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Supporting Preservice Mathematics Teachers' Culturally Responsive Teaching: A Focus on Teaching for Social Justice

Jordan Register [1,*], Anthony Fernandes [2,*] and David Pugalee [3]

[1] Departments of Mathematics and Statistics and Middle, Secondary, and K-12 Education, University of North Carolina at Charlotte, Charlotte, NC 28223, USA
[2] Department of Mathematics & Statistics, University of North Carolina at Charlotte, Charlotte, NC 28223, USA
[3] Center for Science, Technology, Engineering, and Mathematics Education, University of North Carolina at Charlotte, Charlotte, NC 28223, USA; david.pugalee@uncc.edu
* Correspondence: jtrombly@uncc.edu (J.R.); anthony.fernandes@uncc.edu (A.F.)

Abstract: This paper reports on how 10 middle and high school preservice teachers (PSTs) designed a social justice focused lesson using the culturally responsive mathematics teaching (CRMT) tool. Results from our analysis indicate that most of the PSTs were able to select appropriate social justice topics, though not all the PSTs integrated mathematics and social justice throughout their lessons. The results show that most of the PSTs need more experience with mathematization, handling controversial discussions, and developing transformative student action. Our work also led to a modification of the tool (CRMT-M). We discuss the implications of the study for mathematics teacher preparation.

Keywords: culturally responsive mathematics teaching (CRMT); social justice; lesson planning

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1. Introduction

The difference between the knowledge and experiences valued outside school versus those that are privileged in school are a commonly cited cause for the underperformance of culturally and linguistically diverse students (CLD), who are a majority in urban schools [1–6]. U.S. schools typically restrict access to CLD students by privileging Eurocentric instructional practices, curricula, and learning contexts [7,8]. Scholars argue that teachers must understand, and build their instruction on, CLD students' cultural characteristics, experiences, and perspectives [7,9,10]. By situating their learning within their lived experiences, such culturally responsive pedagogy (CRP) leads to meaningful learning and positive academic outcomes for CLD students [9].

A key element of CRP includes exploring sociopolitical issues that are relevant to students' realities [11,12]. In the last few decades, scholars concerned with critical mathematics education have formally introduced the concept of teaching mathematics for social justice (TMSJ) as a means for pushing their students to "read and write the world with mathematics", a notion built on the work of Freire's "reading and writing the world" through adult literacy [11,13–17]. Freire's goals were to enable pupils from oppressed groups to develop a critical consciousness of the world around them through a praxis of action, reflection, and agency [13]. While he was mainly concerned with the development of literacy, Freire's ideas translated into mathematics education as well. Critical mathematics education scholars maintain that the development of critical consciousness, which is the ability to identify and critique systems of inequality, and the commitment to take action against these systems can and should be fostered in a mathematical setting [11,17]. In other words, mathematics provides a method for identifying inequities and injustices in society that can be explored and analyzed in a way that allows for solutions to be both considered and acted upon. As such, it seems imperative that teachers foster critical mathematics consciousness; that is, the

awareness of the ethical, sociopolitical and communicative implications of our own and others' mathematical work through intentional instructional design grounded in justice-related contexts [18].

To address the needs of a diverse student body, teachers must have adequate content knowledge, pedagogical content knowledge, and cultural knowledge of their students [19]. Teacher preparation programs concerned with CRP have focused on developing PSTs' sociopolitical knowledge through diversity coursework, CLD student interviews, community/family outreach, and enhanced programs that require field placement in CLD schools and participation in continued professional development [20–23]. Aguirre and Zavala [1] further recommend that CRMT requires that teachers attend specifically to its elements in their lesson designs. Elements like the students' culture and social justice can be easily overlooked as they can be challenging to novice teachers [11,24].

Lesson planning is an important but under-discussed feature of preparing teachers. As preservice mathematics teachers (PSTs) prepare to teach CLD students, it is important that they adapt their lessons to both the cultural and mathematical characteristics of their students [1]. Aguirre & Zavala [1] define culturally responsive mathematics teachers as those who

> "leverage mathematical learning by expanding children's mathematical thinking, building bridges between previous knowledge and new knowledge, supporting bilingualism and academic language development, fostering connections with cultural funds of knowledge and experiences, and cultivating critical mathematical knowledge that enables students to analyze and address authentic problems" (p. 168).

Though PSTs can read about and understand ideas related to CRP, our interactions indicate that they desire more support in planning culturally responsive mathematics lessons. Specifically, they want to see how CLD students' experiences can support mathematical learning objectives in social justice contexts. The purpose of the current study is to examine how PSTs use the CRMT tool to design social justice centered lessons. We draw on Aguirre and Zavala's [1] lesson analysis tool, to improve PSTs' CRMT by embedding systematic analysis, critique, and pedagogical dialogue into their planning. The CRMT tool focuses explicitly on mathematical thinking, language, culture, and social justice. As such, the **research question** that guided this study is:

> How do PSTs implement social justice contexts into their lesson design using the CRMT tool?

2. Literature Review

The literature review that follows will describe research related to lesson design and teaching mathematics for social justice. Specifically, it will call attention to the importance of lesson design and analysis for developing CRMT lessons for CLD students. In addition, scholarship regarding the implementation of TMSJ will be discussed, with specific attention being paid to pedagogical difficulties for teachers and recommendations for teacher preparation. Notably, there is a dearth of literature related to teachers' experiences with TMSJ in secondary classrooms as well as PSTs' experiences with TMSJ in general.

2.1. Lesson Design

Lesson plans are artifacts that mediate the activity of teaching [25]. Given the complex nature of mathematics instruction, the expectation of CRP amplifies the importance of lesson planning for mathematics teachers with backgrounds that differ from those of their students [26]. Research in mathematics and STEM education have shown that when teachers are systematic in their lesson planning and implementation, student learning improves [27,28]. In terms of engaging in lesson design, Rusznyak and Walton [29] demonstrated the positive effects of engaging with a design tool that scaffolds aspects of pedagogical content knowledge. Further, Ding and Carlson [30] show that teachers can

improve their lesson planning skills through professional development that focuses on essential elements of instruction such as worked examples, representations, and questioning. Successful lesson design depends on a deep understanding of students' personal and mathematical experiences and works best in collaboration with colleagues. It includes unpacking learning objectives, situating lessons in students' realities, anticipating student responses and conceptions, sequencing discussions according to learning goals, and deciding what counts as understanding prior to the lesson [27,28].

Research with CLD students shows that there are other aspects of the lesson that are important. For example, CLD students who are emerging multilingual learners in the class benefit from an explicit focus on discussions around language [31,32] and social issues that relate to their lives [11,12]. Similarly, CRP promotes situated learning in meaningful contexts [11,33,34]. Thus, in addition to the mathematical design elements, lesson design must attend explicitly to elements like social justice and students' culture, which can easily be overlooked.

Aguirre and Zavala [1] found that teachers may not naturally include aspects of culturally responsive teachers in their lesson design. Thus, they designed the CRMT tool so that teachers can benefit from making culturally responsive elements explicit through their use of a lesson analysis tool and guiding questions. In a three-year professional development study with six K–8 teachers they found that the CRMT tool helped the teachers to analyze and critique mathematics lessons along multiple dimensions. They also found that the tool generated discussion about the teachers along the various dimensions as they sought to improve their lesson design and implementation. Though scholars in equitable teaching typically discourage the use of superficial check off lists [10], tools which explicate CRMT elements while promoting pedagogical analysis, critique, and discourse [1] may help novice mathematics teachers to attend to aspects of lesson design that they might typically overlook.

2.2. Teacher Experiences with Teaching Mathematics for Social Justice (TMSJ)

Although TMSJ is discussed in the critical mathematics education literature, much of the research has focused on the benefits of TMSJ for students and teachers' difficulties with the method in K–8 classrooms. Relatively little research has been conducted regarding the use of TMSJ in secondary mathematics classrooms, except for Gutstein's [12], Rubel et al.'s [35], and Gutiérrez' [36] studies with high school students. In addition, few empirical studies have examined teacher preparation regarding TMSJ, and more specifically how teacher educators might support PSTs in their adoption of the method. In the remainder of this section, the literature related to teachers' experiences with TMSJ is discussed, followed by the authors' recommendations for teacher preparation.

Studies concerned with teaching mathematics for social justice have demonstrated that although TMSJ seems beneficial for students' learning and identity, teachers experience both pedagogical and idealistic difficulties with this method. For instance, Miescu et al. [37] conducted a study with 22 novice elementary teachers from a large public school district in New England who graduated from a program that focused on social justice. The focus of their study was to test whether TMSJ could be analyzed quantitatively. Using the Teaching for Social Justice Observation Scale (TSJOS) and the results of students' district-wide unit tests, they found that there was a positive and statistically significant relationship between practices related to teaching mathematics for social justice and students' mathematics proficiency. Further, the teachers who used these practices more frequently had greater student outcomes. However, they also found that while the majority of the observed novice teachers attempted to teach social justice, several struggled with the implementation.

Difficulties with the implementation of TMSJ for teachers have been noted to fall under the categories of ideology or pedagogy. In a study by Aguirre et al. [19], 40 PSTs were asked to analyze their own mathematics lessons utilizing the CRMT tool with categories about children's mathematical thinking, academic language supports, cultural funds of knowledge, and critical math/social justice. They found mixed receptivity to integrating

social justice into mathematics lessons that were based more so on pedagogy than ideology. Specifically, they found that 15 of the 40 PSTs were ideologically receptive to teaching for social justice, but pedagogically resistant. They noted that for the PSTs in the study social justice was an important component in their lesson planning, but the PSTs shared concerns related to pedagogy including not knowing how to develop authentic problems grounded in a social justice context for specific mathematical topics or specific grade levels. Similarly, Simic-Muller and Fernandes [38], report on a study exploring the beliefs that PSTs have about teaching in real-world contexts, including those related to injustices, controversial issues, and children's home and cultural backgrounds. PSTs in the study were comfortable with the idea of teaching mathematics through real-world contexts but were apprehensive about bringing up controversial issues, especially with younger students. The PSTs were also challenged to provide concrete or non-trivial examples of real-world contexts.

In addition to difficulties with implementation, Bartell [39] documented tensions specifically related to balancing mathematics and social justice goals in lesson design. In an analysis of teachers' work related to TMSJ, she found that their definitions of TMSJ differed and that these definitions served to both guide and constrain teachers' development of their TMSJ practice. A notable difficulty was the negotiation of mathematical and social justice goals in the lesson that often divided the lesson among these two foci. The mathematical goals for the lesson were concentrated at the beginning, building on the mathematical procedures learned in previous lessons, while the social justice goals were concentrated at the end. This culminating social justice element typically included discussions related to their interpretations of the calculations that they made throughout the lesson and the implications related to these results. Bartell further suggested that because this was her participants' first experience with TMSJ, they focused more on the social justice goals when designing the lesson, potentially explaining their lack of attention to the mathematical goals and the integration of the two.

