad (5-114)$$

5.8 The Mirror Method

The solution of the diffusion equation in shifted coordinates together with the initial condition $p(y, t = 0) = \delta(y - y')$ and different boundary conditions can be obtained as a superposition of solutions.

$$P_{y0}(y,t) = \frac{1}{\sqrt{4\pi\nu t}} e^{-\frac{(y-y')^2}{4\nu t}} \qquad (5-115a)$$

for natural boundary conditions with different positions y' of the delta- peaks at the initial time moment $t = 0$. Any such superposition satisfies the diffusion equation due its linearity. The only problem is to fulfill the initial and boundary conditions. Let us consider the case where there is a reflecting boundary at $y = 0$ and we are looking for the solution within $y \; \epsilon \; [0, \infty)$, so the other boundary is located at $+\infty$. Formally we can extend the y interval from $-\infty$ to $+\infty$ and make a mirror image. If the image point of y' were to be at $-y'$, then the fundamental solution of the diffusion equation with natural boundaries will be :

$$P_{-y0}(y,t) = \frac{1}{\sqrt{4\pi\nu t}} e^{-\frac{(y+y')^2}{4\nu t}} \qquad (5-115b)$$

Obviously, the superposition as the sum of $P_{y0}(y,t)$ and $P_{-y0}(y,t)$ fulfills the initial condition for the interval $y \; \epsilon \; [0, \infty)$ as well as the reflecting boundary condition. $\frac{\partial p(y,t)}{\partial y}$ around this point.

$$P(y,t) = P_{y0}(y,t) + P_{-y0}(y,t) \qquad (5-116)$$

which yields the solution

$$P(y,t) = \frac{1}{\sqrt{4\pi\nu t}} e^{-\frac{(y-y')^2}{4\nu t}} + \frac{1}{\sqrt{4\pi\nu t}} e^{-\frac{(y+y')^2}{4\nu t}} \qquad (5-117)$$

5.9 Diffusion in a Finite Interval with Absorbing Boundaries:

Restating The Diffusion Equation but with finite absorbing boundaries:

$$\frac{\partial p(y,t)}{\partial t} = \nu \frac{\partial^2 p(y,t)}{\partial y^2} \qquad (5-118a)$$

Initial Condition:

$$p(y,t_0) = \delta(y-y') \qquad (5-118b)$$

Absorbing Boundary Condition at y = 0 (bottom):

$$p(y=0,t) = 0 \qquad (5-118c)$$

Absorbing Boundary Condition at y = h (top)

$$p(y=h,t) = 0 \qquad (5-118d)$$

The method of separation of variables can be used to solve the problem $(5-118)$ Let $p(y,t)$ be represented by:

$$p(y,t) = G(t)F(y) \qquad (5-119)$$

Subtituting $(5-119)$ into $(5-118)$ we obtain:

$$\frac{1}{G(t)}\frac{dG(t)}{dt} = \nu \frac{1}{F(y)}\frac{d^2F(y)}{dy^2} \qquad (5-120)$$

Both sides should equal a constant $-\lambda$

Rewriting the left hand side of $(5-120)$

$$\frac{1}{G(t)}\frac{dG(t)}{dt} = -\lambda \qquad (5-121)$$

$$G(t) = G_0\ e^{(-\lambda t)} \qquad (5-122)$$

with $G(t=0) = G_0 = 1$

The right hand side of $(5-120)$ can be written as:

$$\nu\frac{1}{F(y)}\frac{d^2F(y)}{dy^2} = -\lambda \qquad (5-123)$$

Introducing the notion of the wave number k given by :

$$k^2 = \frac{\lambda}{\nu} \qquad (5-124)$$

Equation $(5-123)$ can be written as:

$$\frac{d^2F(y)}{d^2y} + k^2F(y) = 0 \qquad (5-125)$$

The general solution of $(5-118)$ is

$$F(y) = Asin(ky) + Bcos(ky) \qquad (5-126)$$

At the bottom boundary condition:

$$p(y,t) = 0, \Rightarrow F(y) = 0, at\ y = 0 \qquad (5-127)$$

$$F(0) = A\sin(0) + B\cos(0) = 0,\ \ B = 0 \qquad (5-128)$$

Applying the other absorbing boundary condition at $y = h$

$$F(h) = Ak\sin(k(h)) - Bk\cos(k(h)) = 0 \qquad (5-129)$$

$$Ak\sin k(h) = 0 \qquad (5-130)$$

$$\sin k(h) = 0 \qquad (5-131)$$

This yields discrete solutions for k and eigen values :

$$k_m = \frac{\pi m}{h} \qquad (5-132)$$

but $\lambda_m = \nu k_m^2 = \frac{\nu\pi^2}{h^2}m^2$ (5 − 133)

where $m = 0, 1, 2..............n$ *ie positive integers.*

A particular time dependent solution which fulfills the boundary condition and corresponds to the eigenvalue λ_m is thus the eigenfunction $p_m(y,t)$ given by

$$p_m(y,t) = A_m e^{-\lambda_m t}\sin(k_m y) \qquad (5-134)$$

The complete solution of the problem is forced as superposition of these eigenfunctions.

