THE CASE FOR BUILDING ON STUDENTS' PROPORTIONAL REASONING FOR SLOPE-RELATED TASKS

by

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ABSTRACT

CURTIS D. KENDRICK. The case for building on students' proportional reasoning for slope-related tasks. (Under the direction of DR. DAVID K. PUGALEE)

The purpose of this research was to identify the proportional reasoning strategies that seventh-grade students use to solve slope-related problems relating to the origin and nonzero y-intercept. In this qualitative study, students worked in pairs to complete problems and then participated in interviews in which they explained how they had arrived at their answers and why they had used specific approaches. The aim was to contribute to the limited research that has yet been done on connecting proportionality and slope by assessing students' thought processes based on their prior knowledge and collaboration when they encountered graphic and tabular questions relating to slope. The approach here was informed by the work of Skemp (1976), who encouraged building on students' prior knowledge in order to improve their ability to adapt to uncertain situations, reduce their need to memorize rules and heuristics, and enhance their intrinsic motivation to learn mathematics.

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LIST OF ACRONYMS/ABBREVIATIONS

CCSSM	Common Core State Standards for Mathematics
HLT	Hypothetical Learning Trajectory
IRB	Institutional Review Board
IRF	Initiation Response Feedback
NAEP	National Assessment of Educational Progress
NCDPI	North Carolina Department of Public Instruction
NCTM	National Council of Teachers of Mathematics
OECD	Organization for Economic Cooperation and Development
PISA	Program in International Student Assessment
RME	Realistic Mathematics Education
TIMMS	Third International Mathematics and Science Study

CHAPTER ONE: INTRODUCTION

Dr. Martin Luther King Jr. (1947) declared that "The function of education is to teach one to think intensively and to think critically. Intelligence plus character—that is the goal of true education." The first part of this quote seems, however, to be completely at odds with the current state of mathematics education in America. Thus, the Third International Mathematics and Science Study (TIMSS; Silver, 1998) harshly criticized U.S. teachers for tending to "state" ideas rather than to "develop" them and to engage students in tasks involving low-level cognitive activity, such as memorization, rather than high-level thinking, such as reasoning and problem-solving. The TIMMS study further indicted the U.S. mathematics curriculum as being repetitive and unfocused— "a mile wide and an inch deep"—in that more topics are included in the curricula at each grade level in U.S. schools compared with schools in most other countries (Silver, 1998, p. 14).

The need to unearth and invoke a document that is 20 years old is reflected in the fact that research continues to describe the teaching of mathematics in the United States as broad and shallow in terms of both curricula (textbooks and standards) and actual instruction (Polikoff, 2012; Roller, 2016). Thus the 2012 Program in International Student Assessment (PISA) ranked the United States twenty-seventh of the 34 industrialized countries in the Organization for Economic Cooperation and Development (OECD) in mathematics (PISA, 2012, p. 3) and identified weaknesses in the ability of U.S. students to perform cognitively demanding tasks and those involving mathematical literacy. According to the report, students were unable to create effective mathematical models of given situations in the form of terms or equations with variables for geometric or physical quantities.

The distressing fact, then, is that the findings from the two reports, separated by 14 years, are so similar. Mathematics education in the U.S. continues to be viewed as failing to stress higher-level thinking and to make connections to real-world situations. It was accordingly unsurprising when the overall ranking of U.S. math students on the 2015 PISA 2015 dropped to 30 of 35 OECD nations.

Statement of the Problem

The consistent emphasis on procedural mathematics in today's classrooms has deprived students of the opportunity to conceptualize mathematical ideas in terms of their own experiences and contexts. Skemp (1976) advocated building on students' prior knowledge as an effective means of improving their ability to adapt to uncertain situations, reducing their need to memorize rules and heuristics, enhancing their intrinsic motivation to learn mathematics, and stimulating their desire to become independent lifelong learners. Also, cognitive science and brain research point to prior knowledge as the basis for new knowledge; thus, new experiences that build on earlier ones are much better retained (McGowen & Tall, 2010; De Lima & Tall, 2008; Tall, 2004). Prior learning is leveraged when old ideas serve to make sense of new ones (McGowen, 2016).

The PISA, though, as discussed, concluded that U.S. students are deficient when it comes to generating a mathematical model of a given real-world situation. One possible explanation for this deficiency may relate to the specific failure of mathematics education to engage students' prior proportional reasoning. The ability to reason proportionally is of particular importance in that it marks the transition from numerical reasoning to algebraic reasoning; in the words of one team of scholars, it "inherently involves some of the most important algebraic understandings having to do with equivalence, variables and transformations" (Lesh, Post, & Behr, 1988, p. 97).

Research has identified the instructional focus of mathematics as the use of a rule to obtain a correct solution. According to a number of studies, this focus on the mastery of skills, rather than on the use of prior knowledge to build conceptual understanding, has left students ill-equipped to use mathematics in their future careers (Carlson, 1998; McGowen & Tall, 2013; Stigler, Givvin & Thompson, 2010; Stump, 1999). This study was designed to address this problem.

Purpose of the Study

Cai and Sun (2002) argued that proportional relationships play an influential role in students' development of algebraic thinking and function sense. Furthermore, research indicates that those with a solid foundation in proportional reasoning are prepared to understand the concept of slope and in particular are less susceptible to errors when calculating slopes on graphs with nonstandard measurements on the axes (Lobato & Thanheiser, 2002; Lobato, Ellis, & Munoz, 2003). This kind of reasoning further enables students to understand the graph of a line as a collection of points representing an infinite number of equivalent ratios (NCTM, 2013).

The purpose of this study is accordingly to identify proportional reasoning strategies that seventh-grade students use to solve slope-related problems and to investigate the manner in which they connect proportionality with slope. I have followed a contextual approach that takes into account students' experiences, the history of linear relationships, and research-based activities that have been successful in connecting proportional reasoning to notions of linear rates of change. Freudenthal (1977) theorized that, in order for school mathematics to be of value to students, they should learn by developing and applying mathematical concepts and tools in real-life situations that make sense to them (Van Den Heuvel-Panhuizen, 2003).

Unfortunately, the concept of slope tends to be taught as an algorithm that finds the difference between two *y*-coordinate points in relation to the difference between two x-coordinate points with which it is associated as an ordered pair. As a result, when students participate in activities involving slope, it is only seen as a mathematical concept associated with coordinate planes. This approach to teaching slope lacks a key component, namely the development and application of mathematical concepts in the contextual format just referred to (Van Den Heuvel-Panhuizen, 2003).

My principal at the time of this study expressed frustration regarding the struggles of eighth-grade students at a previous school on the North Carolina End-of-Year examination precisely because of the emphasis on problems relating contextually to slope. This frustration had arisen because students were being taught traditional algorithms through non-contextual coordinate-plane problems with ordered pairs that were not meaningful, which naturally meant that they had difficulty applying an algorithm to contextual problems. From this perspective, the absence of context impedes students' success in fully comprehending slope, which involves understanding how one-unit changes in relation to another.

This type of learning is consistent with Freudenthal's notion of proportional reasoning as the comparison of magnitudes of different quantities that share a meaningful connection or of two quantities that are conceptually related but not naturally considered parts of a common whole. Through this type of understanding, students can appreciate such relationships as those between crime and time, food yield and precipitation, pressure and temperature, velocity and time, and less commonly discussed relationships. I therefore designed this qualitative study as a contribution to the literature on the use of students' prior knowledge to make connections and build new mathematical concepts.

A further consideration in the design of this study was that the introduction of formulas and symbols neglects the notion that formalization using symbolic representations should be the last stage of mathematical instruction (e.g., Freudenthal, 1971). Thus Gravemeijer (1997) argued for a bottom-up approach through which models are constructed using manipulatives and graphical representations as intermediate steps that support students' progress from informal knowledge to the abstractions involved in the final stage.

Research Questions

Two research questions accordingly guided this study:

1. What proportional reasoning strategies do participants use when solving slope-related questions that pertain to nonzero *y*-intercepts?
2. How do participants connect proportionality with slope?

Significance of the Study

Relatively little research has been conducted on students' use of proportional reasoning strategies to solve slope-related problems. The aim of this study was to offer insight for math educators regarding the role of proportional reasoning skills in a series of interviews with students to develop an understanding of slope. Furthermore, the research that has been done has been quantitative in nature. One study, that of Cheng (2010) relied on a mixed-methods approach in which students completed ten-question proportion and steepness tests and scored them for accuracy and were then interviewed to assess the connection between proportionality and slope, but the analysis was decidedly more quantitative than qualitative, and the aim was to detect proportional reasoning in solving steepness problems asking such questions as which roof, stairs, or line was steeper, with the steepness of the line pertaining to direct variation.

The reality is that slope relationships do not always commence from the origin and that units play an important part when describing data. The steepness test did not depict units on the axis; therefore, this study's goal was to use contextual slope problems in which students would determine the relationship between the different units and analyze the proportional reasoning skill that they used. Part of the significance of the current study is that it did not focus solely on direct variation, as previous studies have done, but rather examined the techniques used when a nonzero y-intercept comes into play.

A qualitative approach was considered the best means to offer detailed insight into how students' existing proportional reasoning strategies can be used to solve slope-related problems. The students' work and rationales were thus highlighted using a qualitative approach that focused on how they viewed graphic or tabular representations in which the dependent variable of slope changed in relation to change in the independent variable. Conceptualizing this idea required proportional reasoning. The concept of slope, as already noted, tends to be taught as an algorithm that finds the difference between two y-coordinate points in relation to the difference between two x-coordinate points associated with the ordered pairs and, as a result, when students participate in these types of activities, slope is only seen as a mathematical concept associated with coordinate planes. This study accordingly explored how to extend the view of slope.

Furthermore, this study explored how the teaching of proportional reasoning could emphasize the mathematical relationship between two items in problems that did not depend on the physical similarity between the items. In this respect, the study has the potential to illustrate the benefits of teaching students to focus on quantitative relationships between the units of each object in the problem; for research has shown that the unit approach can lead most students to an intuitive understanding of proportional reasoning (Lawton, 1993).

This study also stands to make a substantial contribution to science education. In a study conducted by Woolnough (2000), some students were found to consider it improper to apply mathematical concepts to physics. This notion was solidified as students were analyzing line graphs; thus, some were apprehensive about calculating the slope owing to their perception of it as a mathematical objective. Planinic, Milin-Sipus, Kactic, Susac, and Ivanjek (2012) concluded that student knowledge is highly compartmentalized and that stronger links are need between science and mathematics through greater emphasis on the interpretation of graphs. This disconnect may arise because students fail to build on instruction that lacks contextual relevance.

Additionally, Stump (2001) found that students demonstrated a better understanding of slope as a measure of the rate of change—which graphically illustrates change in one unit relative to a corresponding change in another—rather than of steepness, thus providing context for the phenomenon relating the two units. By using contextual problems with graphs and tables, students were able to use prior knowledge and make learning mathematics more personal and applicable.

It is, then, imperative that students focus on the applicable quantitative relationship involving units of slope, for through this approach they analyze change in one variable in comparison with another. This study offers insights related to the ways in which students who have not been formally introduced to slope, draw on their proportional reasoning to make sense of non-zero y-intercept situations in a non-algorithmic format owing to its focus on proportional relationships in a contextual manner. Reiken (2009) pointed out that, since the slope formula does not provide any visual information regarding how *y*-values change relative to *x*-values, and since using the phrase "rise over run" does not provide any further information for understanding the rate of change, it is not clear whether students see slope as a measure of a rate of change or are simply recalling material learned from their beginning algebra class.

Summary

This first chapter has provided the rationale for this research study. It is clear that there has been a decline in U.S. mathematics education in comparison with other countries. In addition, students have difficulty understanding mathematical concepts relating to proportional reasoning. This understanding is, however, crucial, as researchers have identified proportional reasoning as the cornerstone for understanding higher-level mathematics. These perspectives underscore the importance of slope in relation to linear functions. However, since slope is normally taught as an algorithm rather than a relationship between two units, the opportunity to connect it with proportionality is often lost.

Streefland (1991) argued that students show greater initiative when they are encouraged to construct and produce their own solutions, but traditional teaching focuses on content. It is therefore important that math educators move away from traditional lesson planning in favor of what De Lange (1987) called “conceptual mathematization,” which helps students to choose the mathematical concept appropriate for a concrete situation. In this way, students explore situations schematically and visually to discover patterns that lead to the development of mathematical models.

Thus, through the process of reflecting and generalizing, students solidify their understanding of the concept. This process is referred to as applied mathematization because students apply mathematical concepts to other aspects of their lives, which reinforces and strengthens them. The aim of this study was to observe how students applied their previous proportional reasoning to slope-related problems. The rationale for connecting proportional reasoning to slope has been explained in this introductory chapter; the idea of using contextual problems and students’ experiences as a springboard for this connection is based on Freudenthal’s notion of mathematics as a human activity.

Freudenthal’s ideas are discussed in greater detail in Chapter 2 as the theoretical framework that informs this study. Also discussed is literature pertaining to how proportions and slope are connected, the benefit of this connection for understanding slope, and the theoretical basis for building on students’ prior knowledge. Chapter 3 discusses the research methodology with regard to the type of study, participants, research site, limitations, and the collection and analysis of the data. Chapter 4 presents the findings and the themes that emerged from the collection of the data in relation to the research questions. Finally, Chapter 5 discusses the analysis from Chapter 4, the conclusions of the study, and the implications of the findings and offers recommendations for future research.

CHAPTER TWO: REVIEW OF THE LITERATURE

The purpose of this study was to assess the viability of utilizing students' prior knowledge of proportional reasoning in order to help them understand slope. This chapter describes the relationship between slope and proportional reasoning. In addition, the review of the literature explores use of the instructional design known as realistic mathematics education (RME) for teaching slope through proportional reasoning. Through a discussion of relevant mathematics education research, this review establishes the theoretical framework for the study, laying the foundation for the connection of proportionality to slope. The early history of proportions is then approached from this perspective. Next, research is explored that addresses aspects of the current math curriculum that support the merging of proportionality and slope. Finally, the interdependence of slope and proportionality is examined in the context of other studies of specific tasks relating to these concepts.

Theoretical Framework

Freudenthal (1977) had some dire predictions for the state of mathematics education in the twenty-first century:

> What will mathematics education look like in 2000? The answer is simple. There will be no more mathematics education in 2000, it will have disappeared. There will be no more subject called mathematics, no math program, no math textbook to teach from. . . . It is there to be lived and enjoyed, just as reading, writing, handicrafts, art, music, breathing in integrated education. (p. 294)

This did not, of course, come to pass. Math classes continue to be taught in the traditional manner, even as well as the spread of state-mandated testing has caused students anxiety about math and caused them to feel disconnected from it, leaving them with the age-old question, “When am I ever going to use this?” Freudenthal’s conception of mathematics as a human activity thus appears to be no more than a dream.

It is essential to recognize that the theoretical rationale does not reflect the current status of mathematics education, in which students’ performance is quantified based on the percentage of questions that they answer accurately. Freudenthal discussed how children acquire number sense in the course of their physical and mental activities, which makes it difficult for researchers to assess how this happens in detail. The problem is that mathematics instruction is viewed as a basket of formulas and recipes that on closer examination may be full or empty.

Most people have been taught mathematics as a set of rules of processing, or algorithms (Freudenthal, 2013). The problem with this notion is that, while this approach works for students who learn to master these rules, it is disastrous for those who fail to master them. In part, mathematics is taught this way because that is the tradition. This is the way in which mathematics teachers learned the subject themselves, though they have forgotten it was not the way in which really came to understand mathematics—if ever they did.

Skemp used the terms “relational understanding” and “instrumental understanding” to describe types of pedagogies used in mathematics. Relational understanding is defined by knowing what to do as well as why, while instrumental understanding involves “rules without reasons.” Approached the latter way, students learn mathematics by simply accepting that a rule obtains. They do not question the law but apply it to problems that illustrate it. Freudenthal (2013) considered the inculcation of instrumental understanding to be detrimental to students who are not algorithmically gifted.

It is the mental activity involved rather than the subject matter that characterizes mathematics. This being the case, instruction should begin with common sense ideas. Rules learned in isolation are unlikely to develop a common sense of a higher order; put another way, a set of algorithms is useless unless one understands how and why they work. When students are not taught to use an algorithm in a true-life situation in which common sense counts, they continue to depend on less efficient lower-order operations (Freudenthal, 2013). The development of mathematical common sense originates in the acts of abridging and streamlining, as Skemp (1987) demonstrated in research comparing the performance of students who were given a definition as well associations that built upon prior knowledge with students whose instruction consisted of rote memorization. Such research has shown that schematic learning is twice as efficient as rote learning in promoting the retention of material knowledge.

This being the case, it is necessary to nurture the schema by means of which the individual student organizes past experiences and assimilates new data, for reconstruction is required before a new circumstance can be understood (Skemp, 1987). The focus of instruction needs to shift to observations of everyday situations and to consider that both teaching and learning are best served by vigorous interaction between the guides and those being guided. Mathematics education can no longer afford missed opportunities because of the failure to make clear "why they did what they did" and "why they think what they think."

Skemp (1987) accordingly discussed ways to enhance and develop the schema in terms of intuitive and reflective intelligence. The intuitive level refers to individuals' awareness through receptors (vision and hearing) of data from the external environment. From this perspective, the data automatically are automatically classified and related to other conceptual structures. Skemp described reflective intelligence as the process through which intervening mental activities become the object of introspective awareness. When an individual is able to reflect, to some degree, on his or her own schema and their use, important further steps can be taken. Thus, for example, one can, through reflection, identify and correct errors in existing schema and to make beneficial changes. Intuition, by contrast, can sometimes prove insufficient in the face of critical analysis and lead to inconsistency. Reflection gives the learner the opportunity to formulate ideas explicitly and to justify them by deriving them logically from other ideas. In essence, Skemp was saying that argument and discussion are useful ways of reflecting.

The didactics of a subject area refer to the organization of relevant teaching/learning processes. Learners should be allowed to find their own levels and to explore their own paths with as much or as little guidance as each particular case requires. The implications in this respect are, first, that knowledge is better retained and more readily available when acquired through a student's own efforts than when handed down by others. Second, discovery can be enjoyable and learning by reinvention can be motivational. Third, following Freudenthal (2013), mathematics should be experienced as a human activity.

Traditionally, mathematics is taught as a ready-made subject. Students are given definitions, rules, and algorithms according to which they are expected to proceed, but in fact only a small minority are able to learn mathematics this way. Freudenthal (2013) described learning mathematics as a process of reinvention. His concept of mathematics as a human activity is premised on the key notion that students should be given the opportunity to reinvent mathematics under the guidance of adults.

This notion was also supported by Skemp (1987), who stated that

> The teacher of mathematics has two important tasks: first, to make a conceptual analysis of the material; second, to plan carefully ways in which the necessary schemas can be developed, with particular attention to stages at which restructuring of the learner's schema will be needed. Then when in direct contact with learners, the teacher is responsible for general direction or guidance of the work, for explanation and for correction of errors. The teacher also needs, to a varying extent, to create and maintain interest. (p.163)

From this perspective, teachers need to build on students' knowledge base and guide them to correct solutions rather than simply stating the answers to problems. Furthermore, it is imperative that teachers provide activities that are engaging and that promote a love of mathematics.

Skemp further discussed the intellectual discourse between teacher and student. The discourse has the teacher as the expert who is nourishing and expanding the students' knowledge. It is essential that dialogue between teacher and student be based on sound principles and in-depth analysis of the concepts being covered rather than taking the form of "rules without reasons." The conversation should also go beyond teacher-to-student to include peer-to-peer interactions through which students can benefit from various perspectives in solving problems and thereby further enhance their schema.

There is no simple answer, since guiding reinvention means striking a delicate balance between the freedom of reinventing and the force of guiding, between allowing learners to please themselves and asking them to please their teachers. Freudenthal (2013) realized that the learner's free choice is already restricted by the "re" in "reinvention" but felt that guiding means maintaining just this balance between the force of teaching and the freedom of learning.

To be sure, algorithms allow individuals to act automatically for long stretches of time, avoiding the distracting interference associated with insightful thought; but algorithms are exacting, in that mastery is either complete or completely absent. It is therefore time to move beyond the teaching of mathematics in accordance with a "learn first, understand afterward" approach. Freudenthal described the development of mathematical concepts in terms of *mathematization*, a process through which students solve problems, look for problems, and organize subject matter in relation to mathematical activities. Mathematization is a process that continues as long as reality continues to change, broaden, and deepen under a variety of influences—including that of mathematics, which in turn is absorbed by that changing reality.

Mathematization has two forms, horizontal and vertical. In horizontal mathematization, the learning starts with contextual problems that ask students to describe situations and find solutions using their own language or symbols. Vertical mathematization, by contrast, involves more or less sophisticated mathematical processing. Horizontal mathematization, by leading from the world of life to the world of symbols, offers learners flexibility when it comes to reaching solutions. Vertical mathematization also begins with contextual problems, but in the long run learners are able to construct certain procedures that can be applied directly to similar problems. In either form, mathematization bypasses the mathematical formulas that are commonly applied like recipes to a complex reality in ways that lack any intermediate model to justify their use.

Another essential cognitive factor that eliminates commonly used math recipes is the schema introduced above. Skemp (1987) discussed a schema as “something which we can do to an idea or transformation” (p. 23) that is pertinent to students, asserting that “Our existing schemas are also indispensable tools for the acquisition of further knowledge. Almost everything we learn depends on knowing something else already.” (p.25)

Skemp concluded that it is necessary first to establish a well-structured foundation of basic mathematical ideas on which learners can build, to encourage learners always to be looking for these ideas in new situations, and to teach to them to reconstruct their schemas and to appreciate better approaches to solving problems. This conception coincides with Freudenthal’s notion of retrospective learning, which refers to recalling old material whenever it is fitting to do so. All of these considerations suggest that students should see similarities between proportionality and slope.

Algorithms created the fundamental antinomy within the didactics of mathematics of insight versus drill (Freudenthal, 2013), for which mathematization is the remedy. Rather than focusing on the rules, this approach offers a perspective on one’s intuitions and reflection on what appears to be obvious. This approach, however, requires more patience than most teachers are capable of. Retention is fostered when students are able to create and use prior strategies to solve problems.

Retrospective learning serves a dual purpose: it roots the new matter in the old one, and it strengthens the old roots (Freudenthal, 2013). Learning a new idea often involves nothing more than becoming more aware of a complex of previously little-noticed bits of knowledge and abilities and of their interrelatedness. This is the experience of retrospective learning. Since this study was designed to determine how students use prior schematic proportional strategies to solve slope-related tasks, it is imperative that the theory behind it incorporate the students' thinking rather than imposed rules.

To support the notion of connecting students' proportional reasoning to their understanding of slope, it is imperative to start with the instructional theory of RME for two reasons. First, as Van Amerom (2002) discussed, the underlying educational theory remains underdeveloped; thus, the realization that new developmental research studies can produce a new impetus for theoretical ideas and their implementation in the classroom has not yet come about. I find the notion of a theory that constantly evolves exciting because complacency can be detrimental to academic growth.

The theoretical framework of RME is influenced by Freudenthal's conception of mathematics as a human activity and the associated principle of learning mathematics as a reinvention process, and it also embraces his phenomenological analysis of the concept of number. Gravemeijer (1994) has also discussed common ideas that arise during the developmental stage, including Van Hiele's (1973) domain-specific instructional theories, Treffers' (1978) analysis of the mathematical thinking processes, Ausbel's

(1968) and Skemp's (1987) cognitive psychological approaches, and activity theory as described by van Parreren and Carpay (1972), Gal'perin (1972), and Davydov (1972). RME is therefore more concerned with how students attain knowledge than in how a textbook purveys content.

Constructivism lends itself to RME because it is a model of how learning takes place. Yager (1991) called it a "most promising model" of learning (p.53), because the traditional teaching of rote memorization involves no interpretation and is rarely meaningful. Therefore, most of what students memorize is gone (Cobern, 1992, p.108). However, the constructivist model of learning is built on three premises: (1) learning is always influenced by prior learning; (2) learning involves negotiation and interpretation; and (3) students need to be engaged in negotiation and interpretation of ideas (Cobern, 1992). This means that learning does not occur by transmission but interpretation. Thus, interpretation is always influenced by prior knowledge as well as facilitated by discourse. This ideology supports the notion that students will us prior knowledge to solve slope-related problems.

Cobb et al (1991) described that a constructivist institutes a "problem centered approach in which the teacher and students engage in discourse that has mathematical meaning (p.25). Therefore, constructivism might manifest itself through problem challenges, small-group work, and classroom discussions using curriculum materials. The implementation of meaningful curricula plays an intricate role. The appropriate use of curricula allows students to reinvent and construct their own mathematical knowledge structures (Steffe & Kieren, 1994), which is beneficial in retaining knowledge. Additionally, Steffe and Kieren (1994) discussed how these created mathematical knowledge structures serve as descriptors of constructs such as levels of units pertaining to ratios and constructive mechanisms such as unitizing, partitioning, proportionality operations, unit compositions and decomposition. As a result, these descriptions can serve practicing teachers in two ways. First, they can provide guides for listening and observing students and, second, they can provide potential sources both for content and organization of various mathematical curricula (Steffe & Kieren, 1994).