Previous studies provide several recommendations for helping teachers to learn to implement TMSJ, which include research, professional development, and the use of explicit tools for guidance. For instance, Bartell [39] suggests that more research is needed to understand the type of support needed for teachers who intend to practice TMSJ. That is, scholarship must address how to develop teachers' conceptions of TMSJ as an ongoing and complex process, beyond its introduction. She also suggests that to support teachers' substantive definitions of TMSJ they must engage with readings and sample lessons to understand the fundamental components of lesson study and design. Aguirre et al. [19] further recommend that teacher educators engage PSTs with the CRMT tool to guide their analysis of CRMT lessons. A similar sentiment was described by Miescu et al. [37], who found that TMSJ strategies could be quantified using the Teaching for Social justice Observation Scale (TSJOS) and recommended its use in teacher preparation programs to support PSTs development of the TMSJ method. As such, the purpose of this study is to contribute to the growing literature base around developing teachers' TMSJ practice. Specifically, we aim to explore middle and high school mathematics PSTs' conceptions related to CRMT lesson design with a specific focus on teaching mathematics for social justice.

3. Theoretical Framework

Aguirre and Zavala's [1] framework for Culturally Responsive Mathematics Teaching (CRMT) and their corresponding CRMT lesson analysis tool was used to guide this study. We also draw on scholarship from Critical Mathematics Education and Teaching Mathematics for Social Justice (TMSJ) to guide our analyses and CRMT tool adaptations.

3.1. CRMT

Aguirre and Zavala [1] argue that to define powerful mathematics teaching and to enact it are separate processes. The latter process requires the intentional development of culturally responsive mathematics teachers. They define CRMT as "a set of specific pedagogical knowledge, dispositions, and practices that privilege mathematical think-

ing, cultural and linguistic funds of knowledge, and issues of power and social justice in mathematics education" [1] (p. 163). The CRMT framework is an outgrowth of the CRP frameworks developed by Ladson-Billings [7] and Gay [9] to foster equity and excellence in historically marginalized youth, specifically targeted towards mathematics learners. According to Gay [9], in addition to fostering academic achievement, culturally responsive pedagogy develops "social consciousness and critique, cultural affirmation, competence, and exchange" (p. 43). It emphasizes community-building, personal connections, self-efficacy, ability, and caring. Culturally responsive teachers view their students as intellectually capable and believe that learning is composed of "intellectual, academic, personally, social, ethical, and political dimensions", which are developed simultaneously and in relation to one another [9] (pp. 43–44). Notably, CRMT requires both pedagogical content knowledge and CRP to simultaneously tap into students' mathematical, social, cultural, and linguistic resources [1,40].

Aguirre and Zavala [1] maintain that to promote equitable and meaningful mathematics learning for all students, teachers must be able to attend not only to research informed notions of students' mathematical thinking but also their language, culture, and power relations [2,41–43]. While such variables are typically introduced in general teacher preparation courses on diversity, they have not been considered an essential component of mathematics education until relatively recently [1]. Significantly, CRP was not developed in the context of the mathematics classroom and thus may not explicitly attend to the nuances of mathematics learning. However, pedagogical extensions of CRP have evolved in the mathematics education sphere and serve as a foundation for CRMT; that is, teaching mathematics for social justice [1,2,40,44].

3.2. Critical Mathematics/Teaching for Social Justice

Scholars who prescribe to a critical mathematics perspective view mathematics as a tool for analyzing power structures and relations, as well as social, economic, and civic issues at the local, national, and global levels [1,2,7,42,45–48]. In addition, they emphasize the importance of agency in challenging the injustices that they explore. Importantly, proponents of CM and TMSJ maintain that teachers play an important role in guiding students towards understanding, challenging, and dismantling the power relationships and structures that produce inequitable outcomes [1,2]. As such, the CRMT framework is guided by the core tenets of CM/TMSJ, which emphasize that mathematics teaching should:

1. build on the knowledge that students bring with them from outside of school, and
2. broaden their understanding of how mathematics can be used to interpret their world and act to rectify social injustices.

Gutstein [2,12,49] further elaborates on the role of the teacher in classrooms centered around TMSJ. He developed activities based on themes generated by students and their experiences. Notably, he leveraged class discussions to allow important issues and viewpoints to surface. In one study, he then developed a housing activity based on his students' concern for affordable housing and the impact that the broader policies were having on the housing in the barrio where the students lived [49]. Thus, culturally responsive teachers concerned with teaching mathematics for social justice must engage in discourse with their students intended to elicit broader social issues that are relevant to their lives. Once important issues are selected, the teacher systematically determines how the mathematics and the issue connect and support their mathematical and critical development simultaneously. In his work, Gutstein constantly emphasized both the mathematics and the social justice goals in his role as the teacher.

3.3. The CRMT Tool

Aguirre and Zavala [1] operationalized their approach to lesson design by developing the CRMT tool. The tool is composed of eight dimensions that exist within the four overarching themes of CRMT (i.e., mathematical thinking, language, culture, and social justice). Each dimension is accompanied by a guiding question as well as five potential ratings with

corresponding descriptors [1]. The tool itself is a rubric based on the following seven dimensions, each with five levels of attainment: (1) cognitive demand, (2) depth of knowledge and student understanding, (3) mathematical discourse and communication, (4) power and participation, (5) academic language support for ELLs, (6) cultural/community funds of knowledge, and (7) social justice [1]. The purpose for creating the tool is three-fold: (1) to assist teachers in their planning for culturally responsive teaching in mathematics classrooms, (2) to provide a tool for the analysis of lesson plans and their implementation, and (3) to make the elements of CRMT explicit for teachers [1]. They found that the CRMT tool fostered critical reflection and intentional pedagogical dialogue, which guided the purposeful modification of preservice teachers' lesson plans to be more culturally responsive to their students. In the current study, the CRMT tool was initially used as an analytical framework for the PSTs lesson plans. However, over the course of the three semesters, we adapted the tool to make the essential subcategories of each element more explicit for our PSTs.

4. Methods

4.1. Participants and Program

The study was conducted in a seminar course with fifteen middle and secondary mathematics PSTs who were part of a culturally responsive mathematics teaching scholarship program. The PSTs applied and were selected for this scholarship program by a committee. The program sought to recruit STEM majors to become mathematics teachers. In addition to their regular teacher preparation activities (e.g., course work, clinicals), the PSTs were required to complete additional work related to the focus of the program, which was developing culturally responsive teachers. They also spent time in local classrooms and attended a monthly seminar where they reflected on their classroom experiences with CLD students, completed readings, participated in online discussions related to the readings, engaged in mathematics analysis of real world and social justice contexts (e.g., Gerrymandering, traffic stops), and developed culturally responsive lesson plans.

This paper focuses on the last aspect of lesson planning and pays specific attention to its social justice element. Our aim was for the PSTs to design a lesson that would address both the mathematical and social justice goals. Importantly, we believe that these are intertwined, and we did not want to privilege one over the other. Instead, we used the CRMT tool to encourage the PSTs' planning for rigorous mathematics instruction and learning that builds students' critical consciousness of social injustices. Since the PSTs design other lessons as part of their program that emphasize the mathematical goals, and given the program's focus on culturally responsive teaching, we chose to provide a different experience, one that would support our PSTs in incorporating social justice issues into rigorous mathematics lessons. We also thought about the goals with respect to the students in the class. CRMT has found that students of color engage more in lessons that intersect with their lived experiences and thus it was important for us to provide these experiences. Note that the pandemic restricted all the face-to-face activities, so this activity was conducted solely online. To compensate for their reduced workload due to the pandemic, PSTs also read Kendi's [50] *How to be an Antiracist* and Gutstein's [11] *Reading and Writing the World with Mathematics* and participated in online discussion boards and Zoom discussions with their peers. The effects of these readings will be discussed in a subsequent section.

4.1.1. Previous Semester Activities

The lesson planning activity spanned three semesters. Activities were developed to accommodate our pedagogical goals related to developing the PSTs' understanding of CRMT and lesson planning more generally. Given the scope of this paper, we will focus on the third round of analysis, which occurred in spring of 2021 in the second semester. We briefly summarize previous seminar activities and pedagogical goals in Table 1. We will discuss the corresponding analyses in a later section.

Table 1. Seminar activities.

Semester	Activity	Pedagogical Goals	Analysis
Fall 2020	1. Adapt a given lesson plan from state lesson repository on linear functions to be CR according to their own understanding. 2. Read Aguirre et al.'s [1] article and review the CRMT tool and its eight dimensions. 3. Reflect on and peer review their linear function lesson. 4. Seminars to elaborate on the dimensions of the CRMT tool: 1. Mathematical thinking. 2. Language. 3. Culture. 4. Social justice. 5. Adapt area lesson plan to be culturally responsive according to the CRMT tool. Ongoing Readings: *How to Be an Antiracist* by Kendi [50].	Develop a baseline understanding of PSTs' understanding of CRMT and their lesson planning capabilities to guide instruction.	First round of analysis: develop a baseline. Second round of analysis: analyze PSTs' understanding of the tool.
Spring 2021	1. Choose a topic according to grade level, etc. 2. Explore CRMT Tool during online seminars (using Jamboards, etc.). 3. Explicit focus on 6a and 6b (cultural/community funds of knowledge and social justice). 4. Find real world/social justice context to build their topic around. 5. Write lesson plan based on their topic and context. 6. Analyze their own lesson plan using the original CRMT tool. 7. Present video launch. 8. Adapt lesson plan after multiple rounds of feedback (two rounds from professors, two peer review discussions on Canvas). 9. Turn in final lesson. Ongoing Readings: *Reading and Writing the World with Mathematics* by Gutstein [11].	• Gauge their ability to develop a lesson. • Avoid content/context restrictions found in second round of analysis. • Help them focus on the social justice element. • Scaffolding: provide flexibility in topic/context. • Gauge how PSTs plan to hook/engage students in the topic/context.	Third round of analysis: • Score developed lesson plans according to tool. • Develop holistic summaries of individual lesson plans. • Focus on integration of mathematics and social justice context. • Adapt the CRMT tool into two separate tools (planning and observation). • Use the new CRMT-M lesson planning tool to score lesson plans.
Fall 2021	1. Student Interviews. 2. Co-teach mini lessons. 3. Use CRMT-M observation tool to critique lesson implementations.	• Observe PSTs' execution of their developed lesson plans. • Allow PSTs to critique lessons using the adapted CRMT-M observation tool. • Promote critical reflection on PSTs lesson plans and implementation according to the CRMT-M tool. • See how PSTs integrate the mathematics and social justice elements of the lesson [Ma1].	Fourth round of analysis: • Use the new CRMT-M observation tool to score mini-lessons. • Continue to adapt CRMT-M tools to support PSTs' use of the tools.

