

INTERMITTENCY FOR BRANCHING RANDOM WALKS WITH HEAVY
TAILS

by

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ABSTRACT

ASMAA GETAN. Intermittency for Branching random walks with heavy tails.
(Under the direction of DR. BORIS VAINBERG)

Branching Markov process models populations in which each individual in generation n produces some random number of individuals in the next generation, $n + 1$, according to a certain probability distribution.

Branching processes play important role in the study of the evolution of various population plants, where members of the population may die or produce offspring independently of the rest. They can be used to model reproduction of bacteria where each bacteria generates several offspring with some probability in a single time unit. And they can be used to model other systems with similar dynamics, e.g., the spread of surnames in genealogy or the propagation of neutrons in a nuclear reactor.

In our dissertation, we consider a long time behavior for a model of branching random walk problem of a population of particles on the d - dimensional lattice Z^d .

In this model, the number of particles increases exponentially by duplicating, with a constant rate of birth, (each particle can split into two particles), and the particles spread everywhere by jumping to not necessary a neighbor place, (it could be a faraway distance), under probability of jumps that is described to be, a heavy tailed probability. Branching or jumping of each particle occurs independently of the other particles.

Under these two conditions, (constant rate of birth and heavy tailed probability of jumps), the front of propagation (where local growth occurs) has been found to be

moving exponentially fast.

A well developed non-uniformity concept called intermittency, is used to investigate the uniformity of the distribution of the particles, on, inside and outside of the front. A random field is called intermittent, if it is distributed very non-uniformly, where huge values can appear with a very small probability. For instance, the magnetic field of the sun is highly intermittent, as almost all of its energy is concentrated in black spots which covers only small parts of the surface of the sun.

In our work, we found that particles on, and outside the front exhibit intermittent behavior. We proved that, the same is true for some region inside the front. Despite that the front of propagation itself moves exponentially fast, the front of intermittency moves with a small power rate, $|x| > t^\gamma$, inside the first front. In the area between those two fronts, the particles are concentrated in very sparse spots with clustered density. This means that, the zone of non-intermittency extends with that rate too. This rate has been found exactly.

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CHAPTER 1: INTRODUCTION

1.1 A General Summary

We consider a population of particles on a d -dimensional lattice Z^d , where each particle sits in its place for a while before it either jumps randomly to another place, or it splits in the same place into two particles. After that, each of those particles (the original particle after it jumps to the new place, the original particle after producing offspring particle or the offspring itself) starts again either to jump to another new place or to split into two particles. This process is continued forever, under the condition that each particle jumps and splits independently of the others. For this process, we impose another two conditions. First, we consider constant rate branching, which means that the number of particles is exponentially increasing in time. And second, we assume that the particles can jump not only to neighbor points but also for distant points, with not necessarily small probability. The limit structure of the particle population inside the propagating front (where global or local growth occur) will be investigated. This structure is described in terms of intermittency.

1.2 Description of the Main Problem

Let us provide a little more detailed description of our branching random walk model.

We assume that each particle in this population, at any position $x \in Z^d$, at a given

time t , stays at this point a randomly exponentially distributed period of time which is Δt , with a parameter 1, before it either splits into two particles in the same place, or it jumps with a probability $a(z)$, to a new position $x + z \in Z^d$. In addition, if ν is a constant that represents the rate of particle splitting, then at any time interval $(t, t + \Delta t)$, this particle which is located at any point $x \in Z^d$, may split with the rate $\nu \Delta t$, into two particles located at the same point x . The process of splitting or jumping of any particle is independent of any other particle. Later on, each of those particles (parental one and the offspring) evolve independently of each other by the same law as the initial particle.

Obviously, the probability of the jump $a(z)$, satisfies the two conditions that,

$a(z) \geq 0$ and $\sum_{z \in Z^d} a(z) = 1$. Here we will assume that the distribution of the jump is symmetric, which means that, $a(z) = a(-z)$ for all $z \in Z^d$.

In our work, a main assumption is included in our model. We assume that the probability for any particle to make a long jump is not necessarily small. Thus we are assuming the following behavior of $a(z)$ at infinity:

$$a(z) = \frac{a_0(\dot{z})}{|z|^{d+\alpha}}(1 + o(1)), \quad |z| \rightarrow \infty, \quad \dot{z} = \frac{z}{|z|}, \quad (1)$$

with

$$a_0(\dot{z}) > \delta > 0, \quad 0 < \alpha < 2.$$

Such a probability distribution is called heavy tailed. Under this heavy tailed probability assumption, the second moments $\sum z^2 a(z)$ are not defined when $\alpha < 2$. They would exist only when $\alpha > 2$.