Freudenthal's (1973) concept of mathematics as a human activity is premised on the key idea that students should be given the opportunity to reinvent mathematics under the guidance of an adult. Gravemeijer (1994) understood this view of mathematics education as being highly interactive, in that teachers build on students' ideas, which "is only possible if the teacher reacts to what the student brings to the fore" (p. 13). Thus, RME is concerned with building on the knowledge that students possess and not with the structured outline or the sequence of lessons in a textbook. This emphasis is consistent with Skemp's (1987) definition of schema as a means to symbolize cognitive development. Skemp further justified his preference for a conceptual approach (schema) with an experiment in which one group of students was given a definition of a new concept as well as associations that built upon prior knowledge while another group was instructed through rote memorization; in this study, schematic learning proved to be twice as effective as rote learning in terms of retention of the material.

Second, RME is a vital theoretical framework because it both incorporates socio-constructivist viewpoints and stresses the importance of problem-solving from an emergent perspective. This theoretical point of view reveals the shortcomings of what Freire (1970) called "banking" in reference to the form of teaching that TIMSS associated with mathematics education in the United States, which is utterly inconsistent with the notion that knowledge emerges through invention and reinvention. Banking instead is predicated on the premise that students are containers or receptacles to be filled by the teacher; they memorize information and are only exposed to what the teacher deems important. The downside of this approach is that, the more students work at storing the material deposited with them, the less they develop the critical consciousness that could lead them to reinvent and transform the world.

The socio-constructivist approach is motivated by the desire to understand students' mathematical learning as it occurs in the classroom or other social situations. Additionally, RME constitutes a highly compatible, domain-specific instructional theory that relies on real-world applications and modeling. From the perspectives of both theories, mathematics is a creative human activity, and mathematical learning occurs as students develop effective ways to solve problems (Streefland 1991; Treffers 1987).

In a Vygotskian sense, these approaches do not call for skill and drill instruction but rather serve as means of developing skills that students have not yet acquired; they also lead to an individual evaluation or test of learning. The Vygotskian conception of proximal development emphasized the creation of social contexts in which children actively learn to use, try, and manipulate language in order to make sense or create meaning. The lessons consist of a series of interrelated but diverse learning activities usually organized around a specific theme or topic. Vygotsky (1990) asserted that "The role of the teacher is to foster the necessary guidance and mediations, so that children through their own efforts assume full control of diverse purposes and uses of oral and written language" (p. 9).

In order for students to have the opportunity to reinvent mathematics under the guidance of adults and to benefit from the insights of RME theory, they should first be exposed to a variety of real-world problems and situations (DeLange, 1996). Applied problem-solving is the primary approach to teaching mathematics in the real-world context. According to Hilton (1976), applied mathematics refers to a collection of activities directed toward the formulation of mathematical models, the analysis of the mathematical relations in these models, and the interpretation of the analytical results within the framework of their intended application.

The principles of instruction in RME described by Cobb (1994) and DeLange (1996) can be summarized as follows.

- The starting points of instructional sequences should be experientially real to students so that they can immediately engage in personally meaningful mathematical activities.
- In addition to taking into account students' current mathematical ways of knowing, the starting points should also be justifiable in terms of the potential end points of the learning sequence.
- Instructional sequences should involve activities in which students create elaborate symbolic models of their informal activities.
- The foregoing three tenets can only be effective if they are part of interactive instruction, which involves explaining and justifying solutions, understanding other students' solutions, agreeing and disagreeing, questioning alternatives, and reflecting.

- Consideration of real phenomena in which mathematical structures and concepts manifest themselves can lead to the intertwining of learning strands.

This study concentrated on the starting point because the concern was to identify a connection between proportional reasoning and slope. The instructional starting point was the attempt to understand the consequences of earlier instruction rather than simply to document the typical age-appropriate level of reasoning (Den Akker, Gravemeijer, McKenney, & Nieveen, 2006). Therefore, the Math Common Core Standards discussion will be later in the literature review to indicate how prior instruction supports the connection of proportionality and slope.

Proportionality and Slope

Historic Connections

Consideration of the annals of mathematical history in regard to proportions is important in terms of a principle of RME known as *didactical phenomenology* that, according to Freudenthal, is fundamental for applying the knowledge acquired through mathematics (Bell & Brookes, 1986). Freudenthal (1983) urged mathematics educators to recognize that young learners can benefit from recapitulating a mathematical concept, structure, or idea in relation to the phenomenon with which it was first associated. Therefore, to assure that my subjects were placed in appropriate mathematical situations, I explored the history of proportions and looked for references to slope in an effort to select contextual problems that fell within the framework of RME.

As Rossmeissl and Webber (2006) have observed, the concept of proportions has existed since ancient times, having been discussed in Euclid's *Elements* in the context of commensurability; specifically, two segments are described as being commensurable if there is a segment that "measures" each of them, one that is contained within each segment a whole number of times. Euxodus's discovery of incommensurable quantities, however, led to a new theory of proportions independent of commensurability. The inclusion of incommensurability allowed the concept of irrational numbers to be applied to proportions, thus expanding the coverage of mathematical topics to include

- computational devices for approximating the roots of numbers,
- operations and manipulations of expressions involving radicals,
- the development of analytic geometry, with the attendant need to associate a number with every point on a line, and
- algebraic symbolism, the theory of equations, and the development of calculus and its use with problems of limits and continuity (Rossmeissl & Webber, p. 71).

Thus, proportionality was historically associated with line segments. In modern terms, Walter and Gerson (2007) and Schoenfeld, Smith, and Arcavi (1993) categorized this approach as a local means of comprehending slope. A local level of understanding involves recognition and translation of slope for a specific segment of a line.

Descartes's *La Geometrie* (1637) gave rise to analytic geometry, which brought together geometry and algebra. Church (1851) defined analytical geometry as "that branch of Mathematics in which the magnitudes considered are represented by letters, and the properties and relations of these magnitudes made known by the application of the various rules of Algebra" (p. 2). The system of Cartesian coordinates allowed for the use of letters in algebraic expressions. Thus, equations with two kinds of quantities, *x* and *y*, which are different for different points of the line, are called variables, and *b*, which remains the same, is called a constant. Thus, by using letters to represent magnitudes, this system eliminated the measuring of line segments because it was possible to use a scale with intervals to represent numbers instead of the actual measurements. The notion of magnitude and its application to lines had already been discussed in the proportion section of Euclid's *Elements.* Nevertheless, slope in a classroom setting refers to the coordinate plane and is measured as the relational change in two units, thereby indicating that it is synonymous with proportionality.

This historical perspective sheds light on the modern definition of proportional reason by Karplus, Pulos, and Stage (1983) as a system of two variables between which there exists a linear functional relationship. This perspective is also consistent with an understanding of slope on a global level in a functional context as the rate of change between two quantities (Walter & Gerson, 2007; Schoenfeld, Smith, & Arcavi 1993). It is thus apparent that the similarities from a historical perspective reveal proportionality to be a step toward understanding slope as a relationship between changes in two variables, though the true nature of their coexistence is regulated by passage through the origin. The relationship is considered not to be proportional when a line intercepts the *y*-axis at a point other than zero.

Current Curriculum

Since the focus of this study was on how middle school students used proportional reasoning strategies to solve slope-related problems, it is naturally important to consider the current mathematics curriculum. The Common Core State Standards for Mathematics (CCSSM) call for students in sixth grade to be able to describe the relationship between two quantities, to understand the concept of unit rate, to use ratio and rate reasoning to solve real-world and mathematical problems about tables of equivalent ratios, and to plot pairs of values on a coordinate plane. In algebra textbooks, these concepts would be presented in the sections on analyzing linear equations, but they are located under "Ratio and Proportional Relationships" in the CCSSM.

Furthermore, the Common Core State Standards for Eighth Grade Mathematics listed as one critical area of instruction that students understand connections among proportional relationships, lines, and linear equations by using linear equations to represent, analyze, and solve a variety of problems. Thus, eighth-grade students are expected to

- recognize equations for proportions ($y/x = m$ or $y = mx$) as special linear equations ($y = mx + b$),
- understand that the constant proportionality (m) is the slope and that graphs are lines through the origin, and
- understand that, because the slope (m) of a line is a constant rate of change, the output or y-coordinate changes by the amount m*A when the input or x-coordinate changes by an amount A.

The National Assessment of Educational Progress (NAEP) which found that only 20% of students were able to interpret slope based on a verbal description, only 33% were able to find collinear points in a table (NAEP, 2013), and only 41% were able to solve problems based on a linear graph. These results demonstrate that the conceptual understanding of slope taught in middle school is insufficient. Cheng (2010) concluded that one component of proportional reasoning essential to the transformation of algebra is an understanding of slope in a range of contextual situations, examples including such linear relationships as those between distance and time, cost and the number of items bought, and hours worked and pay.

Benson (2009) stressed that proportional reasoning is found in all strands of the middle school mathematics curriculum as students reason using multiplicative relationships in order to make quantitative, qualitative, or algebraic generalizations (p. 8). Thus, one study has reported that the proportional tasks encountered by students at school are usually formulated in a missing-value format, but that non-proportional tasks are not, so that students tend to develop a strong association between this format and proportionality as a mathematical model (Ainley & Pratt, 2005).

Unsurprisingly, poor utilization of proportional reasoning in this respect prevents students from fully comprehending the aspect of change in one quantity in comparison with another. Instead, this mathematical model approach embeds an equivalent ratio perspective that is less effective when it comes to understanding change in slope relative to another quantity. It is therefore important to grasp that proportionality involves much more than establishing two ratios as equal and solving for the missing term; rather, proportionality involves recognizing related quantities and using numbers, tables, graphs, and equations to think about their relationship (NCTM, 2000).

Students would, then, benefit from discussing how one quantity is changing in relation to another as an exercise in describing the given data and formulating other quantities when the input changes. This method would help to bridge the gap in terms of conceptualizing slope. Moreover, studies suggest that such interventions as activity-based instructional modes and explicit comparisons of correct and incorrect strategies can be more effective than standard instruction. According to Piaget and Inhelder (1958) adolescents' proportional reasoning develops from a global compensatory strategy, often additive in nature, to an organized proportional strategy without generalization to all cases, and finally to the formulation of a law. This last step could play a vital role in the connection of proportional reasoning strategies with slope provided that appropriately sequenced activities guide the way to the discovery.

To be effective, these strategies need to integrate activities that go beyond mere paper-and-pencil computations. Thus, Cloutier and Goldschmid (1978) found that discussing how to solve proportional problems led to significant improvement in comprehension. Also, Freudenthal (1983) argued that

> formulating instructional objectives should be preceded by observing such learning processes as could reveal what is being, and thus what should be, learned; and that for observing learning processes as well as for educational development an indispensable precondition is a didactical phenomenology. (p. 178)

Thus, mathematics educators must seize upon opportunities for their students to experience mathematical concepts (such as proportionality and slope), structures, and ideas that have been invented to serve as tools to organize the phenomena of the physical, social, and mental world (Freudenthal, 1983). This exploration could lead to improvements in the retention of mathematical concepts and in appreciation of their relatedness.

Thus Stump (1999), for example, viewed proportional reasoning as equivalent to linear reasoning (slope) on the grounds that algebra textbooks frequently define the slope of a line as the ratio of vertical rise to horizontal run moving from one point to another along the line. However, frequent difficulties with slope and connecting various representations of it arise on the part of both students and teachers (Postelnicu, 2011).

Nevertheless, most problems in textbooks are non-contextual, failing to connect mathematical phenomena to the real world. Cheng (2010) accordingly concluded that middle school students' study of proportional relationships through the lens of algebra can help them to learn slope. According to the NCTM, algebraic thinking includes recognizing and analyzing patterns, studying and representing relationships, generalizing, and analyzing change. In order to promote such thinking, it is important that the proportional relationships be rooted contextually so that students can use their prior knowledge to make connections.

Stanton and Moore-Russo (2011) emphasized the need to address the proportional approach to slope in middle school given that the concept is a key means to describe the behavior of a function in secondary mathematics. The constant slope of a linear function is the most basic rate of change that students encounter; thus, these scholars referred to it as a "powerful linking concept" to help understand functions and their graphs (p. 271). The conceptualization of slope as a rate thus often requires thinking of it as a functional property within the context of a real-world situation; in this respect, proportional reasoning is considered equivalent to linear reasoning. The teaching of slope must, then, make use of a learning trajectory that accounts for the interdependence of proportional reasoning and slope.

It was from this perspective that Simon and Blume (1994) referred to slope a "ratio-as-measure"—because it involves the construction of a ratio as a given attribute.

Dealing with slope in this manner therefore requires students to reason proportionally. However, when working with the slope formula, students are expected to count squares on a coordinate grid system in order to find rise over run without ever realizing that they are describing a ratio (Lobato & Thanheiser, 2002). Nagle and Moore-Russo (2014) argued for a focus in the sixth grade on proportional relationships so that students can simultaneously build their images of change involving two covarying quantities.

As mentioned earlier, in the seventh-grade CCSSM objectives, terms normally associated with the analysis linear functions are found under the topic of "Ratios and Proportional Relationships." The major terms that students are to understand relating to slope are unit rates, ratios, proportional relationships, and constant proportionality. In seventh grade, students are expected to be able to determine whether two quantities are in a proportional relationship by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph forms a straight line through the origin.

The last portion of the objective, which discusses the graph being a straight line through the origin, concerns what algebra textbooks term "direct variation." In the sequence of these textbooks, this concept is normally encountered right after students have been introduced to slope, though the CCSM refers to it as "constant proportionality." The rationale for this sequencing from the sixth to seventh grades is that students should be able to identify the slope in graphs, tables, and equations in order to compare two proportional relationships represented in different ways. There is some irony in the fact that this objective is no longer found under "Ratios and Proportional Relationships" but instead has been transferred to the algebraic realm of "Expressions and Equations." It thus appears that the curriculum itself makes a strong case for connecting proportionality with slope.

Research Similarities

Numerous peer-reviewed research articles have demonstrated that proportionality and slope are analogous concepts. The latter is a key part of algebra, an aspect of mathematics in which students underperform in the middle and secondary grades (Stump, 2001). Lobato and Thanheiser (2002) recognized that the relationship was not being explored empirically and that curricula were failing to make the connection explicitly. As already mentioned, the Mathematics Common Core Standards use a plethora of words to describe slope. Moreover, the concept has been presented in connection with the proportional reasoning standards in the sixth and seventh grades and then used as an algebra standard in eighth, at which point the term "slope" is first used. Under these circumstances, the connections between proportional reasoning and slope are naturally lost, despite the fact that both have ratios as their basis.

Additionally, algebra as it has traditionally been taught in schools tends to be a very rigid and abstract branch of mathematics with little direct connection to the real world. It is often presented to students in terms of a pre-determined and fixed set of strict rules that leaves no room for their own input. Traditional instruction begins with the syntax of algebra, presenting students with a symbolic language to which they do not relate (van Amerom, 2002).

On the other hand, because the proportional tasks that students encounter at school are formulated in a missing-value format (whereas non-proportional tasks usually are not), students as noted earlier tend to develop a strong association between this problem format and proportionality as a mathematical model (De Bock, Van dooren, Janssens, & Verschaffel, 2002). De Bock et al. (2002) produced empirical evidence for students' disengagement, showing that they improperly applied direct proportional reasoning when problems were stated in a missing value format. Changing problems to a comparison format in turn brought about substantial improvement. It is therefore apparent that the comparative format is preferable in terms of both proportionality and slope, for this format increases students' proportional understanding in the context of specific situations. Nevertheless, U.S. schools still use traditional classroom settings in which students are simply told how to set up problems that correspond to various scenarios without giving them the opportunity to mathematize.

The use of contextual problems can likewise enhance the understanding of slope. In fact, students may have an intuitive understanding of slope based on their real-world experiences with covariation in such contexts as cooking (e.g., an egg for every cup of flour). However, the example used by Hattikudur et al. (2012) involved discussing a rate in order to imply the use of a contextual slope problem, while in a traditional textbook this type of problem would be found in the chapter on proportions. These researchers also stressed that students must be able to interpret and construct graphs of linear functions as a way of understanding slope and suggested that graphs presented that include only qualitative features (as opposed to numerical values) might help students to draw on their common sense and reality-checking strategies. This qualitative aspect of slope can usefully be considered in terms of the three types of tasks described by Cramer, Post, and Currier (1993) for assessing proportional reasoning, namely

- missing value problems in which three pieces of information are provided and the task is to find a fourth,
- numerical comparison problems in which two complete rates or ratios are provided and compared but no numerical answer is required, and
- qualitative predictions and comparison problems that again do not ask for specific numerical values.

The final type of proportional assessment is practically identical to the concept of slope. Furthermore, students' ability to utilize their own strategies to solve problems by constructing graphs with correct slope may be attributable to their successful transfer of prior knowledge about slope in the context of tables over to graphs. This is the crux of RME, in that students are creating rather than having an algorithm forced upon them.

Further discussion of graphs and tables is merited here. Returning to the governing curricular standards, this type of mathematics is again found under the rubric of proportional reasoning standards in the sixth and seventh grades, though it is associated with algebra in this study. This is another revealing example of how slope and proportionality are synonymous, though math educators have developed multiple terms for them.

If contextual problems were used that allowed students to formulate their own strategies through discourse rather than through the dictates of a textbook when seeking to solve problems, it would soon become clear that they become engaged in proportional and slope tasks from the point when they first learn fractions and describe line graphs. Students may also relate problems to their real-world experiences with covariation, for instance relating hills and mountains to the graphing of slope. In light of recent research on students' understanding of covariation (Blanton & Kaput, 2004) as well as efforts to integrate algebraic ideas into elementary school mathematics curricula (Kaput, 1998; Olive, Izsak, & Blant, 2002), it is perhaps unsurprising that students tend to demonstrate a fairly good ability to graph slope even before the topic is addressed formally.

Covariation, however, is not always used to strengthen the understanding of slope. Thus Stump's (1997) interviews with teachers confirmed that they had difficulty both in recognizing slope as a parameter or a constant rate of change in two covarying quantities and in interpreting graphs that required connections among various representations of slope. Instead, secondary teachers again most often conceptualized slope as a geometric ratio, focusing on rise over run. Students' responses suggested that they relied on procedurally based conceptualizations of slope rather than on covariational reasoning. This is unfortunate, because use of these linear graphs and functional property conceptualizations implies that slope is a procedure and reveals little grasp of the significance of covarying quantities involved in physical and real-world applications.

It is precisely the covariational aspects of slope and its applications to physical phenomena that students should develop before entering calculus (Carlson et al., 2010). Instructors need to capitalize on students' dominant conceptualizations and on real-world references as a foundation on which to build more advanced ideas when teaching slope. Once again, slope is revealed as synonymous with proportional reasoning. Returning to the three assessments of proportional reasoning, the relevant one here involves numerical comparison problems, in which two complete rates or ratios are given and compared to explain covariation but no numerical answer is required.

Furthermore, covariational reasoning, including the concepts of rate of change and growth rates of functions, has been identified as a key prerequisite for pre-calculus instruction (Carlson et al., 2010; Confrey & Smith, 1995). Thus, returning to the argument of Lesh et al. (1998) that proportional reasoning is both the capstone of the middle school mathematics program and the cornerstone of all that is to follow, it is perhaps the case that the synonymy of the concepts is why the statement of slope plays such a critical role in the mathematics curriculum as

- an important prerequisite concept for advanced mathematical thinking (Carlson, Oehrtman & Engelke, 2010; Confrey & Smith, 1995) and
- a notion that is represented and conceptualized in many different contexts and settings (Stanton, & Moore-Russo, 2012; Stump 2001), requiring students and instructors to connect various conceptualizations to form a complete and unified concept image.

Task Similarities

Here the aim is to highlight the similarities between proportional reasoning tasks and slope tasks in order to reinforce the importance of building on students' prior proportionality knowledge when they seek to solve slope-related tasks. A good point of departure for this discussion is the notion of "ratio-as-measure" put forward by Simon and Blume (1994), according to which slope involves the construction of a ratio as a measure of a given attribute. Lobato and Thanheiser (2002) described this type of thinking as a complex because it requires students to reason proportionally, whereas the slope formula allows them to count squares on a coordinate grid system and find rise over run without mentally formulating a ratio. The latter researchers also observed that ratio-as-measure tasks involve aspects of modeling, such as focusing on one quantitative relationship or conceiving of an indirect measure when faced with direct measures, throughout their school career.

Although the discussion has thus far focused on the similarities between slope and proportionality in terms covariation, it is important to keep in mind that proportionality is a multiplicative relationship that can be represented on the coordinate plane as linear functions that pass through the origin (Lobato & Ellis, 2010). Therefore, once students have recognized the relationship, they can simply multiply the input value by the rate of change. In the effort to make this connection, it is imperative that mathematics teachers assess students' comprehension of uniform growth problems as well, since these problems illustrate an increase of y as a fixed multiple of x. The fixed factor indicates dependence on the circumstances. Uniform growth is described algebraically by a linear function as an additive phenomenon. Thus, students must determine that growth is proportional to time elapsed. This type of reasoning illustrates students' readiness for algebra and makes clear the feature of graphs that gives them visualizing power equal to or greater than that of a table or a formula.

Freudenthal (1983), as already mentioned several times, described a ratio as an equivalence relation in a set of ordered pairs of numbers or magnitude values as well as a ratio dependent on two sets of data. This conception is consistent with the notion that ratios and slope are synonymous. Ordered pairs are used to illustrate the position on a graph that contains two different units (one being the x and the other the y). The plotting of these points on a graph determines whether there is a relationship between the two sets of data: if the data progresses as a constant ratio, direct variation is observed. This concept tends to be relegated in textbooks to the chapter on linear equations and slope.

Thus, for example, in Lobato and Thanheiser (2002) activity referred to as “same steepness,” students, when asked to create as many ramps as possible with the same steepness as a ramp with a height of 3 cm and a length of 12 cm, generated such solutions as 2 cm high and 8 cm long, 4 cm high and 16 cm long, and 1 cm high and 4 cm long. They thus constructed a multiplicative comparison, as they identified the length as being four times the height. This is the same reasoning by which students find equivalent ratios, but the context used a real-world situation.

Similarly, in a “same speed” activity, students were asked to create a table of values that would make a frog in the Mathworlds environment move at the same speed as a clown moving 10 cm in 4 seconds. Though the problem initially created some difficulties, techniques such as doubling and partitioning were developed and discussed as means of ascertaining solutions; thus, one pair of students discussed how 20 cm in 8 seconds was the same because it is the same distance just walked twice. However, the biggest breakthrough was the student who partitioned and iterated, explaining that 2.5 cm per 1 second was the same speed as 10 cm per 4 seconds. The student recognized that the relationship was one-fourth of the 10 cm and one-fourth of the 4 seconds, and that, if the 2.5 cm and 1 second were multiplied four times, this would re-create the original 10 cm in 4 seconds. Once again, this reasoning demonstrates proportionality, as the students used doubling to create an equivalent ratio and a unit rate to explain equivalence. In order for slope to be meaningful for them across a broad range of situations, students need to develop an understanding of it as a ratio that measures some aspect of a situation (Lobato & Thanheiser, 2002, p. 174).

In these two slope examples, scalar strategies were prominent. Nunes and Bryant (1996) argued that scalar solutions relate to children's informal building up of such strategies as doubling and halving. However, slope is a more functional relationship in that it serves to provide students with the explicit understanding that a fixed multiplicative relationship exists between the two measures. These researchers concluded that the logic of a functional relationship between variables is the crux of modeling, which they defined as "the process of representing the world and operating on the representations to come to conclusions about the world" (Nunes, 2012, p. 2). Ponte (1992) viewed linear functions as the simplest of these models and suggested that an understanding of this concept might support the use of more complex functions in science and mathematics.

Illustrative here is a proportional study by Pydah and Nunes (2012) on the use of schematic representations to improve students' understanding of proportional reasoning. The objective was to help them move from a scalar to a functional appreciation of mathematical relationships. The first problem that introduced a diagram asked, "Emily bought 4 balloons and paid £10 for them. She went back and bought 12 more balloons for her class. How much did she have to pay for 12 balloons?" The schematic diagram showed how to utilize scalar thinking by multiplying the balloons by 3, which was at the top of the diagram, and then the money by 3, which was at the bottom of the diagram. In this way, the students were introduced to tables, which tend to be incorporated into the algebra curriculum as they look for relationships that can help them to devise equations, and taught proportional reasoning that involved setting up cross-multiplicative ratios. This type of problem is normally presented in the context of an algorithm that can solve any proportion problem but that does not offer opportunities to view relationships.

In the second session, the functional approach was introduced as a means of utilizing the linking table, the aim being to express visually the notion of a constant multiplicative relationship between two variables. This approach bears comparison to the discussion of slope, as the aim was for students to see how the two units vary in relation to one another. The specific problem asked, "In school, there is a hamster. It eats 12 scoops of food in 4 days. How much food will the hamster need for 7 days?" The researcher denoted the given pair (4 days and 12 scoops) on corresponding lines, but this time the students were first asked to find the amounts for 2 days and 1 day and then told to focus on the constant vertical relationship between the two lines. Once the students observed the functional link of "times 3," they were able to solve the problem. It is ironic that a proportional study would introduce functional relations, as this topic is normally introduced in connection with algebra, but in any case, proportional reasoning is once more seen to be synonymous with slope.

Summary

Slope and proportional reasoning, then, coincide in terms of meaning as well as the tasks with which they are associated, and are indeed found to be similar from the perspectives of history, appearance in curricula, and various research studies. The ultimate objective, however, whether tasks are considered proportional or in terms of slope, is for students to transfer knowledge and to make connections with other subjects and in particular with their everyday experiences. This goal can be achieved through mathematical explorations in which students learn to construct and interpret scatter plots of bivariate data and to investigate linear patterns. Perfect linearity is of course less frequent in reality than in the problems used in math class, but the ability to apply interpretations and make conjectures based on how units vary is essential. The literature review made clear that this ability can be categorized in terms of either proportionality or slope.