4.1.2. Spring 2021 Semester

Our experiences with CRMT lesson planning in the fall semester resulted in shifts in the seminar design for the spring semester. Initially, we felt that by providing students with a lesson plan to adapt they could focus on understanding the elements of the CRMT tool. We observed, however, that the lessons constrained them in terms of the adaptations they could make. For example, in a lesson centered on area, the PSTs discussed how they would engage the students in a discussions about purchasing a home to embed the mathematics content of algebraic expressions. We felt that there was a loose connection between the context and the mathematics and that it felt forced. Note that the lesson was taken from state resources that are available to teachers. Despite the constraints, we believed that adapting the lesson was part of the scaffolding process that would help the PSTs. Furthermore, because of their lack of attention to issues of social justice, community funds of knowledge, and authentic integration of their students' cultures, we chose to focus explicitly on these areas. Seminar activities included searching for and sharing social justice contexts that could be explored mathematically and engaging in regular discussions about how to incorporate potentially sensitive topics into mathematics instruction.

Our next level of scaffolding included the PSTs developing a lesson that they would eventually teach. From their experience adapting the lessons, the PSTs expressed an interest in seeing what a mathematics lesson for social justice would look like. We decided that reading Gutstein's [11] *Reading and Writing the World with Mathematics* would provide them with tangible examples of lessons that focus on teaching mathematics for social justice. We felt that Gutstein's reflections about his thought process and implementation would give the PSTs a glimpse into the experience from a teacher's point of view. In addition to providing examples of lessons around social justice and mathematics, Gutstein elaborates on his implementation of the lessons and the impact on the students, teachers, parents, and administration. The PSTs read the chapters over the course of 15 weeks and engaged in regular online discussions that drew their attention to important features of teaching mathematics for social justice. Some of our discussions included the challenges of implementing mathematics activities that drew heavily on social justice goals. However, the PSTs also realized the potential of social justice contexts with their future students. We intended this critical reading to prompt PSTs' shift towards lesson designs that specifically drew on students' experiences and addressed social justice.

4.2. Adaptation of the CRMT Tool

Our findings from the first two rounds of analysis prompted us to take a different approach to analyzing the PSTs' lesson adaptations. In using the CRMT tool as is, we felt restricted in our ability to analyze the PSTs' lesson plans. Thus, we modified the CRMT tool to better support our analysis of the PSTs' lesson plans (CRMT-M). Note that these modifications were not included in the CRMT tool that PSTs used to develop their lesson plans; however, these concepts were discussed on a regular basis during the seminars and in regular, individual meetings with the PSTs. Due to the scope of this paper, we will not discuss the modification process here. However, specific limitations and corresponding modifications to the tool are outlined in Table 2 below. Furthermore, while our full study included the adaptation of all seven elements of the CRMT tool, we will report only on the analysis of the social justice element using our modified CRMT tool (CRMT-M).

Table 2. Modifications to the CRMT tool (CRMT-M).

Limitations of the CRMT Tool	Modifications (CRMT-M)
Too granular (five levels of attainment).	Reduced levels of attainment to three.
Targeted toward a lesson observation as opposed to a lesson plan.	Developed two interconnected CRMT tools for lesson planning and lesson observation.
Uses educational terminology that may not be familiar to PSTs.	Adjusted the language to accommodate the PSTs level of education and corresponding access to educational literature.
Essential components of each element are unclear.	Identified essential components of each element and developed explicit subcategories drawn from the literature on critical pedagogy, critical mathematics, and realistic mathematics education (RME) [2,12–14,17,51].
Less emphasis on addressing the integration of mathematical and social justice goals.	Created a specific subcategory addressing the integration of mathematics and social justice goals in the social justice element.

Explicit Subcategories in the Social Justice Element

We first broke each element of the CRMT rubric down into what we felt were its essential components, henceforth referred to as subcategories. Each of these subcategories in the social justice element were drawn from the literature on Critical Pedagogy, Critical Mathematics, and Realistic Mathematics Education (RME) e.g., [2,12–14,17,51]. Integral to this process was the analysis of PSTs' lessons using a holistic approach, as opposed to our previous, thematic coding process. We first developed summaries of the PSTs' lesson plans then analyzed the social justice element using those summaries, adapting the tool as we encountered subcategories that were either not clear or not included in the tool (such as the integration of social justice and mathematics, mathematizing, and controversial topics). The subcategories can be seen horizontally across each element in the CRMT-M tool. For instance, in the social justice element, we identified context, integration of social justice and mathematics, mathematization, controversial topics, and transformative student action as the key components, each with a guiding question at the header of the section (see Tables A2 and A3 for details). The first two subcategories came directly from the original tool, although we adapted the verbiage to highlight the importance of integrating the mathematics and the social justice goals. We added the latter three subcategories to encourage the PSTs to make such considerations since they did not seem to recognize their importance in the lesson adaptations.

We noticed that the PSTs did not explicitly consider how they would encourage their students to mathematize the situation at hand; that is, how attributes of real-world situations could be operationalized mathematically to question the source of the data involved and to explore the meaning of their mathematical findings in relation to the context [50]. We isolated the mathematization subcategory from the literature where students were able to isolate and observe the inequities in the context by examining the appropriate measures. For example, students in Rubel et al. [35] developed the measure of relative spending on the lottery in a neighborhood compared to the median income in the neighborhood. Based on comparisons between the lowest and highest median income areas, the students determined that people in lower-income communities spent more in the hopes of winning. However, they found that the net winnings in the lower income neighborhoods were much less than the net spending. Thus, the mathematization through the appropriate ratios afforded the students opportunities to observe the disproportionate burden on lower-income communities.

In addition to isolating mathematization, we also focused on the influence of controversial issues in the lesson. Controversial issues can bring up sensitive discussions with regards to race, gender, and poverty, for example. As such, PSTs need to be able to handle

possible tensions that could arise among students with different views. Simic-Muller and Fernandes [38] point to K–8 PSTs' preference for avoiding controversial issues in the class, especially in the case of younger K–5 students. As such, we decided to have a separate subcategory devoted to handling difficult conversations, including teachers' anticipation of student experiences related to the context, potential topics that could arise during the lesson, and how they plan to facilitate discussions centered around such topics. Finally, we chose to make attention to transformative student action (agency) more explicit in the tool. According to Freire [13], reflection and action are a reflexive process in building critical consciousness. As such, we sought to incorporate student agency as a subcategory that PSTs should explicitly plan for in the lesson. The social justice element of the CRMT-M lesson planning tool can be seen in Appendix B Table A2.

4.3. Data Collection Procedures

Our data corpus for the overarching study consisted of all the artifacts developed by the PSTs over the three (fall 20, spring 21, fall 21) semesters, which included initial lesson adaption (13), revised lessons (13), final lesson plans (11) (note that the other two PSTs graduated and thus did not submit lesson plans in the fall 2020 semester), launch videos, video recorded co-taught mini-lessons, student interviews, PST completed CRMT tools, final semester reflections, Zoom recordings, seminar materials (Google Jamboards, interactive PowerPoint presentations, Zoom chats), researcher field notes, meeting notes, and tool design notes.

4.4. Analysis Methods

The PSTs' lesson adaptations were analyzed to answer the research question. The analysis was geared toward understanding the PSTs' initial conceptions of CRMT, how these evolved over the course of the three semesters, and what may have fostered or inhibited their development.

4.4.1. Analysis 1 and 2

Given the large data corpus, we started our analysis with the PSTs' lesson plan adaptations. All the lessons were coded using NVivo. The fall semester consisted of two rounds of analysis, one to develop a baseline after their first adaptation and one following their adaptation using the CRMT tool. For the sake of brevity, and because the results were similar, we have only included the results of the second analysis, outlined in Appendix A.

4.4.2. Analysis 3

For our third analysis, we chose to focus on the social justice element of the adapted CRMT-M lesson planning tool. Using a spreadsheet, we classified each lesson plan according to the subcategories of each element. Each component included a description of our holistic analysis of the lesson according to that component. Individual subcategories were then color coded for each PST according to which level we felt the PST obtained. We coded half the PSTs independently, discussed our coding throughout, and resolved our coding to align. Note that we were working with the PSTs as we carried out the analysis, so one or both researchers regularly interacted with the PSTs to clarify any part of the lesson that may not have been clear. Further, we also provided detailed feedback that the PSTs could use to improve their lesson throughout the semester. As part of the coding, we used color in the cells to get a visual sense of the overall data. For a more detailed description of our calculations and coding process, see Appendix C. Example results for one PST, Juliet, are shown in Appendix C Figures A1 and A2. A cumulative analysis of our findings across PSTs will be discussed in the results section and can be viewed in Appendix C Figure A3.

5. Results

Recall that we set out to address the following research question: How do PSTs implement social justice contexts into their lesson design using the modified CRMT tool?

5.1. Cumulative Results of the Social Justice Element

Our third round of analysis, where PSTs created their own lesson plans, yielded significantly different results from that of Analyses 1 and 2. Given that the PSTs were explicitly instructed to create a lesson grounded in a social justice context, the social justice element was the target of our analysis. Results of the social justice element for the ten submitted lesson plans can be viewed in Appendix C Figure A3. A point we would like to iterate about the results is that they are reflective of PSTs who are in the process of developing their knowledge of the content and context and are constantly evolving. We see lower scores and points of transition as the PSTs evolve into critical educators. The low scores should not be interpreted as if the PSTs are lacking; instead, they should indicate areas of teacher preparation that need improvement. They were thus integrated into our modified CRMT tool, which we refer to as the CRMT-M tool. Notably, the CRMT-M tool was not used by the PSTs in developing their lesson plans, as it had yet to be created. However, the subcategories added were discussed regularly during seminars and in individual meetings with the PSTs.