$$P(y,t) = \sum_{m=0}^{\infty} p_m(y,t) = \sum_{m=0}^{\infty} A_m e^{-\lambda_m t}\sin(k_m y) \qquad (5-135)$$

To obtain A_m , we employ the initial condition

$$P(y, t=0) = \sum_{m=0}^{\infty} A_m\ \sin(k_m y) = \delta(y-y') \qquad (5-136)$$

Employing the coefficients of Fourier sine series:

$$A_m = \frac{2}{h}\int_0^h dy\ \delta(y-y')\sin(k_m y) \qquad (5-137)$$

$$A_m = \frac{2}{h}\sin(k_m y') \qquad (5-138)$$

but $f(y) = A\sin(ky) + B\cos(ky)$ and $Y(t) = Y_0 e^{-\lambda t}$

$$p(y,t) = Y(t)f(y)$$

The solution reads

$$p(y,t) = \frac{2}{h}\sum_{m=0}^{\infty} e^{-\lambda_m t}\sin(k_m y')\sin(k_m y) \qquad (5-139)$$

$$p(y,t) = \frac{2}{h}\sum_{m=0}^{\infty} e^{-\nu\frac{\pi^2}{h^2}mt}\sin(\frac{\pi m}{h}y')\sin(\frac{\pi m}{h}y) \qquad (5-140)$$

The probability distribution $p(y,t)$ tends to zero with increasing time.

5.10 First- Passage Time Problem.

The mean time during which the system finds its stable state by overcoming a potential barrier due to stochastic fluctuations. The problem is to find the average time during which a stochastic system reaches for the first time, some given state if started from another state. This time is called the mean first- passage time. The first-passage time distribution(breakdown probability density) follows from the balance condition

$$P(t, y = h) = -\frac{d}{dt}\int_0^h p(y,t)dy \qquad (5-141)$$

By inserting the solution (5- 140) into (5-141) we obtain

$$P(t,h) = -\frac{d}{dt}\Big[\int_0^h \frac{2}{h}\sum_{m=0}^{\infty} e^{-\lambda_m t}\sin\left(k_m(y')\right)\sin\left(k_m(y)\right)\Big]dy \qquad (5-142)$$

$$P(t,h) = \frac{2}{h}\sum_{m=0}^{\infty} \lambda_m e^{-\lambda_m t}\sin\left(k_m(y')\right)\int_0^b \sin\left(k_m(y)\right)dx \qquad (5-143)$$

$$P(t,h) = \frac{-2}{h}\sum_{m=0}^{\infty}\frac{\lambda_m}{k_m} e^{-\lambda_m t}\sin\left(k_m(y')\right)\Big[\cos\left(k_m(y)\right)\Big]\Big|_0^h$$

$$P(t,h) = \frac{-2}{h}\sum_{m=0}^{\infty}\frac{\lambda_m}{k_m} e^{-\lambda_m t}\sin\left(k_m(y')\right)[\cos\left(k_m(h)\right) - 1] \qquad (5-144)$$

$$\lambda_m = Dk_m^2, \quad k_m = \frac{\pi}{h}m \qquad \cos\left(k_m(h)\right) = (-1)^m \qquad (5-145)$$

substituting quantities in (5-145) into (5-144) we obtain

$$P(t,h) = \frac{2\pi D}{(h)^2}\sum_{m=0}^{\infty} m\ e^{-Dk_m^2 t}\sin\left(k_m(y')\right)[1 - \cos\left(k_m(h)\right)] \qquad (5-146)$$

$$P(t,h) = \frac{2\pi D}{(h)^2}\sum_{m=0}^{\infty} m\ e^{-Dk_m^2 t}\sin\left(k_m(y')\right)[1 - (-1)^m] \qquad (5-147)$$

