GEOMETRIC METHODS FOR CONTROL OF NONHOLONOMIC MECHANICAL SYSTEMS WITH APPLICATIONS TO THE CONTROL MOMENT GYROSCOPE AND WHEELED MOBILE ROBOTS

by

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ABSTRACT

YAWO H. AMENGONU. Geometric methods for control of nonholonomic mechanical systems with applications to the control moment gyroscope and wheeled mobile robots. (Under the direction of DR.YOGENDRA P. KAKAD)

The advantage of geometric dynamics analysis over the classical analysis method is that geometric method is independent of the choice of coordinates. The work, presented here, applies differential geometry for analysis and control of underactuated dynamical systems which include mobile robots, aircraft systems, underwater vehicles, satellites and many more systems. In the first part we will model a class of wheeled mobile robots and for which geometric method is applied to trajectory tracking. In the second part of the dissertation, geometric method is applied to the control moment gyroscope mounted on an inverted pendulum. The control moment gyroscope inverted pendulum is originally modeled at Embry-Riddle Aeronautic University by Dr. Douglas Isenberg. Stability analysis and control law design is proposed. The first solution proposed uses collocated partial feedback linearization and then the dynamics are transformed into strict feedback form, a form suitable to apply backstepping method. This work appears in the Springer series Advances in Intelligent Systems and Computing [6]. The application of collocated partial feedback linearization due to Mark Spong, makes it easy to transform the system into a cascade of a linear and a nonlinear subsystems. Peaking phenomenon is an issue which is inherently present in interconnected subsystems; the manifestation of this phenomenon is sometimes observed as finite time escape. Finite time escape can excite unstable modes in the nonlinear subsystem. Peaking phenomenon is studied and a solution is proposed.

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DEDICATION

To my wife and son Bella and Freddy Amengonu, my mother, Grace Akotsu, and our pastors Freddy and Beatrice Shembo, whose support, encouragement, and faithful prayers made the research possible.

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CHAPTER 1: INTRODUCTION

This dissertation is about geometric control of a certain class of mechanical systems, the Control Moment Gyroscope (CMG) inverted pendulum and wheeled mobile robots. CMGs are mostly used on space exploration mission systems such as spacecrafts, satellites, telescopes, international space station to perform maneuvers. Many of them employ more than three CMGs for attitude control. The swift Gamma Ray Burst explorer, for example, employs six wheels to quickly relay location of gamma ray burst to ground stations so that ground stations can observe afterglow. NASA has recently put out call for proposals for High Torque, Low Jitter Reaction Wheels or Control Moment Gyros for its small satellites in the range of 5-100 Kg size range. The need to fully study and understand these actuation systems is necessary. For example in the case of failure of one more of the reaction wheels, would the system still be controllable and stabilizable. This dissertation is concerned with the study of a similar CMG platform with only one momentum wheel which can simulate an actuator failure. The system is mounted on a pendulum and the physical model is at Embry-Riddle Aeronautic University in Arizona. Application of this system could be used for stabilization of ships, underwater vehicles and wheeled mobile robots. The dissertation will first consider modeling and stabilization of wheeled mobile robots in the first part and then in the second part we will study stability and control of the Control Moment Gyroscope inverted pendulum. Geometric mechanics have introduced new tools which have led to better understanding of the structures of dynamical systems especially in mechanical systems. These structures are often times rather difficult to understand using the classical analysis methodologies. Great insights into dynamical systems analysis and control were gained while applying geometric theory

[18], [19], [25], [42], [62]. Geometric methods applied to mechanical systems provide robust control algorithms and stronger background theory to analyze where the classical nonlinear systems theory may fail. It was pointed out in the early 1970's by Brockett [15],[43] that there are many physical systems in engineering and physics that cannot be treated by the classical control theory because the state space (the configuration space) is not a vector space. For example the state space of a rigid body whose attitude is controlled relative to some fixed set of axes is the tangent bundle of $SO(3)$ which is the *group* of orthogonal matrices with determinant equals to unity. This research is mainly about mechanical systems with fewer inputs than the degrees of freedom. Such systems are sometimes characterized by nonintegrable dynamic relations. Research that has been done for systems with nonintegrable relations were mainly about system with nonholonomic kinematics. In chapter I, we will present a summary of the dissertation and an overview of some of the main results obtained in the analysis of systems with both nonintegrable kinematic and dynamics relations. The section on mathematical background concludes Chapter I. Note that the mathematical background section is added for completeness of the dissertation. The mathematical background on geometric mechanics is extracted from an excellent source; Foundation of mechanics in which, the theory is based on point set topology [1]. Another outstanding textbook is Introduction to Mechanics and Symmetry which applies symmetry to reduce mechanical system dynamics [58]. The dissertation is a comprehensive compilation of the research performed as doctoral candidate at the University of North Carolina at Charlotte under the supervision of Prof. Yogendra P. Kakad. The research has yielded several publications and this document is more comprehensive.

1.1 Overview of the Dissertation

The main focus of this dissertation is to apply geometric perspective to stability analysis and control of mechanical systems with fewer inputs than degrees of freedom.

While many analysis techniques were presented for these kind of systems in the past, the control law design for certain underactuated dynamical systems remains an open research problem in most cases. The background material presented in the next section of this chapter is applied in the analysis and control design of systems. The equations of motion of rigid-body and multi-body dynamics are often modeled as a second order linear or nonlinear differential equations. A unifying modeling technique in most cases is to describe the equations of motion in terms of general coordinates and their time derivatives where the generalized velocities are considered to be elements of the tangent space to the configuration manifold, and the generalized forces are taken to be elements of the cotangent or dual space to the tangent space. When the systems under consideration have constraints (holonomic or nonholonomics), the tangent space and the cotangent space do not have a natural metric since the generalized coordinates or the constraints can be expressed as a combination of translational and rotational components hence, the dynamical model formulation depends on the metric selected; in other words the formulation is not invariant. Several researchers have explored different methods in formulating the dynamics in the presence of constraints. One of the most popular methods is the *quasivelocities* technique. Quasivelocities are obtained by exploiting the factorization of the mass matrix, which is a positive define matrix, to obtain a linear combination of the generalized velocities and the generalized coordinates. A survey can be found in [2], [34]. It turns out that the Euclidean norm of the quasivelocities vector is proportional to the square root of the kinetic energy of the system [33], [37], [48], [70]. Quasivelocities technique such as Maggi's and Bolzmann-Hammel methods were applied to model the dynamics of a differential mobile robot and the classical method to model a car-like mobile robots as shown in chapter II. The control methodology proposed exploits *dynamics extension* in the first case to design a trajectory tracking control law because the system lacks relative degree with respect to the proposed output. The design is mainly performed to show that when the system does not have a well defined relative degree, a dynamic extension can be performed to obtain a relative degree. The advantage of Maggi's method is that in the presence of nonholonomic constraints, as it is in the case of wheeled mobile robot, Lagrange multipliers are used to incorporate the constraints in the differential equations that model the dynamics. Doing so, increases the number of variables that one has to solve in order to obtain the equations of motion. But at the end, the Lagrange multipliers are eliminated since they are not needed in the resulting equations. However, Maggi's formulation eliminates the Lagrange multipliers in the beginning and therefore the number of variables is reduced. The work presented in Chapter II appears in the proceedings of the International Conference on CADCAM, Robotics and Factories of the Future 2014, Amengonu and Kakad [4, 5]. Dynamics modeling section of the work we published in Amengonu and Kakad [3] is given the this Chapter as well.

In Chapter III, dynamics and control of the CMG inverted pendulum are presented. Here, the configuration variable are separated into shape and external variables. We use the fact that the external variables define a group action on the configuration manifold and the kinetic energy is symmetric with respect to the external variables. The symmetry of the kinetic energy with respect to external variables was exploited to transform the system into Byrnes-Isidori normal form [36]. One step further is taken to transform the system into two interconnected subsystems: one linear and the other nonlinear which results from the application of collocated partial feedback linearization introduced by Mark Spong [79, 81]. A backstepping control law is proposed in formulating this research problem. In the backstepping procedure, the variable used as a virtual input to the nonlinear subsystem is computed using *implicit function theorem* and saturation function. Saturation function method and saturation stabilization techniques are given in [85], [89], [93]. Peaking is a phenomenon inherently present in interconnected subsystems. Peaking phenomenon is also studied for the resultant interconnected subsystems.

1.2 Mathematical Background and Literature Review

The mathematical background needed for the work proposed here can be found mostly in Foundation of Mechanics [1] and some introductory textbooks in differential geometry for example differential geometry by Boothby [14], and by DoCarmo [26]. For the work to be self contained, we summarize some of definitions and notations that we will be using extensively in the course of the work.

1.2.1 Topological Space

A topological space is a set M together with a collection $\mathcal{O}$ of subsets called *open sets* such that

(T1) $\emptyset \in \mathcal{O}$;

(T2) If $U_1, U_2 \in \mathcal{O}$, then $U_1 \cap U_2 \in \mathcal{O}$;

(T3) The union of any collection of open sets is also open.

A basis $\mathcal{B}$ for the topology is a collection of open sets such that every open set of M is a union of elements of $\mathcal{B}$. The topology is called *second countable* if it has a countable basis.

A topological space is called Hausdorff if and only if each two distinct points have disjoint neighborhoods. This notion is very important because it will help in studying trajectory convergence and distance between neighboring points. Before defining the abstracts objects that are essential to our study, let us make some notations which are largely drawn from [84]. $\mathbb{R}^n = [(a^1, \ldots, a^n), a^i \in \mathbb{R}]$ is the product of ordered n-tuples of real numbers. Let $\xi^i, i = 1, \ldots, n$ a natural coordinate (slot) functions define as $\xi^i : \mathbb{R}^n \to \mathbb{R}$ by $\xi^i(a^1, \ldots, a^n) = a^i$. An open set U of $\mathbb{R}^n$ is a set such that for every point $u \in U$, there is $\delta > 0$ such that $(u - \delta, u + \delta) \subset U$. Thus the distance between two points $a = (a^1, \ldots, a^n)$ and $b = (b^1, \ldots, b^n)$ in $\mathbb{R}^n$ is given by

$d(a,b) = \sqrt{\sum_{i=1}^{n}(a^i - b^i)^2}$ using the standard metric topology induced by the natural metric function d on $\mathbb{R}^n$. A map f from an open subset U of $\mathbb{R}^n$ into $\mathbb{R}$ is called C^r on U if f possesses continuous partial derivatives on U of all orders $\leq r$. So f defined from $\mathbb{R}^n$ into $\mathbb{R}^k$, $k \geq 1$ is C^r if each of its slot functions $f^i = \xi^i \circ f, i = 1, \ldots, k$ is C^r on U.

1.2.2 Manifold

Roughly speaking, a manifold M of dimension n is a space that looks locally as an n-dimensional Euclidean space. For example a sphere in $\mathbb{R}^3$ is defined by $S^2 = \{(x,y,z) \in \mathbb{R}^3, x^2 + y^2 + z^2 = 1\}$. S^2 is not a part of the Euclidean space $\mathbb{R}^2$ but the immediate neighborhood of a point on the sphere can be described by a two coordinates system which can be identified with $\mathbb{R}^2$.

An n-manifold M is Hausdorff space with a countable basis is locally homeomorphic to n-dimensional Euclidean space $\mathbb{R}^n$. Since there is a one-to-one mapping from subsets of M to $\mathbb{R}^n$, we are allowed to specify each point on M by a vector of n real numbers which is called the *coordinates* of the corresponding point.

Let U be a neighborhood of a point p on M. Let V be an open subset of $\mathbb{R}^n$ and suppose a mapping φ from U into V. Then we can write: $\varphi(p) = [x^1(p), \ldots, x^n(p)]$, $x^i = \xi^i \circ \varphi(p)$. These are the coordinates function of p. Each $x^i(p)$ may be viewed as a function $p \to x^i(p), i = 1 \ldots n$ which maps a point p to its i^{th} coordinate. The mapping φ is called differentiable if its coordinates $x^i(p), i = 1 \ldots n$ are differentiable.

Let M be a Hausdorff topological space. An *open* coordinate *chart* on M is a pair (U, φ) where U is a open subset of M and φ is a homeomorphism. (φ and φ^{-1}) are continuous functions of U into an open subset of $\mathbb{R}^n$.

Let M be a Hausdorff space. A differential structure on M of dimension n is a collection of open charts $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ on M where $\varphi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^n$ such that the following conditions are satisfied:

(M1) M$=\bigcup_{\alpha\in A} U_\alpha$;

(M2) For each pair $\alpha, \beta \in A$ the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a differential mapping of $\varphi_\alpha(U_\alpha \cap U_\beta)$ onto $\varphi_\beta(U_\alpha \cap U_\beta)$.

Note that for an analytic structure of dimension n, in $(M2)$ differentiable is replaced by analytic. This is a very important notion used very often in geometric control theory.

1.2.3 Tangent Space

The tangent space at a point p of a manifold is a collection of all tangent vectors at p; in other words, it is the collection of all derivatives at p. The tangent space at a point is a vector space. This is also a very important notion because locally on a smooth manifold, we can apply multilinear algebra. As we have seen above, at every point on a n-dimensional manifold, we can find a local coordinates chart $(x_1, x_2, \ldots, x_n)$. The collection of the tangent spaces at $p \in M$ form a space that is denoted by tangent bundle $TM = \bigcup_{p\in M} T_pM$. The dimension is $2n$ and the elements are (p, v), v is a tangent vector. A vector field is a smooth map defined form the manifold M to TM. This mapping is used to represent ordinary differential equations on the manifold M. Let I be a subinterval of $\mathbb{R}$ and $c : I \to M$ a map defined form I to M. The curve $c(t)$ with $t \in I$ is an integral curve of a vector field X if

$$\frac{dc(t)}{dt} = X(c(t))$$

We say that the vector field X is complete if it is defined for all $t \in \mathbb{R}$ $i.e.I = \mathbb{R}$.

1.2.4 Flow of Vector Fields

The flow of a vector field X, $\phi^X : \mathbb{R} \times M \to M$, $(t, p) \mapsto \Phi^X(t, p)$ form a group with respect to composition of mappings (one-parameter subgroup). If the vector field is said to be not complete for example if $t \geq 0$ then the flow is a one-parameter semigroup; in other words it is not defined for all $t \in \mathbb{R}$ but for only positive time.

1.2.5 Meaning of Lie Bracket

Consider two complete vector fields X, Y in the set $\mathfrak{X}(M)$ of vector fields defined on M and let us consider the associated flows Φ^X, Φ^Y. The Lie bracket is the second order term of the expansion of the flow composition $\Phi_s^{-Y} \circ \Phi_t^{-X} \circ \Phi_s^Y \circ \Phi_t^X$. To justify this, let us flow along the vector field X for a short period of time say ϵ. We know from above that the derivative of the integral curve $c(t)$ is given by

$$\begin{aligned} \dot{c}(t) &= X(c(t)) \\ \ddot{c}(t) &= \frac{\partial X}{\partial c} X(c(t)) \end{aligned} \tag{1.1}$$

Without loss of generality, if we assume $c(0) = p_0 \in M$, a local solution for a small time ϵ from $c(0) = p_0 \in M$ is $c(\epsilon)$. The Taylor series expansion is given by

$$\begin{aligned} c(\epsilon) &= \Phi_\epsilon^X(c(0)) \\ &= c(0) + \epsilon \dot{c}(0) + \frac{\epsilon^2}{2}\ddot{c}(0) + O(\epsilon^3) \\ &= p_0 + \epsilon X(p_0) + \frac{\epsilon^2}{2}\frac{\partial X}{\partial c} X|_{p_0} + O(\epsilon^3) \end{aligned} \tag{1.2}$$

let us flow along the vector field Y for another short period of time ϵ. This is given by

$$\begin{aligned} c(2\epsilon) &= \phi_\epsilon^Y \circ \phi_\epsilon^X(c(0)) \\ &= \phi_\epsilon^Y(p_0 + \epsilon X(p_0) + \frac{\epsilon^2}{2}\frac{\partial X}{\partial c} X|_{p_0} + O(\epsilon^3)) \\ &= p_0 + \epsilon X(p_0) + \frac{\epsilon^2}{2}\frac{\partial X}{\partial c} X|_{p_0} + \epsilon Y(p_0 + \epsilon X(p_0)) + \frac{\epsilon^2}{2}\frac{\partial Y}{\partial c} Y|_{p_0} + O(\epsilon^3) \end{aligned} \tag{1.3}$$

But since the vector fields are assumed complete, we can flow along $-X$ and $-Y$. The flow along X in the opposite direction *i.e.* $-X$ is given by

$$
\begin{aligned}
c(3\epsilon) &= \Phi_\epsilon^{-X} \circ \Phi_\epsilon^{Y} \circ \Phi_\epsilon^{X}(p_0) \\
&= \Phi_\epsilon^{-X}(c(2\epsilon)) = c(2\epsilon) - \epsilon X(c(2\epsilon)) - \frac{\epsilon^2}{2}\frac{\partial X}{\partial c}\dot{c} \\
&= p_0 + \epsilon(X(p_0) + Y(p_0)) + \epsilon^2 \left(\frac{1}{2}\frac{\partial X}{\partial c}X(p_0) + \frac{\partial Y}{\partial c}X(p_0) + \frac{1}{2}\frac{\partial Y}{\partial c}Y(p_0) \right) \\
&- \epsilon X(p_0) - \epsilon^2 \left(\frac{\partial X}{\partial c}X(p_0) + \frac{\partial X}{\partial c}Y(p_0) \right) + \frac{\epsilon^2}{2}\frac{\partial X}{\partial c}X(p_0) \\
&= p_0 + \epsilon Y(p_0) + \epsilon^2 \left(\frac{\partial Y}{\partial c}X(p_0) + \frac{1}{2}\frac{\partial Y}{\partial c}Y(p_0) - \frac{\partial X}{\partial c}Y(p_0) \right) + O(\epsilon^3) \qquad (1.4)
\end{aligned}
$$

and flowing along $-Y$ we have

$$
\begin{aligned}
c(4\epsilon) &= \Phi_\epsilon^{-Y} \circ \Phi_\epsilon^{-X} \circ \Phi_\epsilon^{Y} \circ \Phi_\epsilon^{X}(p_0) \\
&= p_0 + \epsilon Y(p_0) + \epsilon^2 \left(\frac{\partial Y}{\partial c}X(p_0) + \frac{1}{2}\frac{\partial Y}{\partial c}Y(p_0) - \frac{\partial X}{\partial c}Y(p_0) \right) \\
&- \epsilon Y(p_0) - \epsilon^2 \frac{\partial Y}{\partial c}Y(p_0) + \frac{\epsilon^2}{2}\frac{\partial Y}{\partial c}Y(p_0) + O(\epsilon^3) \\
&= p_0 + \epsilon^2 \left(\frac{\partial Y}{\partial c}X - \frac{\partial X}{\partial c}Y \right) + O(\epsilon^3) \\
&= p_0 + \epsilon^2 [X, Y] + O(\epsilon^3) \qquad (1.5)
\end{aligned}
$$

Thus the Lie bracket $[X,Y] = \frac{\partial Y}{\partial c}X - \frac{\partial X}{\partial c}Y$ is an infinitesimal motion (actually of order ϵ^2) that results from flowing along the vector fields X and Y as specified above. This means that after a reparametrization, the tangent vector at zero to the curve $t \rightarrow \alpha(t)$ is proportional to [X,Y]. This fact will become the corner stone while studying controllability properties of nonlinear systems. A detail account and proof are given for example in Spivak [78]. If the Lie bracket is evaluated to be zero then X and Y *commute*. Let us state a proposition on convergence of linear combination of vector fields [39].