In our analysis, we examine the five subcategories in the social justice factor of the CRMT-M tool; that is, meaningful social justice context, integration of social justice and mathematics, mathematizing, handling controversial issues, and student agency. Overall, of the 10 lesson plans we analyzed, one scored a level 3 (Juliet), six scored a level 2 (Martha, Allison, Elizabeth, Jessie, Karen, and Otis), and three scored at a level 1 (Heidi, Joan, and Rachel). In reporting these results, we look closely at the subcategories. Notably, the PSTs difficulties were primarily concentrated in their attention to mathematizing, controversial topics, and student agency (scaled mean score of 2, 1, and 1, respectively). Their scaled mean scores for sustained meaningful social justice context and integration of social justice and mathematics were both 2, indicating that they attempted to develop a lesson through which their students could meaningfully explore a social injustice using mathematics but that they may require further support in integrating the mathematics and social justice context throughout the entirety of the lesson. Notably, a key characteristic of the social justice element is that students will learn more about the issue as well as the mathematics. One cannot be rigorously explored at the expense of the other. A summary of our findings is provided in Table 3 and in Appendix C Figure A3.

Table 3. Analysis results.

PST/Lesson Information			**Social Justice Subcategories and PST Levels of Attainment**					
Pseudonym	SJ Context	Math Topic	*Sustained Meaningful SJ Context*	*Integration of SJ and Math*	*Mathem-atizing*	*Handling Controver-sial Topics*	*Student Agency*	***SJ Element Cumula-tive/Scaled PST Score***
Juliet	Student Representation in Schools	Ratios and Propor-tions	*3*	*3*	*2*	*1*	*3*	***3***
Otis	Minimum wage	Linear functions	*3*	*3*	*2*	*1*	*2*	***2***
Jessie	Incarceration rates	Probability	*3*	*3*	*1*	*2*	*1*	***2***
Martha	Gender wage gap	Proportional reasoning	*3*	*3*	*2*	*2*	*1*	***2***
Karen	Bias in media/ Racial represenat-ation	Measures of center and spread	*3*	*2*	*1*	*1*	*1*	***2***

Table 3. *Cont.*

PST/Lesson Information			Social Justice Subcategories and PST Levels of Attainment					
Pseudonym	SJ Context	Math Topic	*Sustained Meaningful SJ Context*	*Integration of SJ and Math*	*Mathem-atizing*	*Handling Controver-sial Topics*	*Student Agency*	***SJ Element Cumula-tive/Scaled PST Score***
Allison	Minimum wage	Linear functions	*3*	*2*	*2*	*1*	*1*	***2***
Elizabeth	Living wage/ CEO salaries	Measures of center	*2*	*2*	*2*	*1*	*1*	***2***
Heidi	Gender wage gap	Measures of center	*2*	*1*	*1*	*1*	*1*	***1***
Joan	Access to services	Linear functions	*1*	*1*	*1*	*1*	*1*	***1***
Rachel	No single context	Exponential functions	*1*	*1*	*1*	*1*	*1*	***1***
Scaled Mean Score			***2***	***2***	***2***	***1***	***1***	***2***

5.2. Subcategory Results: Sustained Meaningful Social Justice Context & Integration of Social Justice and Mathematics

5.2.1. Sustained Meaningful Social Justice Context

The result of our analysis of PSTs' final lesson plans (spring 2021) indicates that the PSTs in our study were proficient in selecting a meaningful social justice context. The scaled mean score for the sustained meaningful social justice context component was a 2, which we would consider a level 2 for attainment. Eight of the ten PSTs were able to draw on a single meaningful social justice context that grounded the mathematics in the lesson. The eight PSTs drew on contexts like the gender wage gap (2), minimum wage (2), comparison between CEOs and workers in corporations, incarceration rates in a state, the pros and cons of attending a small versus a large school (with respect to available resources), and critical analysis of data presented on social media.

Four PSTs who scored a 1 or 2 on the subcategory of sustained social justice context lacked grounding in the social justice context itself. The PSTs either did not use a context concerned with social injustice or they went back and forth from the context to procedural mathematics. For instance, Joan, one of the two PSTs who scored a 1, examined accessible services in the community but spent a considerable amount of the lesson working out the distance to the various movie theatres in proximity to the school and the time it would take to get there, a context that we did not consider to be a social justice issue. She did, later in the lesson, mention working out the proximity of the community to essential services like hospitals; however, this was not explored further. In the other level 1 case, Rachel mentioned using exponential functions to model COVID numbers, but the lesson focused more on students coming up with the exponential function independent of the context. She instead had her students brainstorm possible real-world examples that could be modeled by exponential functions. Notably, Joan and Rachel did not sustain the context throughout the mathematical elements of the lesson, switching to direct instruction to discuss the mathematical content. Similarly, Heidi, who scored at a level 2, began her lesson by having her students examine a visual graph representing the gender wage gap in the various states, which she followed up by having students perform guided practice on measures of spread. She then returned to the wage gap context to encourage students to discuss the results of their mathematical calculations in relation to the wage gap—but proceeded to tell the students what their inferences should be as opposed to encouraging them to make and discuss their own inferences. Since she elaborated more on the mathematical aspects of the lesson as compared to the social justice context, we interpreted this as a score of 2. Notably, the disconnect between the social justice context and the mathematical topic was also evident in our analysis of her integration of the social justice context and mathematics.

5.2.2. Integration of Social Justice and Mathematics

In addition to sustaining the context, we were also interested in how the PSTs integrated the mathematics and the social justice issue. We checked if the mathematics was appropriate for the students at their grade level, if it would challenge students, and if the mathematics would help the students gain insight and understanding of the context that they may not have had before the lesson. Four of the ten PSTs demonstrated this integration. They were also part of the eight PSTs who sustained the context through the lesson. The integration between the mathematics and the social justice context was seen partially or was missing in the other six PSTs lessons. Specifically, we noticed that for these PSTs, the social justice and mathematics were either (1) disconnected, in that the social justice context and mathematical content were introduced/explored separately, (2) procedural, in that the mathematics was not explored in a way that developed the students' conceptual understanding of the mathematics or context (e.g., the social justice issue is used predominantly as a context for performing calculations), or (3) there was a lack of mathematical rigor in that the context was explored in a way that expanded their understanding of the issue but did not advance their mathematical conceptions in a meaningful or rigorous way.

As an example, we scored Jessie's lesson, examining the incarceration rates broken down by race in the state of NC, at a level 3 for integration of the mathematics and social justice context. Her lesson started with a video of Ronnie Long, a Black man who was wrongly incarcerated at the age of 21, for 44 years. The initial discussion was launched through an autobiographical video of Ronnie Long, followed by a small group activity where the students were asked to explore US incarceration rates disaggregated by race. The students used concepts from probability, independence, and conditional probability to get a sense of the data and draw inferences. At the end, the students discussed their findings, returning to the context of incarceration.

In contrast to Jessie's integrated lesson, we felt that Elizabeth's mathematical exploration was periodically disconnected from the social justice context and was scored at a level 2. In the exploration, students are asked to examine the salaries of workers and the CEO of companies and evaluate the use of the mean or median in the context. The students first investigate if the mean or the median is an appropriate measure through a Desmos activity void of any meaningful context. The students could "move" arbitrary points on a Desmos plot to see how this would impact the mean and median of the collection of points. Students then built on this information as they examined a set of salaries that were provided. We scored this at a level 2 because the Desmos activity focused only on the measures of center of an arbitrary dataset rather than integrating it with the salary data that was the focus of the social justice context. We believe that without integrating the mathematics and context, students may also adopt the separation in their beliefs about mathematics as a subject that can be studied independent of the real world.

Three PSTs were scored at a level 1. Characteristics of these lessons included a separate examination of the social justice issue and the mathematics (disconnected) and a procedural or non-rigorous exploration of the mathematical topic. For example, in Heidi's lesson that explored income differences by gender, students worked with a graphical display of the income inequality in various cities and a teacher generated a word cloud based on the students' feelings about gender pay gaps. Prior to seeing the graph, the students learned about the measures of center through a guided notes activity that was completely disconnected from the context and procedural in nature. After commenting on the graph, the students engaged in more practice centered around measures of spread. The final activity involved dividing the students into two groups, one with a set of salaries for men, and the other with the salaries of women. The students were not told which gender they were given, and they were asked to find the mean, median, mode, range, interquartile range, and the mean absolute deviation. The students shared their findings with the class and were told which group of salaries they had. Though the topic was related to a social justice issue, there was little integration between the mathematics and the context. For example, the students did not explore the impact that mean or median had on outliers.

Here we see that the social justice issues were handled separately to the calculations of the measures of spread and center.

5.3. Subcategory Results: Mathematizing, Handling Controversial Topics, and Student Agency

The PSTs further grappled with the mathematizing, handling controversial topics, and student agency subcategories of the social justice element. Notably, these three elements were not explicitly included in Aguirre and Zavala's [1] CRMT tool as independent subcategories. Thus, the PSTs did not yet have access to a tool that would make these considerations explicit. For this reason, and because we recognized their lack of attention to them, we chose to add them into our revised CRMT tools for future use by PSTs in their planning.

5.3.1. Mathematizing

For the mathematizing subcategory, none of PSTs scored at a level 3, five obtained a level 2, and five scored a level 1. This indicated that the PSTs only sometimes or rarely probed the students to explore/interpret the source of the data and information used in the exploration, how attributes of the context could be operationalized/measured, and the meaning behind their mathematical findings in relation to the context.

Developing the right measures allows the students to gain insight into the context. For instance, in her lesson about ratios and proportions in the context of representation and resources in schools, Juliet scored a level 2. Throughout the lesson there were opportunities for the students to develop measures based on the available information to compare schools. To begin the lesson, students read an article that made a case for small schools and sought to disprove common myths about them. After reading the article, Juliet would share publicly available data on school personnel and resources and the students were directed to choose two schools to make a comparison. The students began by developing a measure for what they would consider a big school and a small school. Next, they had to develop measures to make comparisons. Given the focus on ratios, Juliet wanted the students to examine possible ratios drawn from the data. Here, she mentioned that the students would "discuss the proportion of class sizes, student population, teachers, etc.", but was not specific about ratios such as student to teacher ratios or funding per student that could be used in the comparisons. Further, students could also make other direct comparisons like the number of AP classes offered and the overall academic performance of the school on the state tests. In the culminating class discussion, Juliet further prompted her students to discuss the implications of their mathematical findings from the given data.

> *Juliet*: The discussion will start with the mathematical calculations. The students will discuss the proportion of class sizes, student population, teachers, etc. Then the class could begin talking about other proportions they found when comparing the schools such as resources, test results, etc. Then as a class we will discuss where they had consistent data and where there was inconsistency between groups. Discuss if the data could be misleading. The discussion should also lead to why the students' thought schools in the same district varied or did not vary and how schools outside the district looked different. The teacher should ask the students to think about other comparisons they could make that could explain the differences/commonalities between schools. Students who used to attend other schools should get to talk about past experiences and how their experience changed from school to school.