In the case when the particle jumps without being split, the initial particle would perform the symmetric d -dimensional random walk $X(t)$ with the generator

$$(\mathcal{L}f)(x) = \sum_{z \in Z^d} [f(x+z) - f(x)] a(z), \quad (2)$$

which is a bounded operator on the space $\ell^2(Z^d)$.

The symmetry of the jumps (namely $a(z) = a(-z)$), implies that $\mathcal{L} = \mathcal{L}^*$ is self-adjointed operator in $\ell^2(Z^d)$.

1.3 Previous Studies

The mathematical study of branching processes goes back to the work of Galton and Watson [22] who were interested in the probabilities of long-term survival of family names. Later similar mathematical models were used to describe the evolution of a variety of biological populations, in genetics [11, 12, 13, 14], and in the study of certain chemical and nuclear reactions [20, 15]. The branching processes (in particular, branching diffusions) play important role in the study of the evolution of various populations such as bacteria, cancer cells, carriers of a particular gene, etc., where each member of the population may die or produce offspring independently of the rest.

1.4 *KPP* Problem on R^d

The original classical model in branching diffusion processes belongs to Kolmogorov, Petrovski and Piskunov. It was published in their famous paper in 1937, see [16]. Their model is called the *KPP* model. This model was applied to biological prob-

lems and underlines processes in the case of Brownian motion in R^d . This diffusion process can be described by Poisson kernel $p(t, x, y)$, namely the probability density to find particles at the point $x \in R^d$ at a time t when the starting point at $t = 0$ is $y \in R^d$. It is given by

$$p(t, x, y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{(x-y)^2}{4t}}. \quad (3)$$

The generating operator in *KPP* model is the pure Laplacian $\Delta\psi$ in the Euclidean space R^d , and the governing equation of this model is the heat equation

$$\frac{\partial p}{\partial t} = \Delta p \quad (4)$$

$$p(0, x, y) = \delta(x - y) \quad \text{on } R^d.$$

In the case that *KPP* model splitting with constant rate ν , the expectation density to find particles at x is denoted by m_1 . This function satisfies the equation

$$\frac{\partial m_1}{\partial t} = \Delta m_1 + \nu m_1 \quad (5)$$

$$m_1(0, x, y) = \delta(x - y) \quad \text{on } R^d.$$

In this case we have

$$m_1(t, x, y) = p(t, x, y)e^{\nu t}. \quad (6)$$

In this model, it can be easily found that the front (where $m_1 \approx 1$) is growing linearly in time, since equations (3) and (6) with $m_1 \approx 1$ imply that

$$-\frac{|x-y|^2}{4t} + \nu t \approx 0. \quad (7)$$

So by starting from the origin $y = 0$, the front is the region defined by

$$|x| \approx 2\sqrt{\nu t}.$$

The regions where $\frac{m_2}{m_1^2} \rightarrow \infty$ are considered in our thesis. These regions are called the regions of intermittent behavior. Here m_1 and m_2 are called the first moment and the second moment respectively.

For *KPP* model on R^d , Korolov and Molchanov in ([18]) studied different levels of intermittency. They are the regions where, $\frac{m_j}{m_1^j} \rightarrow \infty$ as $t \rightarrow \infty$ and m_j is the j th moment in this model.

1.5 *KPP* Problem on the Lattice Z^d

In the direct analog of *KPP* model on the lattice, the right hand side of the heat equation (4) will be replaced by the Laplacian operator, which is defined on lattice Z^d and given by

$$(\Delta\psi)(z) = \frac{1}{2d} \sum_{|e|=1} [\psi(z+e) - \psi(z)]. \quad (8)$$

This operator is somewhat simpler than (4) since ψ is defined on the lattice instead of R^d . On the other hand it is much more difficult to handle since it is not spherically symmetric.

1.6 The Difference Between *KPP* Model and Our Model

1- The most important difference between our problem and the *KPP* problem is related to the fact that the generating operator in our model, given in (2), allows

particles to jump not only in neighbor points, but also to make long jumps on the lattice Z^d with relatively not small probability. Any particle x after it splits, can jump to the point $x + z$ with a probability $a(z)$. The point $x + z$ can be found as faraway from x as we please. Note that in the *KPP* model, the probability to find a particle at the time $t = 1$, with a distance d from the parent particle is exponentially small, it is of order $e^{-d^2/4}$.

2- The front propagates linearly in *KPP* model, while in our model, the front will be proved to be propagated exponentially fast, due to the assumption of a long jump probability.

1.7 The First and Second Moments in Our Model (m_1 and m_2)

1.7.1 The First Moment m_1

Let $n(t, x, y)$ be the random variable of the number of particles at a point $x \in Z^d$, at the time $t \geq 0$, under the condition that $n(0, x, y) = \delta(x - y)$, which means that the process starts at the initial time $t = 0$, with a single particle located at the point $y \in Z^d$.