Proposition: Suppose we are given two vector fields X and Y defined on U, and let

$p \in U$. Let λ_1 and λ_2 be real constants. Define the following local diffeomorphism of U

$$\eta_t = \Phi^X_{\lambda_1 t} \circ \Phi^Y_{\lambda_2 t}, \quad \psi_t = \Phi^X_{-t} \circ \Phi^Y_{-t} \circ \Phi^X_t \circ \Phi^Y_t. \tag{1.6}$$

Then the families of curves shown

$$\begin{aligned} \alpha_k(t) &= \eta_{t/k} \circ \cdots \circ \eta_{t/k}, \;\; k - times \\ \beta_k(t) &= \psi_{t/k} \circ \cdots \circ \psi_{t/k}, \;\; k^2 - times \end{aligned} \tag{1.7}$$

converge to the trajectories of the vector fields $\lambda_1 X + \lambda_2 Y$ and [X,Y], respectively. More precisely, we have the convergence $\beta_k(t) \to \Phi_t^{\lambda_1 X + \lambda_2 Y}$ and $\beta_k(t) \to \Phi_{t^2}^{[X,Y]}$ as $k \to \infty$. A version of this proposition is given in [1], Corollary 2.1.27.

1.2.6 Properties of Lie Brackets

Let f, g and h three smooth vector fields defined on $\mathbb{R}^n$ and α, β smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$. The Lie brackets satisfies the following properties.

1. skew-symmetry:

$$[f, g] = -[g, f] \tag{1.8}$$

2. Jacobi identity:

$$[f, [g, h]] + [h, [f, g]] + [g, [h, f]] = 0 \tag{1.9}$$

3. chain rule:

$$[\alpha f, \beta g] = \alpha\beta[f, g] + \alpha(L_f \beta)g - \beta(L_g \alpha)f, \tag{1.10}$$

where $L_f\alpha$ and $L_g\beta$ are the Lie derivatives of the functions α, β along the vector fields f and g respectively.

1.2.7 Lie Algebra

A vector space V (*over* $\mathbb{R}$) is a *Lie algebra* if there exist a bilinear operation from $[\ ,\ ]\ V \times V \to V$, $[\ ,\ ]$ is the operation satisfying the skew-symmetry and Jacobi identities enumerated above.

1.2.8 Distribution

A distribution is a map Δ on M assigning to $p \in M$ a linear subspace of T_pM: $\Delta(p) \subset T_p(M)$. The distribution Δ is smooth if it is spanned at each point p by a set of smooth vector fields $X_1, \ldots, X_m \in \mathfrak{X}(M)$.

$$\Delta(p) = span\{X_1(p), \ldots, X_m(p)\} \subset T_p\mathbb{R}^n \tag{1.11}$$

The distribution Δ is regular if its dimension is constant everywhere on the manifold M. Δ is said to be *involutive* if the Lie bracket of any pair of vector fields defined at a point in Δ is a vector field in Δ. We say that Δ is closed under bracketing. Frobenuis theorem is a consequence of this. Which basically says that *a regular distribution is integrable if and only if it is involutive.* This theorem is fundamental to geometric theory of control. In fact one could say that it is to control theory what the Caushy-Lipschitz existence theorem for ordinary differential equation is to the theory of autonomous model found in analytical mechanics, electrical theory etc [16]. A constant dimension k distribution Δ is said to be *integrable*, if for every point $p \in \mathbb{R}^n$, there is a set of smooth functions $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n-k$ such that the row vector $\frac{\partial h_i}{\partial p}$ are linearly independent at p and for every vector field f in Δ

$$L_f h_i = \frac{\partial h_i}{\partial p} f(p) = 0 \quad i = 1, \ldots, n-k. \tag{1.12}$$

The hypersurfaces defined by the level sets

$$\{p : h_1(p) = c_1, \ldots, h_{n-k}(p) = c_{n-k}\}$$

are called *integral manifolds* for the distribution. An immersed submanifold S is an integral manifold of an involutive manifold Δ if the tangent space $T_p(S)$ at every point of S coincides with the distribution Δ at the point p. Thus Δ is integrable if there exists an integral manifold. The involutive closure $\overline{\Delta}$ of Δ is the closure of Δ under bracketing. $\overline{\Delta}$ is the smallest distribution that contains Δ such that if $X, Y \in \Delta$ then $[X, Y] \in \overline{\Delta}$. The notion of distribution and codistribution is a key concept in geometric control. Everything is almost done around it.

1.2.9 Orbit of Points on a Manifold

An orbit of a point p on a manifold M under the set $\mathfrak{D}$ of vector fields is the set of all points reached starting from p by flowing along the integral curves in either forward and backward in time in all possible sequences. Let us define an integral curve for a vector field X as curve $c : [0, T] \mapsto M$ such that $\dot{c}(t) = X(c(t))$, $c(0) = p$. The flow of X was also defined as $\Phi_t^X = \Phi(t, x) = c(t)$. Here we slightly change notation of the flow. We write the flow of X as $exp_X(t, x) = e^{tX}(x) = c(t)$. It is just a notation that is easier to manipulate. Let $X_1, \ldots, X_k$ be vector fields of $\mathcal{D}$. A point p on the manifold M is sent to a point obtained by flowing a short period of time t_k along the vector field X_k, and then for time t_{k-1} along X_{k-1} and so on down to X_1 for time t_1 defines a subgroup of M generated by elements of the form

$$e^{t_1 X_1} \circ \cdots \circ e^{t_k X_k}(p), \quad t_1, \ldots, t_k \in \mathbb{R}, \;\; k \in \mathbb{N}. \tag{1.13}$$

Each group transformation is a diffeomorphism where it is defined [69]. For a family of vector fields, a subgroup of the diffeomorphism group of M generated by the elements of the form given in (1.13) is denoted by Diff($\mathcal{D}$). The orbit through a point p on M is denoted by

$$\mathcal{O}rb(p, \mathcal{D}) = \{\zeta(p) | \ \zeta \in Diff(\mathcal{D})\}. \tag{1.14}$$

But since in control system analysis, time is counted to be positive, the orbit in this case is restricted to positive time and denoted by

$$\mathcal{O}rb^{+}(p, \mathcal{D}) = \{\zeta(p) | \ \zeta \in Diff^{+}(\mathcal{D})\}. \tag{1.15}$$

where $Diff^{+}(\mathcal{D})$ is the semi-group generated by

$$e^{t_1 X_1} \circ \cdots \circ e^{t_k X_k}(p), \quad t_1, \ldots, t_k \geq 0, \ \sum_{k=1}^{m} t_k = T \ \ X_1, \ldots, X_k \in \mathcal{D}, \ \ k \in \mathbb{N}. \tag{1.16}$$

From (1.5), if $\epsilon = \frac{T}{4}$ we have

$$c(T) = p + \frac{T^2}{16}[X, Y] + h.o.t \tag{1.17}$$

where $h.o.t$ are higher order terms.

1.3 Lagrangian Mechanics

Let V be a finite dimensional vector space over $\mathbb{R}$ and V^* the dual space to V. V^* is the set of all linear maps form V to $\mathbb{R}$. Since V is finite dimensional there always exists a basis and therefore there always exists an inner product on V which is a metric that is bilinear, symmetric and positive definite. An inner product thus defines a map from $V \times V$ to $\mathbb{R}$. With respect to some basis chosen in V, the inner product can be defined as $g(v, w) = v^T A w$, where A is a symmetric positive definite matrix. It is always possible to define the sum of two vector spaces V and W at a point as the set of pairs (v, w) with v, w elements of V and W respectively. Operations such as addition and multiplication by a scalar α are defined component-wise as:

$$\begin{aligned} (v, w) + (v', w') &= (v + v', w + w') \\ \alpha(v, w) &= (\alpha v, \alpha w) \end{aligned} \tag{1.18}$$

If we consider two n-dimensional vector spaces V and W with bases $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ then $A \cup B$ is a basis of $V \oplus W$.

It is sometimes convenient to manipulate vectors fields together with their covector fields which is extremely useful in control algorithm design. For a vector space V; the dual space V^* is defined as the set of linear maps $V \to F$, where F is the base field $\mathbb{R}$ or $\mathbb{C}$. Naturally, V^* is isomorphic to V. We will mostly be concerned with the case $F = \mathbb{R}$. If $e_1, \ldots, e_n$ is a basis of V and $e^1, \ldots, e^n$ is a basis of V^* then we have:

$$\langle e_i, e^j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

A Riemannian manifold (M, g) consists of a smooth manifold $C^\infty - manifold\ M$ and an Euclidean inner product g_p defined on each tangent spaces T_pM of M. So for any two smooth vector fields X,Y, the inner product $g_p(X(p), Y(p))$ is a smooth function of p. Whenever we have a finite dimensional vector space V with an inner product defined on it, we can construct a Riemannian manifold by defining

$$\begin{aligned} g : V \times V &\to \mathbb{R} \\ ((p, v), (p, w)) &\mapsto g((p, v), (p, w)) = v \cdot w \end{aligned} \tag{1.19}$$

Let Q be the configuration manifold of a dynamical system of dimension n. We assume that Q is smooth. We write TQ the tangent bundle (the set of all possible velocities on this configurations) and $(q, v) \in T_qQ$ is a point in the bundle; v is a tangent vector at q. The cotangent bundle T^*Q of Q is defined as the dual of TQ.

We define a Riemannian metric on a Q as a smooth function that associates to each tangent space T_qQ at a point q an inner product noted by $\langle\langle,\rangle\rangle_q$. We define the kinetic energy K of the system is an inner product from $T_qQ \to \mathbb{R}$ with metric $M(q)$, the inertia matrix (*a symmetric positive definite matrix*) of the system defined on Q. A mechanical system on Q is described by the *Lagrangian* L, which is the kinetic energy $K : T_qQ \to \mathbb{R}$ minus the potential energy $V(q)$. K is defined by

$$K(\dot{q}, q) = \frac{1}{2}\dot{q}^T M(q)\dot{q} \tag{1.20}$$

The Lagrangian is then defined as $L : TQ \to \mathbb{R}$ by

$$L(q, v) = \frac{1}{2}v^T M(q)v - V(q). \tag{1.21}$$

In the presence of external forces $F : TQ \to T^*Q$, the equations of motion for a Lagrangian system are given by

$$\sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)\delta q^i = \sum_{i=1}^{n} F_i(q, v)\delta q^i \qquad \delta q \in T_qQ \tag{1.22}$$

Equation (1.22) can be found in any classical mechanics book for example [29] [64]. By substituting equation (1.21) in (1.22) we obtain the following equation in terms of the inertia matrix $M(q)$ given by

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \frac{\partial V}{\partial q} = F, \tag{1.23}$$

where F is a column $n-$*components* vector of element in $\mathbb{R}$ and $C(q, \dot{q})$ is the coriolis and centrifugal matrix given by

$$C_{ij}(q, \dot{q}) = \left(\tfrac{\partial M_{i,j}}{\partial q_k} + \tfrac{\partial M_{i,k}}{\partial q_j} - \tfrac{\partial M_{k,j}}{\partial q_i}\right)\dot{q}_k$$

The assumption that was made during this derivation is that the sum of work done internally by the constraints forces is equal zero. A total account can also be found for example in [61], [72].

In geometric control, the notion of symmetry plays an important role in the development of control algorithms. Sometimes *Lie group* theory is essential. There are some good references for example [1], [58] and [69].

Definition: A *group (G,*)* is a set G with a binary operation

$$* : G \times G \to G,$$

and a unit $e \in$ G, possessing the following properties:

1. Unital: for g $\in$ G, we have $g * e = e * g = g$
2. Associative: for $g_1, g_2, g_3 \in G$, $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$
3. Inverses: for g$\in G$, there exists $g^{-1} \in G$ so that $g * g^{-1} = g^{-1} * g = e$.

Let G be a Lie group, the essential feature of Lie theory is that for any Lie group G one can associate a *Lie algebra* $\mathfrak{g}$, a vector space equipped with [,], a bilinear product is called *Lie bracket*. The bridge between a Lie algebra $\mathfrak{g}$ and its Lie group G is the exponential map $exp : X \in \mathfrak{g} \to exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!} \in G$. Lie groups appear mostly as matrix groups. A very good source is Application of Lie groups to Differential Equations by Olver [69]. Let $\gamma : [t_1, t_2] \to M$ a curve on the configuration manifold Q and $\dot{\gamma}(t)$ is the tangent vector to γ at a point q on Q. The trajectories are the solutions of the variational equations $\delta I[\gamma] = 0$. $I[\gamma]$ is the functional defined as [10],[59]

$$I[\gamma] = \int_{t_1}^{t_2} L(q, \dot{q}) dt \tag{1.24}$$

where $\gamma(t_1)$ and $\gamma(t_2)$ are fixed endpoints. These well-known variational equations are equivalent to the Euler-Lagrange equations

$$\begin{aligned} \frac{dq^i}{dt} &= v^i \\ \frac{d}{dt}\frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial q} &= 0 \end{aligned} \tag{1.25}$$

$$I[\gamma] = \int_0^1 L(q, \dot{q})dt = \int_0^1 \tilde{\gamma}^*(L)dt. \tag{1.26}$$

Note the parametrization of the segment $[t_1, t_2]$.

The curve $t \to \tilde{\gamma}(t)$ is the *lift* of $\gamma(t)$ on Q to the tangent space TQ. In the same spirit we can lift a group action from a manifold to a tangent bundle to that manifold.

Let G be a Lie group and $\mathfrak{g}$ its associated Lie algebra as we can *always do.* Let us define a left group action of G on the configuration manifold Q by $\Phi_g : Q \to Q$ and $\Phi_{g^*} : TQ \to TQ$ the *lifted* action of G on TQ and $\xi_Q : Q \to TQ$ infinitesimal generator defined by

$$\xi_Q(q) = \frac{d}{ds}\Phi_{e^{\xi s}}|_{s=0} \quad \xi \in \mathfrak{g} \tag{1.27}$$

In order words for $\xi \in \mathfrak{g}$, we can write $\xi_Q(q)$, a vector field on Q for the corresponding infinitesimal generator which is obtained by differentiating the flow $\Phi_{exp(t\xi)}$ with respect to time at $t = 0$ [11] [59]. $\mathfrak{g}$ is the Lie algebra of the Lie group defined above. The orbit of a point q on the configuration manifold under the action of G is denoted by $Orb(q)$ and $T_qOrb(q)$ the tangent to group orbit through q, which is given by the set of infinitesimal generators at q is given by

$$T_qOrb(q) = \{\xi_Q(q)| \quad \xi \in \mathfrak{g}\} \tag{1.28}$$

Assuming that the group action of G on Q is free *i.e.* the maps $\Phi : G \times Q \to Q$, $\Phi(g, q) = gq$ has no fixed point and the maps are proper *i.e.* inverse image of compact sets are compact. Bloch *et.al* defined the quotient space $\Xi = Q/G$ as the space whose points are the group orbits, and the quotient space is called *shape space*. With the assumption made about free action and proper, the shape space is a smooth manifold and the projection map $\pi : Q \to Q/G$ is a smooth map. The projection is surjective otherwise the quotient space is not possible. The surjective derivative $T_q\pi$ is the infinitesimal generators of the group action at the point q. The Lagrangian L is said to be invariant under the group G action when

$$L(\Phi_g(q), \Phi_{g^*}(v)) = L(q, v), \quad \forall g \in G \tag{1.29}$$

In classical mechanics, symmetry is defined as invariance of the Lagrangian under a group action. This implies (via Noether's theorem) the existence of conserved quantities. This conservation law is expressed as

$$\frac{d}{dt}\langle\frac{\partial L}{\partial \dot{q}}, \xi_Q\rangle = 0 \quad \xi \in \mathfrak{g} \tag{1.30}$$

For the general momentum $p_j = \frac{\partial L}{\partial \dot{q}_j}$, be a conserved quantity, the corresponding torque τ_j and $\frac{\partial V}{\partial q_j}$ must vanish. Let us explain what (1.30) basically means. The Lagrangian is defined on the tangent space TQ of the configuration manifold of Q so the generalized momentum $p = \frac{\partial L}{\partial \dot{q}}$ is defined on cotangent space T^*Q which is the dual space of the tangent bundle TQ. Naturally we know that if V is any vector space and $F : V \mapsto \mathbb{R}$, a smooth real-valued function, then the gradient $\frac{\partial F}{\partial v}$ at any point of V is an element of the dual space V^* consisting of all continuous linear functions on V. By definition we have [69]

$$\langle\tfrac{\partial F(v)}{\partial v}, w\rangle = \lim_{t\to 0} \tfrac{F(v+tw)-F(v)}{t}$$

So applying this naturally to the momentum map we have

$$\langle \frac{\partial L}{\partial \dot{q}}, \xi_Q \rangle = \mu = constant \quad \xi \in \mathfrak{g}, \tag{1.31}$$

which is a constant because of conservation and its derivative must vanish. The invariance of the Lagrangian under a group action is a technique used in [11],[59] to reduce a high dimensional mechanical system to a lower order for analysis. This is a powerful method of reduction of dynamical systems. This is done by constructing the *shape space* by taking the quotient of the configuration space of a mechanical system by the symmetry group. The reduction method is treated in [58]. In the analysis of underactuated mechanical systems which is applied to the CMG, we will use the symmetry of the kinetic energy with respect to *external* variables. The notion of shape variables and external variables are also defined in the study of multi-body and interconnected mechanical systems in [49],[50],[51],[54].

1.4 Underactuated Systems

Tracking problem for robot manipulators has been studied extensively since the early 1980's. The contributions [7, 75, 92], where asymptotic, exponential and adaptive tracking made their way into standard control system texbooks such as [64, 67] and reference therein. Analysis and control of fully actuated dynamical system have received a tremendous amount of attention. In the past two decades, geometric methods applied for tracking control of these systems were considered [20]. When the number of configuration variables is greater than the number of inputs, the system is called underactuated. Such systems are for example satellites, car-like mobile robots. Mechanical systems are sometimes also subject to constraints. Constraints are of all types. Here we are concerned with kinematic and dynamics constraints. Oftentimes, the kinematic constraints cannot be integrated to obtain configuration variables. Systems subject to these nonintegrable constraints are called *nonholonomic* systems. In the 1990's, studies were limited primarily to systems with nonintegrable *kinematic constraints*; for example [13, 47]. There are many cases where *nonintegrable dynamic* relations arise. The control problem of such systems were open problems until a solution to their controllability and stability was proposed in [71] where the authors use the collocated partial feedback linearization proposed by Spong [81]. In the section that follows, we will lay the ground to study controllability of nonlinear systems.

1.5 Controllability of Nonlinear Systems

The most common problems considered in the design of control systems are the *controllability,observability* and *stabilization* and have been extensively studied. A control system consists of 4-tuple $(M, C, f, \mathcal{U})$ where

(a) M is a smooth manifold, which is the state space of the system

(b) C is a set called the control space

(c) f is a mapping which assigns to $x \in M$, $u \in C$, a tangent vector f(x,u) to M at x,

(d) $\mathcal{U}$ is a class of functions defined on [0, T] which is the time interval. The elements of $\mathcal{U}$ are called admissible controllers.