Since she prompted her students to question both the mathematics and their analysis of the issue but did not plan to discuss how the attributes from the context are operationalized, Juliet was scored at level 2. Notably, Juliet shared her struggle to find school-related data, constraining her to some extent in the design of the activity. Finding data and then cleaning them to be useful and understandable for middle and high school students was an additional challenge that was shared by several PSTs; something to note when we expect PSTs to engage in TMSJ.

Martha, another PST who was scored at level 2, examined the gender pay gap across immigrant groups (Figure 1). In her activity, she shared the following visual demonstrating the earnings of women relative to a dollar earned by men. In addition, she asked the students to work out problems such as determining the amount of time that an undocumented woman would have to work to earn $50 as compared to an undocumented man (note that the hourly wage was suggested to be $10 h for a White male worker). Here we see that the students get a chance to investigate the measures themselves, supporting their sense of the context. However, we felt that more could be done to unpack representation.

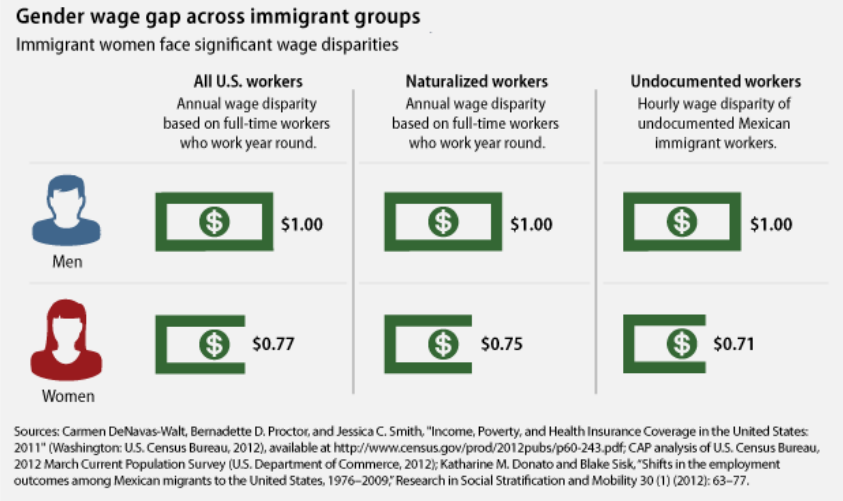

Figure 1. Gender pay gap image from Martha's lesson.

In contrast to Juliet and Martha, The PSTs who scored level 1 typically paid little attention to the mathematization of the context and how the developed measures could reveal something about the phenomena. Specifically, these PSTs either focused more on the mathematical procedures than their findings in relation to the context or provided a vague description of how they would facilitate their students' mathematization of the context and their findings. Earlier, we mentioned Heidi who, in her lesson covering measures of center and variability in the context of the gender wage gap, asked her students to look at a visual depicting the wage gap between men and women. However, the lesson did not encourage her students to examine the mathematization through the mean or median or why the given data provided the median pay rather than the mean pay. Notably, Heidi's exploration (scored at a level 1) was focused primarily on guiding their calculations of measures of spread.

> *Heidi:* We will begin working through a guided practice on IQR and MAD. These topics are just being introduced to them and *I am more focused on how proficient they are at computing mean, median, mode, and range at this point.* We will have more time to practice MAD and IQR. I want the students to understand what IQR and MAD means for a data set and how we can use those values to compare between two data sets.

Following her guided notes, the students were asked to calculate the measures of central tendency for a given dataset of either male or female wages but with less of an emphasis on a discussion of the measures themselves. Furthermore, like the other PSTs who scored at a level 1, Heidi did not consider her students' conceptions related to the context itself. Thus, she was unable to document how she planned to foster her students' growth in reasoning related to the source of the data, how it was collected, how the attributes were operationalized, and what their mathematical findings indicate in relation to the gender wage gap.

5.3.2. Handling Controversial Topics

Only two of the ten lessons included plans on how they might handle controversial topics that may come up in the lesson (scored at level 2). We find this to be an essential aspect of planning for CRMT lessons, given that the exploration of social injustices may be uncomfortable for those students and teachers who have experiences related to the context. Martha (level 2) handled potentially controversial topics by anticipating what her students' concerns might be and how she would address them with the class.

> *Martha:* For slide 2, students may have a large range of answers. Students may point out the difference between the ethnicity of women. This can bring up a sensitive issue for students as it may be upsetting to hear how their ethnicity is treated compared to a white male. Students could also discuss the undocumented pay percentage, which shows men who are undocumented still earn more than undocumented and us working women. For slide 3, students could discuss how this is a problem as women are not going to be given the same opportunities as men and therefore will struggle to become leaders in America. Females in the class may discuss their fears of this and how this is unfair if they are putting in the same work and effort as a male at a job.

Martha could have scored a 3 but was allotted a 2 because of her reference to potentially changing the subject rather than approaching it.

> *Martha:* If during the discussion, a controversial topic arises, such as someone stating how men work harder than women or are strong and can work fast pace compared to women, I could turn to talking about women figures who have had a large impact on the world *or change the topic into discussing the mathematics content* or talk about how men of other races may not earn the same amount as a white man. However, in my classroom I hope for a positive classroom culture, where a student would not try and put others down and remind the students of that during the discussion.

Jessie (level 2) also addressed how she would handle controversial topics during the launch of her lesson by not shying away from difficult topics. Rather, she highlights the importance of being a facilitator when it comes to sensitive topics, as well as developing classroom norms that include empathy and respect for their peers and their diverse experiences.

> *Jessie*: What questions do they have? I expect that some students will be shocked by what they have just heard while others will already know about this particular case or one similar. I will facilitate this sensitive topic by first reminding students of the classroom norms. These norms include being respectful of their classmates, understanding that these are real people's lives we are discussing and that there might be students in the class for who this topic is hard to discuss.

Jessie did not anticipate or address any specific questions or comments that would come from students related to the context, and thus may not be prepared to handle such instances.

In contrast to Jessie and Martha, the remaining PSTs did not discuss how they would plan to handle students' questions or comments that may be sensitive or made comments about classroom norms and respect. For instance, during her culminating class discussion, Heidi (level 1) states that students should remember the classroom norms of respect but does not elaborate on how she would address sensitive topics that may come up from students in relation to the context of the gender pay gap. In fact, a few PSTs indicated during our seminar discussions that they might shy away from certain topics that they felt might make their students feel uncomfortable. Note that the literature supports K–8 PSTs' tendency to avoid controversial topics in the classroom [52].

5.3.3. Transformative Student Action (Agency)

Although transformative student action resulting from the mathematical exploration of social justice issues is a key component of the development of critical mathematics consciousness, few PSTs facilitated meaningful opportunities for their students to act on the issue at hand. For instance, while eight of the PSTs did not address student agency at all, one PST, Otis, scored at level 2 and one, Juliet, at level 3. Otis hinted, in the last sentence of the class discussion portion of his lesson plan, at exploring minimum wage through linear functions.

> *Otis*: The role of the student in the discussion is to bring their conclusions about how minimum wage can be seen in linear equations so that they are "reading the world" through math. Then, after reading it they can form their own conclusions on how minimum wage should be reformed, abolished, or kept the same based on their findings.

While he alluded to pushing his students to make an informed decision about the state of minimum wage, he did not facilitate an activity that might allow them to create an action plan to transform the issue. In contrast, Juliet provided the most rigorous opportunity for student action in that she facilitated a distinct opportunity for her students to brainstorm ways to improve the social injustices created by disparate resources and representation in schools.

> *Juliet*: At the end of the discussion, students will return to their groups to brainstorm ways to improve equity between schools. They will summarize their reports by explaining their conclusions and their ideas about how to distribute resources more equitably.

The remaining PSTs did not address how they would encourage their students to promote change. However, in our interactions with Allison, another PST who designed a lesson around minimum wage, we found that she was thinking about the context of the problem from the point of view of small business owners rather than the workers. She argued that the big jump in the minimum wage would mean that less workers are hired and would affect the workers and businesses. We anticipate that her thinking may have been different if she considered the effect on corporations rather than small businesses.

6. Discussion and Conclusions

6.1. Overview of the Research Question and Key Results

This study sought to answer the question: How do middle and secondary mathematics PSTs use the CRMT tool to design a social justice focused lesson? During the study, the CRMT tool was modified and used to evaluate the lesson in terms of five subcategories that were drawn from the CRMT tool and the social justice literature. Most of the PSTs picked appropriate topics with the aim of highlighting inequities and were able to sustain these issues throughout the lesson. Though the topic was sustained by most of the PSTs, fewer than half were able to integrate the mathematics and social justice issue throughout the lesson. The PSTs focused either on the mathematics or the issue separately, with a few lessons where there was a disconnect between the mathematics and the social justice issue. The PSTs grappled with mathematizing, with none scoring at level 3. The PSTs also overlooked how they would handle divisive opinions and statements as they arose in the class as part of the discussion. Finally, there were only two PSTs who mentioned student agency in their lesson and scored at levels 2 and 3 respectively. We note that mathematization, handling controversial issues, and student agency, were not a part of the CRMT tool when they designed the lesson, so it is understandable that there was less focus on those aspects; however, we believe that the readings and corresponding discussions related to *Reading and Writing the World with Mathematics*, as well as the adapted CRMT tools, have encouraged PSTs to think about these aspects in their lessons.

The results of this study are reflective of findings in the literature related to PSTs planning for social justice lessons. For instance, like Bartell [39], our PSTs initially struggled

to balance the mathematical and social justice goals of the lesson. However, like the PSTs and teachers in Aguirre and Zavala's [1] study, they found the CRMT tool to be helpful for planning culturally responsive and social justice-oriented lessons. Feedback from the PSTs indicated that they appreciate how the tool makes such considerations explicit, a point that also resonated with the participants in Aguirre and Zavala's study.

In addition, we found that the PSTs felt that the CRMT tool and lesson planning experience helped them develop a better understanding of equity and social justice, as well as how to incorporate such considerations into mathematics learning. A notable implication of the study for our PSTs was that they began to understand that equitable mathematics teaching must go beyond making accommodations for the students, a common perception of PSTs. Rather, they understood that equity, and social justice in particular, must connect to the lived experiences of their students and that mathematics provides a unique lens for understanding these experiences. As such, this intertwining of the mathematics and social justice goals was an important aspect of their learning.