For each arbitrary constant rate of splitting, $\nu > 0$, let $m_1 = m_1(t, x, y) = E(n(t, x, y))$, be the expected value of $n(t, x, y)$. It is called the first moment of the random variable $n(t, x, y)$. Let $p = p(t, x, y)$ be the expected value of the random variable $n(t, x, y)$ in the case when $\nu = 0$. Then m_1 and p satisfy the relations (9)

and (10) given below respectively:

$$\frac{\partial m_1}{\partial t}(t, x, y) = \mathcal{L}m_1(t, x, y) + \nu m_1(t, x, y) \quad , \quad t \geq 0 ; \quad (9)$$

$$m_1(0, x, y) = \delta(x - y).$$

$$\frac{\partial p}{\partial t}(t, x, y) = \mathcal{L}p(t, x, y) \quad , \quad t \geq 0 ; \quad (10)$$

$$p(0, x, y) = \delta(x - y).$$

It is clear that if $p(t, x, y)$ is a fundamental solution of (10), then

$m_1(t, x, y) = p(t, x, y)e^{\nu t}$ is a fundamental solution of (9).

1.7.2 The Second Moment m_2

The second moment $m_2(t, x, y)$ of the random variable $n(t, x, y)$ is defined by

$m_2(t, x, y) = E(n^2(t, x, y))$. It satisfies the following differential equation

$$\frac{\partial m_2}{\partial t}(t, x, y) = (\mathcal{L} + \nu)m_2(t, x, y) + 2\nu m_1^2(t, x, y) \quad (11)$$

$$m_2(0, x, y) = \delta(x - y).$$

The Derivations of (9), (10) and (11) will be given in the next chapter.

1.8 Front of Propagation

The concept of front of propagation is essential in our work. The region in Z^d which separates the large and small values of $m_1(t, x, y)$ is called the **front**. In our work we define the front as the boundary of the set where $m_1 < 1$.

1.9 Intermittency

The notion of intermittency (or intermittent random fields), is popular in natural sciences, (astrophysics, biology, etc). From the qualitative point of view, intermittent random fields are distinguished by the formation of sparse spatial structures such as high peaks, clumps, patches, etc., giving the main contribution to the process in the medium. For instance, the magnetic field of the Sun is highly intermittent as almost all its energy is concentrated in the black spots, which cover only a very small part of the surface of the Sun. Many bio-populations also exhibit strong clumping (clustering). A random variable is called intermittent if it is distributed very non-uniformly. It means that huge values can appear with a very small probability.

Intermittency is a well developed non-uniformity concept. For physicists, the magnetic field of the Sun is intermittent since, say, 99% of its magnetic energy is concentrated on less than 1% of the surface. For mathematicians, 0.1, 0.01 or 10^{-6} are not necessarily small numbers, and a limiting process must be considered instead.

The definition of intermittency based on the progressive growth of the statistical moments was proposed in the review [24], a more formal presentation can be found in [5]. In the simplest form, a field $n(t, x)$, $x \in \mathbb{Z}^d$, is intermittent as $t \rightarrow \infty$ on a non-decreasing family of sets, $\omega(t)$, if

$$\lim_{t \rightarrow \infty} \frac{En^2(t, x)}{(En(t, x))^2} = \infty$$

uniformly in $x \in \omega(t)$.

Let us provide an example to illustrate the meaning of intermittency. Consider a

random variable X with expectation equal to one.

Let this random variable takes the values α_n , $1 \leq n \leq 100$, with probabilities p_n , $\sum p_n = 1$. Then $m_1 = E(X) = \sum \alpha_n p_n$, and $m_2 = E(X^2) = \sum \alpha_n^2 p_n$.

Let $\alpha_1 = \frac{c}{\epsilon}$, and $p_1 = \epsilon$, for $0 < c < 1$ and ϵ is a very small number. Let $\alpha_n \approx 1 - c$ for all $n > 1$ with $\sum_2^{100} p_n = 1 - \epsilon$. Then it is clear that $m_1 \approx 1$ and $\frac{m_2}{m_1^2} \approx c^2/\epsilon \rightarrow \infty$ as $\epsilon \rightarrow 0$. Thus X is an intermittent random variable, as one can see that X is distributed non-uniformly with large values $\alpha_1 = \frac{c}{\epsilon}$ having a small probability.

1.10 Our Goal

In this thesis, we consider the random variable $n(t, x, 0)$, which is the number of particles at a point $x \in Z^d$ at the time t when the process started from a single particle located at the origin 0.