More details are given in [87]. If a function $\overline{u} : [0, T] \to C$ is an admissible controller, if a trajectory of the system corresponding to $\overline{u}$ is an absolutely continuous curve $t \to \overline{x}(t)$, $0 \le t \le T$ such that

$$\frac{d\overline{x}}{dt}(t) = f(\overline{x}(t), \overline{u}(t)) \tag{1.32}$$

for almost every $t \in [0, T]$. And if $t \to \overline{x}(t)$ is a trajectory corresponding to the controller $\overline{u}$, with $\overline{x}(0) = x_0$, $\overline{x}(T) = x_1$, then we say that $\overline{u}$ steers x_0 into x_1. If there exists some controller $\overline{u}$ which steers x_0 to x_1 then we say that x_1 is reachable from

x_0. $\mathcal{R}(x_0)$ is the set of all points in M reachable from x_0. The system $(M, C, f, \mathcal{U})$ is *completely controllable* if $\mathcal{R}(x_0) = M$.

In order to make general controllability statements, we will make the following assumptions [87]:

(A) The class $\mathcal{U}$ of admissible controllers contains the class $\mathcal{U}_0$ of all piecewise constant C-valued functions on intervals of the form [0, T]. If $\overline{u}$ is an admissible controller defined on [0, T], and if $0 < \mathrm{t} < \mathrm{T}$, then the function $\overline{v}$ defined on $[0, T - t_0]$ by $\overline{v}(\mathrm{t}) = \overline{u}(\mathrm{t} + t_0)$ is also an admissible controller.

We also assume that the function f is regular in the sense that equation (1.32) have solutions at least when the controller $\overline{u}$ is constant or piecewise constant. If the function f is real analytic, we have the following:

(B) M is a real analytic manifold and, for each $u \in C$, the map $x \in f(x, u)$ is a real analytic vector field on M.

It is also possible that trajectories corresponding to an arbitrary controllers can be approximated by trajectories corresponding to piecewise controllers. Therefore the following is also true

(C) If $t \to x(t)$, $0 \leq t \leq T$ is a trajectory corresponding to an admissible controller $\overline{u}$, then there is a sequence $\{\overline{x}_n\}$ of curves, defined on [0, T], such that $\overline{x}_n(0) = \overline{x}(0)$, that each $\{\overline{x}_n\}$, is a trajectory corresponding to a piecewise constant controller $\{\overline{u}_n\}$, and that $\overline{x}_n(t) \to \overline{x}(t)$ as $n \to \infty$ uniformly, for $0 \leq t \leq T$.

The following theorem results from assumptions made above

Theorem: Let $\Sigma_1 = (M, C, f, \mathcal{U})$ be a control system that satisfies hypotheses (A), (B), (C) above. Let $\Sigma_2 = (M, C, f, \mathcal{U}_0)$, i.e., Σ_2 is the same as Σ_1 , except for the fact that the class $\mathcal{U}$ of controllers is replaced by the class $\mathcal{U}_0$ of piecewise constant C-valued functions. Then Σ_1 is completely controllable if and only it Σ_2 is completely controllable. For proof of this theorem please refer to [87]. Let us redefine the reachable set $\mathcal{R}(z)$ from z as the set of all states that can be reached starting from z using all possible piecewise constant controllers. If $T > 0$ is specified and time can be arbitrary small, then we have *small-time controllability.* If the states are constrained to be closed to an equilibrium point, one talks about *local controllability* at that equilibrium point $(x_e, u_e) \in M \times C$. In this case we have $f(x_e, u_e) = 0$. So if time T is small for local controllability, we will speak about *small-time local controllability.* The problem of stabilization can roughly be defined as follows. Let us consider the system in equation(1.32) and let us assume zero (0,0) is an equilibrium point (f(0,0)=0). The problem of stabilization is to find a feedback control law $x \to u(x)$ such that the equilibrium point 0 is asymptotically stable for the closed loop system $\dot{x} = f(x, u(x))$.

1.5.1 State and Output Feedback

There are many situations where all states are not available for feedback so state feedback cannot be performed; in this case a fraction of the states are available for measurement. However, in this case, one can possibly recover the observable states by observing the output. We denote the measured states by $z = h(x)$ and call it the output of the system. In this case only output feedback is allowed. Roughly speaking a control system is observable if all its states can be recovered by applying some suitable controls and only by looking at the output for a period of time.

In the early 1960's, Kalman and others studied controllability and observability of linear systems. A tremendous amount of work has been done in that area since then. Controllability and observability of linear system are now a matured subject. The study of these concepts for the nonlinear analog systems started in the early 1970's. For linear systems, controllability implies stabilization which is not necessarily the case for nonlinear systems. However, for nonlinear systems, the global version does not hold. This was proved in 1979 by Hector Sussmann [87] who gave an example of a nonlinear system which is globally controllable but fails to be globally asymptotically stabilized by means of continuous stationary feedback laws. In 1983, Brockett [17] showed that the local version fails to hold even for analytic control systems. The first theorem in latter paper is given below.

Theorem:[17] Let $\dot{x} = f(x,u)$ be given with $f(x_0, 0) = 0$ and f(.,.) continuously differentiable in a neighborhood of $(x_0, 0)$. A necessary condition for existence of a continuously differentiable control law which makes $(x_0, 0)$ asymptotically stable is that:

(i) the linearized system should have no uncontrollable modes associated with eigenvalues whose real parts are positive.

(ii) there exists a neighborhood N of $(x_0, 0)$ such that for each $\xi \in N$ there exists a control $u_\xi(.)$ defined on $[0, \infty)$ such that this control steers the solution of $\dot{x} = f(x, u_\xi)$ from $x = \xi$ at t=0 to $x = x_0$ at $t = \infty$.

(iii) the image of the mapping $\gamma : (x, u) \to f(x, u)$ contains some neighborhood of zero.

The last condition states that there exists $\delta > 0$ such that $\forall\ |\zeta| \leq \delta$, there exist x, u such that $f(x, u) = \zeta$. Intuitively this means that no energy should need to be pumped into the system when it reaches the equilibrium point. Because of the importance of this theorem, we will elaborate a little bit on the conditions and even give some examples. Before considering asymptotic stabilization by a *time-invariant* control law, one has to check that this stabilization is even possible and that at what conditions. For the first condition in the theorem, it is well known from Lyapunov's method that an equilibrium point is unstable if $\partial f/\partial x$ at that equilibrium point has any eigenvalue with real part which is positive. However, it is worth mentioning that if the first condition is not satisfied, only feedbacks that have continuous partial derivatives and which vanish at the equilibrium point can be excluded. If the linearized control system $\dot{x} = Ax + Bu$ is controllable, then we say that $\dot{x} = f(x, u)$ is

Small Time Locally Controllable (STLC) at (x_e, u_e). The system $\dot{x} = f(x, u)$ is small time locally controllable (STLC) at zero if starting from zero at $t = 0$ and using an arbitrary small control, one can reach in an arbitrary small time a neighborhood of zero [23]. STLC is of interest to control theory researchers for a few reasons given by Hector Sussmann in [86]. The condition (ii) in theorem 1.5.1 is obviously necessary if one assumes f and u continuous. It was proven by Roger Nussbaum in [77] and J. Zabczyk in [94] that condition (iii) is also necessary if f and u are only continuous provided that (i) is satisfied. Consider the following example [8];

$$\begin{aligned} \dot{x} &= (2x_1 - x_2)u \\ \dot{x}_2 &= x_1 u + x_2 \end{aligned} \tag{1.33}$$

The input matrix of the linearized model at the equilibrium point is zero $(x, u) = (0, 0)$. So for each K we have

$$A + BK = A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.34}$$

which has a positive eigenvalue. Nevertheless the system can be asymptotically stabilized by a constant $u = u_0 < -2$. Kawski [44] has shown that even when the first condition does not hold, one cannot exclude the existence of continuous stabilizing feedbacks. We spent time on the first condition because we will rely on it for the study of controllability and stabilization of the CMG in the upright equilibrium position. The third condition is a topological condition. Please refer to [8] for more details and examples. Another necessary topological condition for continuous feedback stabilization of nonlinear control system is given by Coron in [23]. Coron's main results show that instead of considering *time-invariant* control law $x \to u(x)$, it is better to consider *time-varying* feedback laws $(x, t) \to u(x, t)$ which are *periodic* with respect to time and this condition is given for driftless control affine systems. An affine control system is given by

$$\dot{x} = f(x, u) = \sum_{i=1}^{i=m} u_i f_i(x). \tag{1.35}$$

the following theorem

Theorem:[24] Assume that

$$\{g(x); g \in Lie\{f_1, \ldots, f_m\}\} = \mathbb{R}^n, \forall y \in \mathbb{R}^n \backslash \{0\} \tag{1.36}$$

Then, for every T>0, there exists u in $C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^m)$ such that

$$u(0, t) = 0, \forall t \in \mathbb{R}, \tag{1.37}$$

$$u(x, t + T) = u(x, t), \forall y \in \mathbb{R}^n, \forall t \in \mathbb{R}, \tag{1.38}$$

and 0 is globally asymptotically stable for the closed-loop system

$$\dot{x} = f(x, u(x, t)) = \sum_{i=1}^{i=m} u_i f_i(x). \tag{1.39}$$

The equation(1.36) is the global controllability of the driftless control system which is Chow's theorem [22]. This condition is also a necessary condition for small-time locally controlability at the equilibrium.

CHAPTER 2: MODELING AND CONTROL OF WHEELED MOBILE ROBOTS

In this chapter we will provide techniques used to model the mechanical systems considered here. When a mechanical system is subject to constraints, additional variables are introduced in the equation of motion to enforce the constraints. The introduction of new variables complicate further the differential equation. Quasivelocity method is used to reduce the number of variables introduce in the presence of constraints. Quasivelocities modeling techniques are used to model a class of dynamical system considered in this chapter.

2.1 Quasivelocities Modeling Techniques

Often-times, the motion of a mechanical system is subject to restrictions in its configuration space. Sometimes, the constraints are imposed on the velocity . It often happens that the kinematic constraints cannot be written as time derivatives of some functions of the generalized coordinates. Mechanical systems with such constraints are called nonholonomic mechanical systems. There is an extensive literature on techniques to derive equations of motion of nonholonomic systems. Because of the nonintegrability, the constraints cannot be imposed on both the curves in the configuration space and the variational curves. The use of the fundamental Lagrange's techniques introduce additional variables to account for the constraints and thus increases the dimension of equations to solve. Quasivelocity techniques such as Maggi's equation and Boltzmann-Hamel's equation eliminate Lagrange multipliers from the beginning as opposed to fundamental Lagrange method where it is required to solve for the configuration variables and the multipliers as functions of time. Thus quasivelocity techniques produce fewer dynamical equations of motion [30]. Using

Boltzmann-Hamel's method, the kinetic energy is formulated for the unconstrained system first and then the kinetic energy is rewritten as a function of configuration variables and quasivelocities. Quasivelocity methods are the generalization of the Lagrangian and the Hamiltonian systems where only position constraints are allowed. While using quasivelocity techniques, it is preferable to select the free quasivelocities as simple as possible and at the same time assure that they span the constraint distribution as there is no set procedure for the selection. For a system of n generalized coordinates, and m nonholonomic constraints, $m < n$, one first defines the m quasivelocities μ_j , $j = 1, \ldots, m$ such that they span the constraint distribution. The constraints then reduce to $\mu_j = 0$, and one is left to solve for the $n - m$ independent quasivelocities in addition to the n integrable kinematic relations to produce the motion curve. It thus requires a total of $2n - m$ differential equations compared to $2n + m$ equations using the classical Lagrange's equations [30]. For nonholonomic mechanical systems, Boltzmann-Hamel's equations are generalized to a form suited for kinematic optimal control for kinematically actuated systems. The development of the Boltzmann-Hamel optimal control equations can be thought of as a generalization of the Euler-Poincaré method [60]. As stated above, because of nonintegrability of the constraint distribution, it is necessary to redefine the velocity along the variational curves and this is called extended velocity [65], [66].

2.1.1 Maggi's Equation of Motion

Consider the diagram of the Differential Wheeled Mobile Robot (DWMR) depicted in figure 2.1. There are three bodies involved; the platform of the robot and the wheels. There are four degrees-of-freedom associated with the DWMR. The coordinates (x_p, y_p) of the midpoint of the axle, the heading angle ϕ, the mobile robot's right wheel's rotation through θ_r, and the robot's left wheel's rotation through angle θ_l. A vector of generalized coordinates for the mobile robot is thus given by $\underline{\gamma} = [x_p, y_p, \phi, \theta_r, \theta_l]^T$, where the subscript T stands for transpose. Using *Koenig's* *theorem*, the kinetic energy is written as in (2.1). Here, it is convenient to select the midpoint P of the axle of the mobile robot as the reference point.

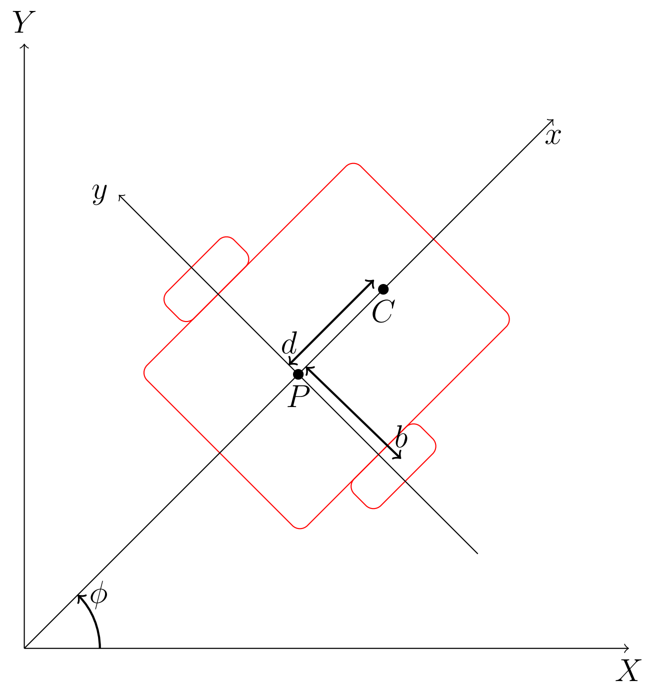

Figure 2.1: Diagram of a differential mobile robot.

$$T = \frac{1}{2}\sum_{i=1}^{3} m_i v_i^2 + \frac{1}{2}\sum_{i=1}^{3} \omega_i I_i \omega_i + \sum_{i=1}^{3} m_i v_i.\dot{\rho}_i. \tag{2.1}$$

where v_i is the velocity of the reference point of the ith body, ρ_i is the position of the center of mass of the ith body relative to P expressed in a translating frame placed at P, and ω_i is the angular velocity of the body measured in the body's coordinate frame. The rotation matrix ${}^I T_B$ which transforms body coordinates to inertial coordinates is given by

$${}^I T_B = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.2}$$

The coordinates of the left wheel W_L and right wheel W_R expressed in the inertial frame are given respectively by

$$
{}^{I}W_L = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & x_p \\ \sin(\phi) & \cos(\phi) & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ b \\ 1 \end{bmatrix} = \begin{bmatrix} x_p - b\sin(\phi) \\ y_p + b\cos(\phi) \\ 1 \end{bmatrix}. \tag{2.3}
$$

$$
{}^{I}W_R = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & x_p \\ \sin(\phi) & \cos(\phi) & y_p \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -b \\ 1 \end{bmatrix} = \begin{bmatrix} x_p + b\sin(\phi) \\ y_p - b\cos(\phi) \\ 1 \end{bmatrix}. \tag{2.4}
$$

The velocity of the left and right wheels are computed respectively and are given by

$$
{}^{I}\dot{W}_L = \begin{bmatrix} \dot{x}_p - b\dot{\phi}\cos(\phi) \\ \dot{y}_p - b\dot{\phi}\sin(\phi) \\ 0 \end{bmatrix}. \tag{2.5}
$$

$$
{}^{I}\dot{W}_R = \begin{bmatrix} \dot{x}_p + b\dot{\phi}\cos(\phi) \\ \dot{y}_p + b\dot{\phi}\sin(\phi) \\ 0 \end{bmatrix}. \tag{2.6}
$$

To find the kinematic constraint equations, the velocities in equations (2.5) and (2.6)are transformed back to the body coordinate frame by multiplying equations (2.5) and (2.6) by the transpose of ${}^{I}T_B$. They are given by

$$
{}^{B}\dot{W}_R = \begin{bmatrix} \dot{x}_p\cos(\phi) + \dot{y}_p\sin(\phi) + b\dot{\phi} \\ -\dot{x}_p\sin(\phi) + \dot{y}_p\cos(\phi) \\ 0 \end{bmatrix}. \tag{2.7}
$$

$$ {}^{B}\dot{W}_L = \begin{bmatrix} \dot{x}_p \cos(\phi) + \dot{y}_p \sin(\phi) - b\dot{\phi} \\ -\dot{x}_p \sin(\phi) + \dot{y}_p \cos(\phi) \\ 0 \end{bmatrix}. \quad (2.8) $$

The constraints of rolling without slipping implies that the velocity of a wheel is equal to the angular velocity of the wheel about its axis multiplied by the radius of the wheel and the side velocities of the wheels must equal zero. In other words the velocity of the wheels expressed in the body coordinates frame must be given by

$$ \begin{aligned} \dot{x}_p \cos(\phi) + \dot{y}_p \sin(\phi) - b\dot{\phi} &= r\dot{\theta}_l \\ \dot{x}_p \cos(\phi) + \dot{y}_p \sin(\phi) + b\dot{\phi} &= r\dot{\theta}_r \\ -\dot{x}_p \sin(\phi) + \dot{y}_p \cos(\phi) &= 0. \end{aligned} \quad (2.9) $$

The constraints can be written in matrix vector form as (Pfaffian constraints with $q = [x_p, y_p, \phi, \theta_r, \theta_l]^T$)

$$ A(q)\dot{q} = 0 $$

From the constraints equation, we see that $\dot{q}$ is in the null space of $A(q)$. $\dot{q}$ can be written as a linear combination vector fields which span the null space of $A(q)$. To find vector fields which span the null, one usually uses reduced row echelon form and compute linearly independent vector fields. The vector fields obtained can be arranged as columns of $S(q)$ so that we can write

$$ \dot{q} = S(q)v \quad (2.10) $$

where v is a set of independent velocity variables. With insight into the system to control, the velocity variables are chosen as control input. In the present case, the inputs are chosen as the driven velocity $v_1 = v_x$, the forward velocity, and the steering velocity $v_2 = \dot{\phi}$. It is shown that two of the three constraints are nonholonomic and one is holonomic because taking the difference between the first two equation(2.9), one can integrate the resulting equation to get a function of configuration space variables. By transforming the constraints matrix A into a reduced row echelon form we can deduce two vector fields that span the space orthogonal to the space spanned by the constraints. The computation of these vector fields will be done later in the manuscript too and will be integrated into dynamics to design a robust control law.