All of this raises the question of why these concepts continue to be treated as different algorithms into which numbers are inserted in order to generate solutions. By way of a working definition, either slope or proportionality can be said to denote reasoning in the context of a system of two variables between which there exists a linear functional relationship, reasoning that leads to conclusions about a situation or phenomenon that can be expressed as a constant ratio (Karplus, Pulos, & Stage, 1983). The research discussed in this chapter supports students' use of proportional reasoning strategies to carry out slope-related tasks. Thus, Lobato and Thanheiser (2002) observed that, while some educators have held that the teaching of slope should not go beyond the slope formula, this approach is only useful in solving textbook problems and cannot be easily applied to real-world situations involving rates of change, which are usually messier and more complex (Lobato & Thanheiser, 2002).

CHAPTER THREE: METHODOLOGY

The purpose of this study was to describe how a group of seventh-grade students used prior proportionality strategies to solve slope-related problems. It was hypothesized that mathematics educators can promote students' understanding by approaching slope problems in a way that draws on their prior knowledge rather than from the traditional "change in *y* over the change in *x*" perspective. An old Chinese proverb that reads "Tell me and I will forget; show me and I may remember; involve me and I will understand" well describes students' experiences with proportional reasoning and slope-related problems.

This chapter details the research design, the procedures employed, and the interpretive framework for the data analysis, offers a subjectivity statement, and discusses possible limitations to the study, which, again, was designed to answer the following research questions:

1. What proportional reasoning strategies do participants use when solving slope-related questions that pertain to nonzero *y*-intercepts?
2. How do participants connect proportionality with slope?

Research Methodology

This qualitative research incorporated a case study approach, the rationale being that case studies shed light on relatively large-scale phenomena through extensive examination of specific instances, depicting events, processes, and perspectives as they unfold (Rossman & Rallis, 2012). Additionally, case studies are considered useful because of the rich description that they provide and their heuristic value. Four separate case studies were included in this study, each involving two students. This design enabled cross-case analyses and comparisons that identified the commonalities and differences listed in the discussion section.

A qualitative methodology involving discourse analysis was also used to determine the effectiveness of drawing on students' prior experiences with proportional reasoning and strategies as a means to advance their understanding of slope. A qualitative study was also considered the best approach in this respect because the focus was on the meaning of the experiences, actions, and events for the participants and researchers and their subcultures (Henwood, 1996). Flick (2002) has asserted that qualitative studies "Do more justice to the object of research than is possible in quantitative research" (p. 8), the idea being that such studies gather details on individuals' thinking processes rather than quantifying participants based on their scores on a survey. In the present context, qualitative approaches enable education researchers to explicate how students acquire false beliefs, thereby providing insight into their reasoning processes and the assumptions that contribute to their misconceptions (Kalinowski, Lai, Fidler, & Cunningham, 2010).

In order to assess the viability of the chosen approach and to optimize the procedures, I carried out a pilot study involving two participants. The rationale for the pilot study was two-fold. First, as Simon (2011) observed, a pilot study is a strategy that allows researchers to address resolvable issues prior to commencement of the main study, such as checking that

- the instructions are comprehensible,
- the investigators and technicians are sufficiently skilled in the various procedures,

- the questions are properly worded,
- the results are reliable and valid, and
- the statistical and analytical processes are effective.

In sum, pilot studies help to identify flaws in research designs. They also serve as retrospective analysis in the context of RME. Although this study did not follow the full research design approach, I nevertheless considered it imperative to conduct a pilot study to reflect on my findings and refine my questions and analysis for the larger study.

The implications of the pilot study were that a connection was indeed discernible between slope and proportional reasoning, as the two participants experienced great success in ascertaining the appropriate solution using math skills that had been acquired previously. However, it was also evident that students were unlikely to have any idea why the cross-multiplication algorithm $a/b = c/d$ is valid, as the participants in the pilot study themselves admitted. Thus, in order to draw firm conclusions regarding any connection between proportionality and slope, I determined that the main study must follow the principles outlined by DeLange (1996), which can be summarized as follows:

- the starting points of instructional sequences should be experientially real to students so that they can immediately engage in personally meaningful mathematical activities;
- in addition to taking into account the students' current mathematical ways of knowing, the starting points should also be justifiable in terms of the potential end points of the learning sequence; and
- instructional sequences should involve activities in which students create elaborate symbolic models of their informal activities.

With the help of the pilot study, then, I identified proportional problems that were relatable for the students while also accounting for their current mathematical knowledge. I selected problems that built on their prior knowledge but with slope in mind as the endpoint. These problems were selected from peer-reviewed mathematical studies—specifically as well as the National Assessment of Educational Progress (NAEP) and the North Carolina Department of Instruction (NCDPI)—in order to ensure their reliability and validity.

Following the lead of Lawton (1993), the problems used for the study allowed students to observe the relationships among units rather than being based on an algorithmic model, affording them the freedom and flexibility to utilize their prior knowledge to arrive at solutions. According to the NCTM (2000), capability with proportionality involves much more than simply establishing two ratios as equal and solving for the missing term, including recognizing quantities that are related proportionally and using numbers, tables, graphs, and equations to think about the quantities and their relationship.

Following the pilot, the study was scaled up to include eight students (those from the pilot study did not participate in the main study). In addition, the questions asking students about the meaning of proportional reasoning were omitted. The goal of RME is for students to mathematize problems and not to concern them with terms or levels that determined by the teacher and textbook. The final study included seventh graders from a suburban charter school enrolled in Math 7 as well Pre-Math, the thinking being that, if the bottom-up approach is in fact used, there should be no discrimination in terms of math classification.

The participating students also worked in pairs, using discourse as a means to find solutions to the various problems. This approach is in keeping with the social constructivism that provides part of the theoretical framework for this study. Thus, RME is based on the notion that mathematical development is socially situated and that knowledge is constructed through interactions with others. It was therefore essential that I be guided by and reflect on the importance of discourse in understanding mathematical concepts and connections.

Barnes (1976) discussed the ways in which

> Speech unites the cognitive and the social. The actual (as opposed to the intended) curriculum consists in the meanings enacted or realized by a particular teacher in the class. In order to learn, students must use what they already know so as to give meanings to what the teacher presents them. Speech makes available a reflection the processes by which they relate new knowledge to old. But this possibility depends on the social relationships, the communication system, which the teacher sets up. (cited in Cazeden, 2001, p. 2)

This profound statement is crucial to mathematics reform, for too many educators continue to dictate the dialogue in the classroom, asking students merely to regurgitate the solutions or information presented.

Ideally, discourse in the mathematics classroom should involve whole-class discussions of mathematics that reveal students' understanding of concepts by engaging them in mathematical reasoning and debate (Macguire & Neill, 2006). It is imperative to pose premeditated questions that force them to make clear both how a problem was solved and why a specific method was selected. In this way, students learn to critique their own and others' conjectures as well as to identify plausible mathematical solutions.

The theory behind classroom discourse evolved, as alluded to earlier, from constructivist views of learning according to which knowledge is created internally through students' interactions with the environment in the context of various socio-cultural perspectives on learning. Following this approach, students work together to reach understandings that they could not have reached working alone (Macguire & Neill, 2006). Moreover, the Principles and Standards for School Mathematics advocated by the NCTM (2000) called for the replacement of traditional classrooms with communities of learners who communicate with each other and by groups of students voicing their opinions in whole class discussions. For such a situation to come about, talking about mathematics in the classroom must be considered acceptable, indeed essential, so that verbal explanations and defenses of ideas become the defining features of quality mathematical pedagogy (Walshaw & Anthony, 2008).

Furthermore, researchers have found that both the cognitive and material decisions that teachers make in relation to classroom discourse significantly influence learning (Walshaw & Anthony, 2008). To be more specific, classroom work is enriched when discussion involves the construction of mathematical knowledge through the respectful exchange of ideas as teachers build inclusive partnerships and ensure that the ideas put forward are, or become, commensurate with mathematical conventions and curricular goals. Brophy (2001) argued that practice and conditions that engage students in thoughtful and sustained discourse can facilitate learning provided that the discourse is centered on solid mathematical notions and that teachers motivate students to develop explanations, make predictions, debate alternative approaches to problems, and to clarify or justify their assertions.

From this perspective, education is best viewed as the development of shared understanding. Expressed in negative terms, mathematical skills are not best inculcated when teachers view their students as little pitchers waiting to be filled up with facts. Discourse is the antidote to this failed model, for when people communicate, their combined experiences may allow them to reach a higher level of understanding than would have been possible for individuals in isolation.

Various research methods have been used to investigate classroom discourse. One well-known linguistic-based approach is discourse analysis; it is associated with Sinclair and Courthard (1975), who devised a scheme for analyzing and categorizing discussions of teaching and learning in secondary classrooms. They demonstrated that the formal social order of a typical secondary classroom is embodied in a linguistic order, a pattern of talk that reflects how education is pursued in such a setting, that they categorized under the hierarchical headings of *lesson*, *transaction*, *move*, and *act*. Stubbs and Robinson (1979) simplified the process to *initiation* by the teacher, which elicits a *response* from a pupil, followed by an evaluative comment or *feedback* from the teacher (thus generating the acronym IRF).

Stubbs (1983) further argued that discourse sequencing made it possible to study empirically and in detail how teachers select the sorts of knowledge that they present to their students and how they group topics and order their presentation. Nevertheless, because this approach was devised to reveal linguistic structures, as opposed to educational or cognitive processes, the emphasis remains on the form rather than the content of discourse. My own opinion is that the manner delivery is a key aspect of the effectiveness of discourse; if it is not fluid or articulate, meaning can be lost. Thus, while the linguistic approach may not emphasize content knowledge, it nevertheless plays a vital role in how information is exchanged. If, for example, students cannot express them themselves well verbally, they may lack access to discourse that provides crucial information.

Research in the fields of psychology and sociology as well as education has been conducted on the effectiveness of this kind of classroom discourse. Initially, researchers were concerned with the *input* and *output* characteristics of the education system, such as the relationship between pupils' social class backgrounds and their eventual levels of achievement and occupational destinations. Then, with advances in the field of ethnography, anthropologists developed a method to describe and understand groups and individuals that has proved useful in the study of classrooms and their local cultures. This research emphasizes the conflict between the interactional norms of other cultures within the classroom and those of students' home communities, and it explains why some children behave in ways considered inappropriate by teachers who do not share their backgrounds. In any case, the ethnographic approach requires researchers to make detailed observations of what is said and done and, more problematically, to suspend their own common-sense interpretations of what is going on when making an analysis.

Another approach is educational psychology, though the predominant concern in this subfield has been with measurement making accurate assessments of individuals' abilities, aptitudes, and attitudes. Some scholars have suggested that educational psychologists' preference for relatively "hard" quantitative methodologies reflects insecurities regarding their professional status in relation to other psychologists and to teachers. Additionally, the prescriptions for the design of ideal learning environments put forward by educational psychologists often include detailed analyses of learners' performance limitations of learners without considering the performance characteristics of teachers.

A favored theme in such research is the development of efficient ways to control disruptive or nonconforming behavior in school and, through positive reinforcement, to direct or shape the actions of children toward the furtherance of acceptable goals; less important are discourse processes and the development of knowledge. Furthermore, such research holds to a kind of Vygotskian theory, acknowledging as it does that children undergo quite profound changes in their understanding by engaging in joint activity and conversation with other people in the process of their intellectual development. Vygotsky (2005) for his part was of the opinion that human learning presupposes a specific social nature and process through which children grow into the intellectual life of those around them.

The NCTM (2000), as the governing body of mathematics in the United States, spoke with authority when it asserted that communication in the mathematics classroom is vital to students' understanding of concepts and skills and their ability to benefit from their peers' approaches to problem-solving. The council defined classroom discourse as written and oral ways of representing, thinking, communicating, agreeing, and disagreeing used by teachers and students. The NCTM further discussed ways in which teachers orchestrate and promote discourse and the interplay of intellectual, social, and physical characteristics that shape knowing and working in the classroom.

The NCTM implied that teachers should orchestrate discourse by listening carefully to students' ideas, asking them to clarify and justify their ideas orally and in writing, selecting which of their ideas to pursue in depth during a discussion, and deciding when to provide information and when to let a student struggle with a difficulty (cf. Piccolo, Capraro, Capraro, Harbaugh, & Carter, 2008). These techniques were further implemented in the Common Core math curriculum, which highlights under the topic of mathematical practices the construction of viable arguments and critiquing the reasoning of others, activities that of course necessarily involve discourse.

As discussed in Chapter 1, mathematics teaching has traditionally involved teachers telling students facts that the latter then regurgitate verbatim. This practice of conveying *ritual knowledge* needs to be replaced with a *principled knowledge* built on explanations of how procedures work and why certain conclusions are necessary or valid rather than on arbitrary statements intended to please the teacher. Math educators accordingly need to concern themselves with identifying ways in which both ritual and principled knowledge are communicated in the situated discourse of lessons through joint understanding. This kind of discourse is, again, consistent with a Vygotskian perspective. In order to realize its benefits, educators must abandon pedagogy that concentrates on fulfilling our lesson plans and eliciting a predetermined set of responses from the pupils without engaging their thoughts. Teachers must ensure that their pedagogic aims downplay progress through an organized lesson plan and make room for spontaneous understandings on the part of the pupils of the principles underlying their activities. Otherwise, the opportunity for a genuine negotiation of understanding can be missed when pupils interpret things differently from the teacher.

Pirie and Schwarzenberger's (1988) study of the contribution of mathematical discussions to students' mathematical understanding provided valuable insight, particularly in terms of a relationship that may exist between the kind of language that students use (i.e., ordinary or mathematical) and the type of statements that they make (reflective or operational). These researchers, following Skemp (1976) associated reflective statements with relational understanding and operational statements with instrumental understanding in a manner consistent with such current innovations in the classroom as a more fluid discourse environment that may or may not include the teacher. Current research makes clear that students need to take responsibility for their own learning by using their own words rather than those of their teachers.

Cazden (2001) described four intellectual roles that students take on when collaborating with their peers: spontaneous helping, assigned teaching or tutoring, reciprocal critique, and collaborative problem-solving. As an educator, I have found peer-tutoring beneficial because it helps to reveal students' conceptual understanding of the material; moreover, the words of their peers are sometimes easier for students to comprehend than those of their teachers. Also relevant in this context is the argument by Edwards and Mercer (1987) that a major function of education is *cognitive socialization* and the eventual *handover* of control over knowledge and learning from the teacher to the student, who achieves autonomy.

Slavin (1996) proposed from a socio-cultural perspective that the joint knowledge of the members of a group exceeds the individual knowledge of any one member and that a group operates as an interacting system. An individual learns better with a peer because the peer can provide an audience, promote meta-cognition, and help maintain focus on a task. Also, Yackel et al. (2000) concluded that classroom discourse affords students opportunities to explain and justify their thinking to others and in the process to develop intellectual autonomy, including mathematical skill.

Description of Participants and Setting

The research for this study was conducted at a suburban charter school located in the Charlotte, North Carolina metropolitan region. The demographics of the school at the time of the study were 85% white, 9% Black, 4% Asian, and 2% Hispanic. The school did not have a free and reduced population because students were required to bring their own lunches; thus, the socio-economic status of the students was middle-class or higher.

There were two buildings on the school property, one serving as the middle school and the other as the high school, with each housing some 800 students. I chose to focus on the middle school campus for my study and in particular the classroom where I teach, for that was where the participants were receiving their math instruction and were comfortable and familiar. The eight seventh-grade students who took part included four boys from my all-male Pre-Math 1 class and four girls from my Math 7 class. The students were purposefully selected based on their good communication skills, how well they demonstrated discourse during collaborative problem-solving in class, and their ability to meet the minimum standards for seventh-grade math without exceeding eight-grade (Pre-Math 1) standards. Purposive sampling or judgment sampling of this sort allows researchers to select samples based on their experience and knowledge of the group being sampled (Gay & Airasian, 2003). For this study, it was imperative that the students be able to communicate well that they could describe their thought processes while solving problems.

Each pair of students participated once a week on a Monday, Tuesday, Wednesday, or Thursday throughout the three-week duration of the study. In the first week, the pairs solved traditional proportional problems pertaining to missing value and equivalent ratios. In the following week, each pair followed the same schedule but solved constant proportionality problems, which are linear relationships that begin at the origin. In the third week, students again followed the same schedule but solved traditional slope-related problems that had nonzero *y*-intercepts.

Data Collection Methods and Procedures

Prior to conducting the study, I made sure that my classroom created an atmosphere conducive to discourse so that students would become comfortable collaborating with each other. This meant doing away with the classroom layout in which students sat in single-file rows facing me; instead, I grouped the students in quads, allowing them to select their own groups so as to ensure that they would be comfortable with the group work. I made clear to the students from the first day of class that small-group discussions were expected, explaining to them that this approach offered opportunities to gain new perspectives on solving problems from their peers as well as to debate solutions constructively. At the same time, I established ground rules that promoted respect for the ideas, feelings, and thoughts of all.

After obtaining the participants' agreement to participate and informed consent from their parents (Appendix A), I interviewed them after school in their math classrooms using a three-phase, semi-structured protocol (Appendix B) that included four to five math problems per interview session. At the time of the study, the students had not received instruction on constant proportionality or slope, so they had not been exposed to the standard approaches to solving these types of problems. The study proceeded through three phases, with each session taking place after school and lasting for between 30 and 45 minutes. Again, the sessions took place in my and the students' classroom, so they were in a familiar setting. The pairs of students worked together on problems by discussing their strategies and coming up with a consensus solution. They were then asked about their approaches to solving the problems, including their reasons for selecting them.

A visual record of all phases of the study was kept digitally so as to capture any interactions that might otherwise have been missed. The students received the problems on sheets paper on which they were allowed to write. One pair each participated on Mondays, Tuesdays, Wednesdays, and Thursdays for the three weeks of the study. The day's separation between the individual sessions allowed me to view each pair in isolation from the others.

After each interview session, I collected the data (that is, the participants' work on the sheets of paper), which was kept in a locked cabinet, the students having been assigned numbers in order to maintain their anonymity during the collection and analysis of the data. I pass-coded the device used for the recording so that only I would be able to access the data. All documents were kept confidential and used only for the purposes of the study. There was no risk involved in this study, no cause for emotional distress nor possibility of compromising the participants' integrity. In no way were any of the participants placed in any danger of physical or emotional harm.

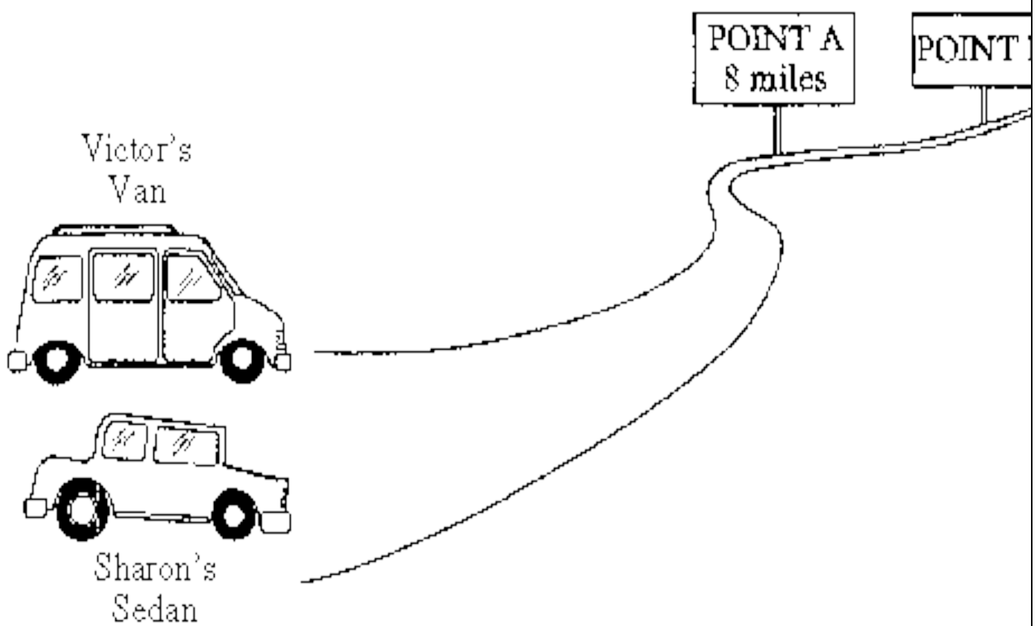
The first phase, as indicated earlier, looked at how students solve traditional proportional problems pertaining to missing-value or equivalent ratios. This phase consisted of four problems that pair of students solved on their assigned day. In this way, I was able to observe the extent of the participants' grasp of traditional proportionality problems. I accordingly selected three traditional proportion problems that could be solved using the ratio $a/b = c/d$ and one equivalent ratio problem. This selection of problems was intended to convey the notion that proportionality involves equivalent ratios as the students found the missing value. I sent the problems to a committee member before the study was conducted in order to ensure that they were consistent with the criteria and my interview protocol. After I had received feedback and edited the problems and the protocol, the data collection began. The participants were allowed to sit wherever they wanted in the classroom. An iPad located behind the pair of students was used to record the interview, beginning when I asked the warm-up questions. Figure 1 (Phase I problems) below presents the problems used in the first phase.

1. Ellen, Jim, and Steve bought three helium-filled balloons and paid $2.00 for all three. They decided to go back to the store and get enough balloons for everyone in their class. How much did they have to pay for 24 balloons?

2. Lisa and Rachel drove equally fast along a country road. It took Lisa 6 minutes to drive 4 miles. How long did it take Rachel to drive six miles?

3. Rule: One food bar can feed 3 aliens.

a.) How many aliens would be fed with 15 food bars?
b.) How many aliens would be fed with 16 food bars?
c.) How many food bars are needed for 63 aliens?



4. Victor's van travels at a rate of 8 miles every 10 minutes. Sharon's sedan travels at a rate of 20 miles every 25 minutes.
   If both cars start at the same time, will Sharon's sedan reach point A, 8 miles away, before, at the same time, or after Victor's van?
   Explain your reasoning.

   ____
   __

   If both cars start at the same time, will Sharon's sedan reach point B (at a distance further down the road) before, at the same time, or after Victor's van?
   Explain your reasoning.

   ____
   __

   Did you use the calculator on this question?

○ Yes ○ No

Figure 1. Problems Used in Phase I

After the students were asked the warm-up questions, they were provided with a calculator or allowed to access their own. Each student in the pair then received the first problem on a separate sheet of paper with an abundant amount of workspace. The students then read the problem and worked to achieve a consensus on the solution. After such a consensus had been reached, the students were asked to explain their approaches to solving the problem, including their reasons for using the strategies that they did. This process was repeated for all problems used in the first phase with each pair over the course of the first week of the main study.

In the second phase, five additional proportional problems were selected that coincided with the seventh-grade Common Core Standards. Under these standards, students are expected to be able to determine whether two quantities are in a proportional relationship (by testing equivalent ratios in a table), to identify constant proportionality in tables, and to represent proportional relationship by equations. I again sent the problems to a committee member to ensure that they fit the criteria and to edit my interview protocol. After these edits, the second phase of data collection began.

Using the same schedule and procedures, the pairs of students solved five questions that introduced linearity from the origin. Although this lesson is found under proportional reasoning in the Common Core Standards, it tends to be taught in the slope chapter of algebra textbooks under the topic of direct variation. This allowed me to see how students described data in situations in which two units varied. Figure 2 (Phase II problems) below presents the problems used in the second phase as well as describe the rationale for their inclusion.

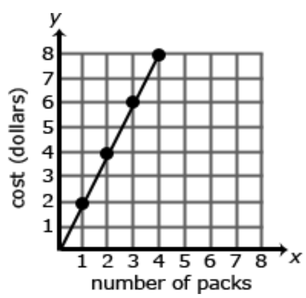
<table>
<tr><th>Problem</th><th>Rationale</th></tr>
<tr><td>

1. The table below gives the price for different numbers of books. Do the numbers in the table represent a proportional relationship?

<table>
<tr><th>Number of Books</th><th>Price</th></tr>
<tr><td>1</td><td>3</td></tr>
<tr><td>3</td><td>9</td></tr>
<tr><td>4</td><td>12</td></tr>
<tr><td>7</td><td>18</td></tr>
</table>

</td><td>To identify proportional strategy used to determine tabular constant proportionality.</td></tr>
<tr><td>

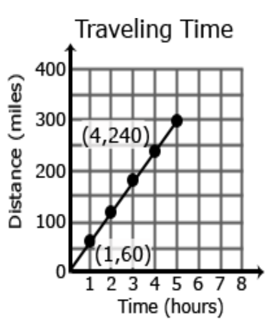
2. The graph below represents the cost of gum packs as a unit rate of $2 dollars for every pack of gum. Represent the relationship between the cost of gum and the number of packs using a table and an equation.



</td><td>To identify proportional strategy used to create an equation and table illustrated by constant proportionality.</td></tr>
<tr><td>

3. Compare the scenarios to determine which represents a greater speed. Explain your choice including a written description of each scenario.