Finally, like Miescu et al. [36] and Aguirre et al. [1], we found that it is possible to measure practices of teaching for social justice. Using the CRMT-M observation tool, we were able to both gauge and provide detailed feedback related to teaching mathematics for social justice. Beyond this, we also found that it is possible to support PSTs' development of social justice mathematics lessons by providing a tool with explicit guidelines for culturally responsive and social justice-focused mathematics learning.

6.2. Limitations

The notable limitations of this study are centered primarily around disruptions to the PSTs' teaching and learning due to the COVID-19 pandemic and their differential levels of access to the seminar materials. We recognize that preparing teachers demands their access to classrooms and developing culturally responsive teachers requires direct access to the students that they will eventually teach. Unfortunately, the pandemic yielded conditions that are not ideal for developing teachers in that they were not allowed into classrooms and were restricted to learning 100% online for two semesters. This constrained both their ability to get to know their students on a personal level and their ability to engage with the seminar materials in a face-to-face manner. Despite these limitations, however, we feel that our adaptation to the seminar curriculum was effective in developing PSTs' understanding of CRMT, albeit on a more theoretical level than initially desired.

Regarding their differential access to the seminar materials, scholars typically join the program in the junior year of their undergraduate teacher preparation program, resulting in 2–3 cohorts of PSTs at any given time (those student teaching, those in year-long internships, and those who have yet to be assigned to their clinicals). Depending on their area of concentration (middle grades mathematics plus an additional subject area versus mathematics major with a secondary education minor), they also enter with varied teaching experiences and subject matter understanding. As a result, some of the PSTs will have had more experience with discourses related to lesson planning and CRMT. On this note, we would like to add that most of the PSTs who scored in the lower ranges on the tool had entered the program a semester later than the others, and so their scores indicate that they may require more time to develop their understanding of CRMT. Furthermore, in terms of handling controversial issues and student agency, we understand that the PSTs have had little to no experience in their preparation in their program, or opportunities to see this in the classes of their clinical educators. Despite these constraints, we feel that our development of the corresponding CRMT tools with the explicit subcategories proved helpful for our PSTs in that it provided explicit considerations for them to attend to in their planning and fostered a reflective teaching practice. The development of the tool further served us as PST educators and researchers in that it provided a systematic and descriptive means of evaluating our PSTs' lesson plans, planning process, and implementation of developed instruction.

6.3. Adapted CRMT Tool

In their reflections on the CRMT-M tool (all categories, not just social justice), all the PSTs found it useful for planning lessons in a way that ensures that their work pays attention to the students, mathematics, and important issues. This was similar to what Aguirre and Zavala [1] found from their teachers after using the CRMT tool. Some PSTs appreciated the balance that they could maintain between discussing the content and the social justice issue. Given that the teachers in Bartell's [38] study identified balancing the mathematical and social justice goals as a significant challenge, we feel that this is a notable point. Specifically, we believe that by including the integration of the mathematics and the social justice subcategory, PSTs were better supported in addressing these goals throughout the lesson planning process. In addition, some PSTs expressed that the tool might be useful for them in their future collaborations with colleagues in that they can use it to better communicate ideas related to planning for CRMT and social justice lessons. Teachers in Aguirre and Zavala's [1] study also reported that the tool grounded collaborative conversations around culturally responsive teaching practices. One PST mentioned that the tool could be used to screen existing materials to see where they could be modified to address the categories in the tool. Finally, the PSTs gained a better understanding of the categories with the description that was provided in the tool. We found that the PSTs asked questions as they sought details about the subcategories. Hence the CRMT-M tool served as an educative instrument for the PSTs.

While the PSTs did not mention any drawbacks of the tool, from our experience and observations, the PSTs found it challenging to address all the categories in the lesson. We attribute this to their lack of experience with lesson planning, with the ideas in the CRMT tool, and with teaching indirect mathematics lessons in general. Notably, they did express that they would like to see more examples of CRMT lessons, including demonstrations of their implementation in classrooms. As such, we believe that further use of the tool in their lesson planning practice, coupled with growth in their content and pedagogical knowledge, will facilitate a more expansive integration of the categories into the lessons.

The PSTs' major concerns were related to the implementation of how the social justice topics and corresponding lessons would accommodate the vast quantity of mathematical standards that they would be expected to implement in their classes. Note that this was a similar concern shared by the PSTs in Aguirre et al.'s [19] study. Our parallel experiences imply that PSTs believe that their designed lessons must address one or two standards at a time, typically outlined in their pacing guides as daily objectives. However, in discussing issues of social justice it is likely that the lesson should address a broader range of learning goals, including multiple interconnected mathematical standards, social justice standards, and standards for practice. As such, we feel that PSTs require additional preparation in developing lessons that address multiple, interconnected goals, including those that they may have addressed previously.

6.4. Implications for Teacher Preparation and Teacher Professional Development

Overall, we see that the PSTs in our program are continuously developing in their growth as TMSJ practitioners and that this development has been facilitated through their use of the CRMT-M tool. As such, we believe that an explicit focus on the elements of CRMT-M could support PSTs in their development of a CRMT practice. In addition, we propose that the five subcategories in the social justice element of the CRMT-M tool serve as useful themes for guiding PSTs' professional development as critical mathematics educators. For instance, by designing instruction for PSTs and teachers aimed at understanding the key components of CRMT and TMSJ, they are given the opportunity to engage in discourse and reflection related to the importance of integrating the mathematical and the social justice goals of the lesson and what it means to mathematize, handle, and sustain productive conversations around sensitive issues, as well as promoting student agency. In addition, PSTs and teachers also need to experience social justice lessons as learners themselves to get a sense of all the aspects of the CRMT-M tool.

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Appendix A. Analysis 2 Results

Table A1. Analysis 2 results.

	CRMT Codes	Code Description	#Codes	# PSTs
1	Cognitive Demand	Encourages high level mathematical thinking and reasoning. Students are mathematically engaged.	5	3
	• Removing Autonomy	Describes students as being potentially unable to complete the task and/or does the task for them.	3	1
2	Depth of Knowledge and Student Understanding	Activity promotes deep understanding (probing/questioning to get at deeper understanding).	15	7
3	Math Discourse and Communication	Develops collective understanding (not just sharing but engaging in collective discourse that enables everyone to understand what's going on).	15	12
4	Power and Participation	States something explicit about power relations and/or giving voice to historically marginalized students. Explicitly valuing students' contributions over the teacher's. Explicit recognition of power in the classroom.	20	9
	• Student Contributions	Explicitly valuing students' contributions over the teacher's.		
	• Marginalized Student Contributions	Explicitly valuing marginalized students' contributions.		
5	Academic Language Support ELL	Academic language support for ELLs.	33	8
6a	Culture/Community Funds of Knowledge	Connects mathematical activity/uses mathematics to explore experiences shared within a group/community/culture.	1	1
6b	Social Justice	Connects mathematical activity/uses mathematics to explore issues of social justice.	0	0

Table A1. *Cont.*

Other Codes	Code Description	Codes	Students
Teacher Moves	Pedagogical moves that are made that are not explicitly connected to the mathematics. Note that there are moves that are made to enhance the student's thinking that are not considered teaching moves.	36	11
Caring	Affectively demonstrating consideration for students' personal and mathematical emotions. Includes accommodating different types of students and valuing multiple ways of knowing.	15	9
Deficit Thinking	Making an activity less rigorous because of perceived or anticipated student difficulties.	12	7
Student Experiences	Making an aspect of the lesson more relatable to students through their individual experiences (not tied to cultural practices in the community).	13	7
General Academic Language Support	Vocabulary support for understanding academic language in general.	24	9

Appendix B. CRMT-M Tools for PSTs

Table A2. PST CRMT-M lesson planning tool, social justice element.

6b. Use of critical knowledge/social justice support
How does the lesson support students' use of mathematics to understand, critique, and change an important equity or social justice issue in their lives?

- Context: Does the chosen context address a social injustice? Is it meaningful to the students in the class?
- Integration: Are the social justice (SJ) context and mathematics explored in an integrated way which is sustained throughout the lesson?
- Mathematization: How are the students encouraged to mathematize the context at hand?
- Controversial Topics: How does the teacher plan for the exploration and discussion of controversial topics in the lesson?
- Transformative Student Action: How are students encouraged to transform the issue?

1	2	3
[Superficial SJ context]	[Meaningful SJ context]	[Meaningful integration of Math/SJ context + student agency]
Context: A social justice context is not used or is not relevant to the students in the class.	Context: A meaningful SJ context (An existing social injustice) is used that is relevant to the students but may not be sustained throughout the lesson.	Context: A meaningful SJ context (An existing social injustice) is used that is relevant to the students and is sustained throughout the lesson.
Integration of SJ and math: The math is imposed and does not reveal anything new about the context. The PST could easily have switched the context without much change in the lesson.	Integration of SJ and math: The mathematics and SJ context are explored in such a way that: • [Disconnected]: The social justice context and mathematical content are introduced/explored separately. • [Procedural]: The mathematics is not explored in a way that develops their conceptual understanding– The social justice issue is used predominantly as a context for performing calculations. • [Lack of Mathematical Rigor]: The context is explored in a way that expands their understanding of the real-world issue BUT does not advance their mathematical conceptions in a meaningful or rigorous way.	Integration of SJ and math: The mathematics and SJ context are simultaneously explored throughout the lesson in such a way that: 1. The math reveals something new to the students about the SJ context, AND 2. The context enables students to conceptually understand the mathematics.

Table A2. *Cont.*

6b. Use of critical knowledge/social justice support How does the lesson support students' use of mathematics to understand, critique, and change an important equity or social justice issue in their lives? • Context: Does the chosen context address a social injustice? Is it meaningful to the students in the class? • Integration: Are the social justice (SJ) context and mathematics explored in an integrated way which is sustained throughout the lesson? • Mathematization: How are the students encouraged to mathematize the context at hand? • Controversial Topics: How does the teacher plan for the exploration and discussion of controversial topics in the lesson? • Transformative Student Action: How are students encouraged to transform the issue?		
1	**2**	**3**
	Mathematizing: The teacher sometimes/rarely probes the students to explore/interpret: • The source of the data and information used in the exploration. • How attributes of the context can be operationalized/measured. • The meaning behind their mathematical findings in relation to the context.	Mathematizing: The teacher continuously probes students to explore/interpret: • The source of the data and information used in the exploration. • How attributes of the context can be operationalized/measured. • The meaning behind their mathematical findings in relation to the context.
Controversial topics: The teacher avoids anything controversial, including the context itself.	Controversial topics: Teacher does not anticipate (or plans to avoid) how they will manage controversial topics and discussions associated with the context.	Controversial topics: Teacher plans for how they will manage controversial topics and discussions associated with the context. (Ex: men working harder than women → gender pay gap; more crime in black communities → more police presence).
Transformative student action: Do not believe that the students have any agency.	Transformative student action: students complete and discuss the lesson, but do not follow up the lesson with any sort of plan for action.	Transformative student action: Students are given opportunities to plan and/or make a meaningful attempt to transform the issue using mathematics as their justification (action plan, letter/email to policy makers, organization, etc.).