The kinetic energy of the mobile robot using eq.(2.1) is given by

$$T = \frac{1}{2}(m_c + 2m_w)v_p^2 + \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}I_w\dot{\theta}_r^2 + \frac{1}{2}I_w\dot{\theta}_l^2 - m_c d\dot{\phi}(-\dot{x}_p \sin(\phi) + \dot{y}_p \cos(\phi)). \tag{2.11}$$

where $I = I_c + 2I_m + 2m_w b^2 + m_c d^2$, and m_c, m_w are the masses of the platform and the mass of the wheels respectively.

2.2 Dynamics of the Differential Wheeled Mobile Robot

In general, the dynamical equations of motion of a mobile robot with n generalized coordinates $\underline{\gamma} = [\gamma_1 \dots \gamma_n]^T$ subject to m constraints can be written as shown in (2.12) by applying Euler-Lagrange's equation

$$\underline{H}(\underline{\gamma})\ddot{\underline{\gamma}} + \underline{D}(\underline{\gamma}, \dot{\underline{\gamma}}) + \underline{G}(\underline{\gamma}) = \underline{B}(\underline{\gamma})\underline{\tau} - A^T(\underline{\gamma})\underline{\lambda} \tag{2.12}$$

The description of each component can be found in any classical dynamics textbook. In the case considered here the gravitational vectors are equal to zero. The equation of motion of the DWMR can be written as

$$\underline{H}\left(\underline{\gamma}\right)\ddot{\underline{\gamma}} + \underline{D}\left(\underline{\gamma}, \dot{\underline{\gamma}}\right) = \underline{B\tau}. \tag{2.13}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\gamma}_i}\right) - \left(\frac{\partial T}{\partial \gamma_i}\right) = Q_i + \sum_{j=n-m+1}^{n} \lambda_i \psi_{ji} \quad (i = 1 \ldots n). \tag{2.14}$$

In general, there is no set procedure on selecting quasivelocities. For a mechanical system of n generalized coordinates with m nonholonomic constraints, n quasivelocities are selected such that m of them span the constraint space, and (n-m) are independent quasivelocities. During the selection, one keeps in mind that the matrix Ψ obtained as in (2.15) is invertible with an inverse Φ.

$$\underline{u} = \Psi \underline{\dot{\gamma}} \tag{2.15}$$

where $\underline{u} = [u_1 \ldots u_n]^T$, is a quasivelocity vector. Equation (2.15) can be rewritten as

$$u_j = \sum_{i=1}^{n} \psi_{ji}(\underline{\gamma})\dot{\gamma}_i \quad (j = 1 \ldots n - m) \tag{2.16}$$

$$u_j = \sum_{i=1}^{n} \psi_{ji}(\underline{\gamma})\dot{\gamma}_i = 0 \quad (j = n - m + 1 \ldots n), \tag{2.17}$$

and

$$\dot{\gamma}_i = \sum_{j=1}^{n-m} \phi_{ij}(\underline{\gamma}) u_j \quad (i = 1 \ldots n). \tag{2.18}$$

Note that these equations do not explicitly involve time otherwise there must be additional terms.

Using equation(2.17), virtual displacements are given as

$$\delta\theta_j = \sum_{i=1}^{n} \psi_{ji}(\underline{\gamma})\delta\gamma_i \quad (j = 1 \ldots n - m) \quad \text{or}$$

$$\delta\gamma_i = \sum_{j=1}^{n-m} \phi_{ij}(\underline{\gamma})\delta\theta_j \quad (i = 1 \ldots n). \tag{2.19}$$

The first (n-m) δ_j are selected independently. Using Lagrange's principle of virtual displacement we have

$$\sum_{i=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\gamma}_i} \right) - \left(\frac{\partial T}{\partial \gamma_i} \right) - Q_i \right] \delta\gamma_i = 0 \tag{2.20}$$

with $T(\underline{\gamma}, \underline{\dot{\gamma}})$, the unconstrained kinetic energy defined above. After substituting for the virtual displacement as defined in (2.19), we have

$$\sum_{j=1}^{n-m} \sum_{i=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\gamma}_i} \right) - \left(\frac{\partial T}{\partial \gamma_i} \right) - Q_i \right] \phi_{ij}(\underline{\gamma}) \delta\theta_j = 0. \tag{2.21}$$

Using the fact that the first (n-m) $\delta\theta_j$ are independent, the coefficients of (3.74) must equal zero and we have

$$\sum_{i=1}^{n} \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\gamma}_i} \right) - \left(\frac{\partial T}{\partial \gamma_i} \right) \right] \phi_{ij}(\underline{\gamma}) = \sum_{i=1}^{n} Q_i \phi_{ij}(\underline{\gamma}) \quad (j = 1 \ldots n-m) \tag{2.22}$$

The resulting equation (2.22) is Maggi's equation. Let us explicitly define quasivelocities as

$$\begin{aligned}
u_1 &= \dot{x}_p \cos(\phi) + \dot{y}_p \sin(\phi) \\
u_2 &= \dot{\phi} \\
u_3 &= -\dot{x}_p \sin(\phi) + \dot{y}_p \cos(\phi) \\
u_4 &= \dot{x}_p \cos(\phi) + \dot{y}_p \sin(\phi) + b\dot{\phi} - r\dot{\theta}_r \\
u_5 &= \dot{x}_p \cos(\phi) + \dot{y}_p \sin(\phi) - b\dot{\phi} - r\dot{\theta}_l.
\end{aligned} \tag{2.23}$$

$$\Phi = \begin{bmatrix} \cos(\phi) & 0 & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & 0 & \cos(\phi) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{r} & \frac{b}{r} & 0 & -\frac{1}{r} & 0 \\ \frac{1}{r} & -\frac{1}{r} & 0 & 0 & -\frac{1}{r} \end{bmatrix}. \tag{2.24}$$

Applying (2.22) to the unconstrained kinetic energy and using the first and second columns of Φ, we have

$$\begin{aligned} m\cos(\phi)\ddot{x}_p + m\sin(\phi)\ddot{y}_p + m_c d\dot{\phi}^2 + \frac{I_w}{r}\ddot{\theta}_r + \frac{I_w}{r}\ddot{\theta}_l &= \frac{\tau_r}{r} + \frac{\tau_l}{r} \\ m_c d\cos(\phi)\ddot{x}_p - m_c d\sin(\phi)\ddot{x}_p + \frac{I_w b}{r}\ddot{\theta}_r - \frac{I_w b}{r}\ddot{\theta}_l &= \frac{b\tau_r}{r} - \frac{b\tau_l}{r}. \end{aligned} \tag{2.25}$$

The time derivative of the constraint equations together with (2.25) form the second-order differential equations of motion of the differential mobile robot and is given by

$$\underline{H}\left(\underline{\gamma}\right)\underline{\ddot{\gamma}} + \underline{D}\left(\underline{\gamma}, \underline{\dot{\gamma}}\right) = B\underline{\tau}. \tag{2.26}$$

where

$$\underline{H}\left(\underline{\gamma}\right) = \begin{bmatrix} m\cos(\phi) & m\sin(\phi) & 0 & I_w/r & I_w/r \\ m_c d\sin(\phi) & -m_c d\cos(\phi) & I & I_w b/r & -I_w b/r \\ -\sin(\phi) & \cos(\phi) & 0 & 0 & 0 \\ \cos(\phi) & \sin(\phi) & b & -r & 0 \\ \cos(\phi) & \sin(\phi) & -b & 0 & -r \end{bmatrix}, \tag{2.27}$$

$$\underline{D}\left(\underline{\gamma}, \underline{\dot{\gamma}}\right) = -\begin{bmatrix} -m_c d\dot{\phi}^2 \\ 0 \\ \dot{x}_p\dot{\phi}\cos(\phi) + \dot{y}_p\dot{\phi}\sin(\phi) \\ \dot{x}_p\dot{\phi}\sin(\phi) - \dot{y}_p\dot{\phi}\cos(\phi) \\ \dot{x}_p\dot{\phi}\sin(\phi) - \dot{y}_p\dot{\phi}\cos(\phi) \end{bmatrix}. \tag{2.28}$$

and

$$\underline{B} = \begin{bmatrix} 1/r & 1/r \\ b/r & -b/r \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.29}$$

Let us define the state vector $q = [x_p, y_p, \phi, \theta_r, \theta_l, \dot{x}_p, \dot{y}_p, \dot{\phi}, \dot{\theta}_r, \dot{\theta}_l]^T$ by choosing the state space variables $q_i, 1 \leq i \leq 10$ as

$$\begin{cases} q_i = \gamma_i & \text{for } 1 \leq i \leq 5 \\ q_i = \dot{\gamma}_{i-5} & \text{for } 6 \leq i \leq 10 \end{cases}$$

The system of dynamical equations can be expressed in the form

$$\underline{\dot{q}} = f(\underline{q}) + \underline{g_1}(\underline{q})\tau_1 + \underline{g_2}(\underline{q})\tau_2. \tag{2.30}$$

More explicitly

$$\underline{\dot{q}} = \begin{bmatrix} q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ f_6(q_3, q_6, q_7, q_8) \\ f_7(q_3, q_6, q_7, q_8) \\ f_8(q_3, q_6, q_7, q_8) \\ f_9(q_3, q_6, q_7, q_8) \\ f_{10}(q_3, q_6, q_7, q_8) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ g_{61}(q_3) & g_{61}(q_3) \\ g_{71}(q_3) & g_{71}(q_3) \\ g_{81}(q_3) & -g_{81}(q_3) \\ A & B \\ B & A \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}. \tag{2.31}$$

The variables shown in the matrices above will be given explicitly in the appendix.

2.2.1 Control System Design

By taking the difference between the first and second equations of (2.48), we have

$$\dot{\phi} = \frac{r}{2b}\left(\dot{\theta}_r - \dot{\theta}_l\right). \tag{2.32}$$

Taking time derivative of (2.32), the resulting equation is equal to row 8 of (2.31). From this , we can solve for τ_1 which is given by

$$\tau_1 = \frac{r}{2bg_{81}}\left(\dot{\theta}_r - \dot{\theta}_l\right). \tag{2.33}$$

Upon substitution in (2.31) and noticing that the substitution does not affect the upper half of the equation in (2.31), the lower half of (2.31) is given by

$$\begin{bmatrix} 1 & 0 & 0 & -g_6K & g_6K \\ 0 & 1 & 0 & -g_7K & g_7K \\ 0 & 0 & 1 & -\frac{r}{2b} & \frac{r}{2b} \\ 0 & 0 & 0 & 1-g_{91}K & g_{91}K \\ 0 & 0 & 0 & -g_{10}K & 1+g_{10}K \end{bmatrix} \begin{bmatrix} \dot{q}_6 \\ \dot{q}_7 \\ \dot{q}_8 \\ \dot{q}_9 \\ \dot{q}_{10} \end{bmatrix} = \begin{bmatrix} f_6 - \frac{g_{61}f_8}{g_{81}} \\ f_7 - \frac{g_{71}f_8}{g_{81}} \\ 0 \\ f_9 - \frac{g_{91}f_8}{g_{81}} \\ f_{10} - \frac{g_{10}f_8}{g_{81}} \end{bmatrix} + \begin{bmatrix} 2g_{61} \\ g_{71} \\ 0 \\ g_{91}+g_{92} \\ g_{10}+g_{11} \end{bmatrix} \tau_2 \ , \tag{2.34}$$

where $K = \frac{r}{2bg_{81}}$. g_{81} is a constant so the division by it is well defined. The first half of (2.31) together with (2.34) give a system of "single input". Upon computing the control law for τ_2 one can compute τ_1 and $\ddot{\phi}$ explicitly.

The control law derived in the rest of the dissertation involves the two inputs system shown in (2.31) using dynamic extension.

2.3 Tracking Control Design of the Two Inputs System

The dynamics of the mobile robot are formulated in the state space representation as dynamics of a standard nonlinear system as

$$\underline{\dot{q}} = \underline{f}(\underline{q}) + G(\underline{q})\underline{u} \tag{2.35}$$

Immediate calculations show that $rank\{\underline{g_1}, \underline{g_2}\} = 2$ *i.e.* the vectors $\underline{g_1}, \underline{g_2}$ are linearly independent. The distribution G=$\{\underline{g_1}, \underline{g_2}\}$ is also involutive because simple calculations show that $rank[\underline{g_1}, \underline{g_2}, [\underline{g_1}, \underline{g_2}]]$=2 for all q. where $[\underline{g_1}, \underline{g_2}]$ is the Lie bracket of $\underline{g_1}$ and $\underline{g_2}$ defined by

$$[\underline{g_1}, \underline{g_2}] = \frac{\partial g_2}{\partial \underline{q}}\underline{g_1} - \frac{\partial g_1}{\partial \underline{q}}\underline{g_2} \tag{2.36}$$

Thus by Frobenius' Theorem, we deduce the existence of *8 real-valued functions* that span the space orthogonal to the space spanned by $\{\underline{g_1}, \underline{g_2}\}$. However, it is not always very easy to find these functions because it involves solving partial differential equations. It is well known that systems with at least one nonholonomic constraints are not input-state linearizable [21] and since the system cannot be made asymptotically stable by a smooth feedback [12], we seek feedback control which achieves input-output stability. A natural choice of outputs are the coordinates of the reference point P figure 2.1 which defines a position of the mobile robot in the plane. The output vector is defined as

$$\underline{y} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{2.37}$$

With respect to these outputs, the system does not have a relative degree because immediate calculations show that

$$L_g h = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.38}$$

and

$$A(\underline{q}) = L_g L_f h(\underline{q}) = \begin{bmatrix} g_{61}(q_3) & g_{61}(q_3) \\ g_{71}(q_3) & g_{71}(q_3) \end{bmatrix} \tag{2.39}$$

which has rank 1 for all q. We apply the dynamic extension algorithm. Relative degree r=*{4,3}* is achieved after two iterations of Dynamic Extension Algorithm when the following compensator is cascaded with the system. (Refer for example to [36, 46] for Dynamic Extension).

$$\begin{aligned} \tau_1 &= \frac{1}{g_{61}(q_3)} \left(-f_6(\underline{q}) + \eta\right) - v_2 \\ \dot{\eta} &= \zeta \\ \dot{\zeta} &= v_1 \\ \tau_2 &= v_2 \end{aligned} \tag{2.40}$$

2.4 Simulations of the DWMR Trajectory Tracking

A simulation is developed to demonstrate the effectiveness of the control law designed above. As subsequent figures demonstrate, the tracking control which takes the dynamics based on previous analytical development of the mobile robot into account perform good tracking. One should also note that the control law is valid for $g_{16}(q_3) \neq 0$, i.e.,$q_3 \neq \frac{\pi}{2}(2k+1)$, where k is an integer.

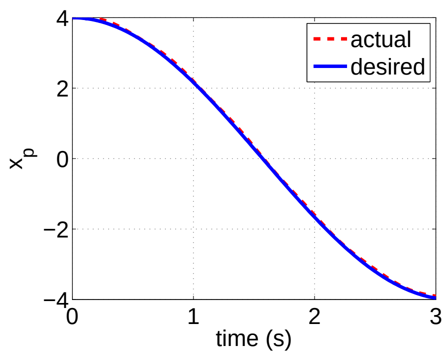

Figure 2.2: x_p and the desired trajectory.

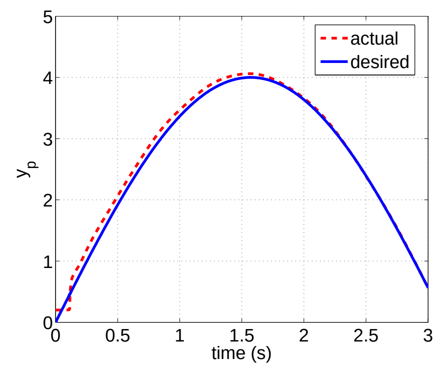

Figure 2.3: y_p and the desired trajectory.

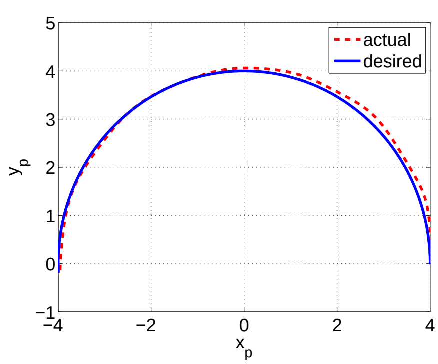

Figure 2.4: Circular path.

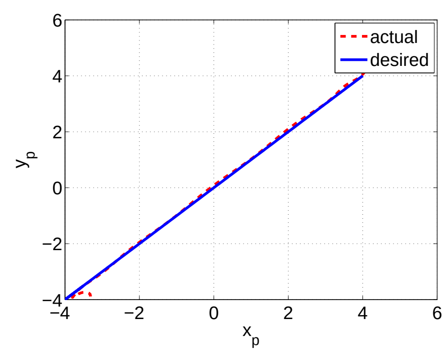

Figure 2.5: Straight line path.