5.11 Diffusion in a Finite Interval with Mixed Boundaries:

If the fluid flow were to be in a medium between two infinite plates with absorbing boundary condition on one end and a reflecting boundary on the other end. The flow field may be solved as follows:

$$\frac{\partial p(y,t)}{\partial t} = \nu \frac{\partial^2 p(y,t)}{\partial y^2} \qquad (5-148a)$$

Initial Condition:

$$p(y,t_0) = \delta(y-y') \qquad (5-148b)$$

Absorbing Boundary Condition at y = 0 (bottom):

$$p(y=0,t) = 0 \qquad (5-148c)$$

Reflecting Boundary Condition at y = h (top)

$$\frac{\partial p(y,t)}{\partial y} = 0 \qquad (5-148d)$$

The method of separation of variables can be used to solve the problem $(5-148)$ Let $p(y,t)$ be represented by:

$$p(y,t) = G(t)F(y) \qquad (5-149)$$

Subtituting $(5-149)$ into $(5-148a)$ we obtain:

$$\frac{1}{G(t)}\frac{dG(t)}{dt} = \nu \frac{1}{F(y)}\frac{d^2F(y)}{dy^2} \qquad (5-150)$$

Both sides should equal a constant $-\lambda$ Rewriting the left hand side of $(5-150)$

$$\frac{1}{G(t)}\frac{dG(t)}{dt} = -\lambda \qquad (5-151)$$

$$G(t) = G_0 \; e^{(-\lambda t)} \qquad (5-152)$$

with $G(t=0) = G_0 = 1$ The right hand side of $(5-150)$ can be written as:

$$\nu \frac{1}{F(y)} \frac{d^2F(y)}{dy^2} = -\lambda \qquad (5-153)$$

Introducing the notion of the wave number k given by :

$$k^2 = \frac{\lambda}{\nu} \qquad (5-154)$$

Equation $(5-153)$ can be written as:

$$\frac{d^2F(y)}{d^2y} + k^2F(y) = 0 \qquad (5-155)$$

The general solution of $(5-153)$ is

$$F(y) = Asin(ky) + Bcos(ky) \qquad (5-156)$$

At the bottom boundary condition:

$$p(y,t) = 0, \Rightarrow F(y) = 0, at \; y = 0 \qquad (5-157)$$

$$F(0) = A\sin(0) + B\cos(0) = 0, \;\; B = 0 \qquad (5-158)$$

Applying the reflecting boundary condition at $y = h$

$$\frac{\partial F(y)}{\partial y} = Ak\cos(ky) - Bk\sin(ky) = 0 \qquad (5-159)$$

$$Ak\cos hk = 0 \qquad (5-160)$$

$$\cos hk = 0 \qquad (5-161)$$

This yields discrete solutions for k and eigen values :

$$k_m = \frac{\pi}{h}(\frac{1}{2} + m)) \qquad (5-162)$$

but

$$\lambda_m = \nu k_m^2 = \frac{\nu\pi^2}{(h)^2}(\frac{1}{2} + m)^2 \qquad (5-163)$$

where $m = 0, 1, 2..............n \quad ie\ positive\ integers$. A particular time dependent solution which fulfills the boundary condition and corresponds to the eigenvalue λ_m is thus the eigenfunction $p_m(y,t)$ given by

$$p_m(y,t) = B_m e^{-\lambda_m t} cos(k_m y) \qquad (5-164)$$

The complete solution of the problem is the superposition of these eigenfunctions.

$$P(y,t) = \sum_{m=0}^{\infty} p_m(y,t) = \sum_{m=0}^{\infty} B_m e^{-\lambda_m t} cos(k_m y) \qquad (5-165)$$

To obtain B_m , we employ the initial condition.

$$P(y, t=0) = \sum_{m=0}^{\infty} B_m e^0 cos(k_m y) = \delta(y - y') \qquad (5-166)$$

multiply by $cos(k_n y)$ and integrate over y from 0 to h

$$\sum_{m=0}^{\infty} B_m \int_0^h dy\ cos(k_m y)\ (cos k_n y) = \int_0^h dy\ \delta(y-y')\ cos(k_n y) \qquad (5-167)$$