Scenario 1: **Scenario 2:**

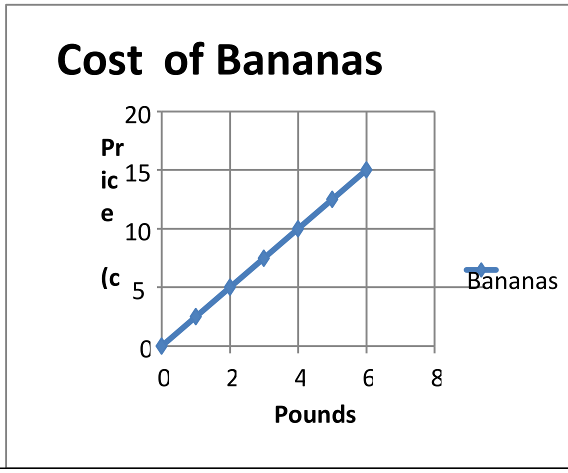


$y = 55x$
x is time in hours
y is distance in miles

</td><td>To identify proportional strategy used to compare proportional relationships in different formats.</td></tr>
</table>

<table>
<tr>
<td>

4. The graph below represents the price of the bananas at one store. What is the cost per pound?

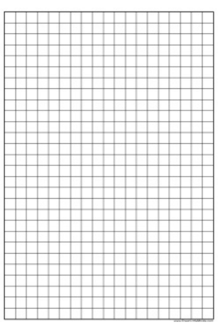


</td>
<td>To identify proportional strategy used to identify the unit rate from a graph.</td>
</tr>
<tr>
<td>

5. A student is making trail mix using the information in the table.
   A. Does the recipe represent a proportional relationship?

<table>
<tr><th>Serving Size</th><th>1</th><th>2</th><th>3</th><th>4</th></tr>
<tr><td>cups of nuts (x)</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>cups of fruit (y)</td><td>1.5</td><td>3</td><td>4.5</td><td>6</td></tr>
</table>

B. Where does the unit rate show up in the graph?

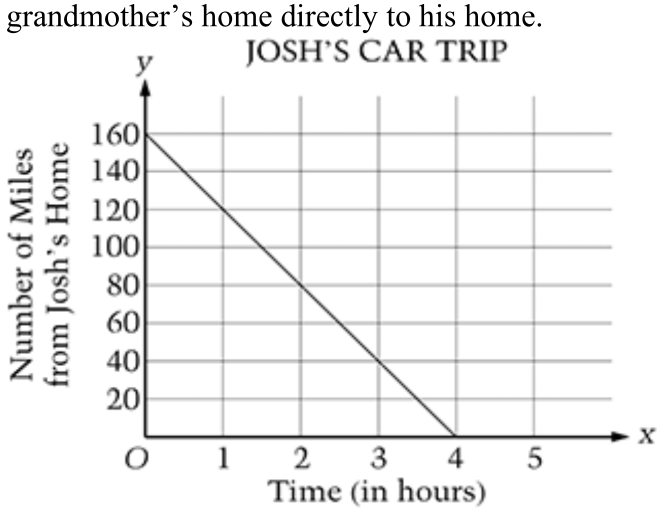


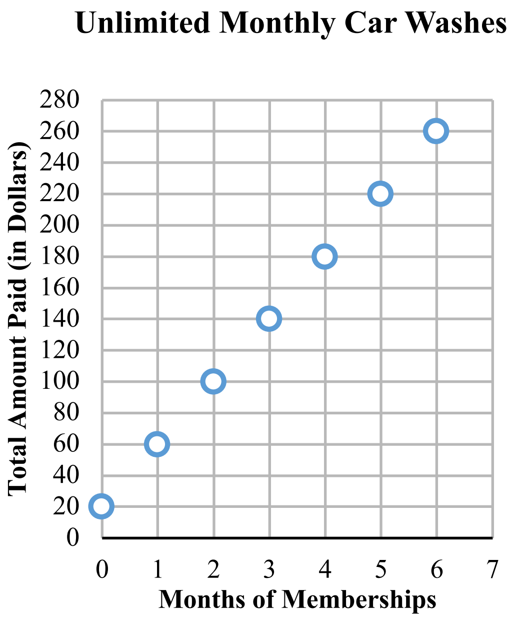
</td>
<td>To identify proportional strategy used to identify the unit rate from a table and then create it on a graph.</td>
</tr>
</table>

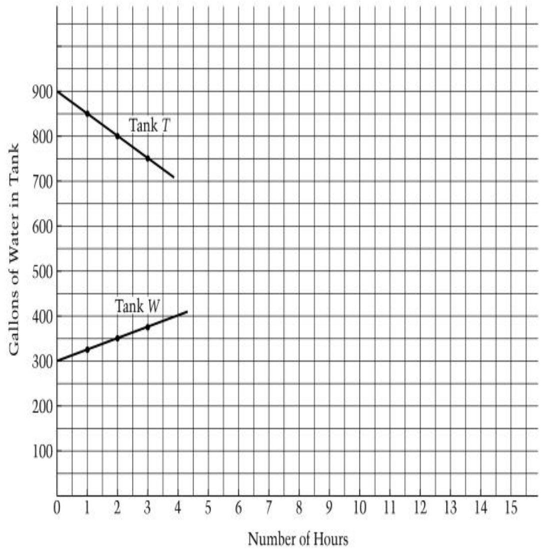
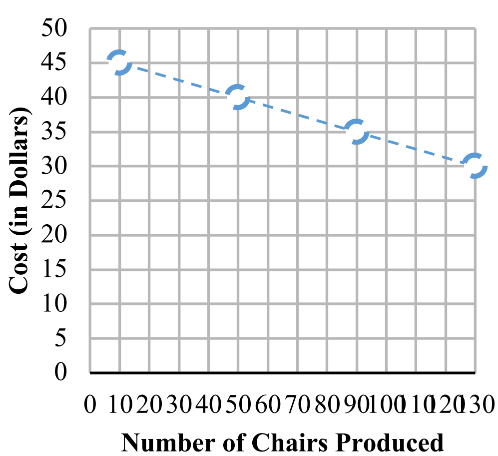
Figure 2. Phase II Problems

For the third and final phase, I selected four-slope related problems that coincided with the eighth-grade Common Core Standards and had nonzero *y*-intercepts. Once again I sent the problems to a committee member to ensure that they fit the criteria and to edit my interview protocol. After these edits, the third phase of data collection began.

Following the same schedule and procedures used in the previous two phases, the pairs of students solved four questions. In the eighth grade, students are expected to build on their understanding of proportional relationships and to interpret the unit rate as the slope of a graph as well as to compare two distinct proportional relationships represented in different ways. In addition, the Common Core Standards expect students to be able to describe how a graph represents a relationship between two quantities as increasing or decreasing, whether it is in the format of $y = mx$ or $y = mx + b$. I used the recordings of these sessions to identify the strategies participants used in solving constant proportionality problems and traditional slope-related problems. Figure 3 (Phase III problems) below presents the problems used in the third phase and the rationale for their inclusion.

Problems	**Rationale**
1. The graph below describes Josh's car trip from his grandmother's home directly to his home.  *a)* Based on this graph, what is the distance from Josh's grandmother's home to his home? *b)* Based on this graph, how long did it take Josh to make the trip? *c)* What was Josh's average speed for the trip? Explain how you found your answer.	To identify if and what proportional strategy is used to interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.

d) Explain why the graph ends at the *x*-axis.	
2. Membership for unlimited monthly car washes at a local car wash costs $20 plus a monthly fee, as shown on the graph below.  *a)* How much does it cost per month to belong to the car wash club? *b)* Write an equation that describes the graph. Explain how you determined the equation that represents the graph.	To identify if and what proportional strategy is used to create a non-proportional linear relationship using the equation $y = mx + b$ for a line intercepting the vertical axis at b on a graph.
3. Two large storage tanks, T and W, contain water. T starts losing water at the same time additional water starts flowing into W. The graph below shows the amount of water in each tank over a period of hours. Assume that the rates of water loss and water gain continue as shown. At what number of hours will the amount of water in T be equal to the amount of water in W?	To identify if and what proportional strategy is used to solve simultaneous non-proportional linear relationships on a graph.

	
4. The production manager of a furniture manufacturing company plotted values on the graph below to show how the production cost per chair decreases as the number of chairs produce increases. The rate of change of the graph below is –1/8. Two students had an argument on what the rate of change of the graph meant. **Production Cost Per Chair** 	To identify if and what proportional strategy is used to describe qualitatively the functional relationship between two quantities by analyzing a graph where the function is increasing or decreasing, linearly.

• Charity said that the rate of change represents each chair produced decreases costs by \$8. • Benjamin said that the rate of change represents for every 8 chairs produced, costs decrease by \$1. Who do you agree with? Justify.	

Figure 3. Phase III Problems

Data Analysis

The students' work samples were collected at the end of each session. The data obtained was then reviewed by observing each student's written work, watching, listening to, and taking notes on the recordings, and creating verbatim transcripts and paraphrasing the highlights of the session to ensure accuracy. Afterward, key phrases and words that evolved into themes during the interviews were transcribed and paraphrased to illustrate types of strategies. I further reread the students' work and again watched the video recordings and determined that there was no need to contact any of the participants for clarification of their responses. Following this step, I created charts with brief synopses of the ways in which the students' described their problem-solving methods for each question represented in each phase and with a section that described their views of collaborative work. Four charts made for each phase because each group represented its own case study.

In the first phase of the study, I had observed how students solved traditional proportional missing-value problems and how they recognized proportionality as equivalent ratios. Also, during this phase I was able to grow accustomed to my role an observer and learned not retreat into teaching mode during the interview process. I also analyzed the work and interviews to observe whether students used strategies other than proportional algorithms on the problems.

In second phase, I had observed the proportional strategies used by the participants to solve constant proportionality problems in tabular, graphic, or equation form. In the charts for this phase, a column was included that identified the proportional strategy was used to solve the problems. Following Lamon (1993), I again paraphrased portions of the interviews to create brief synopses that illustrated the strategies used to solve the problems. I further cross-checked the solutions with the solution strategies designated by the North Carolina Department of Instruction (NCDPI) and with the erroneous strategy identified by Tourniare and Pulos (1985), as can be seen in Figure 4 below.

Strategy	**Description**
Build-up Strategy	Students use the ratio to build up to the unknown quantity.
Unit-rate strategy	Students identify the unit rate and then use it to solve the problem.
Factor-of-change strategy	Students use a "times as many" strategy.
Fraction strategy	Students use the concept of equivalent fractions to find the missing part.
Ratio Tables	Students set up a table to compare the quantities.
Cross multiplication algorithm	Students set up a proportion (equivalence of two ratios), find the cross products, and then solve by using division.
Erroneous Strategy	Students using an inappropriate strategy or misusing a correct strategy (ignore part of the data in the problem).

Figure 4. Problem-solving Strategies in Phase 2 and Phase 3

After I had coded the proportional strategies used to solve the problems in the second phase, I coded the proportional strategies used to solve the slope problems with nonzero *y*-intercepts in the third phase. I created a chart for each pair with headings indicating summarization of their responses and the proportional strategy used to solve each nonzero *y*-intercept problem, and I highlighted how the strategy was used to solve the problem. In this way, I was able to document whether participants had followed similar thought processes and deduced commonalities when solving problems relating to constant proportionality and with nonzero *y*-intercepts. This information helped to demonstrate the importance of students' prior proportional knowledge when they tried to solve-slope related problems. I also analyzed the data using the impressionist tale approach, through which I was able to describe both my own thought process and the participants' actions. The purpose of creating an impressionist tale is to convey in a very specific and high detailed way how it felt to engage in the research.

Finding a connection was vital to my research because, while educators and curricula use the terms "slope" and "proportionality," students seek to solve problems using whatever is their "math rolodexes." My focus was accordingly more on the RME's horizontal mathematization, through which students come up with mathematical tools that can help to organize and solve a problem in a real-life context (Treffers, 1987). The math problems used in the study coincided with various examples of horizontal mathematization, in particular identifying or describing the specific mathematics in a general context, schematizing, formulating, and visualizing a problem in various ways, discovering relations and regularities, recognizing isomorphic aspects of different problems, transforming a real-world problem into a mathematical one (Treffers, 1987). This information could be beneficial in connecting proportionality and slope.

Risks, Benefits, and Ethical Considerations

I am fully aware that any research into human activity, especially research involving subjects under the age of eighteen, has the potential to raise concerns. I therefore familiarized myself with the guidelines and standards of the Institutional

Review Board (IRB) for conducting research and completed the necessary steps to obtain IRB approval for this study. In addition, I explained the project in detail to the parents of the participants and drafted an informed consent letter as well as an assent letter for the students, all of whom were assured that their participation and answers would have no impact on their grades in math classes. I moreover explained the project in detail to the administrative staff and obtained permission to carry out my research at the school.

I assured all parties involved that there were no known risks associated with the study and that participation was strictly voluntary. I also informed the students and parents that they could withdraw from the project at any time with no penalties or harboring of ill will on my part. I explained that all information about their participation, including their identities, would be kept completely confidential. The following steps were taken to ensure this confidentiality and anonymity. First, the recordings were collected by myself and contained no identifying information. All of the recordings were housed in a locked cabinet. The data will be kept indefinitely but will remain confidential and used for educational and research purposes only.

This study offered insights for math teachers into how students' existing proportional reasoning skills can help them to develop an instructional sequence that leads to an understanding of slope. The instructional sequence has the potential assist teachers and their students with activities and tools that develop an understanding of the concept of linear rate of change.

Subjectivity Statement

As my observations in the first two chapters have made clear, I am not a proponent of the traditional approach to teaching mathematics. It is my considered opinion that asking students to complete mathematical problems that bear no relation to the real world can generate disdain in them for the subject. I rather advocate the use of authentic problem-solving in mathematics and admire educators who teach in progressive ways. In my own experience as a math teacher, my students seem to have enjoyed my classes because of my passion, knowledge, and ability to show them why they are being taught the material.

Nonetheless, I am myself a product of a traditional mathematics education, so I know from experience that this approach can be effective for some students in attaining math proficiency. My goal as a math teacher, however, is not simply to inculcate mathematical proficiency, but also to inspire and encourage students to pursue careers in STEM-related fields.

Summary

The intent of this chapter was to provide the reader with information on the methodology used in the study by describing the research design, participants, procedures, data collection method, data analysis, rationale for questions selected in phase 2 and phase 3, inform the reader of proportional solution strategies recognized by NCDPI, ethical considerations, limitations of the study, and my subjectivity statement. The next chapter contains the findings of how students' connected proportionality to slope and what proportional strategies students used in solving slope-related tasks from the origin as well as a nonzero y-intercept.

CHAPTER FOUR: FINDINGS

This chapter presents individual case studies of the four pairs of students that participated in the study, which are referred to as A, B, C, and D. The data consisted of the participants' work samples and responses to interview questions pertaining to traditional missing-value problems, constant proportionality problems, and nonzero y-intercept problems. The following research questions were accordingly formulated:

1. How do students connect proportionality to slope?
2. What proportional strategies do students use to solve slope-related problems with nonzero y-intercepts?

This chapter presents the findings and describes the manner in which the data were collected in order to answer these research questions. The study was divided into three phases for each case study. The main purpose of the first phase was to allow me as the researcher and the participants to become acclimated to involvement in a research study. The second and third phases formed the core of the study, focusing on problems in $y = mx$ (constant proportionality) *and* $y = mx + b$ (nonzero y-intercepts). According to the CCSSM, students should understand that the slope of a line is a constant rate of change or a proportion represented by $y/x = m$ or $y = mx$, where m is the slope. This study accordingly highlighted how the four pairs of participants answered questions that fit the *criteria* $y = mx$ and the special linear *equation* $y = mx + b$ and determined whether the pairs recognized this proportional relationship.

Case 1: Pair A

The two seventh-grade students in Pair A were Caucasians enrolled in my all-boys Pre-Math 1 class. This class was a prerequisite for advancement to Math 1, meaning that the next step in the students' math career would be enrollment in a high school course. Student A described himself as happy, curious, and energetic and identified as his favorite aspect of mathematics the formulas "because it allowed him to plug in any number and then gives the answer." Student B described himself as happy, athletic, and cooperative, saying that he "loved working with people," and identified programming as his favorite feature of mathematics "because it made it faster and the process easier."

Banana Problem

This pair of students had been identified as being strong in mathematics, as reflected by their placement in an advanced math class. Pre-Math 1 is taught as an eighth-grade class to advanced seventh-graders. I was therefore greatly interested in how these students would respond to learning about slope, a concept to which they had not yet been introduced, particularly in terms of their use of their prior knowledge. The CCSSM expects that eighth-grade students will understand that the form $y = mx$, where m is slope, is a line that goes through the origin. One of the questions (Question 4) in Phase 2 of the study addressed this issue by asking the cost of the price of bananas based on Figure 5 (Banana question) below:

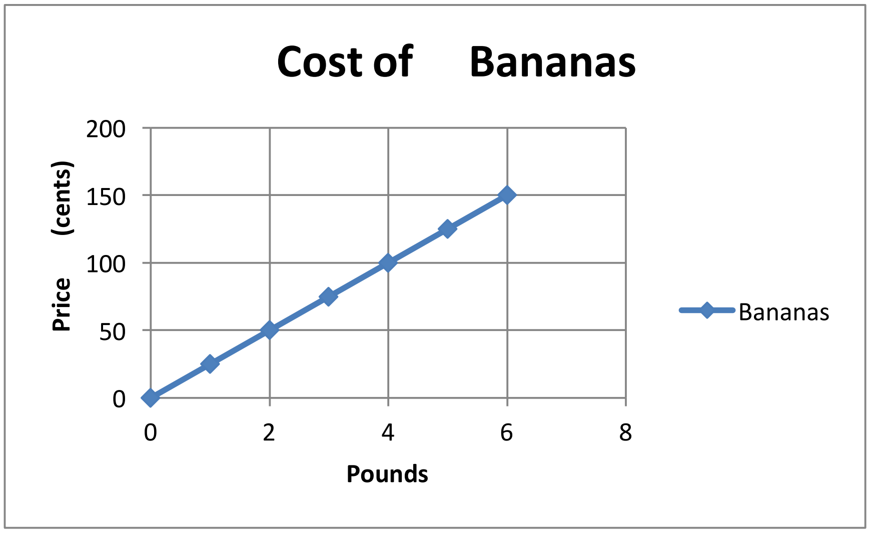


Figure 5. Banana question.

The following dialogue took place after the students had solved the problem and discussed the solution with one another:

Teacher: "How did you guys come up with your answer?"

Student B: "So, I just looked at it because it was right on the dot at 2. So, 2 pounds equals 50 dollars."

Teacher: "Fifty dollars?"

Student A: "Fifty cents."

Student B: "Fifty cents. So, I did 2 divided by 2 to get to one and you have to do the same thing for 50 cents. So, it would be 25 cents."

Teacher: "Twenty-five cents per what?"

Student B: "Per banana."

Student A: "Per pound."

Student B: Yeah, per pound.

If we follow the dialogue from above it is evident that the Pair A used proportional strategies to solve the problem. The pair observed that the point on the line indicated that two pounds bananas would cost 50 cents and divided by 2 in order to arrive at the correct solution of 25 cents per pound. Thus, the students used the fraction strategy coupled with the unit rate strategy. The fraction strategy was employed to simplify the fraction to an equivalence of 25 cents to 1 pound. This also exemplified the unit rate because the quantity of 25 cents is expressed to one pound and unit rates are expressed a quantity of 1. The pair connected the problem to slope because they found a vertical change in comparison to a horizontal change. This exemplified an understanding of covariation because the students compared how the quantity of cents changed in comparison to the quantity of bananas. Confrey and Smith (1995) discussed how this was optimal in the development of functional thinking. The function conveyed in the problem is $y = mx$ because the graph goes through the origin. The students exhibited this idea by stating the cost (y) is 25 cents (m) per pound (x).

Car Wash Problem

The question was now how the students would handle a problem that started from a nonzero y-intercept ($y = mx + b$). The following week, the students were excited, though it was our last session, and I was wondering whether they would a similar or a different strategy when faced with a nonzero y-intercept. I therefore used Question 2 from the third phase because it exuded the $y = mx + b$ format, asking about a car wash club that offered its members unlimited monthly car washes for \$20 plus a monthly fee, as shown on Figure 6 (Car wash question) below:

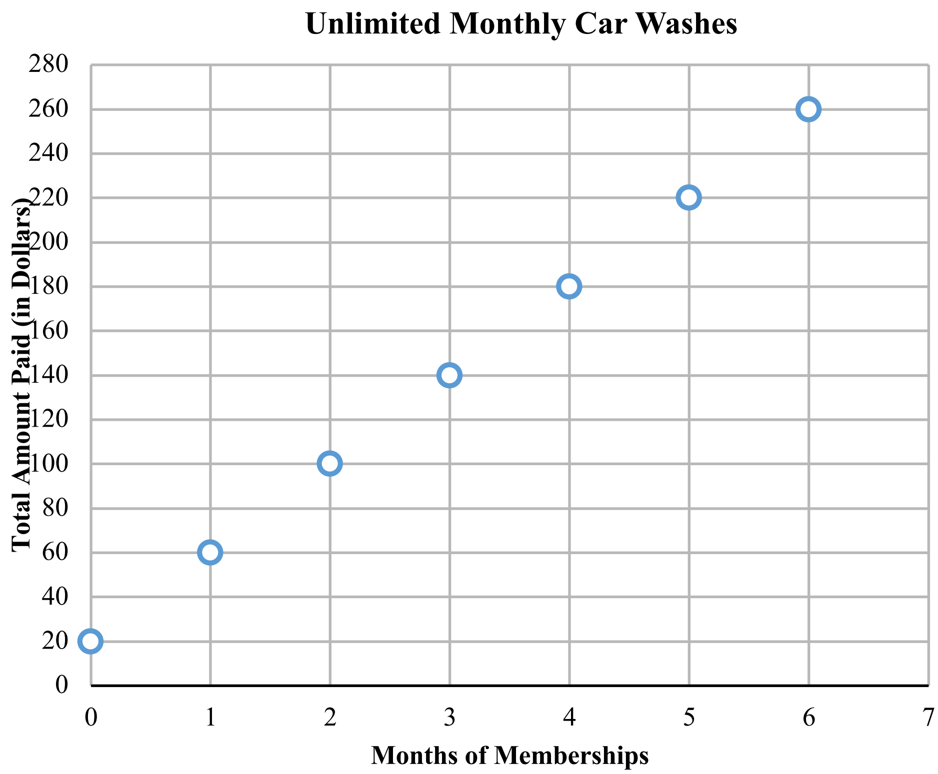


Figure 6. Car wash question.

The students were then asked to calculate the monthly cost for belonging to the car wash club, to write an equation that described the graph, and to explain how they arrived at the equation.

The following discussion then took place:

Teacher: "How much does it cost per month to belong to the car wash club?"

Student B: "Forty dollars."

Teacher: "How did you come up with 40 dollars?"

Student B: "If you look at the graph first, like just to buy the monthly, to buy like the, the plan is like $20. But then there is an additional fee per month and it goes from 20 to 60 to 100, so you know it would be $40."

Teacher: "So, what let you know that it started off at a $20 fee?"

Student B: "Because it's like the zero month, so that the first step of the problem."

Student A: “That’s when the graph begins.”

Teacher: “So, write an equation that describes the graph.”

Student A: “Um, so the equation is 40 dollars a month plus the initial fee of 20 bucks or 20 dollars. Um, for the equation, I did 20 plus in parentheses month times 40. Then that would give you, um, how much it cost.

Teacher: “All right, so explain how you determined the equation that represents the graph again.”

Student A: “Well the initial fee is 20 bucks you have to add that onto the whole equation. So, um, then you just do 40 times the number of months because it’s 40 dollars a month. And then you add it 20 to it.”

Teacher: “So, what is your variable representing?”

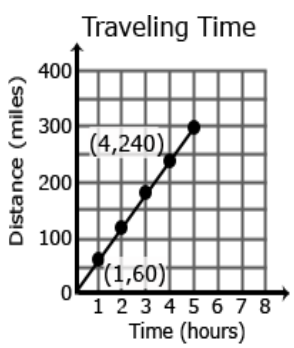
Student A: “Months.”

Despite the problem being a nonzero *y*-intercept, the pair still employed the unit strategy coupled with the build-up strategy. Thus, the pair discussed how the price went up from 20 to 60 to 100 dollars, identifying the consistent increment of 40 dollars, recognizing that the x-axis or month unit is increasing by 1. The pair recognized that the twenty-dollar was the starting fee and the rate of change from that starting fee was forty-dollars. The students then built from the initial 20 to 60 to 100 to see if the rate of change was consistent. The students once again connected proportionality to slope by finding a vertical change in comparison to the horizontal change. This illustrates a creation of the slope formula through guided reinvention of the contextual problems. Furthermore, the students, though they had not encountered the slope-intercept form, correctly calculated the twenty-dollar starting fee and forty-dollar monthly amount. Thus, they created an equation of cost = 20 + (40 x *m*), which is in the form of $y = mx + b$. The students recognized the monthly cost (y) is equal to $40 ($m$) times the number of months (x) from an initial point of $20. The proportional strategies used helped the pair describe a linear function in the same manner teachers utilize slope.

Traveling Problem

Additionally, the CCSSM expects students to use slope to compare relationships represented in various ways (graphs, tables, and equations). I therefore used Question 3 of Phase II to assess the connections and strategies that the students used to solve this type of problem, which asked them to compare two traveling scenarios in order to determine which represented greater speed and to explain their choice using a written description of each scenario.

Scenario 1:



Scenario 2:

$y = 55x$
x = time in hours
y = distance in miles

Figure 7. Traveling question.

The following discussion then took place:

Teacher: "How fast is the car going in Scenario 1?"

Student B: "Scenario 1; they are going 60 mph."

Teacher: "How fast are they going in Scenario 2?"

Student B: "55 mph."

Teacher: "So, which one is going faster?"

Student B: "Scenario 1."