Table A3. PST CRMT-M observation tool, social justice element.

6b. Use of critical knowledge/social justice support (Focus on integration of math/SJ/agency) How does the lesson support students' use of mathematics to understand, critique, and change an important equity or social justice issue in their lives? • Context: Does the chosen context address a social injustice? is it meaningful to the students in the class? • Integration: Are the social justice (SJ) context and mathematics explored in an integrated way which is sustained throughout the lesson? • Mathematization: How are the students encouraged to mathematize the context at hand? • Controversial Topics: How does the teacher plan for the exploration and discussion of controversial topics in the lesson? • Transformative Student Action: How are students encouraged to transform the issue?		
1	**2**	**3**
[Superficial SJ context]	[Meaningful SJ context]	[Meaningful integration of Math/SJ context + student agency]
Context: A social justice context is not used.	Context: A meaningful SJ context (An existing social injustice) is used, but may not be relevant to the students in the class, but may not be sustained throughout the lesson.	Context: A meaningful SJ context (An existing social injustice) is used that is relevant to the students and is sustained throughout the lesson.

Table A3. *Cont.*

6b. Use of critical knowledge/social justice support (Focus on integration of math/SJ/agency) How does the lesson support students' use of mathematics to understand, critique, and change an important equity or social justice issue in their lives? • Context: Does the chosen context address a social injustice? is it meaningful to the students in the class? • Integration: Are the social justice (SJ) context and mathematics explored in an integrated way which is sustained throughout the lesson? • Mathematization: How are the students encouraged to mathematize the context at hand? • Controversial Topics: How does the teacher plan for the exploration and discussion of controversial topics in the lesson? • Transformative Student Action: How are students encouraged to transform the issue?		
1	**2**	**3**
Integration of SJ and math: The math is imposed and does not reveal anything new about the context. The PST could easily have switched the context without much change in the lesson.	Integration of SJ and math: The mathematics and SJ context are explored in such a way that: • [Disconnected]: The social justice context and mathematical content are introduced/explored separately. • [Procedural]: The mathematics is not explored in a way that develops their conceptual understanding (e.g., SJ issue is used predominantly as a context for performing calculations). • [Lack of Mathematical Rigor]: The context is explored in a way that expands their understanding of the issue BUT does not advance their mathematical conceptions in a meaningful or rigorous way.	Integration of SJ and math: The mathematics and SJ context are simultaneously explored throughout the lesson in such a way that: 1. The math reveals something new to the students about the SJ context, AND 2. The context enables students to conceptually understand the mathematics.
Mathematizing: Students are not encouraged to mathematize the explored context (if there is one).	Mathematizing: The teacher sometimes/rarely probes the students to explore/interpret: • The source of the data and information used in the exploration • How attributes of the context can be operationalized/measured. • The meaning behind their mathematical findings in relation to the context.	Mathematizing: The teacher continuously probes students to explore/interpret: • The source of the data and information used in the exploration • How attributes of the context can be operationalized/measured. The meaning behind their mathematical findings in relation to the context.
Controversial topic: The teacher avoids anything controversial, including the context itself.	Controversial topics: The teacher discusses some controversial topics that are built into the lesson but avoids those brought up by the students.	Controversial topics: Controversial topics (which may be brought up by the students) are explored within the context of the lesson. The teacher does not shy away from such topics but handles them with intentionality and consideration for the experiences of their students. (Ex: men working harder than women → gender pay gap; more crime in black communities → more police presence).
Transformative student action: Students are not encouraged to have any agency in the lesson.	Transformative student action: Students complete and discuss the lesson, but do not follow up the lesson with any sort of plan for action.	Transformative student action: Students are given opportunities to plan and/or make a meaningful attempt to transform the issue using mathematics as their justification (action plan, letter/email to policy makers, organization, etc.).

Appendix C. Analysis 3

Scoring Procedures:

Color Codes:

- red = level 1
- yellow = level 2
- green = level 3

Subcategory weights based on importance:

- Meaningful social justice context = 2
- Integration of social justice and mathematics = 2
- Mathematizing = 1
- Handling controversial topics = 1
- Student agency = 1

Formula for weighted average score for the social justice element:

$$\text{Instructor Score} = \frac{\begin{array}{c}2 \times (\text{meaningful social justice context score}) \\ +2 \times (\text{integration of social justice and mathematics}) \\ +(\text{mathematizing score}) + (\text{student agency score})\end{array}}{7}$$

Mean scores were rounded up to the next level of attainment if they earned above a half a point and to the lower level of attainment if below half a point.

B	C	D	E	F	G
Student Pseudonym	**Context**	**Math Topic**	**Self Reported Student Score [1-5]**	**Student Response to SJ Element**	**Instructor Score [1-3]**
Juliet	Student Representation in Schools	Ratios and Proportions	4	This lesson offers students a chance to critique and discuss a social justice issue in their life by looking at schools in the same and different districts. It asks them to think about causes and effects of these differences and how it will affect their future. They must consider why they attend the school they are at and what it would require to attend a different school. The exploration and discussion give an opportunity to think specifically about the benefits or drawbacks of a school. It does not yet have a way for students to address the issue to create change. The goal is not for students to feel negatively about their own school, but to critically consider the importance their education will have on their life and how it is different from the education other students may be receiving.	
					2.6

Figure A1. Social justice element analysis for Juliet.

Instructor Score [1-3]	Sustained Meaningful SJ Context 2 [R:+1, Y:+2, G:+3]	Integration of SJ & Math 2 [R:+1, Y:+2, G:+3]	Mathematizing [R:+1, Y:+2, G:+3]	Handling Controversial Topics [R:+1, Y:+2, G:+3]	Student Action (Agency) [R:+1, Y:+2, G:+3]
	Meaningful social justice context that is sustained throughout the lesson. SJ Goals: Explain why schools have inconsistent resources from a sociopolitical perspective. 2.Explore solutions/action that can be taken with mathematical and sociopolitical perspective.	The SJ context of large vs. small schools and their associated resources is sustained throughout the lesson from the launch through the class discussion. There is potential for the math to enlighted students about the context and for the context to enlighten students about the mathematics with some structure added to the exploration and discussion.	Student are prompted to question both the mathematics and their analysis of the issue. They do not necessarily discuss how attributes are operationalized/measured from the context. Main Questions to Prompt Discussion: Mathematical How do we set up ratios? What challenges did we have making proportional relationships? What kind of information can we get from ratios and proportional relationships? How do we get that information? Where was the data the most consistent across groups? When was the data inconsistent? Could our data be misleading or be misused? Sociopolitical Why do schools in the same district look so different? What determines where a student goes to school? Why might a student change schools? What differences can be made to improve equality between schools?	PST does not discuss or plan for how she will address controversial topics that may come up in the lesson.	Students are given the following opportunity to transform the issue: At the end of the discussion, students will return to their groups to brainstorm ways to improve equity between schools. They will summarize their reports by explaining their conclusions and their ideas about how to distribute resources more equitably
2.6	6	6	2	1	3

Figure A2. Social justice element analysis for Juliet.

Student Pseudonym	Context	Math Topic	Self Reported Student Score Conversion [1-3]	Instructor Score [1-3]	Level of Attainment Description	Sustained Meaningful SJ Context 2 [R:+1, Y:+2, G:+3]	Integration of SJ & Math 2 [R:+1, Y:+2, G:+3]	Mathematizing [R:+1, Y:+2, G:+3]	Handling Controversial Topics [R:+1, Y:+2, G:+3]	Student Action (Agency) [R:+1, Y:+2, G:+3]
Martha	Gender wage gap	Proportional reasoning	3	2.43	Meaningful SJ context	6	6	2	2	1
Allison	Minimum wage	Linear functions	2	2.00	Meaningful SJ context	6	4	2	1	1
Elizabeth	Living wage/CEO salaries	Measures of center	2	1.43	Superficial SJ Context	4	2	2	1	1
Heidi	Gender wage gap	Measures of center (though she claims variability too)	3	1.29	Superficial SJ Context	4	2	1	1	1
Joan	Access to services	Linear functions	2	1.00	Superficial SJ Context	2	2	1	1	1
Jessie	Incarceration rates	Probability	3	2.43	Meaningful SJ context	6	6	1	3	1
Juliet	Student Representation in Schools	Ratios and Proportions	2	2.57	Meaningful integration of Math/SJ context + student agency	6	6	2	1	3
Karen	Bias in media sources/representation in schools based on race	Measures of center and spread	2	1.86	Meaningful SJ context	6	4	1	1	1
Otis	Minimum wage	Linear functions	2	2.43	Meaningful SJ context	6	6	2	1	2
Rachel	Student Loans	Exponential functions	2	1.00	Superficial SJ Context	2	2	1	1	1
Mean Score Weighted:			2.30	1.84	Meaningful SJ context	4.80	4.00	1.50	1.30	1.30
Scaled Mean Score:			2.30	1.84	Meaningful SJ context	2.40	2.00	1.50	1.30	1.30

Figure A3. Final lesson plan analysis results for social justice element.