2.5 Kinematic Model of Nonholonomic Robot without Pneumatic Wheels

We start by deriving the kinematic model of a car-like mobile robot shown in figure2.6. Let us select a set of inertially fixed reference frame denoted by $\{I, X, Y\}$, a body fixed attached to the center of gravity of the robot is $\{C, X_c, Y_c\}$ and θ is

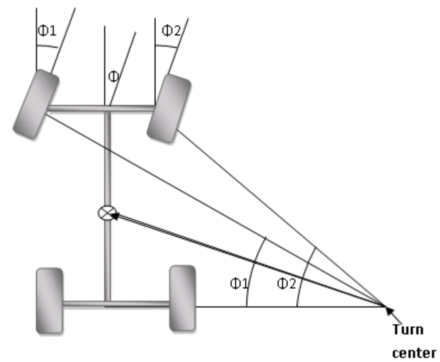

Figure 2.6: car-like Mobile Robot.

the orientation of the frame with respect to the inertial frame and ϕ is the steering angle of the front wheels and the robot is a rear wheel drive robot. The position of the robot is determined by the generalized coordinates $q = [x_c, y_c, \theta, \phi]^T$. Let us number the wheels from left to right and front to rear. The parameters of the robot are defined as follow: l_1 : distance from C to the midpoint of the front axle; l_2 : distance front the C to the midpoint of the real axle; l half the length of front and rear axle and w_1, w_2, w_3, w_4 are the front left, front right, rear left, and rear right wheels respectively. So in the body frame, $w_1 = [l_1, l, 1]^T$, $w_2 = [l_1, -l, 1]^T$, $w_3 = [-l_2, l, 1]^T$, $w_1 = [-l_2, -l, 1]^T$. The homogeneous transformation of the inertial frame $\{I, X, Y\}$, relative to the body frame $\{C, X_c, Y_c\}$ is

$$ {}^{I}R_C = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & x_c \\ \sin(\theta) & \cos(\theta) & y_c \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.41} $$

The wheels are considered as rigid bodies in their own right. Let us locally attach reference frames to their center in order to derive kinematic constraints. As shown in figure (2.6) the axes attached to the front wheels rotate by ϕ_1 , ϕ_2 and the relation between them with ϕ is given later in the text. The homogeneous transformations of the wheels local frames relative to reference frame attached to the center of gravity

$\{C, X_c, Y_c\}$ are given by

$$
{}^C R_1 = \begin{bmatrix} \cos(\phi_1) & -\sin(\phi_1) & l_1 \\ \sin(\phi_1) & \cos(\phi_1) & l \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^C R_3 = \begin{bmatrix} 1 & 0 & -l_2 \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^C R_2 = \begin{bmatrix} \cos(\phi_2) & -\sin(\phi_2) & l_1 \\ \sin(\phi_2) & \cos(\phi_2) & -l \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^C R_4 = \begin{bmatrix} 1 & 0 & -l_2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.42}
$$

The constraints of rolling without slipping implies that the velocity of a wheel is equal to the angular velocity of the wheel about its axis multiplied by the radius of the wheel. Then velocities of the wheels in terms of the frames attached to them are obtained. Expressing w_1 ($w_i, i = 1, \ldots, 4$ is the position of wheel i expressed in the frame attached to the center of mass of the robot) in the inertial frame, we have

$$
{}^I w_1 = {}^I R_C w_1 \begin{bmatrix} l_1 \cos(\theta) - l \sin(\theta) + x_c \\ l_1 \sin(\theta) + l \cos(\theta) + y_c \\ 1 \end{bmatrix}. \tag{2.43}
$$

The velocity of wheel w_1 is the time derivative of (2.43), and when transformed back in the frame attached to wheel w_1, we have

$$
{}^1 \dot{w}_1 = \begin{bmatrix} \cos(\theta + \phi_1)\dot{x}_c - l \cos(\phi_1)\dot{\theta} + l_1 \sin(\phi_1)\dot{\theta} + \sin(\theta + \phi_1)\dot{y}_c \\ -\sin(\theta + \phi_1)\dot{x}_c + l \sin(\phi_1)\dot{\theta} + l_1 \cos(\phi_1)\dot{\theta} + \cos(\theta + \phi_1)\dot{y}_c \\ 0 \end{bmatrix}. \tag{2.44}
$$

Similarly, for the remaining wheels we have

$$
{}^{2}\dot{w}_2 = \begin{bmatrix} \cos(\theta+\phi_2)\dot{x}_c - l\cos(\phi_2)\dot{\theta} + l_1\sin(\phi_2)\dot{\theta} + \sin(\theta+\phi_2)\dot{y}_c \\ -\sin(\theta+\phi_2)\dot{x}_c + l\sin(\phi_2)\dot{\theta} + l_1\cos(\phi_2)\dot{\theta} + \cos(\theta+\phi_2)\dot{y}_c \\ 0 \end{bmatrix} \tag{2.45}
$$

$$
{}^{3}\dot{w}_3 = \begin{bmatrix} \cos(\theta)\dot{x}_c - l\dot{\theta} + \sin(\theta)\dot{y}_c \\ -\sin(\theta)\dot{x}_c - l_2\dot{\theta} + \cos(\theta)\dot{y}_c \\ 0 \end{bmatrix} \tag{2.46}
$$

and

$$
{}^{4}\dot{w}_4 = \begin{bmatrix} \cos(\theta)\dot{x}_c + l\dot{\theta} + \sin(\theta)\dot{y}_c \\ -\sin(\theta)\dot{x}_c - l_2\dot{\theta} + \cos(\theta)\dot{y}_c \\ 0 \end{bmatrix} \tag{2.47}
$$

The condition of the wheels rolling without slipping is that in the reference frame attached to the wheels the x-component will be equal to the angular velocity of the wheel about its axle multiplied by the radius of the wheel. We then have the following equations

$$
\begin{aligned}
\cos(\theta+\phi_1)\dot{x}_c - l\cos(\phi_1)\dot{\theta} + l_1\sin(\phi_1)\dot{\theta} + \sin(\theta+\phi_1)\dot{y}_c &= r\dot{\Omega}_1 \\
\cos(\theta+\phi_2)\dot{x}_c - l\cos(\phi_2)\dot{\theta} + l_1\sin(\phi_2)\dot{\theta} + \sin(\theta+\phi_2)\dot{y}_c &= r\dot{\Omega}_2 \\
\cos(\theta)\dot{x}_c - l\dot{\theta} + \sin(\theta)\dot{y}_c &= r\dot{\Omega}_3 \\
\cos(\theta)\dot{x}_c - l\dot{\theta} + \sin(\theta)\dot{y}_c &= r\dot{\Omega}_4.
\end{aligned} \tag{2.48}
$$

We note that the last two equations of (2.48) can be combined to one equation by taking the difference of them. The resulting equation is given by

$$
\dot{\theta} = \frac{r}{2l}\left(\dot{\Omega}_3 - \dot{\Omega}_4\right) \tag{2.49}
$$

Equation (2.49) can be integrated and gives

$$\theta = \frac{r}{2l}\left(\Omega_3 - \Omega_4\right); \tag{2.50}$$

Since (2.49) is integrated to obtain a generalized coordinate it is a holonomic constraint. We then have three constraints equations one holonomic (2.50) and two nonholonomic which are the first two equations of (2.48).

The condition that wheels do not have lateral motion means that the y-components of the above velocities are zero, we have

$$\begin{aligned}
-\sin(\theta+\phi_1)\dot{x}_c + l\sin(\phi_1)\dot{\theta} + l_1\cos(\phi_1)\dot{\theta} + \cos(\theta+\phi_1)\dot{y}_c &= 0 \\
-\sin(\theta+\phi_2)\dot{x}_c + l\sin(\phi_2)\dot{\theta} + l_1\cos(\phi_2)\dot{\theta} + \cos(\theta+\phi_2)\dot{y}_c &= 0 \\
-\sin(\theta)\dot{x}_c - l_2\dot{\theta} + \cos(\theta)\dot{y}_c &= 0
\end{aligned} \tag{2.51}$$

The components of the velocity of the center of mass of the robot can be written as

$$\begin{aligned}
\dot{x}_c &= v_x\cos(\theta) - v_y\sin(\theta) \\
\dot{y}_c &= v_x\sin(\theta) + v_y\cos(\theta)
\end{aligned} \tag{2.52}$$

and replacing (2.52) in (2.51) and noticing that

$$\begin{aligned}
\tan(\phi_1) &= \frac{l_1+l_2}{(l_1+l_2) - l\tan(\phi)}\tan(\phi) \\
\tan(\phi_2) &= \frac{l_1+l_2}{(l_1+l_2) + l\tan(\phi)}\tan(\phi)
\end{aligned} \tag{2.53}$$

As we performed in the previous chapter, the constraints can be written in matrix vector form as

$$A(q)\dot{q} = 0$$

It is possible to find a basis of vector fields that span the null space of A such that we can express the generalized velocities as

$$\dot{q} = S(q)v \tag{2.54}$$

where v is a set of independent velocity variables. For this particular case, the inputs are chosen as the forward velocity $v_1 = v_x$, and the steering velocity $v_2 = \dot{\phi}$. We have

$$\begin{bmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos(\theta) - \frac{l_2}{l_1+l_2}\sin(\theta)\tan(\phi) & 0 \\ \sin(\theta) + \frac{l_2}{l_1+l_2}\sin(\theta)\tan(\phi) & 0 \\ \frac{\tan(\phi)}{l_1+l_2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

So we are able to express $\dot{q} = g_1(q)v_1 + g_2(q)v_2$. $g_1(q)$ and $g_2(q)$ are columns of $S(q)$. A kinematic control is design and integrated into the dynamics Amengonu and Kakad [3].

CHAPTER 3: MODELING AND CONTROL OF CMG INVERTED PENDULUM

3.1 Application of Implicit Function Method and Saturation Function

There are perhaps only a few simple systems that are better than the inverted pendulum at demonstrating the ability to accomplish a seemingly difficult task through the use of feedback control. It is therefore no surprise that the inverted pendulum has been extensively utilized as a prototype system for both the study and practical demonstration of many types of controllers. Inverted pendulum systems are often attached to a cart or rotating arm that in which case the angle of the pendulum is controlled via the coupling between the translational motion of the pendulum's pivot point and its angle, a consequence of the conservation of momentum. An interesting variation on this problem is the momentum wheel inverted pendulum [79],[80],[82]. In this case, the pendulum's pivot point is inertially fixed and actuation is accomplished via the controlled rotation of a massive disk attached to the pendulum about an axis parallel to the pendulum's pivot axis. Here, a similar configuration is utilized, however, the massive rotating disk is allowed to also rotate about an axis that is parallel to the length of the pendulum. Such a mechanism forms a simple Control Moment Gyroscope (CMG) which can be utilized to provide a torque on the pendulum. The dynamics of the control moment gyroscope are first presented. The equations of motion are derived from the Euler-Lagrange formulation. From these equations, some statements can be made concerning the control requirements. These are addressed in the subsequent discussion. A stabilizing controller for the pendulum is then examined later.

Consider the diagram of the control moment gyroscope inverted pendulum that is depicted in figure 3.1. There are three bodies in this system, body 1, body 2, and body 3. There are also three degrees-of-freedom associated with the pendulum's rotation through angle θ_1, the CMG's rotation through angle θ_2, and the CMG disk's rotation through angle θ_3. A vector of generalized coordinates for this system is thus, $\underline{\gamma} = [\theta_1, \theta_2, \theta_3]^T$, the subscript T stands for the transpose. The system dynamics are easily obtained from Hamilton's principle via the Euler-Lagrange equation. However, this requires an expression for the total system kinetic energy and potential energy. The total kinetic energy is obtained as the sum of the kinetic energies of each of the bodies comprising the system,

$$K = \sum_{B=1}^{3} K_B \ , \tag{3.1}$$

where

$$K_B = \frac{1}{2} m_B \underline{{}^I\dot{R}_B^T} \underline{{}^I\dot{R}_B} + \underline{{}^I\dot{R}_B^T} {}^I\dot{T}_B \underline{\Gamma_B} + \frac{1}{2} \underline{{}^B\omega_B}^T J_B \underline{{}^B\omega_B} \ . \tag{3.2}$$

The potential energy of the system is obtained as

$$U = \sum_{B=1}^{3} U_B \ , \tag{3.3}$$

where

$$U_B = g \left(m_B \underline{{}^I R_B} + {}^I T_B \underline{\Gamma_B} \right) |_z \ . \tag{3.4}$$

In both of these expressions, m_B is the mass of the body, $\underline{\Gamma_B}$ is the vector of first-mass moments of the body measured in the body's frame, J_B is the inertial matrix of the body measured in the body's frame, ${}^I T_B$ is the rotation from body coordinates to inertial coordinates, $\underline{{}^B\omega_B}$ is the angular velocity of body measured in the body's frame, $\underline{{}^I R_B}$ is the position of the body's frame measured in the inertial frame, and g is the gravitational acceleration.

Formulating the Lagrangian as $L = K - U$ and applying the Euler-Lagrange equa-

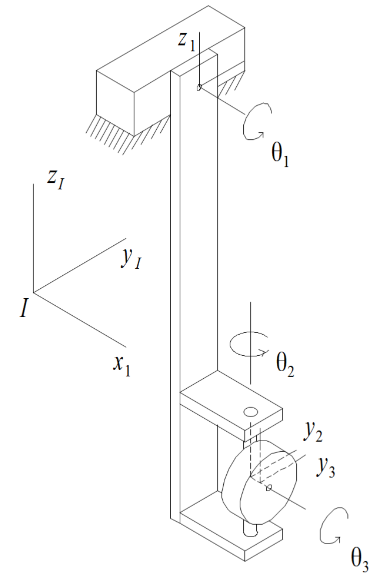

Figure 3.1: Control Moment Inverted Pendulum.

tion results in a system of second-order differential equations of the form

$$H\left(\underline{\gamma}\right)\underline{\ddot{\gamma}} + \underline{D}\left(\underline{\gamma},\underline{\dot{\gamma}}\right) + \underline{G}\left(\underline{\gamma}\right) = \underline{\tau} \ , \tag{3.5}$$

where the components of the symmetric positive-definite system mass matrix, $H\left(\underline{\gamma}\right)$, are

$$\begin{aligned}
H_{1,1} &= a + b\sin(\theta_2)^2 \\
H_{1,2} &= H_{2,1} = c\cos(\theta_2) \\
H_{1,3} &= H_{3,1} = J_{3xx}\cos(\theta_2) \\
H_{2,2} &= d \\
H_{2,3} &= H_{3,2} = 0 \\
H_{3,3} &= J_{3xx} \ .
\end{aligned} \tag{3.6}$$

The components of the vector of the generalized Coriolis and Centripetal forces,

$\underline{D}\left(\underline{\gamma}, \underline{\dot{\gamma}}\right)$, are

$$\begin{aligned} D_1 &= 2b\dot{\theta}_1\dot{\theta}_2\cos(\theta_2)\sin(\theta_2) + J_{3xx}\dot{\theta}_2\dot{\theta}_3\sin(\theta_2) - c\dot{\theta}_2^{\ 2}\sin(\theta_2) \\ D_2 &= -b\dot{\theta}_1^{\ 2}\cos(\theta_2)\sin(\theta_2) + J_{3xx}\dot{\theta}_1\dot{\theta}_3\sin(\theta_2) \\ D_3 &= -J_{3xx}\dot{\theta}_1\dot{\theta}_2\sin(\theta_2)\ \ . \end{aligned} \tag{3.7}$$

The components of the vector of generalized gravitational forces, $\underline{G}\left(\underline{\gamma}\right)$, are

$$\begin{aligned} G_1 &= Ag\sin(\theta_1) + Bg\cos(\theta_1)\sin(\theta_2) \\ G_2 &= Bg\sin(\theta_1)\cos(\theta_2) \\ G_3 &= 0\ \ , \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} a &= J_{1xx} + J_{2xx} + J_{3xx} + d_1^{\ 2}m_3 \\ b &= J_{3yy} - J_{3xx} + d_3^2 m_3 \\ c &= d_1 d_3 m_3 \\ d &= m_3 d_3^2 + J_{2zz} + J_{3yy} \\ A &= d_1 m_2 - \Gamma_{1z} + d_1 m_3 \\ B &= d_3 m_3\ \ . \end{aligned} \tag{3.9}$$

In (3.6)-(3.8), d_1 is the distance along the z-axis of frame 1 from the origin of frame 1 to the origin of frame 2, d_2 is the distance along the x-axis of frame 1 from the origin of frame 1 to the origin of frame 2, and d_3 is the distance from the origin of frame 2 to the origin of frame 3.

3.2 Analysis and Control of the CMG's Reduced Dynamics

The input $\underline{\tau}$ is a vector of external generalized forces, $\underline{\tau} = [0, \tau_2, \tau_3]^T$. The system is therefore under-actuated. The torque τ_3 is utilized to maintain a constant $\dot{\theta}_3$, thus only τ_2 is available for the control of θ_2.

Assuming a control has been applied to regulate $\dot{\theta}_3$ about a set point such that $\ddot{\theta}_3 = 0$, the equations of motion are approximated as

$$\begin{aligned} H_{1,1}(\theta_2)\ddot{\theta}_1 + H_{1,2}(\theta_2)\ddot{\theta}_2 + D_1\left(\underline{\theta}, \dot{\underline{\theta}}\right) + G_1\left(\underline{\theta}\right) &= 0 \\ H_{2,1}(\theta_2)\ddot{\theta}_1 + H_{2,2}(\theta_2)\ddot{\theta}_2 + D_2\left(\underline{\theta}, \dot{\underline{\theta}}\right) + G_2\left(\underline{\theta}\right) &= \tau_2 \ . \end{aligned} \tag{3.10}$$

3.3 Jacobian Based Linearization for the Reduced Dynamics of the CMG

The equations of motion of rigid bodies are mostly converted from second order to first order differential equations. These equations are linearized about an operating point. The Jacobian model obtained is an exact representation of the nonlinear model only at that point. Using smoothness of the vector fields and continuity of the system trajectory at the point about which the linearization is performed, one can usually find a small neighborhood of the operating point where the linearized model is still valid. This approximation destroys important geometric information such as Centrifugal and Coriolis components sometimes used to improve the global behavior of the closed loop system. Since the goal in this part of the design is to design separately swing up and balance control, we will first approximate the system at the equilibrium point and if the approximation is stabilizable to that equilibrium point, then a small neighborhood of the operating point can be found where the nonlinear system can be stabilized by linear feedback. Let $(x_0, u_0) \in \mathbb{R}^4 \times \mathbb{R}$ denote the equilibrium point of the reduced dynamics of the CMG. The linearization about (x_0, u_0) is given by

$$\dot{z} = Az + Bv, \tag{3.11}$$

where

$$z = x - x_0 \qquad v = u - u_0$$

$$A = \frac{\partial}{\partial x}\left(f(x) + g(x)u\right)|_{(x_0,u_0)} \qquad b = g(x_0)$$

where we define $x \triangleq [\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2]^T$, and

$$f(x) \triangleq \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -\underline{H}^{-1}(\underline{D} + \underline{G}) \end{pmatrix} \quad \text{and} \quad g(x) \triangleq \begin{pmatrix} 0 \\ 0 \\ -\underline{H}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

The linearized model is completely controllable if and only if

$$det[B \;\; AB \;\; A^2B \;\; \ldots \;\; A^3B] \neq 0$$

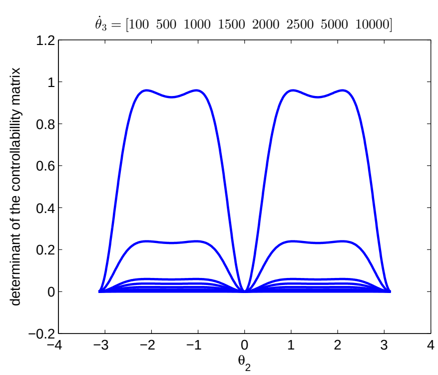

Figure 3.2: Controllability matrix determinant.

The controllability of the pendulum upright depends very much on the speed of the wheel. A plot of the determinant controllability matrix with respect to θ_2 and angular velocity of the wheel is shown in figure3.2

In the neighborhood of the unstable equilibrium point $\theta_1 = \pi, \theta_2 = 0, \dot{\theta}_1 = \dot{\theta}_2 = 0$, the linearized model is given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{AgJ_{311}-Bgc}{J_{311}a-c^2} & \frac{BgJ_{311}}{J_{311}a-c^2} & 0 & 0 \\ \frac{Bga-Acg}{J_{311}a-c^2} & \frac{-Bgc}{J_{311}a-c^2} & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{-c}{J_{311}a-c^2} \\ \frac{a}{J_{311}a-c^2} \end{bmatrix}. \tag{3.12}$$

3.3.1 Transfer Function Representation

The transfer function between θ_1(s) and the input V(s) using (3.17) is given as

$$H_1(s) = \frac{\theta_1(s)}{V(s)} = \frac{-as^2 + Ag}{(c^2 - J_{311}a)s^4 + (AJ_{311}g - 2Bcg)s^2 + B^2g^2} \tag{3.13}$$

and the transfer function between θ_2 and θ_1 is given by

$$H_2(s) = \frac{\theta_2(s)}{\theta_1(s)} = \frac{cs^2 - Bg}{-as^2 + Ag} \tag{3.14}$$

The inertia coupling between the first and the second link can explain the presence of the right half plane zero in the transfer function between the internal torque exerted by the second link and the angle of the first link. Numerical analysis shows that the system is open loop unstable (when the actual values of the parameters of the CMG are replaced in the transfer function equations above).