Teacher: "How did you guys determine the rate of Scenario 1?"

Student A: "Well, we looked at the graph and figured, um, just look how long you would go in one hour. Which one is the simplest number to use for that and then use that to get a greater number by multiplying?"

Teacher: "Oh, you saw that one represented hours and 60 represented miles?"

Student A and B: "Yes."

Teacher: "How would you write your equation for the graph?"

Student B: "$y = 60x$."

Teacher: "y represents what?"

Student A and B: "Distance in miles."

Teacher: "And x represents what?"

Student B: "Time in hours."

I was struck by the fact that the students, having encountered another slope-related task, had not only used proportionality to arrive at a solution but also had used the unit rate strategy again. I noted one student's statement that "you just looked at the graph and see how long it would take in one hour," despite the fact that the problem was in the format of $y = mx$ and the term constant proportionality was also used illustrate the concept. This conception still represented an algebraic notion associated with slope, and the students were still able to compare differing views of a linear function by means of the unit rate. The connection to slope again was that the pair utilized the concept covariation by recognizing that miles where changing per hour. The pair has been consistent in illustrating traditional slope ideology by the covariance describing how the *y* unit is changed in comparison to the *x* unit.

Chair Problem

Pair A had, then, made the connection with and utilized the unit rate strategy as a means to address slope-related problems in the $y = mx$ and $y = mx + b$ formats. I was accordingly interested in their reaction to a problem that used a term synonymous with slope, such as rate of change, in the nonzero y-intercept format. I accordingly examined the students' approach to solving the following problem (Phase 3, Question 4), which presented a scenario in which the production manager of a furniture-manufacturing company plotted values on a graph to illustrate how production cost per chair decreased as the number of chairs produced increased. The rate of change on the Figure 8 (Chair question) was –1/8, and the participants were asked whether they agreed with either Charity, who said that the rate of change meant that each chair produced decreased costs by $8, or with Benjamin, who said that the rate of change meant a decrease of $1 for every 8 chairs produced, and to justify their answers.

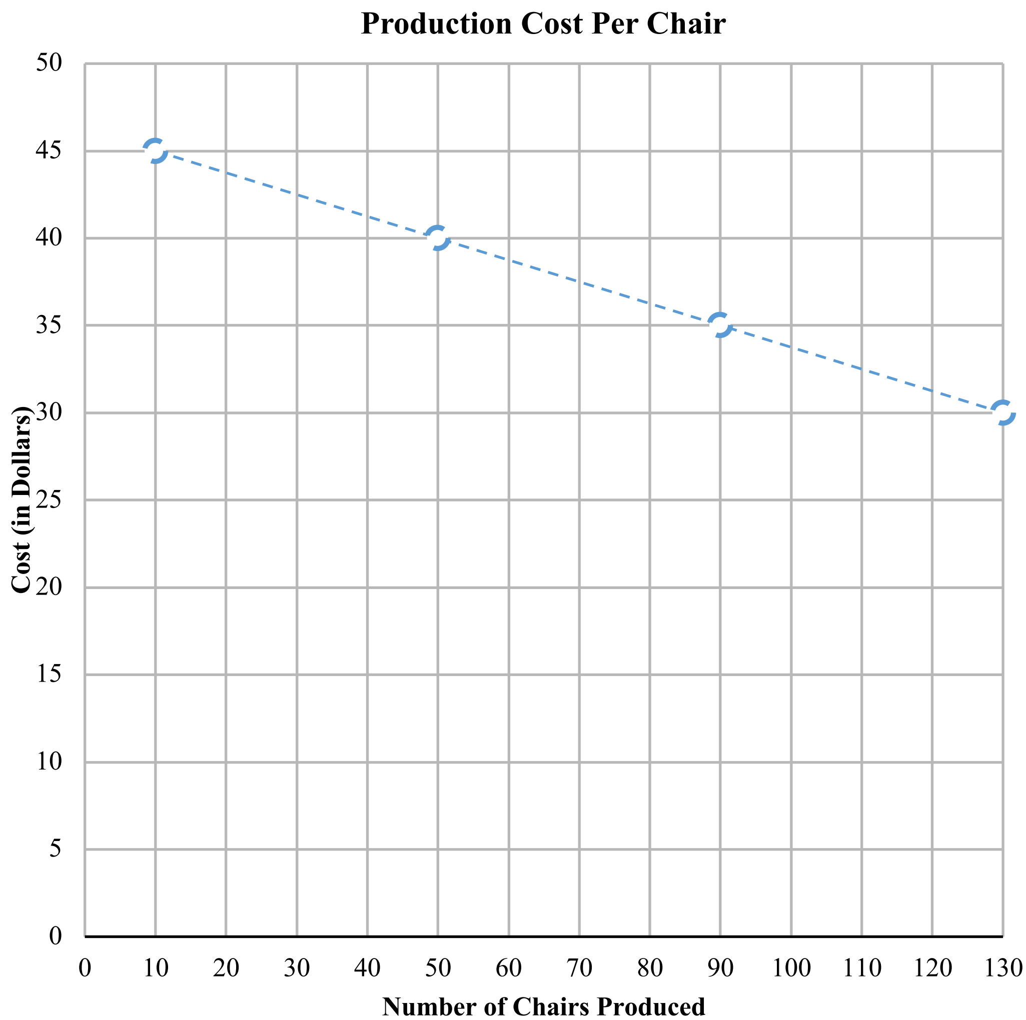


Figure 8. Chair question.

The following conversation then took place.

Teacher: "So, who is right, Charity or Benjamin?"

Student B: "We got Benjamin."

Teacher: "I heard your process. You were going through some stuff, but it seems like you came up with an agreement. So, why don't you tell me about that. Who do you agree with and justify?"

Student B: “So, I did Benjamin. First, I did 45-40 because those were the first two dots I saw. So, I got 5 and that was the dollars. Then I looked at how many sets were between 10 and 50, which is 40. Um, I simplified it, and I got a dollar for every 8 chairs.”

Teacher: “Now, if you look at the problem it says a -1/8. Why do you think the graph has a -1/8?”

Student A: “Um, well for every 8 chairs it is going down 1 dollar.”

Teacher: “And how do you know it is going down?”

Student A: “It’s a negative. It says -1/8.”

Teacher: “Is there any way you can look at the graph and tell that it is negative?”

Student B: “The line is slanting down.”

Teacher: “So, what does it mean if the line is slanting down?”

Student A: “The cost is getting cheaper as the amount of chairs are going up.”

The unit rate strategy coupled with the fraction strategy was used as the student honed in on the correct answer, that the cost of 8 chairs decreased by 1 dollar. The student described the relationship the number of chairs per one dollar which implies unit rate. This differs from the -1/8 figure, which placed the dollar amount on top. Yet, the students understood how the units are varying amongst one another. This illustrated a connection to slope again as the pair recognized that the vertical change was dollars decreasing and the horizontal change was chairs increasing. The pair also recognized that a line slanting down is negative. This traditionally is taught by teachers showing 4 graphs of slope relationships and explaining what the directions represent. Yet, the students demonstrated comprehension. Additionally, the pair used the fraction strategy, reducing 5 dollars over 40 chairs to 1/8. In any case, the key point is the manner in which the students found the difference between each respective unit and the other. Although they spoke of "dots," they employed the algorithm that students are usually compelled to employ without context. However, since these students mathematized this process, it became more personal to them and was added to their schemas for future use. The implication is that it is not necessary to ply students with algorithms, for they are capable of creating strategies from their prior mathematical experiences to solve problems. Thus Pair 1 here used the graph and implemented a proportional strategy that was effective in solving a nonzero y-intercept problem.

Conclusion

Lawton (1993) suggested that students could benefit from focus on quantitative relationships between the units of each object in mathematics problems, arguing that the unit approach appears to lead most students to an intuitive understanding of proportional reasoning. This view seemed to be confirmed by the findings presented here, as the unit approach was the one most often used. The unit approach has the additional advantage of helping students to see how one unit is changing in comparison to another. This is the premise of how slope is presented in books and by teachers, as the change in y over the change in x. Pair A unitized in all of the problems solved. Yet, they connected proportionality to slope through recognition of the vertical change of the graph in comparison to a one-unit change on the horizontal axis (x-axis). Furthermore, the connection of proportionality was exuded when the pair stated "for every 8 chairs it goes down one-dollar." Yet, they understood this was synonymous to -1/8 because they recognized how the units varied to one another. Additionally, the proportional strategies that were used by the pair was build-up, unit rate, and fraction. The pair used the build-up strategy to check if the rate of change was consistent. This was followed by the fraction strategy as the reduced to the fraction to represent per one unit. This signified the unit rate. Furthermore, the pair consistently used a formula (slope) that had not been covered in class by finding the rate of change of the y-axis in comparison to the x-axis.

Therefore, the use of contextual graph problems elicited the pair to use proportionality to make since of slope-related tasks. This is why NCTM (2000) urged that comfort with proportionality involves much more than setting two ratios as equal and solving for the missing term; there is a need to recognize quantities that are related proportionally and to use numbers, tables, graphs, and equations to think about these quantities and their inter-relationships.

The participants in this case were very successful in arriving at the appropriate solution and using math skills that they had previously acquired. The focus for this pair was on the use of the unit rate, which emphasized the expectations of the CCSSM by simplifying the slope a unit rate, whether in the $y = mx$ format or $y = mx + b$. My observation was that these students appeared to the context of the graph to establish how the unit on the y-axis varied in comparison to the x and thus to create equations in the slope-intercept-form and to demonstrate that the expression $y/x = m$ is equivalent to $y = mx$; they also appropriately added the nonzero y-intercept to the rate of change. DeLange (1996) argued that the teaching of concepts should begin with exposure to a variety of real-world problems and situations. When applied problem-solving is the primary approach to teaching mathematics in the real context, students are able to make connections because they see math as a human activity (Freudenthal, 1983).

Case 2: Pair B

Pair B consisted of two seventh-grade students enrolled in my all-boys Pre-Math 1 class, the eighth-grade math class taught to advanced seventh-graders. One was African-American (Student C) and the other Caucasian (Student D). Again, this class was the prerequisite for Math 1 and the next step in their math careers would have been a high school course. Student C described himself as smart and a lover of mathematics, his favorite part of which was "how it is related to everything in the real world." Student D also described himself as smart and rationalized his thought process for participating in the study; his favorite aspect of mathematics was "learning new ways to do things, such as different formulas to do the same thing." Also, again, these students had been identified as strong in mathematics given their placement in an advanced math class and had yet to be introduced to the concept of slope or problems related to it. This pair went through the same steps as described in Case 1.

Banana Problem

The following discussion ensued regarding the banana question (Figure 5).

Teacher: "Explain how you got your answer, 25 cents per pound."

Student D: "So, I made like a chart or a table I mean. And I saw, um, 6 would be 150 cents, 8 would be 200 cents, and 4 would be 100 and 2 is 50. So, I found out it would be like dividing it by 2."

Teacher: "How did you get to 25 cents?"

Student D: "Wait! I just thought of something; you could like take 150, divide it by 6, and get 25."

Teacher: "So, you took the price and divided it by pounds?"

Student D: “Yes!”

Teacher: “Alright sir! You are up; explain your process.”

Student C: “Okay, so I saw the 2:50 ratio, then 4:100 and then the 8:200. So, then if you’re going up by 2’s, also going down by 2’s. So, I divided 2 by 2 and 50 by 2. So, that gives you a 1:25 ratio. So, I checked it because if you take 25 divided by 1 you get 25, and 50 divided by 2 is 25 and if you keep going it always will be equal.

Teacher: “So, you also did cents divided by pounds?”

Student C: “Yes!”

Student D recognized that various strategies could be used to connect proportionality and slope. The initial strategy used was involved deconstruction of the build-up strategy with the aid of a ratio table. The student initially started building his ratio, decreasing it back to 50. However, after using this initial strategy, the students realized that they could simply have used a unit rate, so they took 150 and divided it by 6 to get $0.25 per pound. Thus, the student used the fraction strategy as well to find the equivalence to one pound. This student did not, however, limit himself to one strategy to solve the problem. Student C also used multiple strategies, including a build-up strategy, but recognized that the factor of change was “times 2.” He deconstructed and realized that if the table was increasing by this factor, the inverse of dividing by two was applicable. Yet, in the end, in order to illustrate the cost per pound, the realization that it must be per 1 illustrates knowledge of the unit rate. These students thus also used proportional strategies to solve slope-related tasks. Additionally, they connected proportionality and slope through covariation as well. The students understood that cents varied to pounds and used both units to find and describe their solution.

Car Wash Problem

When Pair B was presented with the car wash question (Figure 6), the following discussion took place.

Teacher: “How much does it cost per month to belong to the car wash club and why?”

Student C: “Forty dollars, because it starts with a $20 fee and it’s adding 40 each time. If you do 60 minus 20 it’s 40, 100 minus 60 is 40. 140 minus 100 is 40, so it’s 40.”

Teacher: “What do you think? How did you come up with your answer?”

Student D: “So, I saw like you had to pay $20 and then for the first month it is 60, so I subtracted them.”

Teacher: “So, similar to what he did?”

Student D: “Uh huh!”

Teacher: “What equation did you guys come up with?”

Student D: “$40x + 20 = y$.”

Teacher: “What does y represent for y’all?”

Student C: “Total amount paid.”

Teacher: “What does x represent to you guys?”

Student C and D (simultaneously): “The months of membership.”

Teacher: “So, in prior math classes have you worked with equations like that or you just came up with it?”

Student D: “I don’t know.”

Student C: “I don’t where I actually learned this.”

Teacher: “It’s just there, huh?”

Student C and D (simultaneously): “Yeah!”

To solve this nonzero y-intercept problem, the students used multiple proportional strategies and knew to place the graph in the equation form $y = mx + b$ format. They understood that $20 was the initial fee that was charged for the club, which is important because interpretation of the y-intercept in contextual problems is vital when it comes to creating equations. Traditionally, when the slope-intercept form is taught, b is defined as the y-intercept or the starting point. The students used prior experiences and context of the problem to make this distinction. Students are also traditionally taught to find the change in y over the change in x, but Pair B demonstrated this approach without having been introduced to the algorithm, understanding that from the initial point went up $40 and then the pattern continued. The students thus concluded that the monthly change must be $40, thereby demonstrating understanding of the units and how they varied, since the change of $40 was per month. In other words, they understood covariation, how one-unit changes in comparison to another.

Furthermore, the students were able to communicate that y represented the cost and x the number of months thus to formulate the equation $40x + 20 = y$ by using such proportional strategies as unit rate and build-up. The build-up served to confirm that the cost was $40 a month: the unit rate was employed because the students understood that this change was happening monthly. In creating their equation, however, they exhibited the factor-of-change because they understood that multiplying months by 40 and adding the initial fee described points on the graph. The strategies employed to solve this problem demonstrated the capacity to use prior experiences to create linear equations or $y = mx + b$. From the perspective of Skemp (1987), the students used their schemas to solve problems in ways that could prove beneficial in solving future problems. Also, relevant here is Freundenthal's (1987) notion of mathematization and his argument that math is more meaningful when students create and use their own tools.

Traveling Problem

Next, the problem involving traveling scenarios (Figure 7) occasioned the following discussion.

Teacher: "I heard you say Scenario 1 is faster, so explain how you know."

Student C: "So, I looked at the graph and it said the time is one hour and the distance is 60. So, if you look up to where it says 4 and 240, it follows the same ratio if it goes 4 hours at the constant speed that would be the 60mph. So, when you go to scenario 2, 55 times the hour equals the distance, so it would be 55mph."

Student C: "Also, I like tried to envision it on a graph. And it would have been at (1,55). Yeah, one on the x- axis and 55 on the y-axis. So, I envisioned it going up and it like kept following that ratio."

Teacher: "Did you solve the problem differently? Explain how you solved it."

Student D: "I didn't see how it worked like the (1, 60). So, I did like 5 and 30. So I divided like 30. I mean 300 divided by 5 and that's 60. So, it is 60mph."

Student D: "And then for Scenario 2, I did y divided by 55 equals x and then I wanted it to be one mile per hour to find out how many miles per hour it was, so I did $y/55=1$ and $55/55=1$. So, it's 55 miles per hour. And 60 is greater than 55."

Both students used the unit rate strategy in various forms to compare slopes. Each took the mileage for the graph and divided miles by hours to determine how fast the car was traveling per hour. Student C recognized that the ordered pairs illustrated the relationship of the x and y axis in the problem. Therefore, the students used the other ordered pair (4,

240) to check whether the 60-to-1 ratio was constant, thereby truly illustrating the covariation concept that intertwines proportionality and slope. Additionally, the student used the fraction strategy because the student simplified 240 over 4 to see if it was equivalent to 60-to-1. The reducing of this fraction demonstrated the unit strategy as well to determine the speed per one hour. The pair understood that one quantity was changing in comparison to the other, as well as how slope-intercept form is used to write linear functions. Student D also utilized the unit rate and fraction strategy by taking a different mileage, such as 300, and dividing it by 5 and in Scenario 2, where it was written in y = mx format, perceiving that the distance was equivalent to y and dividing it by 55 to equal 1. Given that the initial equation it was $y/55 = x$, it was understood that x was time and that, to find the speed per hour, x must equal 1. An understanding of covariation was evident because the change of distance in relationship to time was used to find the speed per hour. The students used the unit rate and fraction strategy to solve this slope-related tasks. However, the connection of proportionality to slope was once again understanding how the vertical unit (y) change in comparison to the horizonal axis (x). The contextual problems have elicited the students naturally describing the linear data in the terms of slope (change of y/ change of x).

Chair Problem

We turned next to the furniture manufacturing question (Figure 8), which occasioned the following discussion.

Teacher: “I see both of you agree that it is Benjamin [who is correct regarding the rate of change in the scenario]. Explain why you think it is Benjamin. So, justify.”

Student C: “So, I looked at it. So (10, 45) and then 50 was 40, so the difference of those is 40 chairs. The price is going down by 5 so 40 chairs divided by 5 is 8. This was also the same for 90. From 50 + 90 is 40 and 40 to 35 is 55.”

Teacher: “What did you say? Justify.”

Student D: “I saw for 40 chairs it went down like 15. So, I did 5 like over 40 and then like I was trying to simplify it to 1/8. So, they like lose $1 for every 8 chairs.”

Teacher: “So, you took the 5 and put it over 40 and saw it matched the –1/8. Well, why is it negative? 1/8 why isn’t it just 1/8?”

Student C: “It’s going down.”

Teacher: “How do you know it’s going down?”

Student D: “Cost of dollars is going down.”

Student C: “Because the cost of chairs is depreciating and the number of chairs being produced is going up.”

Teacher: “Is yours similar because he said 5/40 and you said 40/5?”

Student C: “Yes, because 40 is chairs and 5 is the amount of dollars it went down.”

For this particular nonzero y-intercept problem, multiple strategies were used, with the fraction strategy completing the problem. As noted earlier, traditionally, the slope algorithm would have been taught to solve this problem, but this second pair of students derived this strategy from their prior mathematical experiences without every seeing the formula, identifying a decrease of $5 because the y-coordinate went from $45-40. Utilizing the same strategy, they found that the x-coordinates went from 10 to 50, which illustrated a change of 40 chairs. Once again, comprehension of covariation was evident, since the students understood how two different units, dollars and chairs, varied in relation to one another and employed the fraction strategy to arrive at the -1/8. They recognized that it was negative because the dollar amount decreased. Although Student D transposed the ratios, understanding of the units allowed Pair B to comprehend the solution -1/8, thus demonstrating the unit rate strategy.

Conclusion

The students in Pair B connected proportionality to slope by consistently recognizing a vertical change in comparison to horizontal change. Therefore, they created the slope formula through contextual slope-related problems. The pair naturally found the rate of change of each unit to describe the data of the graph. They used proportional strategies such as factor-of-change, fraction, and unit to solve the problems. The pair was able to create a linear function without using a long algorithmic approach. The students created a nonzero y-intercept problem by recognizing the initial amount as the starting point. The pair then understood that the rate of change was consistent from that initial amount. Therefore, the pair observed how the unit on the y-axis varied to the unit on the x-axis. This sequence of events exemplified how they connected proportionality to slope. It also allowed them to create a linear function that described the graph. Although the students have not been exposed to the slope formula the contextual problems allowed for guided reinvention.

Furthermore, they demonstrated this knowledge by using the multiplicative properties to make sure that the graphs and tables were proportional. The importance of context in the conceptualization of slope, as described by Lobato and Sieberts (2002), was evident in that the students had no difficulty transferring knowledge between contexts. Research tends to emphasize the difficulties that students experience in making connections between the physical aspect of slope and its functional aspect as a rate of change. These students, though, used their prior proportionality to describe the data on a graph, create tables, create equations.

The students, then, invented the slope algorithm naturally, without ever being exposed to the formula and recognized the y-intercept as the essential starting point for building linear equations and functions. The ability to use their prior experiences goes a long way in their being able to transfer this knowledge to other, similar problems. Thus Freudenthal (1983) and Skemp (1987) both concluded that mathematical learning is more beneficial to students when they create or use their own strategies. Furthermore, the students demonstrated that multiple proportional strategies can be used to solve problems with nonzero y-intercepts as well as a connecting proportionality to slope is naturally finding the difference of the unit on the y-axis to the difference of the unit on the x-axis and understanding how they vary to describe the data.

Case 3: Pair C

The two students who comprised Pair C were seventh-grade girls enrolled in my Math 7 class, one of whom was African American female (Student E) and the other Caucasian (Student F). This class was the usual one for seventh-grade mathematics, the next step in their math career being Pre-Math I, the required eighth-grade class. Student E described herself as average; her favorite thing about mathematics was formulas "because you just have to substitute into the formula." Student F described herself as average because she could multiply and divide; multiplication and division were also her favorite aspects of math "as well as shapes and stuff."

This pair of students who had been identified as on grade level in that they were not placed in advanced math classes. They were aware that there were higher math classes from interactions with friends in those classes, and they expressed doubt when faced with certain types of problems. These students had yet to be introduced to the concept of slope or problems related to it. Thus, the aim was once more to assess how they solved problems in the $y = mx$ (constant proportionality) and $y = mx + b$ (nonzero y-intercept) formats, in particular their use of proportional strategies.

Banana Problem

Pair C was asked the same set of questions that Pairs A and B were asked. In response to the banana question (Figure 5), the following discussion ensued:

Teacher: "So, what do you feel is the cost per pound and why?"

Student E : "Twenty-five cents, because I did a proportion because it seems like it's easier."

Teacher: "So, tell me the proportion you used."

Student E: "We did cents over pounds. So, we did 150 over 6 equals 1 over x. To find x, we did 150 times 1 divided by 6. And then you could just do 150 divided by 6, which will give you 25 cents."

Teacher: "So, let me hear more about your proportion. The proportion you used was what over what?"

Student F: "One hundred fifty over 6."

Student E: "What I did was cents over pounds."

Teacher: "So, you did cents over pounds equals?"

Student E: “Cents over pounds, so, well, it equals pounds over x, x over pounds because you are trying to find the other side.”

Teacher: “So, what did you say the answer was again?”

Student F: “Twenty-five cents.”

The students attempted to use the cross-multiplication algorithm to solve the problem, setting up a proportion with equivalence of two ratios, which were equivalent to cents over pounds. If you look up the pair’s setup the first ratio is cents-to-pounds, but their second ratio is cents-to-pounds ratio. However, they did not use the usual cross-multiplication technique but rather implemented the fraction strategy coupled with the unit rate strategy. The pair simply took 150 and divided it by 6 which would have simplified the ratio to 25 to 1. It is in this respect noteworthy that the students multiplied 150 times 1 and divided it by 6 to determine the number that, when multiplied by 6, yields 150 (i.e., 25). Thus, they used the fraction strategy, for of 150/6 simplified is 25/1.

The use of the number one in their initial proportion demonstrates that the pair understood that the aim was to find the price per pound. Although from the dialogue it appears that the pair did not setup equivalent ratios, their work sample illustrated differently. Therefore, the pair used multiple proportional strategies to solve a slope-related problem from the origin. Despite their inappropriate cross-multiplication setup the students connected proportionality and slope by their recognition of a proportional relationship between the quantities (cents-to-pounds) and their relationship based on a linear graph. However, as Stump (1997) recognized, an understanding of covariation on the part of students and educators can strengthen their understanding of slope, the students did not fully display this connection.

Car Wash Problem

The next question, about the car wash (Figure 6), occasioned the following discussion:

Teacher: “You guys got it? So, how much does it cost per month to belong the car wash club, and how did you come up with your answer?”

Students E and F: “Uh, we got 40 dollars per month.”

Teacher: “Alright, how did you come up with that answer?”

Student E: “Because if you look at the graph, at first, we thought it was 20 because we was like looking at the problem. But if you look at the graph, it starts at like 60, and then it goes up to 100 and like between that is 40 and then it goes up to 140, then it goes up to 180 and it keeps that going on. So, it just keeps going up by 40 dollars.”

Teacher: “Did you have anything different or do you agree?”

Student F: “I agree, it’s just like 20 right here [points to the 20 on the y-axis], like we were thinking that [points to $20 in the problem], but we looked at this where it started off at 20 [points back to 20 on the y-axis], and then we just like noticed that it goes up 2, 4, 6, and it is 40 and it kept going.”

Teacher: “So, now your next task was to write an equation that describes that graph. Explain how you determined the equation that represents the graph.”

Student E: “ Um, the equation we wrote is like $N + 40 =$ monthly fee. So, we did that equation because, um, for N you can substitute anything in. So, let’s say you like have 100, like 100 plus 40 will equal up to 140. Which is like one of your monthly fees. It will be your three-month monthly fee, so that is our equation.”