The integral on the left hand side can easily be calculated using the orthogonality of the eigenfunctions

$$\int_0^h dy\, cos(k_n y)\ (cos k_m y) = \frac{1}{2}\int_0^h dy \Big[\cos\left[(k_n+k_m)y\right] + \cos\left[(k_n-k_m)y\right]\Big] \qquad (5-168)$$

$$= \frac{1}{2}\Big[\frac{1}{k_n + k_m}\sin(k_n + k_m)y + \frac{1}{k_n - k_m}\sin(k_n - k_m)y\Big]_0^h$$

$$= \frac{h}{2}\Big[\frac{1}{k_n + k_m}\sin(k_n + k_m)y + \frac{1}{k_n - k_m}\sin(k_n - k_m)y\Big] \qquad (5-169)$$

$$= \frac{h}{2}\delta_{mn} \qquad (5-170)$$

where the limit $lim_{x\to 0}\frac{\sin x}{x} = 1$ has been used. Hence (5-167) reduces to

$$\sum_{m=0}^{\infty} B_m \frac{h}{2}\delta_{mn} = \int_0^h dy\ \delta(y - y')\cos(k_n y) \qquad (5-171)$$

which yields

$$B_m = \frac{2}{h}\cos(k_m y') \qquad (5-172)$$

but $f(y) = A\sin(ky) + B\cos(ky)$ and $X(t) = X_0 e^{-\lambda t}$

$$p(y,t) = X(t)f(y)$$

The solution reads

$$p(y,t) = \frac{2}{h}\sum_{m=0}^{\infty} e^{-\lambda_m t}\cos(k_m y')\cos(k_m y) \qquad (5-172)$$

which is the final probability distribution. The probability distribution $p(y,t)$ tends to zero with increasing time.

First- Passage Time Problem.

The mean time during which the system finds its stable state by overcoming a potential barrier due to stochastic fluctuations. The first-passage time distribution(breakdown probability density) follows from the balance condition

$$P(t, y = h) = -\frac{d}{dt}\int_0^h p(y,t)dy \qquad (5-173)$$

By inserting the solution (5- 172) into (5-173) we obtain

$$P(t,h) = -\frac{d}{dt}\Big[\int_0^h \frac{2}{h}\sum_{m=0}^{\infty} e^{-\lambda_m t}\cos(k_m(y'))\cos(k_m(y))\Big]dy \qquad (5-174)$$

$$P(t,h) = \frac{2}{h}\sum_{m=0}^{\infty}\lambda_m e^{-\lambda_m t}\cos(k_m(y'))\int_0^h \cos(k_m(y))dy \qquad (5-175)$$

$$P(t,h) = \frac{2}{h}\sum_{m=0}^{\infty}\frac{\lambda_m}{k_m}e^{-\lambda_m t}\cos(k_m(y'))\Big[\sin(k_m(y))\Big]\Big|_0^h$$

$$P(t,h) = \frac{2}{h}\sum_{m=0}^{\infty}\frac{\lambda_m}{k_m}e^{-\lambda_m t}\cos(k_m(y'))\sin(k_m(h)) \qquad (5-176)$$

$$\lambda_m = Dk_m^2, \quad k_m = \frac{\pi}{h}(\frac{1}{2}+m) \quad \sin(k_m(h)) = (-1)^m \qquad (5-177)$$

substituting quantities in (5-177) into (5-176) we obtain

$$P(t,h) = \frac{2\pi D}{(h)^2}\sum_{m=0}^{\infty}(-1)^m(\frac{1}{2}+m)e^{-Dk_m^2 t}\cos(k_m(y')) \qquad (5-178)$$

CHAPTER 6 : CONCLUSIONS AND DISCUSSIONS

In broad terms, this research shows that fluid-based transport of diffusive quantities for example, heat, mass, momentum and vorticity can be studied from both classical continuum standpoint using conservation principles of mass and momentum as well as from the perspective provided by the theory of stochastic processes. The diffusion equation can be derived from the continuum limit using potentials, forces, fluxes and transient quantities of the fluid and from a microscopic point of view where the individual probabilistic motions of particles lead to diffusion. The study also exploits an important mathematical equivalence that exists between the Green's function, $G(x, t/x', t')$, associated with a given continuum transport problem, and the transition density, $p(x, t/x', t')$, associated with the corresponding stochastic transport problem.

In detail, the adjoint equation governing a given transport problem's Green's function, G(x,t/x',t'), is shown to correspond exactly to an associated Fokker-Panck equation governing the evolution of a transition density, $p(x, t/x', t')$ In order to complete the stochastic description of a given transport problem, we must hypothesize a physically reasonable entity whose continuum transport is governed by the continuum transport equations and whose stochastic evolution is described by an appropriate stochastic. differential equation. The combined approach to studying transport entities is given the name Green's function stochastic methods framework(GFSM). This method provides a large mathematical tool box for investigating both deterministic and random transport problems. Once a choice of the physical entity is made the physics and mathematical framework lead to deep physical insight into how the entity evolves.

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