3.3.2 Feedback Linearization

Feedback linearization is the process of transforming a nonlinear system into a simpler form which in a sense is a linear system. A nonlinear system written as

$$\dot{x} = f(x, u) \quad \text{or} \tag{3.15}$$

$$\dot{x} = f(x) + g(x)u, \tag{3.16}$$

into a linear system

$$\dot{z} = Az + Bv, \tag{3.17}$$

via a diffeomorphism

$$(z, v) = (\Phi(x), \Psi(x, u)),$$

which is called feedback transformation. A question is at what condition(s) is it possible to find the change of coordinates $z = \Phi(x)$ and a control law $u = \alpha(x)+\beta(x)v$ for which the linear dynamics

$$\dot{z} = Az + Bv,$$

can be found for the second equation in equation(3.16) for example. An insight can be gained by considering a single input linear control system of the form

$$\dot{x} = Ax + bu, \tag{3.18}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$. Let us assume that the linear system of equation(3.18) is controllable *i.e.*

$$rank\left(b, Ab, \ldots, A^{n-1}b\right) = n.$$

Assume that we can find a row vector $\mathcal{C}$ orthogonal to the first $n-1$ columns of the controllability matrix

$$\mathcal{C} = [b, Ab, \ldots, A^{n-2}b, A^{n-1}b]. \tag{3.19}$$

We then have

$$cb = cAb = \cdots = cA^{n-2}b = 0 \quad \text{and} \quad cA^{n-1}b = d \neq 0.$$

We can introduce the linear coordinates

$$\begin{aligned} z_1 &= h = cx \\ z_2 &= cAx \\ &\vdots \\ z_n &= cA^{n-1}x. \end{aligned} \tag{3.20}$$

We have

$$\begin{aligned} \dot{z}_1 &= c\dot{x} = cAx + cbu = z_2 \\ \dot{z}_2 &= cA\dot{x} = cA^2x + cAbu = z_3 \\ &\vdots \\ \dot{z}_{n-1} &= cA^{n-2}\dot{x} = cA^{n-1}x + cA^{n-2}bu = z_n \\ \dot{z}_n &= cA^{n-1}\dot{x} = cA^n x + cA^{n-1}bu = \sum_{i=1}^{n} a_i z_i + du \end{aligned} \tag{3.21}$$

where $a_i \in \mathbb{R}$. We can introduce a new control variable

$$v = \sum_{i=1}^{n} a_i z_i + du$$

So calculating the control v in the new state space, and then pulling the control back to the original space, we can control the system. The above transformation showed that any single input controllable system can be brought to n-fold integrator

$$\dot{z}_1 = z_2, \;\; \dot{z}_2 = z_3, \ldots, \dot{z}_{n-1} = z_n, \;\; \dot{z}_n = v.$$

Coming back to our earlier question, when is similar transformation possible for a single input nonlinear system of the form

$$\dot{x} = f(x) + g(x)u. \tag{3.22}$$

Given a point x_0 in the state space, if it is possible to find a feedback control

$$u = \alpha(x) + \beta(x)v, \tag{3.23}$$

defined in the neighborhood of the point x_0 and a coordinates transformation $z = \Phi(x)$ defined in the same neighborhood, the closed loop system is given by

$$\dot{x} = f(x) + g(x)\alpha(x) + g(x)\beta(x)v. \tag{3.24}$$

In the coordinates $z = \Phi(x)$, the system is transformed into a linear and controllable system

$$z = Az + Bv,$$

with

$$\begin{aligned} \left[\frac{\partial \Phi}{\partial x}(f(x) + g(x)\alpha(x))\right]_{x=\Phi^{-1}(z)} &= Az \\ \left[\frac{\partial \Phi}{\partial x}(g(x)\beta(x))\right]_{x=\Phi^{-1}(z)} &= B \end{aligned} \tag{3.25}$$

For the single input system we consider here, sometimes it is not any easy task to find the feedback control given in equation(3.23). The process of obtaining the control law above is the state space exact linearization problem; and it is possible for the single input case if and only if there exists in the neighborhood of x_0 a real-valued function $h(x)$ such that

$$L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{n-2} h(x) = 0 \ \ \forall x \ \text{ near } \ x_0 \tag{3.26}$$

$$L_g L_f^{n-1} h(x) = d(x) \neq 0, \tag{3.27}$$

where $d(x)$ is a smooth function. This basically means the relative degree is equal n, the dimension of the state space. If around the point x_0, the functions $h, L_f h, \ldots, L_f^{n-1} h$ are independent in the neighborhood of x_0, then in a neighborhood U of x_0 the map

$$\begin{aligned} z_1 &= h \\ z_2 &= L_f h \\ &\vdots \\ z_n &= L_f^{n-1} h. \end{aligned} \tag{3.28}$$

defines a local coordinates system $(z_1, z_2, \ldots, z_n)^T$. The dynamics are then transformed into

$$\begin{aligned} \dot{z}_1 &= L_f h + u L_g h = z_2 \\ &\vdots \\ \dot{z}_{n-1} &= L_f^{n-2} h + u L_g L^{n-1} h = z_n \\ \dot{z}_n &= L_f^n h + u L_g L_f^{n-1} h = L_f^n h + u d(x). \end{aligned} \tag{3.29}$$

As we have done for linear case we introduce a new control variable

$$v = L_f^n h + ud(x),$$

a transformation in the control space which bring the nonlinear system into a linear system

$$\dot{z}_1 = z_2, \;\; \dot{z}_2 = z_3, \ldots, \dot{z}_{n-1} = z_n, \;\; \dot{z}_n = v.$$

The exact feedback linearization above was obtained under two assumptions. First we assume that h exist and $L_g h(x) = L_g L_f h(x) = \cdots = L_g L_f^{n-2} h(x) = 0$ and second the functions $h, L_f h, \ldots, L_f^{n-1} h$ are independent in the neighborhood of x_0. In the first assumption, let us consider the first two elements $L_g h(x) = L_g L_f h(x) = 0$. Since $L_g h(x) = 0$ we can rewrite it as $L_f L_g h(x) = 0$ so then we can write this equality

$$L_f L_g h(x) - L_g L_f h(x) = 0$$

Using the Lie bracket notation we have

$$L_f L_g h(x) - L_g L_f h(x) = L_{[f,g]} h(x) = 0$$

This is a first order partial differential equation. So the first assumption is $n-1$ first order partial differential equations written as

$$L_g h(x) = L_{ad_f g} h(x) = \cdots = L_{ad_f^{n-2} g} h(x) = 0$$

here we introduce the notation

$$ad_f g = [f, g] \;\text{ and inductively }\; ad_f^k g = [f, ad_f^{k-1} g].$$

The existence of the function $h(x)$ is a consequence of Frobenius theorem. The conditions under which a general nonlinear system can be converted in linear system were formulated by in [35],[38] independently. A necessary condition for the above system of partial differential equations to admit a nontrivial solution is that the Lie bracket $w = [ad_f^j g, ad_f^k g]$, $0 \leq j, k \leq n-2$ belong to the vector space generated by $\{ad_f^i g \;\; 0 \leq i \leq n-2\}$. This condition is to say that the distribution spanned by $\{ad_f^i g \;\; 0 \leq i \leq n-2\}$ is *involutive*. The theorem below summarize the conditions under which one can find a local coordinates transformation and a feedback control so that the nonlinear system(3.22) can be transformed in a linear controllable system.

Theorem: [35],[36],[38] There exists a local change of coordinates $z = \Phi(x)$ and a feedback of the form $u = \alpha(x) + \beta(x)v$, where $\beta(x) \neq 0$, transform the nonlinear system 3.22 defined in a neighborhood U of a point x_0 into the linear controllable system

$$\dot{z} = Az + Bv$$

if and only if the 3.22 satisfies in the neighborhood of x_0:

(C_1) g(x),$ad_f g(x), \ldots, ad_f^{n-1} g(x)$ are linearly independent

(C_2) the distribution D={ g(x),$ad_f g(x), \ldots, ad_f^{n-2} g(x)\}$ is involutive in the neighborhood of x_0 and (n-1)-dimensional around x_0.

The condition (C_1) is the controllability test that agrees with linearization. The second condition (C_2) implies that since the state space is n-dimensional and the distribution D is $n-1$-dimensional and involutive, then we can find a real valued function $h : \mathbb{R}^n \to \mathbb{R}$ defined in the neighborhood of x_0 such that its differential dh spans the codistribution of D. In other words we have

$$dh(x)[g(x), ad_f g(x), \ldots, ad_f^{n-2} g(x)] = 0. \tag{3.30}$$

The above conditions implies that the solutions of the n first order partial differential equations

$$\begin{cases} L_w h = 0 & w \text{ defined above} \\ L_{ad_f^q g} h = 0 & \text{if } 0 \leq q \leq n-2, \end{cases}$$

h are constant. The real-valued functions h which are the solutions of these partial differential equations, maps a neighborhood of x_0 into a constant value $a \in \mathbb{R}$. The inverse image $h^{-1}(a)$ in an (n-1)-dimensional space which are the leaves. Equation(3.30) indicates that the gradient of h is orthogonal to the leaves. In feedback linearization h is used to define the required change of coordinates. For the single input system considered, the change of coordinates is given by (3.28). When an exact linearization is performed, we do not need to operate around an equilibrium point. We need to stay around an equilibrium point when we perform an approximation of the system such as Jacobian linearization. A trajectory tracking is possible because we performed an exact linearization of the system. So in the neighborhood of a point where the feedback linearization equations are satisfied, it is possible to achieve exponential trajectory tracking.

3.3.2.1 Exact Feedback Linearization of the Reduced Dynamics of the CMG

Let us consider the reduced dynamics of the CMG (3.10) rewritten as

$$\dot{x} = f(x) + g(x)u,$$

where

$$f(x) \triangleq \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ -\underline{H}^{-1}(\underline{D}+\underline{G}) \end{pmatrix} \quad \text{and} \quad g(x) \triangleq \begin{pmatrix} 0 \\ 0 \\ -\underline{H}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \tag{3.31}$$

Symbolic toolbox in MatLab was used to check the rank of matrix D=$[g(x), ad_f g(x), ad_f^2 g(x), ad_f^3 g(x)]$ which is essentially condition (C_1). In MatLab the rank of this matrix is 4.

Next we check the involutivility of the distribution

$$\Delta = \{g, ad_f g, ad_f^2 g\} \tag{3.32}$$

in other words we must verify that the vector fields

$$[g, ad_f g] \quad [g, ad_f^2 g] \quad [ad_f g, ad_f^2 g]$$

lie in the distribution Δ. Symbolic calculations show that $[g, ad_f g] \quad [g, ad_f^2 g]$ lie in Δ but $[ad_f g, ad_f^2 g]$ does not. The distribution Δ is not involutive, hence the reduced dynamics of the CMG is not input/state linearizable.

3.3.2.2 Input/Output Linearization of the Reduced Dynamics of the CMG

We have seen in the previous section that the CMG is not exactly linearizable by state feedback. An output function can be selected and will be used to perform the linearization. An issue associated with this method is that of introducing unstable internal dynamics called *zero dynamics*. Since we do not have an ouput that is predefined, we can defined an output such that the system is input/output linearizable and has stable zero dynamics. However, finding an output such that the resulting linearized system has stable dynamics is not an easy process. θ_1 and θ_2 are already excluded because when the Jacobian linearization was performed above we have zeros of the transfer functions on the right hand s-plane. To proceed we will design an output function in order to achieve stable zero dynamics. This effect is also present in the nonlinear system. In this section the goal is to construct an output function which is used to construct vector fields closed to the original vector fields but which satisfy the exact linearization conditions and then a controller will be designed and apply to the actual CMG system. We will follow the method developed primarily in [32]. Since the conditions of exact feedback linearization are not satisfied, we can still use the method to generate vector fields $\overline{f}$ and $\overline{g}$ that approximate the actual vector fields f and g of the CMG. The idea is that the approximate vector fields $\overline{f} + \overline{g}u$ should agree to the first order with the original system when evaluated at the equilibrium point. Thus the relative degree of the approximate system should be the same as the relative degree of the linearization.

3.3.2.3 Approximation Thechnique Overview [32][63]

Let us consider the nonlinear single input system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y(x) &= h(x) \end{aligned} \tag{3.33}$$

By proceeding as before, we will take successive derivative of the output until the input appears explicitly. We have

$$\begin{aligned} \dot{y} &= L_f h(x), \\ \ddot{y} &= L_f^2 h(x), \\ &\vdots \\ y^r &= L_f^r h(x) + L_g L_f^{r-1} h(x) u \end{aligned} \tag{3.34}$$

3.3.3 Collocated Partial Feedback Linearization

Many researchers, in the past, have considered the analysis and control design of underactuated mechanical systems. One of the complexities of these systems is that often they are not fully feedback linearizable. In this work we will partially linearize the system using a change of control which transforms it into a strict feedback form[52] and then into a normal form which is a special case of the famous Byrnes-Isidori normal form [36]. This form is suitable for the backstepping procedure. However, after applying this change of control, the new control appears in both the linear and nonlinear subsystems. Another change of variable renders the analysis and design less complicated because the control appears only in the linear subsystem. The global change of control is proposed in [81] as

$$\begin{aligned} \tau_2 &= \alpha\left(\underline{\theta}\right)u + \beta\left(\underline{\theta}, \dot{\underline{\theta}}\right) \\ \ddot{\theta}_2 &= u \ . \end{aligned} \tag{3.35}$$

Equation (3.35) is obtained by solving for $\ddot{\theta}_1$ in the first equation in (3.10) and then replace it in the second equation. Applying this technique, (3.10) is partially linearized and we have

$$\begin{aligned} \alpha(\theta_2) &= \left(H_{2,2}(\theta_2) - \frac{H_{1,2}(\theta_2)H_{2,1}(\theta_2)}{H_{1,1}(\theta_2)}\right) \\ \beta(\underline{\theta}, \dot{\underline{\theta}}) &= D_2(\underline{\theta}) + G_2(\underline{\theta}) - \frac{H_{1,2}(\theta_2)}{H_{1,1}(\theta_2)}\left(D_1(\underline{\theta}) + G_1(\underline{\theta})\right) \ . \end{aligned}$$

The reduced system (3.10) may be written as

$$\begin{aligned} \ddot{\theta}_1 &= -\frac{1}{H_{1,1}(\theta_2)}(D_1(\underline{\theta}, \dot{\underline{\theta}}) + G_1(\underline{\theta})) - \frac{1}{H_{1,1}(\theta_2)}H_{1,2}(\theta)u \\ \ddot{\theta}_2 &= u \ . \end{aligned} \tag{3.36}$$

In general, this change of control is invertible because $det\,(H(\underline{\theta}))$ is not zero because $H(\underline{\theta})$ is a positive definite matrix. In (3.10), $H_{1,1}(\theta_2)$ is positive for all values of θ_2. Denoting $\underline{p} = \dot{\underline{\theta}}$, (3.36) can be expressed as

$$\begin{aligned} \dot{\theta}_1 &= p_1 \\ \dot{p}_1 &= f(\underline{\theta}, \underline{p}) + g(\underline{\theta})u \\ \dot{\theta}_2 &= p_2 \\ \dot{p}_2 &= u \ . \end{aligned} \tag{3.37}$$

In (3.37), it is shown that the control input u appears in both the (θ_1,p_1) and (θ_2,p_2) subsystems. It is interesting to note that the inertia matrix depends only on the actuated variable. The variables that appear in the inertia matrix are called shape variables. The configuration variables that do not appear in the inertia matrix are called external variables.

3.3.4 Shape Variable and Kinetic Symmetry

The fact that the Lagrangian has a kinematic symmetry with respect to external variable i.e $\frac{\partial K(\theta,\dot{\theta})}{\partial \theta_j} = 0$, $j = 1, 3$, and the normalized momentum conjugate to θ_1,

$$\nu_1 = H_{1,1}(\theta_2)^{-1} \frac{\partial L}{\partial \dot{\theta}_1} = \dot{\theta}_1 + H_{1,1}(\theta_2)^{-1} H_{1,2}(\theta_2)\dot{\theta}_2 \tag{3.38}$$

is integrable. This defines an interesting group action in the configuration manifold. This action is defined in a more general way in [68].

Let

$$\psi(\theta_2) = \int_0^{\theta_2} \frac{H_{1,2}(s)}{H_{1,1}(s)}\, ds \ ,$$

we note that the one-form $d\psi(\theta_2) = \frac{H_{1,2}(\theta_2)}{H_{1,1}(\theta_2)}d(\theta_2)$ is exact and the above fact is exploited to perform a global change of coordinates as

$$\begin{aligned} z_1 &= \theta_1 + \psi(\theta_2) \\ z_2 &= H_{1,1}(\theta_2)\dot{\theta}_1 + H_{1,2}(\theta_2)\dot{\theta}_2 = \frac{\partial L}{\partial \dot{\theta}_1} \ . \end{aligned} \tag{3.39}$$

The above global change of variables transform the dynamics of the reduced system into a strict feedback form as

$$\begin{aligned} \dot{z}_1 &= \frac{z_2}{H_{1,1}(\theta_2)} \\ \dot{z}_2 &= g_1(z_1 - \psi(\theta_2), \theta_2) \\ \dot{\theta}_2 &= p_2 \\ \dot{p}_2 &= u \ , \end{aligned} \tag{3.40}$$

where

$$g_1(z_1 - \psi(\theta_2), \theta_2) = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = \frac{\partial L}{\partial \theta_1} = -\frac{\partial U}{\partial \theta_1} \ ,$$

due to kinetic symmetry with respect to θ_1, $\frac{\partial K}{\partial \theta_1} = 0$. We also note that this change of variables is possible because $H_{1,1}(\theta_2)$ is strictly positive for all values of θ_2 and by multiplying (3.38) by $H_{1,1}(\theta_2)$ and setting $y_1 = z_1$ and $y_2 = {H_{1,1}}^{-1}(\theta_2)z_2$, one obtains a special case of the normal form [36] with double integrators as shown in (3.41)

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= f(y, \zeta_1, \zeta_2) \\ \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= u \ . \end{aligned} \tag{3.41}$$

Clearly, the control input appears only in the actuated subsystem. This decouples the two subsystems with respect to the control input u. If a globally stabilizing smooth state feedback exists for (z_1, z_2)-subsystem in (3.40) then a globally stabilizing state feedback can be found for (θ_1, θ_2)-subsystem using backstepping procedure [52]. In this case, θ_2 is considered as virtual input connecting both subsystems.

3.4 Controller Design Using Backstepping

Inertia-wheel pendulum is a planar inverted pendulum with a revolving wheel at the end that was first introduced in [83]. Due to the fact all the Christoffel Symbols associated with the inertia matrix vanish and the inertia matrix is constant, the dynamics and control of the inertia-wheel pendulum is a particular case of the design procedure outlined in this paper.