Teacher: “So, what does the *N* represent to you guys?”

Student E: “The total amount paid.”

Teacher: “So, N represents the total amount?”

Student E: “Oh, wait! No!”

Teacher: “Because you said $N + 40$.”

Student F: “The 20.”

Student E: “Oh yeah, you are right.”

Student F: “Because it says $20 plus a monthly fee. The monthly fee is $40.”

Teacher: “So, it looks like you changed your equation; what is your equation now?”

Student E: “It is still the same, but the N equals 20 dollars because it starts at the 20.”

Teacher: “So, what is your equation again?”

Student F: “Twenty plus 40 equals the monthly fee.”

Student E: “For that month.”

Teacher: “For that month! So, what happens if it’s two months? How would I do it?”

Student F: “If it is 2 months, it would be. . .”

Student E: “You would go off the number you had last month.”

Student F: “That would be 60 plus 40.”

Teacher: “So, I just want to make sure. So, now you changed it to 20 plus 40?”

Student E: “Yeah!”

In solving this nonzero y-intercept problem, the students seem to have relied on the build-up strategy, using the graph and finding that the first month was 60 and then observing that the second month was 100 and the third 140 for a plus-$40 ratio. They initially miscalculated the figure to be 20 because they had only read the problem, though the graph was crucial as it assisted the students in seeing the relationship between the month and total dollar amount and in finding the change in y over the change in x. Though the students had not encountered the slope formula, they were able to build up the constant change per month after the initial $20 fee. Furthermore, the graph seems to have helped them to recognize the unit rate. Clearly, they saw that the change of $40 was monthly. The conversation also revealed that they recognized the y-intercept as the starting point, but where not able to create a linear equation in the $y = mx + b$ format. Although they intended their variable *N* to represent $20, it actually became the prior month's amount added to $40 to find each additional amount. Therefore, the students actually created more of a recursive equation because the next month equals the now month plus 40. The students used the build-up strategy to solve a nonzero y-intercept. The students observed that the increase after the initial twenty-dollar fee would consistently be $40. Thus, they connected proportionality to slope by finding that the y-axis unit changed $40 each month after the initial fee. The pair naturally created the slope formula via guided reinvention through a contextual problem. Although they did not create a linear equation in the nonzero y format of $y = mx + b,$ they still were able to connect proportionality to slope by recognizing the price was changing forty-dollars per month.

Traveling Problem

Next, Pair B tackled the question regarding traveling scenarios (7):

Teacher: "So, which one is faster then? Which has the greater speed?"

Student F: "Scenario 1."

Teacher: "Why do you feel Scenario 1 is faster? Explain."

Student E: "Because in Scenario 2 we figured out that we are going to keep the same hours so it can be like fair. So, x in scenario 2 is going to be like 55 times 5. Which is

275. Okay, we got that. But in Scenario 1, if you look and you see like, okay, you have to have the same number of hours. So, you look at 5 and you go up and see where the point stops at. And it stops at 300. Three hundred is obviously like greater than 275."

Teacher: "Could you tell me how fast it is traveling in Scenario 1?"

Student E: "So, I'd think it would be going probably like 60."

Teacher: "How did you guys come up with that?"

Student F: "Because we did 300 divided by 5 and we get 60. And 60 is right there" [points to ordered pair on the graph].

Student E: "I guess every dot is going up like 60."

Teacher: "So, what does 60 represent to you guys?"

Students E and F: "The speed."

In this case, the students used the equation from Scenario 2 to guide their thinking. The graph was again beneficial, as they observed that it showed 5 hours to be equivalent to 300 miles. Thus, they used the $y = 55x$ and substituted 5 for x to arrive at 275 miles and realized that the mileage was not greater if they traveled for the same amount of time. Therefore, the initial proportional strategy used to solve the slope-related task is factor-of-change. The pair knew the speed was 55 miles per hour, so they multiplied 55 by 5. They initially connected proportionality to slope by understanding that a point on the line represented 300 miles in 5 hours. Therefore, they recognized that miles units were in direct comparison to the hours unit.

However, I delved a little deeper to see whether they could attain the actual rate of speed for Scenario 1. The students took 300 and divided it by 5 to obtain 60 miles per hour, but they also discovered the meaning of ordered pairs from the finding the unit rate, understanding that the ordered pair (1, 60) actually represented the point on the graph. Therefore, at the end of the questioning the students implemented the fraction strategy as well as the unit strategy because they simplified the ratio to represent one hour. Furthermore, they were not aware of what the ordered pair represented until they unitized, but again as Skemp (1987) predicted they added it to their schema for another mathematizing moment (Freudenthal, 1983) as they were able to create meaning from a contextual problem.

Chair Problem

Moving on to the chair factory problem (Figure 8), we had the following discussion:

Teacher: “Okay, are you ready? So, who was correct [regarding the rate of change in the scenario] and why?”

Student F: “We think it is Benjamin.”

Teacher: “Why do you think it is Benjamin?”

Student F: “Because it says negative 1/8 and like Charity’s doesn’t really make sense because it says 8 dollars and his makes sense because his says 8 chairs and then it decreases by 1 dollar. So, we just thought that like his would make more sense than Charity.”

Teacher: “So, you are just saying his sounds like it makes more sense. You did not use any math or the graph to help make that assertion.”

Student E: “Because like if you look at the graph it says, um, so 8 chairs produced causes a decrease of 1 dollar. So, the first is 10 and it decreases by a dollar. So, it would decrease and go like down to 45.”

Student F: “Wait, wait! Can we redo it?”

Student E: “Yeah!”

Teacher: “Well, can I ask a question? Why do you think it says negative 1/8?”

Student E: “Because the graph is going down and it is taking away from the cost.”

Teacher: “So, you guys still agree it is Benjamin, but you are having a hard time articulating why it is Benjamin?”

Students E and F: “Yeah!”

Teacher: “Okay, just give me your final synopsis again on why it is Benjamin.”

Student F: “We thought it was Benjamin because it says it decreases by negative 1/8. And like his says like 8 chairs produced caused by a decrease of 1 dollar. So, I mean we just thought like . . .”

Student E: “It’s like the way he puts his words kind of like matches up with the graph if you think about it.”

Teacher: “Okay! Expand on that. How does it match up with the graph?”

Student E: “Because it goes down like a 1 dollar. Okay, so like if we produced 10 chairs and it goes down by 8 so it would closest to 10, right? So, it would just equal back up to 40, I mean not 40. It would just like equal. I can’t put it in words, it just equals back up.”

Teacher: “Okay!”

For this particular nonzero y-intercept problem, the students had difficulty conveying why they selected the answer. Although the session ran slightly over 45 minutes, the students were still actively engaged. The graph seems to have played a significant part in their thinking as they recognized that the starting amount was $45 and that it decreased to

$40 as stated in the problem. The students even drew attention to the 10 chairs, but this time they did not apply the second point at 50 to see that it had changed 40; then they could have reduced the solution to 1/8. They did demonstrate understanding the negative expression, -1/8, specifically that the decreasing graph signified the negative. They also stated that Benjamin's solution simply made more sense. It thus appeared to me that, as they continued to look at the graph, they perceived that the units of dollars were decreasing while the number of chairs was increasing, given their assertion that Charity's solution did not make since the graph did not display a decrease of $8. Thus, the pair connected proportionality to slope by being able to differentiate that a loss of eight dollars was to great. Therefore, they still demonstrated an understanding how the units on each axis varied in relationship to one another. Although they did not use their invented slope formula this time. This still understood that the vertical change was five dollars and that Chasity's eight-dollar change did not make since. They were unable to articulate a complete proportional strategy for this nonzero y-intercept problem.

Furthermore, the students' articulation of their answer was influenced by the fact that the unit rate was not in the denominator; they unitized the other graphs when the denominator was one. They also used intuition in solidifying their mathematical assertions. Skemp (1987) discussed the intuitive stage as being largely dependent on the way in which the material is presented to students. In this case, the material was presented in graph format, and the students throughout the study exhibited knowledge of how one unit changed in comparison to the other. Thus, they seem to have understood intuitively that Benjamin's solution made more sense, but the fact that the unit rate was in the numerator complicated their articulation of why this was so.

Conclusion

The students demonstrated that they recognized a connection between proportionality and slope by using the graph to discuss how one unit was changing in relation to another. The graphs allowed the students to see how the units varied and thus to explain how they solved the problems. However, complications occurred when the unit rate was not located in the denominator. The students appeared to use their intuition to discard solutions that did not seem reasonable. Another theme that became apparent was the significance of the context of the problems in the students' discussions of their solutions in terms of the change of y over the change of x. This is an important concept in slope because it is conveyed in the slope formula that is presented to them. Nevertheless, the contextual problems allowed students to utilize this format. Furthermore, the students deployed a variety of proportional strategies to solve slope-related tasks, including the unit rate or fraction strategy to simplify the solutions at the end of the problem, while they turned to the build-up strategy to initiate the problem-solving process. Thus, they built up the numbers on the y-axis to make sure that the change was consistent; and this move in turn demonstrated knowledge of the x-axis increasing by 1, as was exemplified in the carwash problem. This approach can be described as "Piagetian," for the well-known educator Jean Piaget (1958) discussed how adolescents' proportional reasoning develops from global compensatory strategies that are often additive in the formulation of thoughts. Thus, the students in the car wash question added up 40 each time, recognizing that the monthly change must be 40.

The students in Pair C also drew on their prior experiences, inventing the slope algorithm naturally without ever being exposed to the formula. They similarly demonstrated knowledge that the y-intercept is the essential starting point for building linear equations and functions and recognized that multiple proportional strategies could be used to connect proportionality as well as to solve problems with nonzero y-intercepts.

Case 4: Pair D

Pair D consisted of two seventh-grade girls in my Math 7 class, one of whom was Caucasian (Student G) and the other African-American (Student H). Again, this class was the standard for seventh-grade mathematics, with the next step in their math career being Pre-Math I, the required eighth-grade class. Students G and H did not see themselves as very strong math students. Both said that what they enjoyed most about math class was the opportunity to work in groups because it allowed them to bounce ideas off of one another and to check solutions. These students were on grade level, as they were not placed in advanced math classes; Math 7 is the class taught to on-grade-level or below-grade-level students. Like the previous pair, they knew that they were not in higher math classes from interactions with friends in those class and expressed doubt when faced with certain types of problems, and they had yet to be introduced to the concept of slope or problems related to it.

Banana Problem

Pair D was presented with the same questions presented to the previous pairs. The banana question (Figure 5) prompted the following discussion:

Teacher: "Why do you feel your answer is 25 cents per pound?"

Student G: "In every other point, yeah, so the pound, 0 pounds, 0 cents, 2 pounds, 50 cents, 4 pounds, 100 cents, 6 pounds, 150 cents. So, if you fill in all those gaps, it'd 0, 1, 2, 3, 4, 5, 6, 7, 8. And then if we filled in the gaps for the price, it would be going up by

25 cents. It would be 0 cents to 25 cents, 25 cents to 50 cents, 50 cents to 75 cents, 75 cents to 100 cents, and on and on. So, 50 divided by 2 is 25. We just went up by 25 each time."

Teacher: "Do you agree, Student H?"

Student H: "Yes!"

Teacher: "And what does filling in the gaps mean again?"

Student G: "Every other point is blank. So, if you filled those in you can see it is going up by 2. So, you just have to fill in the line 1, 3, 5, 7." [They did so on the x-axis.]

Teacher: "And what did you say you did with the 50 cents?"

Student G: "So, with the 50 cents we divided it in half, 50 divided by 2 is 25. And so, we just did 25, 50, 75, 100, 125, 150, and 175." [They did so on the y-axis].

The students used the graph to identify all of the points highlighted on the graph. Next, they filled in the gaps, which they referred to as the missing numbers. Thus, they seem to have been constructing a type of ratio table with blanks to be filled in. The students recognized that the x-axis was increasing by 2 units and the y-axis by 50 and that the numbers in between would be the next in the procession, ascending by 1 on the x-axis and 25 on the y-axis. However, this technique was only used to confirm the solution; the students knew to take the 50 cents and divide it by 2, that is, to unitize the rate to 1 pound. This is the technique that the CCSSM suggested students use to understand slope, building on their prior work with unit rates to identify slope in graphs. The students exemplified the strategy because their division by 2 signified a unit rate. Furthermore, to reiterate that a unit rate was utilized they filled in the gaps by ascending by 1. Thus, they used unit rate strategy and build-up strategy to solve slope-related tasks. However, they connected proportionality to slope by finding rate of change to describe the graph. The students understood that y-axis was increasing by fifty-cents for every two pounds of bananas. Instead of initially just dividing fifty cents by two pounds they filled in the values in between by ones. This was built of the initial value of fifty cents because that divided by two pounds yields twenty-five cents. Therefore, they understood that graph illustrated twenty-five cents increase for every pound.

Car Wash Problem

Pair D next tackled the car wash problem (Figure 6), after which the following discussion took place:

Teacher: “How much does it cost per month to belong to the car wash club?”

Students G and H: “Sixty dollars.”

Teacher: “Okay, and how did you come up with 60?”

Student G: “Because on the line they have the months of memberships on the x-axis and on the *y*-axis it is the total amount of dollars. And if you line it up with the 1, it would be 60 dollars.”

Teacher: “Okay, but what does that 60 represent on the graph? What does that point on the graph represent to you?”

Student G: “The cost per month.”

Teacher: “Okay! Well, how much does it cost for 2 months?”

Students G and H: “One hundred dollars.”

Teacher: “Okay, I was just curious because you said it was 60 dollars per month. Student H, what are you saying?”

Student H: “Well, I thought it would be 40 because you go 20, 30, 40, 50, 60 and that’s a point and you could keep going up 40. So, I thought that would be the cost per month.”

Student G: “What do you mean?”

Student H: “So, like if you keep going. So, if you add 40 to 20 that would give you 60 and that would be the cost for 1 month. Then if you add 40 to 60 that would be the cost for the 2nd month, that would be100.”

Student G: “Oh yeah!”

Teacher: “So, are you guys changing your answer?”

Student G and H: “Yes!”

Teacher: “What made you change your answer?”

Student H: “Because if you do 60 plus 60, you will not get 100 which is the cost of the second month. So, if you did 60 plus 40, it will give you 100 which is the cost of the second month. So, if you keep going you get the cost of the months.”

Student G: “So, how the first point starts at 20, if you add 40 it would be 60. If you keep going it will add up so we are saying that it is 40 dollars a month.”

Teacher: “So, what equation did you guys come up with to describe the graph?”

Student G: “x plus 40 equals the cost. Where x represents the months.”

Teacher: “So, x represents the months?”

Students G and H: “Yeah!”

Teacher: “How would that work if we go to the second month?”

Student G: “So, if you do 1 month and then add 40 to it, that would be the cost for the month. And if you did the second month that would be 80 because 40 plus 40, not including the local car wash.”

Teacher: "So, x represents not the number of months?"

Students G and H: "No, the number of months!"

Teacher: "So, if I have 40, I wouldn't do for the second month 2 plus 40, since you said x represents the number of months."

Student G: "Oh yeah, that doesn't make sense. So, x plus 40 equals the cost, so x is the previous month's total cost plus 40 equals your coming month's cost."

Teacher: "And how did you determine that equation?"

Student G: "If you look at the graph, because if you look at the total for 0 months it would be 20. If you add 40 to the previous month it would be 60. If you go on, it will continue to equal."

To solve this nonzero y-intercept problem, the students used multiple proportional strategies. The graph assisted students in implementing their proportional strategies. They used a build-up strategy again to help come up with their equation and to determine how much it cost per month to be part of the club. The students recognized that the first point was 20 or in the eyes of the slope-intercept from the y-intercept and then realized that it increased by 40 that from the initial amount to the first month and that this was the constant change per month. Once again, the students exhibited covariation, recognizing that the monthly increase was $40.

The creation of their equation in $y = mx + b$ format was thus $x + 40$ = cost, with x representing the previous month's total cost. The students determined that it could not be months, as they initially stated, because it would not yield the solution or point on the graph. This is important because it means that they recognized that the meaning of the points on the graph are directly related to their equation. However, they initially thought was of a unit rate is whatever amount corresponds with an *x*-value of one. Of course, this true if the line originates from the origin but not in a nonzero *y*-intercept. Although their equation did not conform to the traditional slope-intercept format, it worked for them: they were able to mathematize an equation that illustrated the points. In addition to a build-up approach, they also incorporated unit rate, understanding that the $40 was the monthly change.

The students thus perceived that the initial fee for the club was $20, which is significant because interpretation of the y-intercept in contextual problems is vital in creating equations. Traditionally, when the slope-intercept form is taught, b is identified as the y-intercept or starting point. The context of the problem seems to have allowed Pair D to demonstrate the problem-solving attributes, finding the change in y over the change in *x*, like the other pairs, without ever being introduced to the algorithm. The students understood that, from the initial point, there was an increase to 40 and that the pattern continued, so the monthly change must be 40. Their understanding of the units and how they vary exemplified covariation, as one unit changed in relation to another. Thus, they used proportional strategies, such as the unit rate, because they recognized that the change was happening monthly. In by creating their equation, they displayed more of a recursive approach, perceiving that the amount of each previous month must be added to $40.

The pair used a build-up strategy coupled with unit rate to solve a nonzero *y*-intercept slope-related problem. The students recognized that a fee of twenty-dollars is charged first. Then from that initial twenty they continued to build up to ensure the amount was consistent. This when the students recognized that each month increased by forty-dollars from the initial point. However, this notion of forty-dollars was initially problematic because they applied slope from the origin thinking. This meant they assumed you would multiply the forty by the number of months to find the amount for each month. This led them to creating more of a recursive equation when they found out it did not yield the correct answer on the graph. However, the students connected proportionality to slope by naturally creating a slope formula. The pair found the constant change of unit on the *y*-axis and compared it to the one-unit change of the *x*. This further exemplified a connection of proportionality to slope because the students understood how the units varied amongst each other (covariation).

Traveling Problem

The traveling scenarios question (Figure 7) also occasioned a lively discussion:

Student G: “Is the number 2 supposed to be on the 100 line [referring to Scenario 1]?”

Teacher: “There are two different scenarios, Scenario 1 and Scenario 2. And aren’t you supposed to find which represents the greater speed? Is that what the question is asking?”

Students G and H: “Yeah!”

Teacher: “So, it says compare the scenarios to determine which represents the greater speed; Scenario 1 or Scenario 2. Explain your choice including a written description of each scenario.”

Student G: “So, there are two different scenarios.”

Teacher: “Exactly! Different scenarios, which one is faster? Which one has the greater speed?”

Student H: “Scenario 1 would be a greater speed because it is 4,240 miles and Scenario 2 is 2750 miles.”

Teacher: “Where did you get 4,240 from?”

Student G: “From the top [pointing to an ordered pair in Scenario 1].”

Teacher: “Okay, so how did you determine Scenario 1 was faster?”

Student G: “Because right here in parentheses, it says 4,240 [pointing to the ordered pair at the top of Scenario 1], which is the distance in miles.”

Teacher: “So you are saying that the information in parentheses represents the total distance in miles?”

Student H: “Yes!”

Teacher: “Okay, how did you come up with your answer for Scenario 2?”

Student H: “We did 55 times 50, which is 2750.”

Teacher: “Why did you guys do 55 times 50? Where are you getting 50 from in Scenario 1?”

Student G: “The dot, the first point. Isn’t the dot on the line at 50?”

Teacher: “Oh, I can’t tell you. You have two have to determine what that dot means. So, that number next to the dot means nothing to you [referring to the point at ordered pair (1, 60)]. I’m just asking a question. Do you want to move on?”

Student H: “Yeah!”

Teacher: “You don’t have to! Did you come up with something else? Did you change your answer?”

Student G: “Well, I think, since y equals 55 hours. . . .”

Teacher: “So, can I ask you why you are saying that y equals 55 hours?”

Student G: "Because of 55*x*, or is it 55 times the amount of hours. We are trying to figure out that. Because we are trying to figure out if it is distance equals 55 times the amount of hours or if it's just 55 hours or distance equals 55 hours."

Student H: "I think it would be distance equals 55 hours." . . .

Student G: "I think Scenario 2 is greater."

Teacher: "So, you guys are switching your answer?"

Student H: "Yes!"

Teacher: "So, why are you switching your answer? What made you now choose Scenario 2?

Student G: "Because the first point is between 0 and 100, which would be 50. So, then y equals 55*x* and *x* is time of hours, so it is 55 hours. So, I think it is Scenario 2 is greater because the time is 55 hours and Scenario 1 is 50 hours. So, I think it is Scenario 2."

Teacher: "Do you agree with her answer change?"

Student H: "Yes, I think Scenario 2 would be greater."

Teacher: "So, why are you saying Scenario 2 is greater now? I just want to make sure because you changed your answer."

Student G: "Because if y equals 55*x*, and *x* is the time of hours, that would be 55 hours and then the first point is between 0 and 100. So, you have the first point is 50. So, 55 is greater than 50, so it's Scenario 2."

The erroneous strategy that the students used appears to have been related to the context of the problem. After the pair had already connected proportionality to slope through the context of graphs, they still struggled to identify the exact point on the graph. They asked if the 2 was on the 100 because they had already discussed the 1 being located at 50. Thus, while ordered pairs would have been introduced to the students through the CCSSM curriculum, their schematic understanding was missing. Traditionally, ordered pairs are taught as letters associated by points on a coordinate plane, which means without context. However, the other graph problems that the students solved featured points that were clearly on the line, and the points were easily identifiable for the pair, while this time the ordered pairs were given to the students. As a result, the students did not utilize previous connections of proportionality to slope-related problems such as stating how one-unit change in comparison to the other.

They thus seem to have neglected their build-up strategy and application of the unit rate because they assumed that this had something to do with the solution; their initial reasoning for selecting Scenario 1 was that it represented 4,240 miles, and they ascertained 2,750 miles from Scenario 2. The pair apparently used 50 to multiply by 55 because they concluded that Scenario 1 was increasing by 50 and used the same number but changed their solution to Scenario 2 because 55 is greater than 50. The students used the scale of the graph to conclude that 50 is between 0 and 100 and that the point must be lying on 50. The misconception of the ordered paired numbers, as I see it, led them to neglect their traditional proportional strategies and observation of the meaning of the units.

It may appear that the comparison of a graph with an equation caused them problems, but, when the students were faced with another problem, one involving calculating the price of gum using a table and an equation (Figure 9), they were able to create a table that matched the coordinates of the graph and create an analogous equation. Furthermore, the pair communicated that all three views represented the same data.

Again, when the students mathematized and created their own equations and tables, the experience was more meaningful to them than it would have been if they had been provided with these materials.

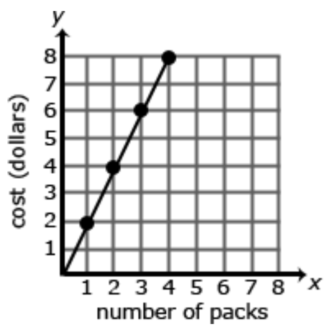


Figure 9. Gum question; the graph represents the cost of gum as a unit rate of $2 dollars per pack.

Chair Problem

Pair D also tackled the question about the chair-manufacturing company (Figure 8), leading to the following discussion:

Teacher: "Did you guys decide on who's right? Because I want to be respectful of your time."

Student G: "We think the ratio would be 5 to 40. So, we think it would be Charity."

Teacher: "What made you say Charity?"

Student H: "So, we did 40 divided by 5, which gave us 8."

Teacher: "What does 40 represent?"

Students G and H: "Chairs."

Teacher: "And what does 5 represent?"

Students G and H: "The cost."

Student G: "No wait, we think Benjamin is correct. Because when you do 5 divided by 40 it gives you 0.125. Then when you do 0.125 times 8 that gives you 1. Where 1 would be the cost and 8 the number of chairs. So, we think Benjamin is correct."

Teacher: "And why did you do the ratio 5/40?"

Student G: "Because the cost is going down by 5 dollars while the chairs are going up by 40 chairs."

Teacher: "And why do you think it says negative 1/8 instead of positive if you are looking at the graph?"

Student G and H: "Because it is going down."

These students used multiple strategies to approach this nonzero *y*-intercept problem, including the fraction strategy and factor-of-change. They initially thought the solution was the one proposed by the character of Charity in the problem, but they soon recognized the units of the *x*-axis and *y*-axis and favored the solution proposed by the character of Benjamin. Once again, the students employed the traditional strategy based on their prior mathematical experiences without every seeing the algorithmic formula. They determined a decrease of $5 because the *y*-coordinate went from 45 to 40 and, utilizing the same strategy, found that the *x*-coordinates went from 10 to 50, which meant a change of 40 chairs for a ratio of 5/40. This is when they recognized their mistake: the five represented the number of dollars and the 40 represented chairs. However, they simplified their fraction to a decimal of 0.125 multiplied it by 8 to see whether this yielded an answer of 1.

The factor-of-change approach demonstrated that the students had some understanding of the functional approach. Nunes and Bryant (1996) observed that slope is intended to be seen as a functional relationship so that the students gain an explicit understanding of the fixed multiplicative relationship between the two measures. Students G and H displayed this kind of thinking when they multiplied 0.125 by 8. Furthermore, the graph played a crucial role in helping the students to see why the slope was negative once they perceived that a descending line means a negative slope.