References

1. Aguirre, J.; Zavala, M. Making culturally responsive mathematics teaching explicit: A lesson analysis tool. *Pedagog. Int. J.* **2013**, *8*, 163–190. [CrossRef]
2. Carter, N.P.; Hawkins, T.N.; Natesan, P. The relationship between verve and the academic achievement of African American students in reading and mathematics in an urban middle school. *Educ. Found.* **2009**, *22*, 29–46.
3. Heath, S.B. *Ways with Words: Language, Life, and Work in Communities and Classrooms*; Cambridge University Press: Cambridge, UK, 1983.
4. Malloy, C.; Malloy, W. Issues of culture in mathematics teaching and learning. *Urban Rev.* **1998**, *30*, 245–257. [CrossRef]
5. Nieto, S. Black, White, and us: The meaning of Brown v. Board of Education for Latinos. *Multicult. Perspect.* **2004**, *6*, 22–25. [CrossRef]
6. Tate, W.F. Race, retrenchment, and the reform of school mathematics. *Phi Delta Kappan* **1994**, *75*, 477–484.
7. Ladson-Billings, G. Toward a theory of culturally relevant pedagogy. *Am. Educ. Res. J.* **1995**, *32*, 465–491. [CrossRef]

8. Gutiérrez, R. Why (urban) mathematics teachers need political knowledge. *J. Urban. Math. Educ.* **2013**, *6*, 7–19. Available online: http://www.creatingbalanceconference.org/resources/wp-content/uploads/formidable/Gutierrez_2013_Politics-of-Urban-Math-Teaching.pdf (accessed on 1 February 2022).
9. Gay, G. *Culturally Responsive Teaching: Theory, Research, and Practice*; Teachers College Press: New York, NY, USA, 2000.
10. Paris, D.; Alim, H. *Culturally Sustaining Pedagogies: Teaching and Learning for Justice in a Changing World*; Teachers College Press: New York, NY, USA, 2017.
11. Gutstein, E. *Reading and Writing the World with Mathematics: Toward a Pedagogy for Social Justice*; Routledge: New York, NY, USA, 2006.
12. Gutstein, E. "Our issues, our people—Math as our weapon": Critical mathematics in a Chicago neighborhood high school. *J. Res. Math. Educ.* **2016**, *47*, 454–504. [CrossRef]
13. Freire, P. Pedagogy of the Oppressed. Continuum: New York, 1970/2018. Available online: https://envs.ucsc.edu/internships/internship-readings/freire-pedagogy-of-the-oppressed.pdf (accessed on 1 February 2022).
14. Frankenstein, M. Critical mathematics education: An application of Paulo Freire's epistemology. *J. Educ.* **1983**, *165*, 315–339. [CrossRef]
15. Kokka, K. Social justice pedagogy for whom? Developing privileged students' critical mathematics consciousness. *Urban Rev.* **2020**, *52*, 778–803. [CrossRef]
16. Rubel, L. Equity-directed instruction practices: Beyond the dominant perspective. *J. Urban Math. Educ.* **2017**, *10*, 66–105.
17. Skovsmose, O. Towards a critical mathematics education. *Educ. Stud. Math.* **1994**, *27*, 35–57. [CrossRef]
18. Register, J.; Stephan, M.; Pugalee, D. Ethical reasoning in mathematics: New directions for didactics in mathematics education. *Mathematics* **2021**, *9*, 799. [CrossRef]
19. Aguirre, J.; Zavala, M.; Katanyoutanant, T. Developing robust forms of pre-service teachers' pedagogical content knowledge through culturally responsive mathematics teaching analysis. *Math. Teach. Educ. Dev.* **2012**, *14*, 113–136. Available online: https://files.eric.ed.gov/fulltext/EJ1018658.pdf (accessed on 1 February 2022).
20. Akiba, M. Identifying program characteristitcs for preparing pre-service teachers for diversity. *Teach. Coll. Rec.* **2011**, *13*, 658–697. Available online: https://www.researchgate.net/publication/286805295_Identifying_Program_Characteristics_for_Preparing_Pre-Service_Teachers_for_Diversity (accessed on 1 February 2022). [CrossRef]
21. Downey, J.A.; Cobbs, G.A. "I actually learned a lot from this": A field assignment to prepare future preservice math teachers for culturally diverse classrooms. *Sch. Sci. Math.* **2007**, *107*, 391–403. [CrossRef]
22. McCollough, C.; Ramirez, O. Cultivating culture: Preparing future teachers for diversity through family science learning events. *Sch. Sci. Math.* **2012**, *112*, 443–451. [CrossRef]
23. Williams, D.L.; Edwards, B.; Kuhel, K.A.; Lim, W. Culturally responsive dispositions in prospective mathematics teachers. *Dis. Commun. Sustain. Educ.* **2016**, *7*, 17–33. [CrossRef]
24. Cochran-Smith, M. *Walking the Road: Race, Diversity, and Social Justice in Teacher Education*; Teachers College Press: New York, NY, USA, 2004.
25. Brown, M. The Teacher-Tool Relationship: Theorizing the Design and Use of Curriculum Materials. In *Mathematics Teachers at Work: Connecting Curriculum Materials and Classroom Instruction*; Remillard, J., Lloyd, G., Herbel-Eisenmann, B., Eds.; Routledge: New York, NY, USA, 2009; pp. 17–36.
26. Ruys, I.; Van Keer, H.; Aelterman, A. Examining pre-service teacher competence in lesson planning pertaining to collaborative learning. *J. Curric. Stud.* **2012**, *44*, 349–379. [CrossRef]
27. Stephan, M.; Pugalee, D.; Cline, J.; Cline, C. *Lesson Imaging in Math Science: Anticipating Student Ideas and Questions for Deeper STEM Learning*; ASCD: Alexandria, VA, USA, 2017.
28. Smith, M.; Stein, M. *5 Practices for Orchestrating Productive Mathematics Discussions*; The National Council of Teachers of Mathematics: Reston, VA, USA, 2018.
29. Rusznyak, L.; Walton, E. Lesson planning guidelines for student teachers: A scaffold for the development of pedagogical content knowledge. *Educ. Chang.* **2011**, *15*, 271–285. [CrossRef]
30. Ding, M.; Carlson, M. Elementary teachers' learning to construct high-quality mathematics lesson plans: A use of the IES recommendations. *Elem. Sch. J.* **2013**, *113*, 359–385. [CrossRef]
31. Chval, K.B.; Khisty, L.L. Bilingual Latino students, writing and mathematics: A case study of successful teaching and learning. In *Multilingualism in Mathematics Classrooms*; Multilingual Matters: Berlin, Germany, 2009; pp. 128–144.
32. Moschkovich, J.; Zahner, W. Using the academic literacy in mathematics framework to uncover multiple aspects of activity during peer mathematical discussions. *ZDM* **2008**, *50*, 999–1011. [CrossRef]
33. Martin, D.B.; Gholson, M.L.; Leonard, J. Mathematics as gatekeeper: Power and privilege in the production of knowledge. *J. Urban Math. Educ.* **2010**, *3*, 12–24.
34. Turner, E.; Gutiérrez, M.V.; Simic-Muller, K.; Díez-Palomar, J. "Everything is math in the whole world": Integrating critical and community knowledge in authentic mathematical Investigations with elementary Latina/o students. *Math. Think. Learn.* **2009**, *11*, 136–157. [CrossRef]
35. Rubel, L.H.; Lim, V.Y.; Hall-Wieckert, M.; Sullivan, M. Teaching mathematics for spatial justice: An investigation of the lottery. *Cogn. Instr.* **2016**, *34*, 1–26. [CrossRef]
36. Miescu, E.; Pedulla, J.; Cannady, M.; Cochrane-Smith, M.; Jong, C. Measuring Practices of Teaching for Social Justice in Elementary Mathematics Classrooms. *Educ. Res. Q.* **2011**, *34*, 15–39.

37. Gutierrez, R.J. [Rodrigo]. Building "Consciousness and Legacies": Integrating Community, Critical, and Classical Knowledge Bases in a Precalculus Class. (Order No. 3592727, The University of Arizona). ProQuest Diss. Theses 2013. p. 282. Available online: https://www.proquest.com/openview/0822799c487af4aa955be690bf61dba2/1?pq-origsite=gscholar&cbl=18750&diss=y (accessed on 1 February 2022).
38. Simic-Muller, K.; Fernandes, A. Preservice Teachers' Understanding of 'Real world': Developing a Typology. *Int. J. Math. Teach. Learn.* **2020**, *21*, 31–53.
39. Bartell, T.G. Learning to Teach Mathematics for Social Justice: Negotiating Social Justice and Mathematical Goals. *J. Res. Math. Educ.* **2013**, *44*, 129–163. [CrossRef]
40. Turner, E.E.; Drake, C.; Roth McDuffie, A.; Aguirre, J.M.; Bartell, T.G.; Foote, M.Q. Promoting equity in mathematics teacher preparation: A framework for advancing teacher learning of children's multiple mathematics knowledge bases. *J. Math. Teach. Educ.* **2012**, *15*, 67–82. [CrossRef]
41. Gutiérrez, R. Beyond essentialism: The complexity of language in teaching mathematics to Latina/o students. *Am. Educ. Res. J.* **2002**, *39*, 1047–1088. [CrossRef]
42. Gutiérrez, R. Helping students to play the game and change the game. *Teach. Equity Excell. Math.* **2009**, *1*, 4–8.
43. Martin, D.B. Researching race in mathematics education. *Teach. Coll. Rec.* **2009**, *111*, 295–338. [CrossRef]
44. Moll, L.; González, N. Engaging life: A funds-of-knowledge approach to multicultural education. In *Handbook of Research on Multicultural Education*, 2nd ed.; Banks, J.A., Banks, C.A., Eds.; Jossey-Bass: Hoboken, NJ, USA, 2004; pp. 699–715.
45. Aguirre, J. Privileging mathematics and equity in teacher education: Framework, counterresistance strategies and reflections from a Latina mathematics educator. In *Culturally Responsive Mathematics Education*; Greer, B., Mukhopadhyay, S., Nelson-Barber, S., Powell, A., Eds.; Routledge: New York, NY, USA, 2009; pp. 295–319.
46. Christiansen, I. Some tensions in mathematics education for democracy. In *International Perspectives on Social Justice in Mathematics Education*; Sriraman, B., Ed.; Information Age Publishing: Charlotte, NC, USA, 2008; pp. 69–86.
47. Mukhopadhyay, S.; Greer, B. How many deaths? Education for statistical empathy. In *International Perspectives on Social Justice in Mathematics Education*; Sriraman, B., Ed.; Information Age Publishing: Charlotte, NC, USA, 2008.
48. Turner, E.; Strawhun, B.T. Posing problems that matter: Investigating school overcrowding. *Teach. Child. Math.* **2007**, *13*, 457–463. [CrossRef]
49. Gutstein, E.; Lipman, P.; Hernandez, P.; de los Reyes, R. Culturally relevant mathematics teaching in a Mexican American context. *J. Res. Math. Educ.* **1997**, *28*, 709–737. [CrossRef]
50. Kendi, I. How to be an antiracist. In *One World*; The University of Mississippi: Oxford, MS, USA, 2019.
51. Van den Heuvel-Panhuizen, M.; Drijvers, P. Realistic mathematics education. *Encycl. Math. Educ.* **2020**, *53*, 713–717.
52. Simic-Muller, K.; Fernandes, A.; Felton-Koestler, M.D. "I Just Wouldn't Want to Get as Deep Into It": Preservice Teachers' Beliefs about the Role of Controversial Topics in Mathematics Education. *J. Urban Math. Educ.* **2015**, *8*, 53–86. [CrossRef]