Let us first consider the stabilization of the nonlinear (z_1, z_2)-subsystem

$$\begin{aligned} \dot{z}_1 &= H_{1,1}(\theta_2)^{-1} z_2 \\ \dot{z}_2 &= g_1(z_1 - \psi(\theta_2), \theta_2) \ , \end{aligned} \tag{3.42}$$

where

$$\psi(\theta_2) = \frac{c}{\sqrt{ab}} \arctan\left(\frac{b \sin(\theta_2)}{\sqrt{ab}}\right) \ , \tag{3.43}$$

with $\theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Clearly, the subsystem (3.42) is non-affine in the virtual control input θ_2. The stabilization of (3.42) can be achieved using the following assumption. Consider the above nonlinear system non-affine in control θ_2 in (3.42). If the following condition

$$g_1(z_1, \theta_2) = \frac{\partial L}{\partial \theta_1} \tag{3.44}$$

is a smooth function with $g_1(0,0) = 0$, $H_{1,1}(\theta_2) > 0$ for all values of θ_2, zero is not a critical value for $g_1(z_1, \theta_2)$ and $\frac{\partial g_1(z_1,\theta_2)}{\partial \theta_2} \neq 0$, on the manifold $M = ker(g_1) = \{(z_1, \theta_2) \in \mathbb{R}^2 : g_1(z_1, \theta_2) = 0\}$ and $g_1(z_1, \theta_2)$ has an isolated root $\alpha(z_1)$ such that $g_1(z_1, \alpha(z_1)) = 0$, so there exists a continuously differentiable state feedback law in the following form $\theta_2 = \alpha(z_1) - \sigma(z_1, z_2)$ that globally asymptotically stabilizes the origin of (3.42) ($\sigma(.)$ is a sigmoidal function. Refer to [68] for proof).

$$\begin{aligned}
g_1(z_1, \theta_2) &= Ag\sin(z_1 - \psi(\theta_2)) + Bg\cos(z_1 - \psi(\theta_2))\sin(\theta_2) = 0 \\
\frac{\partial g_1(z_1, \theta_2)}{\partial \theta_2} &= -Ag\frac{H_{1,2}(\theta_2)}{H_{1,1}(\theta_2)}\cos(z_1 - \psi(\theta_2)) + Bg\frac{H_{1,2}(\theta_2)}{H_{1,1}(\theta_2)}\sin(z_1 - \psi(\theta_2)\sin(\theta_2) \\
&\quad +Bg\cos(z_1 - \psi(\theta_2))\cos(\theta_2) \ .
\end{aligned} \tag{3.45}$$

Solving the first line in (3.45) is equivalent to first solving for θ_1 in $Ag\sin(\theta_1) + Bg\cos(\theta_1)\sin(\theta_2) = 0$ which gives

$$\theta_1 = -\arctan\left(\frac{B\sin(\theta_2)}{A}\right) \ , \tag{3.46}$$

and substituting it in the first line of (3.39) we have

$$z_1 = \frac{c}{\sqrt{ab}}\arctan\left(\frac{b\sin(\theta_2)}{\sqrt{ab}}\right) - \arctan\left(\frac{B\sin(\theta_2)}{A}\right) \ , \tag{3.47}$$

with $\theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. In our work, we numerically invert (3.47) and find θ_2 as a function of z_1 . From here a state feedback control law stated in the assumption above can be found to stabilize the (z_1, z_2)-subsystem to the origin $(0, 0)$ as

$$\theta_2 = K_1(z_1, z_2) = \alpha(z_1) - \sigma(c_1 z_1 + c_2 z_2) \ . \tag{3.48}$$

3.4.1 Backstepping Control Design

Let us define a control Lyapunov fuction $V(z_1, z_2, \eta_1)$ and $\eta_1 = \theta_2 - K_1(z_1, z_2)$ as

$$\begin{aligned} V(\eta_1) &= \frac{1}{2}{\eta_1}^2 \\ \dot{V}(\eta_1) &= \dot{\eta_1}\eta_1 \ , \end{aligned} \tag{3.49}$$

where

$$\dot{V}(\eta_1) = \dot{\eta_1}\eta_1$$

$$\dot{\eta_1} = -c_3\eta_1$$

$$p_2 - \dot{K}_1(z_1, z_2) = -c_3(\theta_2 - K_1(z_1, z_2))$$

or

$$p_2 = K_2(z_1, z_2, \theta_2),$$

or

$$p_2 = K_2(z_1, z_2, \theta_2) = -c_3(\theta_2 - K_1(z_1, z_2)) + \dot{K}_1(z_1, z_2),$$

where $c_3 > 0$. Next we define

$$\eta_2 = p_2 - K_2(z_1, z_2, \theta_2)$$
$$V(z_1, z_2, \eta_1, \eta_2) = \frac{1}{2}{\eta_1}^2 + \frac{1}{2}{\eta_2}^2$$
$$\dot{V}(z_1, z_2, \eta_1, \eta_2) = -c_3{\eta_1}^2 + \dot{\eta}_2\eta_2$$
$$\dot{\eta}_2 = -c_4\eta_2$$
$$u - \dot{K}_2(z_1, z_2, \theta_2) = -c_4\left(p_2 - K_2(z_1, z_2, \theta_2)\right),$$

or

$$u = K_3(z_1, z_2, \theta_2, p_2) = -c_4\left(p_2 - K_2(z_1, z_2, \theta_2)\right) + \dot{K}_2(z_1, z_2, \theta_2)\ , \tag{3.50}$$

where $c_4 > 0$.

With $\dot{\theta}_3$ regulated about a setpoint such that $\ddot{\theta}_3 = 0$, the overall system can be controlled using τ_2 which is available for the control of θ_2.

Table 3.1: Simulation parameters.

a	b	c	d	J_{311}	A	B	c_1	c_2	c_3	c_4
2694.6	6.7728	75.5573	7.3916	6.8707	879.6338	22.9022	1	0	10	8

3.4.2 Simulations

The simulation results for the overall dynamics of the Control Moment Gyroscope are shown in figure3.3. An oscillation was noticed around the equilibrium point. To overcome this issue, a nonlinear damping was added. The CMG parameters used for simulation are shown in the table(3.1). From the analysis above, the overall system is an interconnection of two subsystems. Peaking is a phenomenon that is serious issue encountered very often when designing a stabilizing control law for the overall systems. The analysis of peaking phenomenon is considered in the sequel.

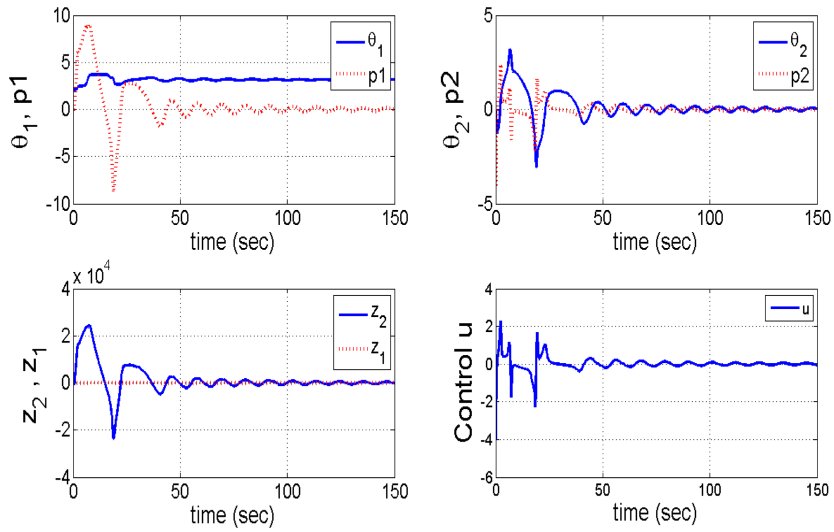

Figure 3.3: Simulation result with initial conditions $[-\frac{\pi}{3}, 0, 0, 0]$, where $p_1 = \dot{\theta}_1, p_2 = \dot{\theta}_2$.

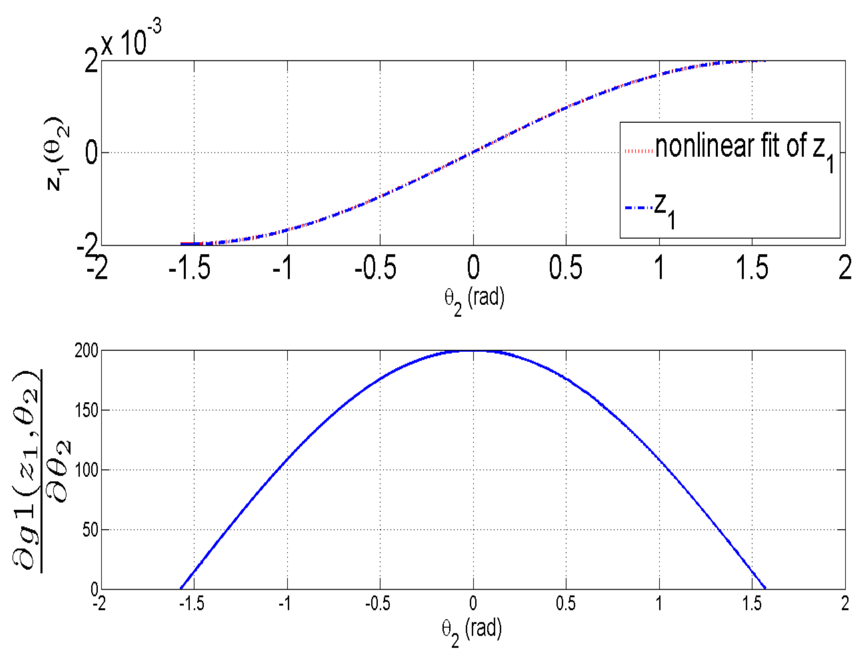

Figure 3.4: Analysis of $z_1(\theta_2)$ and $\frac{\partial g1(z_1,\theta_2)}{\partial \theta_2}$.

3.5 High-gain Observer and Stabilizing Controller for the Composite System

In the section above, the dynamics of the Control Moment Inverted Pendulum CMG was transformed into a partially linear composite system. A smooth dynamic state feedback was designed using a constructive method to stabilize the pendulum. In this section, we take advantage of the decomposition of the dynamics into a partially linear composite system to successfully stabilize the pendulum. The stabilization of the linear subsystem is first designed. Since the states of the linear subsystem are inputs to the nonlinear subsystem, we ensure that the linear variables which enter the nonlinear subsystem do not peak to prevent the nonlinear subsystem which is zero input asymptotically stable to escape to infinity in finite time. The problem of stabilization of a partially linear composite system by means of state feedback is considered. There is a large body of literature on control of cascade systems for example in [90], a nonlinear small gain theorem which provides a formalism to analyze and design control law for systems that contain saturation was proposed. Global stability of partially linear composite systems is also considered by the work presented in [73]. The linear subsystem of the CMG is controllable and the nonlinear subsystem receives its inputs from the states of the linear subsystem. With zero input, the equilibrium of the nonlinear subsystem is globally asymptotically stable. It may seem that since the zero input equilibrium of the nonlinear part is globally asymptotically stable, when the states of the linear subsystem are driven to zero at very fast exponential rate, the controller must stabilize the whole cascade system. However, it is seen in high-gain feedback design that some of the states peak to very large values before they rapidly decay to zero. The rich literature on perturbation of nonlinear systems indicates that they can be destabilized by exponentially decaying inputs. It is proposed in [31], [88] to restrict the nonlinear subsystem by a linear growth condition and then apply total stability theorems. In the design, a control law is first designed to stabilize the linear subsystem of the CMG. The design utilizes linear state feedback from its own states only. A high-gain linear observer is implemented to estimate the rest of the states from measurement and to produce eigenvalues with very negative real parts. To stabilize the overall system, the control law from the first part is modified as globally bounded function of the states estimates such that it saturates during peaking period.

The linear subsystem is given by

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= u \ , \end{aligned} \tag{3.51}$$

Inspired by [45, 27], let us find a control $u(t) = F\zeta(t)$ such that the real parts of the eigenvalues of the closed-loop system are strictly negative. There is a tradeoff between the transient response and the control effort. The approach to this tradeoff is the design of *u(t)* that minimizes the performance index

$$J = \int_0^\infty \left[\zeta(t)^T Q \zeta(t) + u(t)^T R u(t)\right] dt \tag{3.52}$$

where $Q = M^T M$, R is symmetric and positive definite, *(A,B)* is stabilizable and *(A,M)* is detectable. The controller is obtained by solving the Algebraic Riccati Equation

$$PA + A^T P + Q - PBR^{-1}BP = 0, \tag{3.53}$$

where P is a symmetric positive semidefinite solution and the control law is given by

$$u(t) = -R^{-1}B^T P\zeta. \tag{3.54}$$

For the linear subsystem, let us select Q as to solve for P in equation(3.53)

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \rho. \tag{3.55}$$

Solving (3.53) for P defined as

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix},$$

we have $p_{11} = \sqrt{2}\rho^{\frac{1}{4}}$, $p_{12} = \rho^{\frac{1}{2}}$, and $p_{22} = \sqrt{2}\rho^{\frac{3}{4}}$. The eigenvalues of the closed-loop system are $\lambda_{1,2} = \frac{\rho^{-\frac{1}{4}}}{\sqrt{2}}(-1 \pm j)$ which depend on ρ which we will vary to find the values that meet our design requirements.

The proposed observer is given by

$$\begin{aligned} \dot{\hat{\zeta}}_1 &= \hat{\zeta}_2 + \frac{3}{\alpha}(z - \hat{\zeta}_1) \\ \dot{\hat{\zeta}}_2 &= u + \frac{2}{\alpha^2}(z - \hat{\zeta}_1), \end{aligned} \tag{3.56}$$

where $z = \zeta_1$ is the measured output. The observer eigenvalues are assigned at $-\frac{1}{\alpha}$ and $-\frac{2}{\alpha}$. The overall closed-loop eigenvalues are $\lambda_{1,2} = \frac{\rho^{-\frac{1}{4}}}{\sqrt{2}}(-1 \pm j)$, $\lambda_3 = -\frac{1}{\alpha}$ and $\lambda_4 = -\frac{2}{\alpha}$. Using the principle of separation, one can judiciously select the value of ρ and α to meet the observer-based controller requirements and assign the closed-loop eigenvalues properly and keep in mind that in the presence of noise, there is a lower bound on the selection of the value α which affects the selection of the value of ρ also. The design of the linear feedback control law $u = F\zeta$, with the eigenvalues of the closed-loop system $A+BF$ assigned to the left of a values -a help us guarantee that the trajectories resulting from the closed-loop linear system $\dot{\underline{\zeta}} = (A + BF)\underline{\zeta}$ will satisfy $\|\underline{\zeta}(t)\| \leq K\|\underline{\zeta}(0)\|e^{-at}$ for all valid initial conditions $\underline{\zeta}(0)$ and all $t \geq 0$. The number a can be made arbitrary large, however K depends on F. The resulting peaking phenomenon comes from the fact that one cannot make a arbitrary large *i.e.* α designed above arbitrary small without making K large as well. The authors in [88] demonstrate that if the nonlinear subsystem is bounded input bounded state *(BIBS)* and zero input stable *ZIS* then the nonlinear subsystem is bounded input

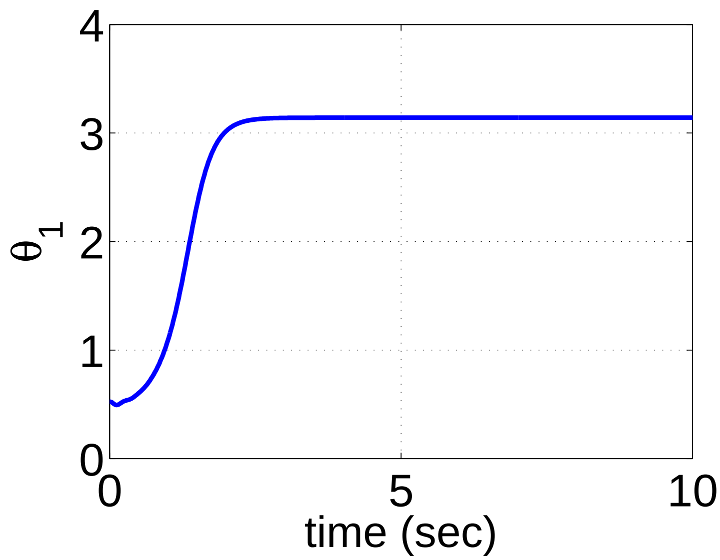

Figure 3.5: Angular position θ_1.

bounded state stable *BIBSS*. This theorem will help us design inputs to the nonlinear subsystem that have appropriate exponential decay. The values of the feedback gain were selected during simulation.

3.6 Simulations

With the controller designed above, we have

$$P(A + BF) + (A + BF)^T P = -Q \leq 0, \tag{3.57}$$

with $F = -R^{-1}B^TP$, and $Q = M^TM$. We selected

$$Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{thus} \quad M = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

It is easily found that *(A,B)* is controllable (*rank(A,B)=2*) and $(Q^{\frac{1}{2}}, A_F)$ is detectable, where $A_F = A + BF$. A composite Lyapunov function for the whole system can be chosen as

$$W(\underline{y}, \underline{\zeta}) = V(\underline{y}) + \zeta^T P \zeta \tag{3.58}$$

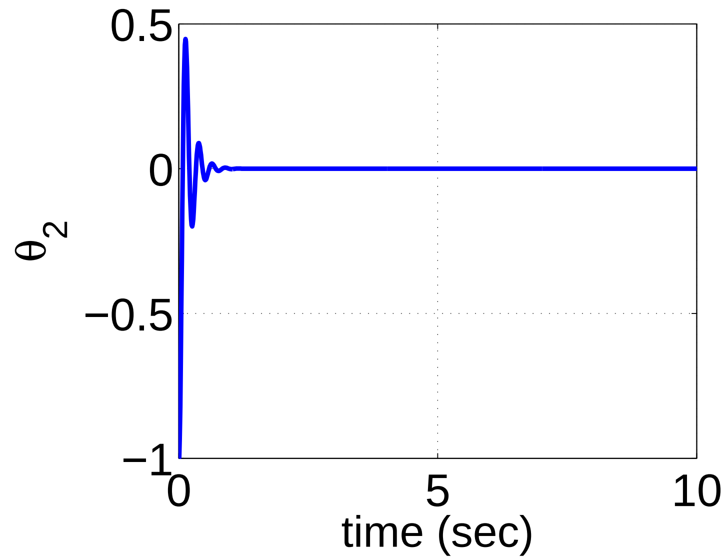

Figure 3.6: Angular position θ_2.

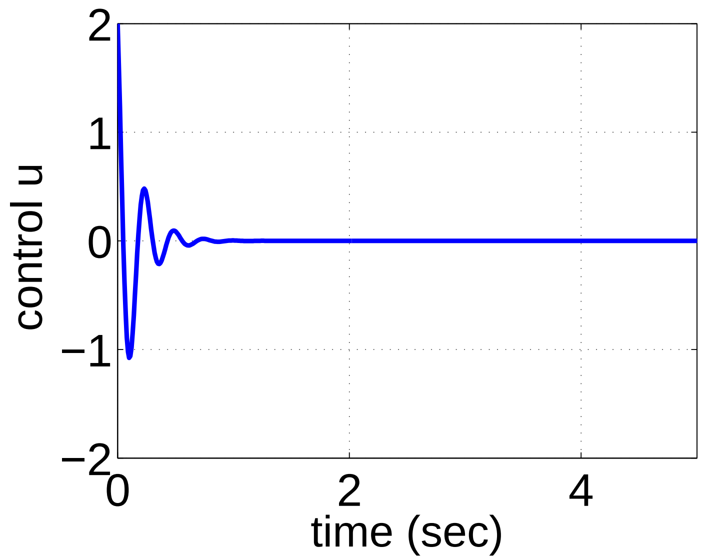

Figure 3.7: The control u.

where $V(\underline{y})$ can be chosen to be a saturation function.