Conclusion

The students in Pair D made a connection between the proportionality and slope-related problems by naturally creating the slope formula. The pair found the change of the unit of *y*-axis as it related to the change of the unit of the *x*-axis. This is how slope is naturally taught for linear representations. Yet, the pair was able to create this formula through the context of the problems. Additionally, the pair described the vertical change of the graph in comparison to the horizontal change of the graph. This is also traditionally how slope is taught. Therefore, the pair made traditional connections to slope base of the proportional strategies they used to solve the problems. Thus, they were able to use the factor-of-change strategy to create their own linear equations and the multiplicative properties to ensure that graphs and tables were proportional. The importance of context in the conceptualization of slope, as noted by Lobato and Sieberts (2002), was once more on display as the students easily transferred knowledge between contexts. Research has shown that students tend to have difficulties in making connections between the physical aspect of slope and its functional aspect as a rate of change. These students, though, used their prior understanding of proportionality to describe the data on a graph and to create tables and equations.

Pair D used multiple proportional strategies to solve slope-related problems. Pair D used the build-up strategy to check that the change in the unit was consistent and recognized that the graph varied by one, so they used the unit rate more as the ending strategy to solve for the nonzero *y*-intercepts. The students employed the slope algorithm naturally, without ever having been exposed to the formula. They also demonstrated knowledge of the *y*-intercept being a starting point, which is essential in building linear equations and functions. However, when an ordered pair was involved, Pair D did not use the strategies that had been successful in solving the other problems, though they connected proportionality to slope by employing proportional strategies to solve slope-related tasks. These students' approaches to these problems once more reinforced the arguments of Freudenthal (1983) and Skemp (1987) regarding the benefit to students when they create and use their own strategies. Furthermore, these students demonstrated the ability to use multiple proportional strategies to connect proportionality and to solve problems with nonzero y-intercepts.

Summary

In this chapter, individual case study reports for four pairs of students (A, B, C, and D) were presented. These reports included the discussions in the course of which each pair solved slope-related contextual graph problems in $y = mx$ form (origin) and $y = mx + b$ (nonzero y-intercept) forms. The main theme that emerged from observation of Pair A was these students' predominant use of the unit rate for graphs originating from the origin as well as for those with nonzero *y*-intercepts. Pair A also employed a form of the build-up strategy to ensure that each unit was consistent, demonstrated knowledge of covariation (i.e., they were able to discuss how one unit varied in relation to another), and recognized that the variable b represented the initial value when the graph did not show constant proportionality. Pair B also made use of the unit rate in solving the graph problems and the fraction strategy for simplification, recognizing that the unit on the *x*-axis increased by 1. These students further demonstrated that the b in the $y = mx + b$ equation represented the starting point and that the ratio started from that point and proceeded to the next. The main theme that emerged from observation of Pair C was the predominant use of the build-up strategy in assessing how the units varied on the graph. When the change on the *x*-axis was other than 1, however, these students found it harder to explain their solution. They were unfamiliar with ordered pairs until discussion of one of the problems led to recognition of this kind of representation. Pair D also employed the build-up strategy in identifying how one unit changed in relation to another and had difficulty translating graphs when the point was not aligned perfectly with the numbers on the *x* and *y* axes.

In Chapter 5, these main themes are discussed along with others that emerged in the context of the math classes in which the students were enrolled at the time of the study. This concluding chapter wraps up with an assessment of the implications of the findings presented here for mathematics education and of areas for future research, as well as a personal reflection.

CHAPTER FIVE: CONCLUSION

The research discussed here focused on the manner in which students recognize the connection between proportionality and slope because educators, and their curricula, treat these mathematical concepts as distinct from one another. The students who participated in this study, however, solved slope-related tasks in part by consulting their "math rolodexes" and employing proportional problem-solving strategies. In the cross-case analysis, I compared and contrasted the mathematical strategies the four pairs in the study used to formulate and solve the highlighted problems. Four contextual graph problems served to answer the following research questions, which were presented in Chapter 1 and are reproduced again here for the sake of completeness and convenience:

1. What proportional reasoning strategies do the participants in the study (i.e., the pairs of students) use when solving slope-related questions that pertain to nonzero *y*-intercepts?
2. How do participants in the study connect proportionality with slope?

The characteristics of the problems identified or described the specific mathematical issues in a general context, an approach that allowed the students to schematize, formulate, and visualize them in various ways. They had the freedom to recognize relationships, regularities, and the isomorphic aspects of the various problems as they translated real-world situations into mathematical ones and vice versa. This freedom proved beneficial in connecting proportionality and slope.

Consultation of the literature relating to proportionality and linear functions facilitated and enriched the cross-analysis of the data. The concept of slope has multiple facets, including the rise/run ratio, rate of change, and change in y over change in x, and it plays a key role when students encounter the slope-intercept form as a means to plot linear functions and seek to describe the relationship between two magnitude values. At the time of the study, the participants had not received instruction on constant proportionality or slope, so they were not influenced by the traditional approaches to solving these types of problems.

The cross-analysis drew on Freudenthal's (1977) idea that in order for school mathematics to be of value to the students they should learn mathematics by developing and applying mathematical concepts and tools in contextual daily-life problems that make sense to them. It also drew on the argument of Skemp (1987) that relational learning of mathematical concepts takes longer than does simply learning the rules and that instrumental schemas have limited adaptability because any rules that are learned amount to ways of manipulating symbols—so that students make connections between symbols rather than connecting symbols to concepts. Application of prior learning to substantially new situations requires the formation of conceptual connections, which is to say of relational schemas (Skemp, 1987).

Cross-Case Analysis

The aim of this study was to determine which proportional strategies seventh-grade students used to solve slope-related problems with a nonzero y-intercept and how they connected proportionality to slope. Although Chapter 4 discussed the findings as individual case studies, this section highlights the commonalities among the case studies and addresses how they contrast with similarly focused studies.

Connecting Proportionality to Slope

A theme that evolved from each pair in the study was the natural creation of a slope formula from the contextual graph problems. Traditionally, the students receive the algorithm $\frac{y2 - y1}{\text{x2} - \text{x1}}$ and then sequence of ordered pairs to solve the problem. However, the pairs in the study created the same formula by taking the vertical change from the problems and then comparing it to the horizontal change. Additionally, the description and verbiage of their ratios related to slope-related tasks consistently exuded the higher mathematical notation of $\frac{\Delta y}{\Delta x}$. Yet, if one were to ask the students what that notation meant, there would be silence.

It is evident that the contextual problems in the study elicited a guided reinvention. RME's guided reinvention describes mathematics as a human activity and associated the principle of learning mathematics as a reinvention process. The students should have the opportunity to reinvent mathematics under the guidance of an adult (Freudenthal, 2013). The problems were selected not only to build upon students' prior mathematical experiences, but also to show a connection to slope. The students' ability to solve the problems relates to Freudenthal's (2013) retrospective learning, which means recalling old learning matter whenever it is necessary. Furthermore, the reasoning of the students illustrates the similarities between proportionality and slope.

Further, the students' creation of their own slope formula supports Lobato and Thanheiser's (2002) disagreement with some educators who believed that the teaching of slope should not go beyond the slope formula. However, Lobato and Thanheiser (2002) disagreed with this rhetoric and believed this approach is only useful in solving textbook problems and cannot be easily applied to real-world situations involving rates of change.

This type of rhetoric is incited by studies such as Larson et al. (2004) because their study involved traditional slope problems found in the traditional beginning algebra curriculum.

Thus, traditional slope problems only allow educators to identify if students are procedurally proficient or understand the algorithm. Algorithms create the fundamental antinomy within didactics of mathematics: insight versus drill (Freudenthal, 2013), but mathematization is the remedy. Rather than teaching rules, mathematization allows students to build on prior mathematical experiences. Yet, as math educators we must use theory that objects to drill because it endangers retention of insight. Retention is fostered when students are able to create and use prior strategies to solve problems (Skemp, 1987 and Freudenthal, 1983). The students in the study contradicted the notion of slope being regulated to just the slope formula. The students exemplified this notion because they were able to apply their formula to real-world rate of change problems. This is why as mathematics educators

Further, the students' use of contextual problems to evoke a connection between proportionality and slope support Cheng's (2010) study, which concluded that one component of proportional reasoning essential to the transformation of algebra is an understanding of slope in a range of contextual situations, examples including such linear relationships as those between distance and time, cost and the number of items bought, and hours worked and pay. The problems in the study had units that the students could easily identify and associate how they changed amongst one another by the questioning.

The use of contextual graphs also allowed students to view the relationship between the varying units on the different axes and create equations more easily. This undergirds Eisenberg's (1992) assertion that students failed to consider the graphical approach a valid solution strategy because of the heavy reliance on algebraic methods. The norm is to translate equations to a graph. However, graphs play an intricate role in the study of linear functions. Freudenthal (1983) argued that the decisive feature of the graph over a table and formula was its visualizing power. The students demonstrated this attribute in the study. The students connected proportionality to slope through recognition of a relationship between the units on each axis.

This relationship between the units on each axis further supported findings by Carlson et al. (2010) and Lobato et al. (2002), who discussed the importance of students' ability to coordinate covarying quantities on their understanding of slope. The findings of the present study support this notion. The students in the study were unfamiliar with the term "slope." However, they displayed an understanding of covariation, which was another way students' connected proportionality to slope. The graph allowed the students to understand how one unit changed in comparison to the other.

Lobato et al. (2002) warned of students who misinterpret slope as a difference rather than as a ratio. The students in the study were not regulated to this pitfall. The students demonstrated knowledge when they discussed graphs as twenty-five cents per pound. Thus, the findings of this study support the notion that contextual graph problems assisted students in understanding that covariation is ratio and if mathematics educators in the future utilize them, it will allow students to generate their own lasting connections. Freudenthal (2013) warned that rules learned by imposition never had a real chance to develop common sense of a higher order because a set of algorithms is worthless if one does not understand how and why it works.

Dugdale (1993) noted that there are certain qualitative aspects of a graph that algebraic equations cannot illustrate. The students in the present study were able to make observable notions that depicted why the slope was negative, supporting Dugdale's claim. The students saw that the graph was descending and associated that there was a negative relationship between the two quantities. Additionally, they were able to solve slope-related tasks whether they derived from the origin or a nonzero *y*-intercept. The graph easily allowed them to create sensible equations for the pairs. The slope is the basis and anchor of linear functions and the format associated with linear functions is slope-intercept form. The meaning of the linear function is denoted by one's understanding of how the *y*-value varies in comparison to the *x*-value. Additionally, the nonzero *y*-intercept states the initial value of the function as well as represent the value when *x* is zero.

The findings of the present study contradict those of Zaslavsky, Sela, and Leron (2002), who found that students who view slope quantitatively (as a ratio) tend to resort to algebraic representations of slope for its calculation, thus perceiving slope as a formula. The students in the present study were not aware of the term slope or a slope formula. However, they were able to create and understand the meaning of the ratio described by the graph because, when students had the opportunity to create their own meaning, it resonated with them in this study. Unlike previous studies that addressed students' procedural understanding of slope, the students in the present study used schematic learning, which Skemp (1987) described as twice as efficient as rote learning in promoting the retention of material knowledge. This schematic learning explains why the students in the study were able to use the same proportional strategies consistently in solving the contextual problems.

Proportional Strategies Used to Solve Nonzero *Y*-intercept Slope-related Tasks

Another theme that evolved from all groups was the use of proportional strategies to solve nonzero y-intercept problems. All pairs recognized a starting point, which would coincide with the y-intercept and then applied a build-up strategy to check if the change from the initial points was consistent. The pairs then associated this change to the one unit it changed over the horizontal axis. The unit rate, coupled with the fraction strategy, played a major role in their problem-solving with nonzero y-intercepts. The students used the context of the graph and easily depicted how the different units varied with one another. The students thinking and creating their own mathematical formulas contradicts the need to give them ready-made definitions, rules and algorithms. RME would suggest that the students are mathematizing as well as reinventing. De Lange (1996) felt in order for students to have the opportunity to reinvent mathematics they should first be exposed to a variety of real-world problems and situations. This is because when we look at traditional slope problems it becomes problematic to students because they are trying to memorize a formula that has no meaning to them.

The practice of unitizing is supported by Lawton (1993), who concluded that students may benefit from a focus on quantitative relationships between the units of each object in a problem. Lawton also asserted that the unit approach could lead to an intuitive understanding of proportional reasoning in most students. This unit approach is the premise behind slope when textbooks and teachers express change in y over change in x. This being the case, the introduction of students to proportional reasoning in this manner could prepare them to make connections to slope in the future. Further, according to the NCTM (2000) and as noted earlier, facility with proportionality involves much more than simply being able to set two ratios as equal and solve for the missing term; it also involves recognizing quantities that are related proportionally and using numbers, tables, graphs, and equations to think about them and their relationships.

Kaput and West (1994) and Tournaire and Paulo's (1985) works described the build-up strategy as a dominant proportional strategy used during childhood and adolescence that enables students to solve ratio problems without recognizing the multiplicative relationship inherent in a proportion. This may be applicable to traditional proportional problems, but I disagree with this rudimentary description when it is applicable to linear graphs. The pairs in study used the build-up strategy as means to check the consistency of the vertical change in comparison to the horizontal change on the graph. Once the students recognized the change was constant in both units, they did employ proportional strategies such as the unit rate or fraction strategy.

The reduction (fraction strategy) of the covarying units demonstrated that the students understood the multiplicative nature of the proportion. They were able to check create linear equations and compare values through this multiplication. Therefore, it is imperative as mathematics educators that we do not diminish students' problem-solving process. The aim of horizontal mathematization is for the learner to start with contextual problems that evoke students to describe situations and find solutions using his or her own language and symbols. This type of mathematization is the type of learning that assist students in retaining knowledge. Further, Misailidou and Williams (2003) described additive structures as a stepping stone to multiplicative structures. The students in the present study displayed this idea as they used additive structures as a means to simplify or create a unit rate when necessitated.

The findings of this study had similarities to those of Bishop (2000) and Lobato et al. (2003), who also found that students often notice constant differences or recursive numeric patterns in linear contexts, but do not tend to see the constant proportion. One of the differences between the pairs in Pre-Math and Math 7 was the creation of a linear function. The students in the Pre-Math class were able to create linear functions from nonzero y-intercepts. They understood multiplicative nature as well as the representation of the nonzero y-intercept. The students recognized that proportionality began at that particular starting point represented on the y-axis. Therefore, if you made an equation you would have to multiply and then add the amount the proportionality started at. However, the students in Math 7 communicated a starting point, but their equation was recursive in nature. This recursive nature was illustrated by the Math 7 pairs describing their equation as adding the prior month to the constant proportion they found from the initial point. Nevertheless, the students were able to describe the meaning of this initial point.

Although the Math 7 students used a recursive type equation, it still yielded the same answer as the linear equation. However, Carraher et al. (2008) alluded to the notion that the didactical history of early algebra lessons introduced letters to represent any number. Despite the Math 7 students manifesting that their variable stood for the nonzero y-intercept of twenty it represented the prior months' value. This prior month representation is because in the recursive equation the constant proportionality occurs as an increment in the repeating condition (Carraher et al., 2008). Carraher et al., (2008) also referred to the term as a recursive sequence, because students treat the *y*-value as a set-in which order matter. The students identified recursion as "[continuing to add] the constant proportionality from the nonzero *y*-intercept" as the principle generates the successive vertical change values (Carraher et al. 2008, p.7). If the horizontal change (independent variable) always increases by one, it can be treated as an indicator of position in the sequence or position in narrative's time dimension (Carraher et al., 2008).

Therefore, the iterative variant had the students start at the nonzero *y*-intercept initially and build their way up until they reached the desired *y*-value that corresponded to the *x*-value. However, the students were able to simplify the process. For example, if they were looking for the cost of the third month, they took the second month's value and added it to the constant proportionality found from the nonzero y-intercept. However, a pitfall with the recursion equation is that it becomes tedious when the variable becomes a large number. Additionally, Martinez and Brizuela (2006) found that this approach typically occurs with tabular data because the *x* and *y*-values are listed as pure numbers with no definitive units of measure. Carraher et al. (2008) suggested, "Children need to start from carefully crafted contexts and situations that may constitute physical analogues for mathematical structures" (p.19), and graphs support these mathematical structures. Although the Math 7 students did not create a linear equation, they were on the cusp by understanding that the initial value was twenty. Therefore, in a classroom setting, further discussion could have served as a "teachable moment" in which the instructor could have help the students restructure their equation.

A study also important to consider in the context of the present study is Davis' (2007) work, in which s/he recognized that the concept of the y-intercept may seem as a less complex concept compared to slope, but it still poses a challenge for students when they must translate between different representations. Further, Davis (2007) recognized that the y-intercept holds the potential to promote the Cartesian connection because of its presence within the slope-intercept form of an equation. The findings of the present study support Davis' findings.

Knuth (2002) found that students with mathematical backgrounds from first-year algebra to calculus did not apply the Cartesian connection to translate a graphical representation to an algebraic one. Knuth's research further indicated that students preferred to move in the opposite direction, from algebraic to graphical representations. I hypothesize that the connection was not made because the graphs were not contextual. By the graphs not being contextual, the students are not able to interpret or make meaning of what is taking place. Furthermore, curriculum is to blame for the students' wants. Traditionally, math classes have students go from equation to graph. Yet, how many equations are truly contextual?

As a math practitioner with 23 years' experience, algebra textbooks highlight the slope-intercept equation by denoting what each variable represents. The teacher then guides the students through the usage of the variables and they find this graphing process extremely easy. My research supports Davis' (2007) assertion that real world contexts hold potential for making students' mathematical investigations more meaningful. The use of real world contextual problems allows students to naturally discover meaningful learning. Therefore, contextual problems allow students to use their prior experiences to make lasting connections.

Implications

DeLange (1996), thus, may have been correct in his assertion, apropos of the theory of RME, that students benefit from opportunities to discover mathematical ideas and concepts for themselves under the guidance of their teachers and that this is best accomplished through exposure to a variety of real-world problems and situations. When applied problem-solving in real-world contexts is the primary approach to teaching mathematics, students are able to make connections or, as Freudenthal would say, to see math as a human activity.

When teaching slope, then, whether from the origin or a nonzero y-intercept, it is useful to keep in mind the following set of principles elaborated by DeLange (1996):

- The starting points of instructional sequences should be experientially real to students so that they can engage in personally meaningful mathematical activities.
- In addition to considering students' existing mathematical knowledge, the starting points should also be justifiable in terms of the potential end points of the learning sequence.
- Instructional sequences should involve activities in which students create elaborate symbolic models of their informal activities.

This means that educators must come up with slope-related problems that are meaningful to the students in their everyday lives and that take into account their current mathematical knowledge. From this starting point, they must create activities that build on their students' prior knowledge while keeping the endpoint, slope, in mind, especially through the use of contextual graphs. Thus, as discussed by Lawton (1993), students must observe the relationships among units instead of being presented with an algorithmic model. Moreover, the activities must afford students the freedom and flexibility to follow their own notions.

Under such a teaching regimen, a strong indicator of students' success would be their tendency to use contextual approaches to math problems. Thus, the participants in this study were able to use previously acquired proportional reasoning to solve slope problems. Their ability to do so demonstrates that it is possible to create effective, meaningful and contextual problems for learning slope through proportional reasoning provided that math teachers promote their students' mathematization through activities in which one-unit changes relative to another.

In mathematics education, the verbal description of a problem situation or word problem delineate contextual mathematical problems. However, many students encounter difficulties in completely descriptive word problems because they take a calculational approach (Thompson et al., 1994). A difficulty commonly discussed with pure word problems is students do not take in consideration common sense considerations about the problem (Greer, 1997; Verschaffel, Corte, & Lasure, 1994). However, Hoogland et al. (2018) found that the use of real life images could counteract the difficulties in understanding pure word problems. The idea is that depictive representations of problem situations stay closer to real problems that are represented, and the students make more sense of a pictorial situation. This pictorial representation was exhibited in the research as the students were able to use contexts of the graphs to help make sense of their solutions.

Furthermore, Carpenter and Shah (1998) found that within-context linear graphs enabled students to perceptually process pattern recognition of encoded graphic patterns, perceptually process operations on those patterns to retrieve or construct qualitative or quantitative meanings, and to process conceptually translations of the visual features into conceptual relations when one interprets titles, labels, and scales, as well as any other keys or symbols that are part of the display. Additionally, Friel et al. (2001) found that graph instruction within a context of data analysis promoted a high level of comprehension because, to interpret graphs, one must seek relationships among specifiers (data values) in a graph or between a specifier and a labeled axis. Therefore, the findings of the present study supported this notion, as the contextual graph problems allowed student to see the covariation of the units and interpret the meaning in a slope ratio without ever encountering slope terminology. Since contextual graph problems depict a real-life situation, the students were able to use prior experiences to make sense of the problems.

Although the study focused on how the pairs solved the problems. The students worked in pairs and communicated throughout the problem-solving process. The socio-constructivist approach is motivated by the desire to understand students' mathematical learning as it occurs in the classroom or other social situations (Cobb et al. 1991). Brophy (2001) argued that practice and conditions that engage students in thoughtful and sustained discourse can facilitate learning, provided that (a) the discourse focuses on solid mathematical notions and (b) that teachers motivate students to develop explanations, make predictions, debate alternative approaches to problems, and to clarify or justify their assertions. Consistent feedback through the series of interviews the majority of the pairs revealed that discourse provided security to students unsure about an answer; each person was important because of their various strengths in different math topics.

Limitations

The major limitation of this study was the sample size, for there were only eight participants. Small sample sizes are, however, valued in qualitative research because of the detailed information that they can yield. Another limitation was the location in which the study was conducted, specifically a suburban school district serving families of middle and greater socio-economic status. The issue here is that students from privileged backgrounds tend to receive exposure to educational opportunities outside the context of their schools.

Therefore, results may not be generalizable to other seventh-grade students in other schools. Further, the fact that some of the students who participated in the study had high mathematical aptitudes may also have biased the findings. As a result, all participants in the study had not encountered the same mathematics education opportunities. Additionally, the four students from the higher math class were boys due to the teacher have a gender-specific boys' class. Therefore, females of the same caliber were not included into the study.

Furthermore, the analysis focused on four graph problems, two from the origin and two from a nonzero *y*-intercept. Thus, there may have been too few problems analyzed as well as administered in order to make concrete generalizations about proportional strategies used to make connections between proportionality and slope, as well as what proportional strategies are used to solve nonzero *y*-intercept slope-related tasks.

Areas for Future Research

The study could expand to the use of a hypothetical learning trajectory (HLT) which is an essential component in the RME instructional design. The HLT is a heuristic within the Realistic Mathematics Education (RME) design research cycle (Stephan et al., 2003). It coincides with the RME heuristic guided intervention. Guided reinvention is based on the premise that the designer starts to develop a sequence of instructional activities or a route the class might take to develop mathematical understanding of an activity. The route is constructed from the historical events surrounding the concept as well as prior research equated with students' mathematical strategies utilized to develop the mathematical concept. Thus, HLT is synonymous with guided reinvention because it is a *taken as shared* learning route for the classroom community. Therefore, it would allow the study to expand from pairs to a classroom setting

A HLT differs from traditional lesson planning because of the following:

1. It is socially situated nature of the learning trajectory.
2. It views planning as an iterative cycle rather than a single shot methodology.
3. It focuses on students' constructions rather than mathematical content.
4. It offers the teacher a grounded theory that describes how a certain set of instructional activities might play out in a given social setting (Stephan et al., p.55).

Therefore, when developing an HLT it is imperative to outline conjectures about the collective development of the mathematical community by focusing on the practices that might emerge at the beginning of the sequence, then creating tools and activities that might support the emergence of other practices that would be based on increasingly sophisticated ways of acting and justifying mathematical practices (Stephan, Gravemeijer, and Bowers, 2003). As a result, HLT requires intense research and planning because one must have an idea of how previous students have grasped concepts, struggled with concepts as well as have a historical idea of the evolution of the concept. From the research, one must then piece together the best possible itinerary of class activities to serve the classroom community. Thus, HLT emphasizes the students' cognitive development instead of the math content which is more consistent with reform recommendations that place high priority on students' mathematical reasoning and justification (as cited in Stephan, Gravemeijer, and Bowers, 2003). The current study found that the use of contextual graphs might be a starting point, but it does not give an in-depth sequence of lessons to develop slope understanding.

Another facet that makes HLT different from traditional lesson planning is the content is not the focal point. Traditional lesson plans offer a methodology that is meant to be followed by all that utilize the lesson. However, HLT is a route that considers the social and is meant to be deviated from because it understands that all classrooms are not the same. Hence, the research embedded into the construction of the HLT takes in account students' past reasoning in the math concept being developed. Therefore, the student is the priority and not the content.

As a result, future research using an HLT could explore how students collectively collaborate and use prior mathematical experiences in their approach to nonzero *y*-intercept tables in comparison to nonzero *y*-intercept graphs since both views of the function can solicit real world contextual problems. This would allow mathematics educators to find a more viable route in nonzero *y*-intercept slope tasks. Meira (1995) found that table representation helped facilitate student thinking about rate of change, yet the current study found graphs useful.

Meira (1995) found as long as the x-values in the table increase or decrease one integer value at a time, the change observed in subsequent y-values will be a constant value, regardless of the value of the y-intercept. This study's findings indicated that structuring the values in the table as described elicited deeper, more mathematical conversations between students and allowed for a depth of numeric analysis not possible with other representations of function. However, the contextual table approach differs from the nonzero y-intercept problems used in my study because the students were able to view the y-intercept and visually see how the graph changed. Yet, since both studies focused on integers it would be imperative that the HLT also sought to describe how students collectively worked together and use their prior mathematical knowledge in association with decimals?