The nonlinear subsystem is explicitly given by

$$f(y_2, y_1, \zeta_1) = \begin{bmatrix} y_2 H_{1,1}^{-1}(\zeta_1) \\ Ag\sin(y_1 - \psi(\zeta_1)) + Bg\cos(y_1 - \psi(\zeta_1) \end{bmatrix} \tag{3.59}$$

where the explicit expression of $\psi(\zeta_1)$ is given in (3.43) which is driven to zero as shown in figure(3.6). This suggests that $\underline{\zeta}$ enters the nonlinear subsystem with very small magnitude and further analysis shows that

$$\|f(t, \underline{y}, \underline{\zeta}) - f(t, \underline{y}, 0)\| \leq L\|\underline{\zeta}\|, \quad L \geq 0 \tag{3.60}$$

The overall control law stabilize the composite system at $(\theta_1 = \pi, \theta_2 = 0)$

3.7 Contraction Theory

Contract theory developed for nonlinear analysis may be viewed as a generalization of linear eigenvalue analysis. Stability analysis is very important in order to successfully design controllers for a dynamical systems and many techniques are now available to tackle this key aspect in dynamical systems analysis and control. Particularly for nonlinear time-varying systems, Lyapunov theory has become a control tool for stability analysis and control design. Lyapunov theory is mostly applied when one considers the stability analysis at an equilibrium of the state space see for example Khalil [46], Isidori *et.al* [76], Marino and Tomei [57], Vidyasagar [91], Nijmeijer and Van der Schaft, Arjan [67]. A novel method for stability analysis emerged in the late 1990's which is also based on the analysis of convergence of system trajectories to one another in the state space [55]. The method does not required explicit knowledge of a specific attractor. Instead it extensively uses virtual displacements. Virtual displacement was originally introduced by Lagrange [53] and its standard notation is δx. The so-called virtual dynamics are introduced by computing the first variation.

Let us consider a time-varying dynamical system given by

$$\dot{x} = f(x,t), \tag{3.61}$$

virtual dynamics are given by

$$\delta\dot{x} = \delta f = \frac{\partial f}{\partial x}\delta x. \tag{3.62}$$

3.7.1 Differential Dynamics and Contraction

Contraction analysis is presented in [55] and was first developed in the context of observers. Here, we will give a brief overview. For the dynamics represented in (3.61), if there exists a nonsigular transformation matrix $\Theta(x,t)$, such that we can perform the change of variables given by

$$\delta z = \Theta(x,t)\delta x, \tag{3.63}$$

the virtual dynamics can be written as

$$\begin{aligned} \delta\dot{x} &= \dot{\Theta}(x,t)\delta x + \Theta\delta\dot{x} = \left(\dot{\Theta} + \Theta\frac{\partial f}{\partial x}\right)\delta x \\ \delta\dot{z} &= \left(\dot{\Theta} + \Theta\frac{\partial f}{\partial x}\right)\Theta^{-1}\delta z \end{aligned} \tag{3.64}$$

Let $M(x,t) = \Theta(x,t)^T\Theta(x,t)$, a uniformly positive definite metric.

$$\begin{aligned} \delta z^T\delta z &= \delta x^T\Theta(x,t)^T\Theta(x,t)\delta x \\ \frac{d}{dt}(\delta z^T\delta z) &= 2\delta z^T\delta\dot{z} = \delta\dot{x}^T M\delta x + \delta x^T M\delta x + \delta x^T M\delta\dot{x} \\ &= \delta x^T\left(\frac{\partial f^T}{\partial x}M + \dot{M} + M\frac{\partial f}{\partial x}\right)\delta x \\ 2\delta x\Theta(x,t)^T F\Theta(x,t)\delta x &= \delta x^T\left(\frac{\partial f^T}{\partial x}M + \dot{M} + M\frac{\partial f}{\partial x}\right)\delta x \end{aligned} \tag{3.65}$$

where $F = \left(\dot{\Theta} + \Theta\frac{\partial f}{\partial x}\right)\Theta^{-1}$ so that if there exists $\beta_M > 0$, we have $F \leq -\beta_M I$ in a region then

$$\left(\frac{\partial f^T}{\partial x}M + \dot{M} + M\frac{\partial f}{\partial x}\right) \leq -2\beta_M M \tag{3.66}$$

is verified in that region. The following theorem then follows

Theorem: (Theorem 2 in [55]) Given the equation $\dot{x} = f(x,t)$, any trajectory, which starts in a ball of constant radius with respect to the metric $M(x,t)$, centered at a given trajectory and contained at all times in a contraction region with respect to $M(x,t)$, remains in that ball and converges exponentially to that trajectories. The proof of this theorem is given in the same reference given above. It is shown that this theorem is not only sufficient but also necessary as stated in the theorem below.

Theorem: If the system which equations are $\dot{x} = f(x,t)$ is exponentially convergent, i.e. its virtual displacements verify the following inequality

$$\delta x^T \delta x \leq k \delta x_0^T \delta x_0 e^{-\beta t}$$

(where $\delta x_0 = \delta x(0)$ and k and β are positive constants), then it is also contracting with respect to a uniformly positive definite and initially upper bounded metric $M(x,t)$ Contraction theory was used as a flow-oriented approach to stability analysis in [41] where virtual and actual systems methodology was used to compare Lyapunov theory and contraction theory. A unified view of both controllers and observers convergence analysis by adopting an observer perspective which is related to dual observers by Brash and is summarized in Luenberger [56]. Suppose S_2 is the given system and S_1 is a system that we construct to control S_2. It is shown by Luenberger that the system S_2 tends to follow S_1 and hence S_1 can be considered as governing the behavior of S_2. Using the above theory, controller stability analysis using contraction theory could be sketched as follows [41]

- write the "target" system equation ($\dot{x}_s = f(x_s, t)$),
- write the controller equation in implicit form,
- define the virtual system whose particular solutions of actual systems are the target system and the controller,
- analyze the virtual dynamics of the virtual system to conclude to contraction behavior.

Vectorial backstepping will be used to design a controller to stabilize the CMG at it unstable equilibrium $(\pi, 0,)$ and the contraction theory summarized above will be used to study the stability analysis of the controller and the contraction behavior of system.

3.7.2 Application of Contraction to the CMG

Consider the reduced dynamics of the control moment gyroscope as shown below

$$\dot{q} = v \tag{3.67}$$

$$H(q)v + C(q, \dot{q})v + G(q) = B\tau, \tag{3.68}$$

where the dynamics are underactuated; $B = [0, 1]$, H(q)>0. The controller design methodology is similar to that in [9], [74]. First we design a *PID* control law and use contraction theory in the sequel to study stability and convergence of the system and the controller. Most of the time PID controllers are designed to drive the joint velocity error to zeros. However, it does not guarantee that steady-state position errors are eliminated as well. The steady-state position errors can be eliminated by requiring them to lie on a sliding surface.

$$s = \dot{\tilde{q}} + \Lambda\tilde{q} = 0,$$

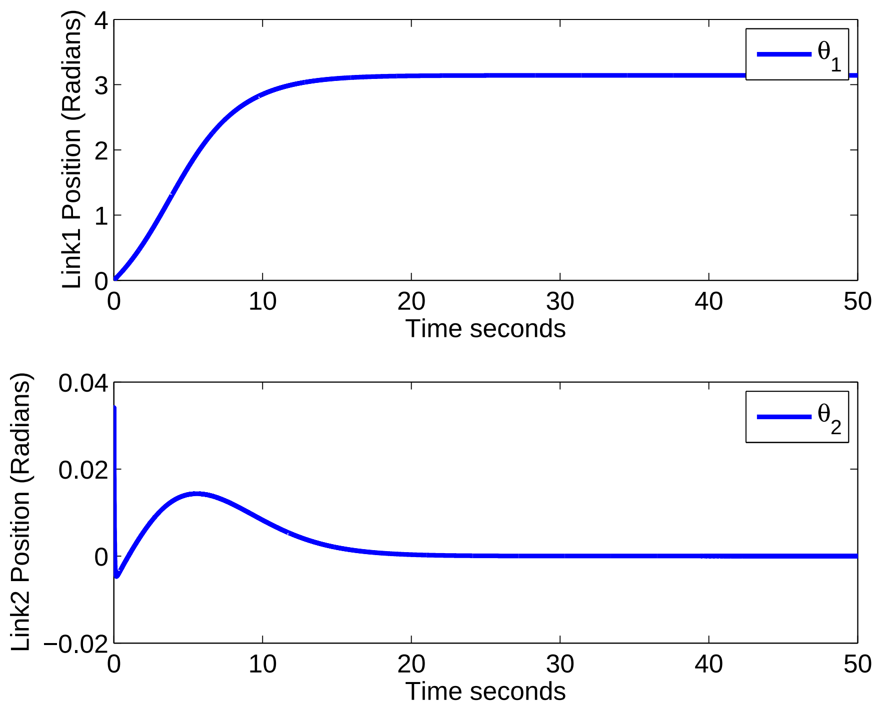

Figure 3.8: PID Control of the CMG.

where Λ is a constant matrix whose eigenvalues lie strictly in the left-hand side of the s-plane and s is a sliding surface. Let $q_d(t)$ be the desired trajectory. We then have the following equations.

$$\dot{q}_r = \dot{q}_d - \Lambda\tilde{q}$$
$$\ddot{q}_r = \ddot{q}_d - \Lambda\dot{\tilde{q}}$$
$$s = \dot{\tilde{q}}_r = \dot{q} - \dot{q}_r = \dot{\tilde{q}} + \Lambda\tilde{q}$$

Let us consider a control Lyapunov function

$$V = \frac{1}{2}\left(s^T H(q) s + \tilde{q}^T K_q \tilde{q}\right), \tag{3.69}$$

where K_q is a positive definite symmetric matrix and $V > 0$. Taking the time derivative of the control Lyapunov function (3.69), we have

$$\dot{V} = s^T \underbrace{\left(\tau - H(q)\dot{v}_r - C(q,v)v_r - G(q) - C(q,v)s + \frac{1}{2}\dot{H}(q)s + K_q\tilde{q}\right)}_{-K_D S} - \tilde{q}^T K_q \Lambda \tilde{q} \tag{3.70}$$

with $v_r = \dot{q}_d - \Lambda\tilde{q}$ and $\dot{v}_r = \ddot{q} - \Lambda\dot{\tilde{q}}$. In essence we have

$$\dot{V} = -\underbrace{\left(s^T K_D s + \tilde{q}^T K_q \Lambda \tilde{q}\right)}_{\geq 0} \leq 0,$$

where the control law is then given by

$$\tau = H(q)\dot{v}_r + C(q,v)v_r + G(q) + C(q,v)s - \frac{1}{2}\dot{H}(q)s - K_q\tilde{q} - K_D s. \tag{3.71}$$

K_D is a symmetric positive definite matrix. Without loosing sight that the system is underactuated, we solve for $\dot{q}_2$ in the first row of τ since it is zero and then replace it in the second to account for the fact that the dynamics are underactuated. We are successfully able to stabilize to CMG in the upright position. Let the equilibrium manifold be $M = \{(q, \dot{q}) : q_1, \dot{q}_1, \dot{q}_2 = 0\}$.

The error vectors of the links are plotted and shown in figure(3.9). The boundedness of q and $\dot{q}$ can be shown as follow. The definition $s = \dot{\tilde{q}} + \Lambda\tilde{q} = 0$, s can be viewed as an input to a stable differential equation in $\tilde{q}$, thus the initial conditions are bounded, and when s is bounded then $\tilde{q}$ is bounded so is $\dot{\tilde{q}}$ and therefore q and $\dot{q}$ are also bounded. So then when $s \to 0$ as $t \to 0$ then $\tilde{q}$ and $\dot{\tilde{q}}$ also tend to zero as t approaches infinity. Since $\dot{V}(t)$ is negative or zero and $V(t)$ is bounded below by zero, $V(t)$ converges to a constant as $t \to 0$.

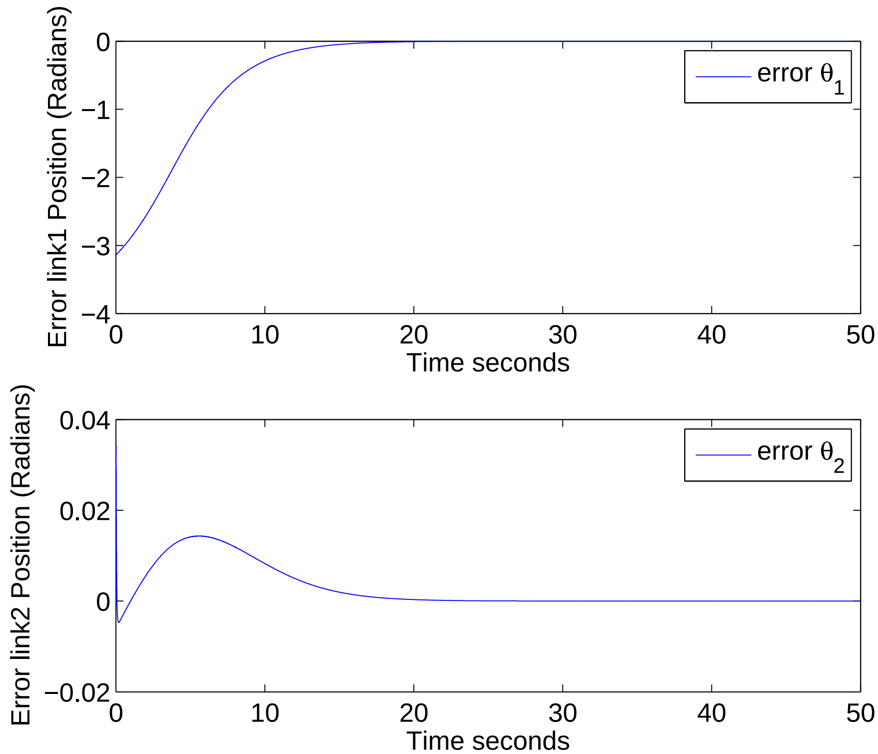

Figure 3.9: Error of $link_1$ and $link_2$.

3.8 Convergence Analysis of Feedback System

The term $C(q,v) - \frac{1}{2}\dot{H}(q)$ is zero so the control law is given by

$$\tau = H(q)\dot{v}_r + C(q,v)v_r + G(q) - K_q\tilde{q} - K_D s \tag{3.72}$$

K_q and K_D are constant Hurwitz matrices. The sliding surface was defined as $s = v - v_d + \Lambda(q - q_d)$, and $v_r = q_d - \Lambda(q - q_d)$, the control law is rewritten as

$$H(q)\dot{v}_d + C(q,\dot{q})\dot{q}_d + G(q) = \tau + [C(q,\dot{q})\Lambda + K_D\Lambda + K_q]\tilde{q} + [H(q)\Lambda + K_D](v - v_d) \tag{3.73}$$

We can deduce the following virtual system equation using the bullet points above in the summary and comparing (3.73) to the CMG dynamics (3.68)

$$\begin{aligned} H(q_s)\dot{v} + C(q_s, \dot{q}_s)\dot{q} + G(q_s) = \tau \quad &+ \quad [C(q_s, \dot{q}_s)\Lambda + K_D\Lambda + K_q](q_s - q) \\ &+ \quad [H(q_s)\Lambda + K_D](v_s - v) \end{aligned} \tag{3.74}$$

where q_s, v_s are the states and the velocities of the actual system respectively and q and v are the states and velocities of the virtual system. It was noted in [55] that for observer convergence one has simply to ensure that the solutions of the actual system are particular solutions of the virtual system. The virtual dynamics are given by

$$\begin{aligned} H(q_s)\delta\dot{v} + C(q_s, \dot{q}_s)\delta\dot{q} \quad &= \quad -[(C(q_s, \dot{q}_s) + K_D)\Lambda + K_q]\delta q \\ &\quad\; -[H(q_s)\Lambda + K_D]\delta v \end{aligned} \tag{3.75}$$

The generalized Jacobian dynamics is given by

$$\begin{pmatrix} I & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} \delta\dot{q} \\ \delta\dot{v} \end{pmatrix} = \begin{pmatrix} -\Lambda & I \\ -H^{-1}K_q & -H^{-1}(C + K_D) \end{pmatrix} \begin{pmatrix} \delta q \\ \delta v \end{pmatrix}$$

after introducing the transformation

$$\begin{pmatrix} \delta\dot{q} \\ \delta\dot{v} \end{pmatrix} = \begin{pmatrix} I & 0 \\ \Lambda & I \end{pmatrix} \begin{pmatrix} \delta q \\ \delta v \end{pmatrix}$$

From the design of the controller above, if we introduce the mapping

$$M = \begin{pmatrix} K_q & 0 \\ 0 & H \end{pmatrix}$$

we have

$$\frac{d}{dt}\left[\begin{pmatrix}\delta q\\ \delta s\end{pmatrix}^T \begin{pmatrix}K_q & 0\\ 0 & H\end{pmatrix}\begin{pmatrix}\delta q\\ \delta s\end{pmatrix}\right] = -2\begin{pmatrix}\delta q\\ \delta s\end{pmatrix}^T \begin{pmatrix}K_q\Lambda & 0\\ 0 & H\end{pmatrix}\begin{pmatrix}\delta q\\ \delta s\end{pmatrix}$$

which less than or equal to zero. This confirms the contraction behavior of the virtual system defined in equation (3.75).

CHAPTER 4: CONCLUSIONS AND RECOMMENDATIONS

We have considered the modeling of the dynamics of a differential mobile robot and the derivation of a nonlinear control law to track a predefined trajectory. Maggi's method, a quasivelocity technique was used to derive the dynamics. This method eliminates the Lagrange multipliers used to enforce the nonholonomic constraints from the start as most of the time one is not interested in the Lagrange multipliers. It thus reduce the number of the variables and the number of equations to solve. Feedback linearization is the technique employed to derive a trajectory control law for the wheeled mobile robot. It is well known that one of the key points to start feedback linearization is the selection of the output vector. Often, some state variable are not available for measurement and the choice is limited. With respect to the output vector selected in this dissertation, there is no well defined relative degree. Dynamic extension is then used to obtain a relative degree vector relative to each component of the output vector. Computer simulations are then added to demonstrate the effectiveness of the control law implemented for this class of wheeled mobile robots. We consider also the dynamics analysis and control of the control moment gyroscope inverted pendulum. The CMG is an underactuated system and has kinematic symmetry with respect of some of its space configuration variables. The system is first partial feedback linearized. Then, we use the kinematic symmetry of the system to perform a global change of coordinates which transforms the original system into a lower order nonlinear subsystem plus a chain of double integrators. The backstepping procedure is used to stabilize the cascade systems and the original system is stabilized at one of its unstable equilibrium points. The design ensures that the states of the linear subsystem which enter the nonlinear subsystem do not peak as it may cause some of the states of the nonlinear subsystem to be unstable. The overall control law is a globally bounded function of the states estimates such that it saturates during peaking period. Contraction theory which does not necessarily required a basin of attraction as opposed to Lyapunov theory is also used to study the stability of the CMG feedback system where vectorial backstepping is used to design a PID stabilization control law. Overall we study stability and control of underactuated systems which are classified as nonholonomic systems. The stability and control techniques can be applied to stabilize systems such as chips, underwater vehicles in case of actuator failures.

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