Also, the current study used students from a higher socio-economic background. A replicated study that focused on how students from a lower socio-economic status solved the problems would be beneficial. It also might offer a different HLT if the full RME research cycle was implemented. Additionally, to extend the results of this study the analysis of proportional strategies used to solve slope-tasks need to include tables and traditional word problems. The inclusion of tables and traditional word problems could alter results. Teuscher and Reys (2010) felt it was imperative that the concepts of slope and proportionality be discussed simultaneously to understand their relationship to one another. The connection of mathematical relationships and the use of students' prior experiences is the aim of RME's HLT. Therefore, the study should further examine which learning sequences would best assist students in understanding the relationship between proportionality and slope.

Personal Reflection on the Research Process

The qualitative research process was arduous and laborious for me, particularly because of the time constraints involved based on program duration. Therefore, I felt somewhat overwhelmed when I saw the timeline and all of the due dates and realized that my initial design research study would probably not be complete within the anticipated time frame. Nevertheless, I gained an appreciation for the way in which qualitative studies allow a researcher to become engulfed and experienced in and enriched by the topic under study. I think that math educators sometimes take for granted students' proficiency in mathematical skills and continue to assign problems only to feel frustration when the students lack the requisite knowledge. The research process has shown me that I need to have conversations with my students on their understanding of mathematical topics. Such conversations can help me both to identify their misconceptions and to create learning trajectories that will allow mathematization to occur.

As the research process progressed, I found the data analysis to be the most intriguing aspect, in part because the researcher must let the data drive the interpretation of findings. I must confess, though, that the analysis can be overwhelming, for it reveals unexpected findings. I learned the importance of research questions as a guide for working through the data in order to identify relevant and interesting results. At the same time, I have come to see how research questions can be altered in response to unforeseen findings that may enrich the research.

In the end, the most vexing part of the qualitative research process was transcribing the discussions with the pairs of students about the various problems. While I enjoyed the opportunity to get to know my students on a more personal level, I found making the transcriptions to be a tedious experience because I am a very slow typist. Nevertheless, listening to these discussions again gave me ideas about ways to improve my research methods in the future.

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APPENDIX A: PARENT CONSENT FORM

UNC CHARLOTTE

Department of Middle and Secondary Education

9201 University City Boulevard, Charlotte, NC 28223-0001

Parental Informed Consent for

The Case for Building on Students Proportional Reasoning for Slope-Related Tasks

Project Title and Purpose

The Common Core State Standards for 8th grade mathematics listed as one of the critical areas of instruction that students understand connections between proportional relationships, lines, and linear equations by students using linear equations and system of linear equations to represent, analyze, and solve a variety of problems. Therefore, parents, I am asking you to grant your child permission to participate in a mini-research project on *The Case for Building on Students' Proportional Reasoning for Slope-Related Tasks*. The purpose of this study is to gain insight from how students solve problems related to proportionality and how they employ these strategies to solve slope related tasks. From an analysis of their work, and responses through the use of videotape and work samples, I hope to find a connection from students' prior proportional reasoning as a means that will help students develop an understanding of slope. The insights gained from this project could lead to ideas to help teachers and students develop an understanding of this pinnacle objective of the Common Core Standards for 8th grade mathematics. Please feel free to contact me if you have any further questions via phone (704-948-8600 ext.1711) or email (ckendrick@lncharter.org).

Investigator(*s*)

This study is conducted by Curtis D. Kendrick a current PhD candidate at UNC Charlotte as well as the students' teacher. The responsible faculty member is Dr. David Pugalee (chair) Director of STEM Education.

Eligibility

Students may participate in this study if they 7th graders currently enrolled in Math 7 or Pre-Math I (8th grade math). These students are eligible because they will have already been exposed to constant proportionality and proportions, but not familiarized with slope or slope intercept-form which is the next progression according to Common Core Math Standards. However, middle grade students in 5th, 6th or 8th grade math classes as well as 7th graders in Math I will not participate in this study. The 5th and 6th graders have not encountered the all of the necessary concepts and the 8th and 7th grade Math I students have already been exposed.

Overall Description of Participation

The study will take place in the teacher-researcher classroom after school from approximately 3:00 p.m. to 4:00 p.m. Participants in this study agree to be interviewed as they solve problems pertaining to proportional reasoning and slope. The participants will explain their thought process in solving the problem as well as offer justification for their solution. The goal of the interviews is to evaluate students' reasoning when solving contextual proportional problems. All interviews will be videotaped and the work samples will be collected. The problem-solving will not count towards their grades, but offer insight on a learning trajectory (sequence of activities) that helps develop an understanding of slope.

The participants are expected to answer questions honestly and try their best in actively engaging during the interview/problem-solving session. Parents of the participants agree to allow the interviews and instructional activities to be recorded and transcribed. The recordings will be collected by the investigator (myself). All video-recordings collected will be passcode-locked as well as their real names will not be used to ensure privacy of students. Furthermore, I will be the only person who has access to the videos and it will not be shown to anyone. In addition, work samples will be housed in a locked cabinet during the study and they will not have any identifiable information (real name or student ID). The data will be kept indefinitely, but will remain confidential and used for educational research purpose only. There are eight participants in the study.

Length of Participation

The duration of the participation for this study is three (no more than 45 minutes) after school class sessions. There will be one session per week, so the total time is 3 weeks. There will be no follow up sessions unless I need to clarify during the analysis. If so, your child will be contacted and we schedule a convenient meeting for after school.

Risks and Benefits of Participation

There are no known risks involved in this study. The direct benefits of your child's participation in the study are miniscule. However, your child's shared insights from the problems and discourse with their peers could possibly assist future teachers and their students with activities and tools that develop an understanding for linear rate of change.

Volunteer Statement

Your child is a volunteer. The decision to participate in this study is completely up to you. If you decide to let your child be in the study, you may have your child stop at any time. Your child will not be treated any differently if you decide not to let them participate in the study or if you stopped once they have started.

Confidentiality Statement

Any information about your participation, including your identity, is completely confidential. The following steps will be taken to ensure this confidentiality: The recordings will be collected by the investigator (myself) and will not contain any identifying information or any link back to the teacher for their participation in this study. All recordings collected will be housed in a locked cabinet during the study. The data will be kept indefinitely, but will remain confidential and used for educational research purpose only.

Statement of Fair Treatment and Respect

UNC Charlotte wants to make sure that you are treated in a fair and respectful manner. Contact the university's Research Compliance Office (704-687-3309) if you have questions about how you are treated as a study participant. If you have any questions about the actual project or study, please contact Dr. David Pugalee (704-687-8887), dkpugale@uncc.edu)"

Participant Consent

I have read the information in this consent form. I have had the chance to ask questions about this study, and those questions have been answered to my satisfaction. I am the parent or guardian, and I agree to let my child participate in this research project. I understand that I will receive a copy of this form after it has been signed by me and the principal investigator of this research study.

____________________________________ ______________________
Participant Name (PRINT) DATE

____
Parent/Guardian Signature

____________________________________ ______________________
Investigator Signature DATE

APPENDIX B: MINOR ASSENT

Assent for Minors

(For subjects under the age of 18 unless emancipated*)

UNC CHARLOTTE

Department of Middle Grade and Secondary Education

9201 University City Boulevard, Charlotte, NC 28223-0001

Assent for the Case for Building on Students Proportional Reasoning for Slope-Related Tasks

My name is Mr. Curtis D. Kendrick and besides being your math teacher, I am currently a doctoral candidate at The University of North Carolina at Charlotte. I am doing a study to determine how students use their prior proportional reasoning to solve slope-related problems. I would greatly appreciate if you participated in this study.

I would like you to take part in my study because you and your partner in math class collaborate extremely well when solving math problems. If you choose to participate, the study requires you to be interviewed as you solve a total of 9 -12 math problems over a three-day session. Each session should last no longer than 45 minutes. The problems will not count towards your grade, but they require you to explain your reasoning on solving the problem. In addition, the after-school sessions will be videotaped and your work samples will be collected for analysis. The interviews will take place after school in my classroom. The responses you give could offer insight on connecting proportionality to developing an understanding of an algebraic math concept known as slope.

Your parents said it was ok for you to be in this study and have signed a form like this one. You do not have to say “yes” if you do not want to be in the study. If you say “no” or if you say “yes” and change your mind later, you can stop at any time and no one will be mad at you.

You may ask questions at any time and you are not required to be in the study. When I am done with the study I will write a report. I will not use your name in the report.

If you want to be in this study, please sign your name.

________________________________ ________________

Signature of Participant Date

________________________________ ________________

Signature of Investigator Date

Emancipated Minor (as defined by NC General Statute 7B-101.14) is a person who has not yet reached their 18^{th} birthday and meets at least one of the following criteria: 1) has legally terminated custodial rights of his/her parents and has been declared ‘emancipated’ by a court; 2) is married, or 3) is serving in the armed forces of the United States.

APPENDIX C: PHASE I INTERVIEW PROTOCOL AND PROBLEMS

Phase I: Interview Protocol

Warming up questions

- List three adjectives that best describe you. Why did you choose them?
- How would you describe yourself as a math student and why?
- What is your favorite thing about mathematics and why?

Interview Questions

- I am going to ask you to solve a few and we are going to discuss your solutions and why you solved the problem that way. Is that okay?
- Explain how you solved the problem. How do you know that your solution is reasonable? (Ask after each question.)

Phase 1: Questions

1. Ellen, Jim, and Steve bought three helium-filled balloons and paid $2.00 for all three. They decided to go back to the store and get enough balloons for everyone in their class. How much did they have to pay for 24 balloons?

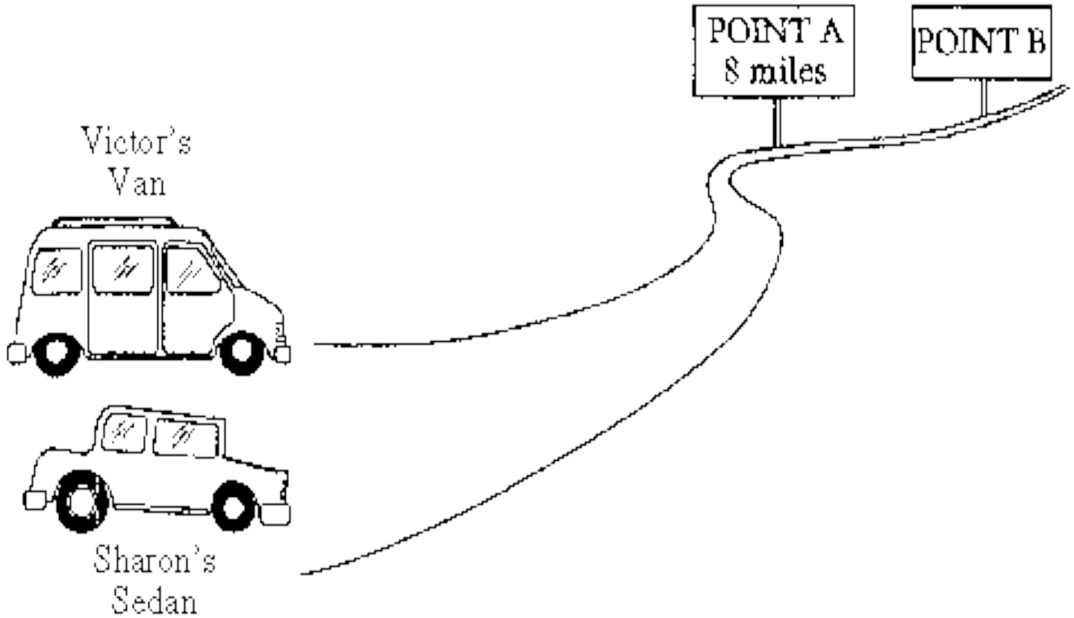
2. Lisa and Rachel drove equally fast along a country road. It took Lisa 6 minutes to drive 4 miles. How long did it take Rachel to drive six miles?

3. Rule: One food bar can feed 3 aliens.

a.) How many aliens would be fed with 15 food bars?

b.) How many aliens would be fed with 16 food bars?

c.) How many food bars are needed for 63 aliens?



4. Victor's van travels at a rate of 8 miles every 10 minutes. Sharon's sedan travels at a rate of 20 miles every 25 minutes.
   If both cars start at the same time, will Sharon's sedan reach point A, 8 miles away, before, at the same time, or after Victor's van?

   Explain your reasoning.

   ____

   ____

   ____

   If both cars start at the same time, will Sharon's sedan reach point B (at a distance further down the road) before, at the same time, or after Victor's van?

   Explain your reasoning.

   ____

   ____

   ____

   ____

   Did you use the calculator on this question?

○ Yes ○ No

APPENDIX D: PHASE II INTERVIEW PROTOCOL AND PROBLEMS

Phase II: Interview Protocol

Warming up questions

- How was school today?
- Tell me something interesting that happened in school today.
- Is your favorite thing in math still ______? Or has it changed?

Questions:

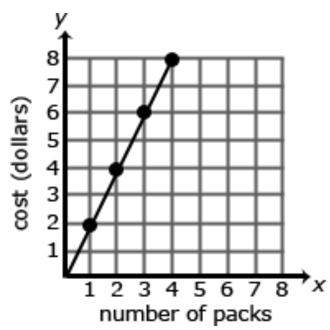
- Explain how you solved the problem
- How did you and your partner discuss the solutions today

Phase II Problems: Slope from the origin ($y = mx$)

1. The table below gives the price for different numbers of books. Do the numbers in the table represent a proportional relationship?

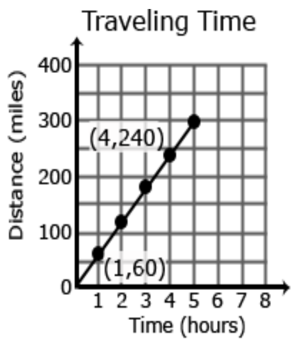
Number of Books	**Price**
1	3
3	9
4	12
7	18

2. The graph below represents the cost of gum packs as a unit rate of $2 dollars for every pack of gum. Represent the relationship between the cost of gum and the number of packs using a table and an equation.



3. Compare the scenarios to determine which represents a greater speed. Explain your choice including a written description of each scenario.

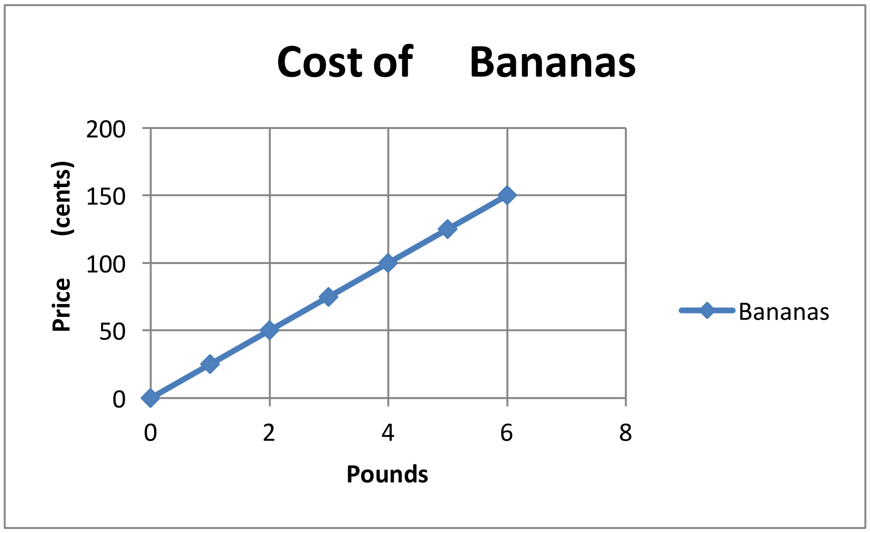
Scenario 1:



Scenario 2:

$y = 55x$
x is time in hours
y is distance in miles

4. The graph below represents the price of the bananas at one store. What is the cost per pound?

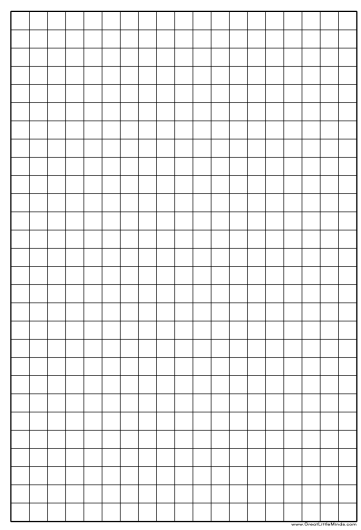


5. A student is making trail mix using the information in the table.

a.) Does the recipe represent a proportional relationship?

Serving Size	**1**	**2**	**3**	**4**
cups of nuts (x)	1	2	3	4
cups of fruit (y)	1.5	3	4.5	6

b.) Where does the unit rate show up in the graph?



APPENDIX E: PHASE III INTERVIEW PROTOCOL AND PROBLEMS

Phase III: Interview Protocol

Warming up questions

- How was school today?
- Tell me something interesting that happened in school today.
- Is your favorite thing in math still ______? Or has it changed?

Questions:

- Explain how you solved the problem
- How did you and your partner discuss the solutions today

Phase III Problems: Nonzero y-intercept slope problems ($y = mx + b$)

1. The linear graph below describes Josh's car trip from his grandmother's home directly to his home.

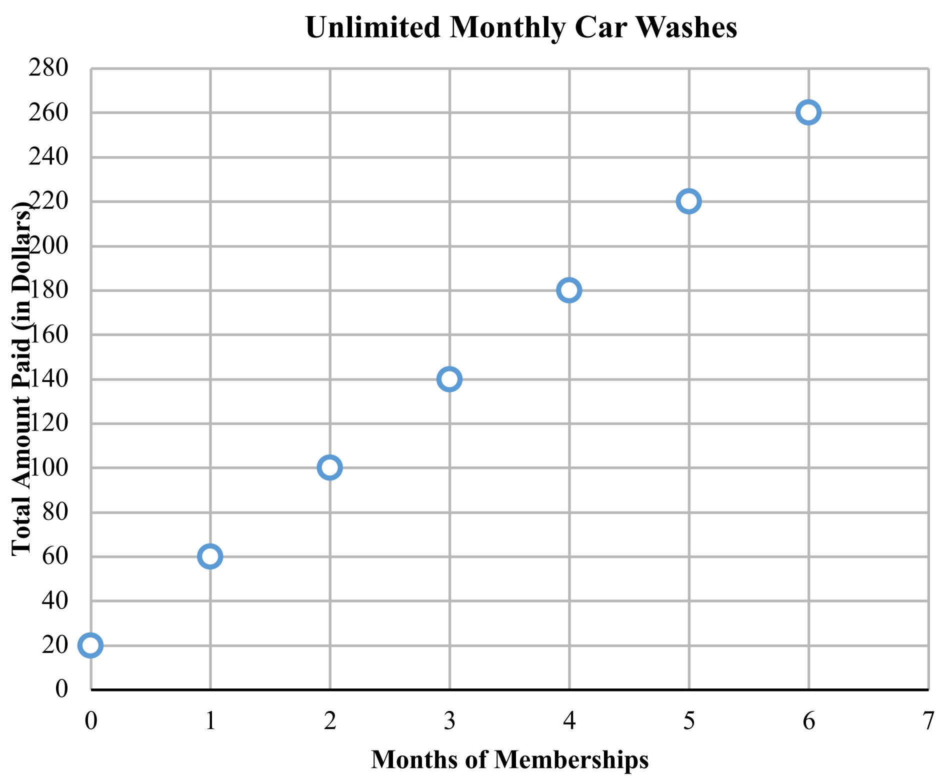
a) Based on this graph, what is the distance from Josh's grandmother's home to his home?

b) Based on this graph, how long did it take Josh to make the trip?

c) What was Josh's average speed for the trip? Explain how you found your answer.

d) Explain why the graph ends at the x-axis.

2. Membership to unlimited monthly car washes at local car wash costs $20 plus a monthly fee, as shown on the graph below.

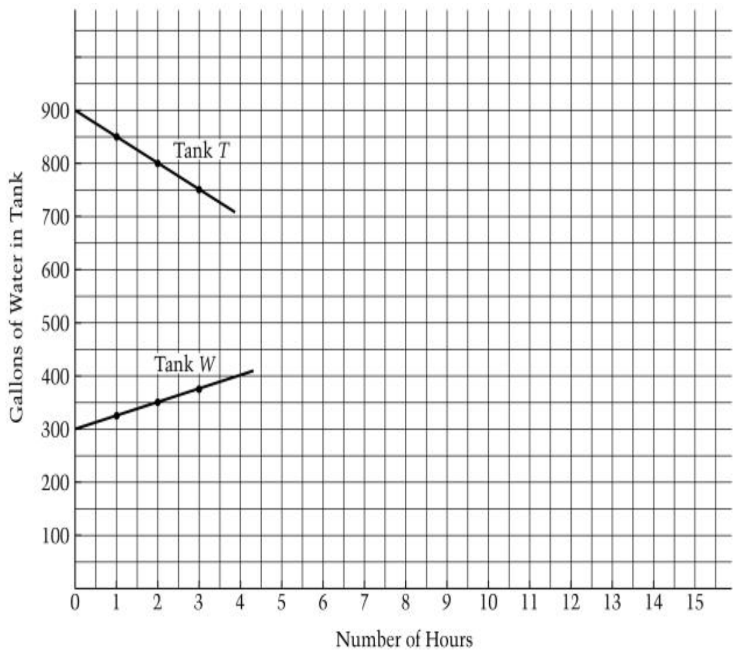


c) Determine the rate of change of a line joining the points on the graph.

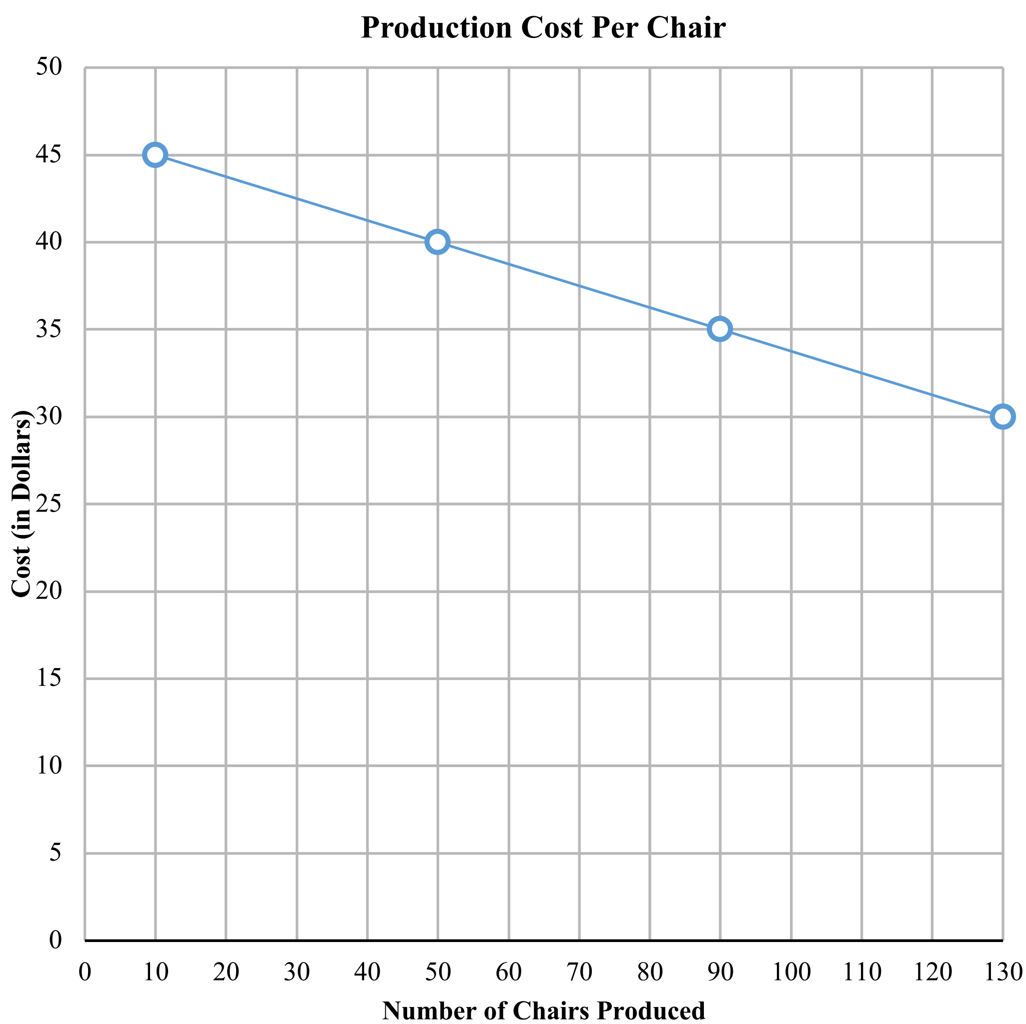
d) Explain what the rate of change represents.

e) Write an equation that describes the graph. Explain how you determined the equation that represents the graph.

3. Two large storage tanks, T and W, contain water. T starts losing water at the same time additional water starts flowing into W. The graph below shows the amount of water in each tank over a period of hours. Assume that the rates of water loss and water gain continue as shown. At what number of hours will the amount of water in T be equal to the amount of water in W?



4. The production manager of a furniture manufacturing company plotted values on the graph below to show how the production cost per chair decreases as the number of chairs produce increases. The rate of change of the line segment joining these points is – 1/8. Two students had an argument on what the rate of change of the graph meant.



- Student A said that the rate of change represents each chair produced decreases costs by $8.
- Student B said that the rate of change represents for every 8 chairs produced, costs decrease by $1.

Which student is correct? Justify.