We will discuss the intermittency of $n(t, x, 0)$ on and inside the propagating front.

Our first goal is to find the asymptotic of $m_1(t, x)$, as $t, x \rightarrow \infty$, in order to find the front of propagation of the particles in our problem.

The second goal is to study if intermittent regions exist inside of that front, and to find in this case the boundary between the intermittent and the non-intermittent regions.

In fact we will show that the front of propagation grows exponentially in time, and our random variable exhibits intermittent behavior on and outside of it.

For the region inside the front, we are going to prove that the intermittent behavior occurs when $|x| > t^\gamma$, with a specific values of $\gamma > 0$ that will be found exactly.

The estimation of $p(t, x)$ given in [1] will play an important role in our study.

CHAPTER 2: THE FIRST AND SECOND MOMENTS

2.1 Derivation of the Equation for m_1

In order to derive the equations (9) that governs $m_1(t, x, y)$, we need to evaluate $m_1(t + \Delta t, x, y)$.

For this reason, the time interval $(0, t + \Delta t)$ can be split up in the backward form into two successive intervals which are $(0, \Delta t)$ and $(\Delta t, t + \Delta t)$, of the lengths Δt and t respectively. Then $m_1(t + \Delta t, x, y)$ can be represented in the form

$$m_1(t + \Delta t, x, y) \sim \sum_{z \in Z^d} a(z) \Delta t m_1(t, x, y + z) + 2\nu \Delta t m_1(t, x, y) \quad (12) \\ + (1 - \Delta t - \nu \Delta t) m_1(t, x, y).$$

The terms on the right hand side of the relation (12) can be explained as follows: The first term is the probability for the particle to jump from the point y to the point $y + z$ during the time Δt , which is $a(z)\Delta t$, multiplied by the expectation of the number of particles at the point x when the walk starts at a single point $y + z \in Z^d$, which is $m_1(t, x, y + z)$. The second term is the probability $\nu\Delta t$ of branching during the time Δt , multiplied by the expected value of the number of particles at x that are descendant of both the original and the new born particle at y , which is $2m_1(t, x, y)$. The last term is the expectation of the number of particles at the point x under the condition that the initial particle stays at y without splitting and without jumping

during the time interval Δt .

We subtract $m_1(t, x, y)$ from both sides of the equation (12) above.

Using that $\sum a(z) = 1$, we divide by Δt , and pass to the limit as $\Delta t \rightarrow 0$. This implies the following equation

$$\frac{\partial m_1}{\partial t}(t, x, y) = \sum_{z \in Z^d} [m_1(t, x, y + z) - m_1(t, x, y)]a(z) + \nu m_1(t, x, y), \quad t \geq 0 \quad (13)$$

such that

$$m_1(0, x, y) = \delta(x - y). \quad (14)$$

By using the definition of $(\mathcal{L}f)(x)$ given in (2) in the equation (13) above, the derivation of the equations (9) is complete.

The solution of (13) gives the integral representation of $m_1(t, x, y)$. The details are given in the section 2.2 below.

Note that the term $\sum_{z \in z^d} m_1(t, x, y + z)a(z)$ in (13) can be considered as a convolution in y of the two functions $a(y)$ and $m_1(t, x, y)$.

2.2 Integral Representations of p And m_1

The integral representations of p and m_1 are obtained by solving the equation (13) together with the initial condition (14) for the two cases, when $\nu = 0$ and when $\nu \neq 0$ respectively.

One can easily show that m_1 and p depend not on x and y but on their difference, i.e., $m_1(t, x, y) = m_1(t, x - y, 0)$ and $p(t, x, y) = p(t, x - y, 0)$.

Thus we can drop the argument y in both $m_1(t, x, y)$ and $p(t, x, y)$.

In this case $p(t, x)$ denotes the solution of (13) with $\nu = 0$ and $y = 0$, and $m_1(t, x)$

denotes the solution of (13) with $\nu \neq 0$ and $y = 0$.

For the case when $\nu = 0$, we find $p(t, x)$ as follows:

Consider $p(t, x)$ and $a(x)$ as Fourier coefficients (defined at an integer points $x \in Z^d$) of the two functions $\widehat{p}(t, \sigma)$ and $\widehat{a}(\sigma)$ respectively, with $\sigma \in [-\pi, \pi]^d$. In this case

$$\widehat{p}(t, \sigma) = \sum_{x \in Z^d} p(t, x) e^{-i(\sigma, x)} \quad \text{and} \quad \widehat{a}(\sigma) = \sum_{x \in Z^d} a(x) e^{-i(\sigma, x)}. \quad (15)$$

We use that $\sum_{x \in Z^d} a(x) = 1$, and pass to the Fourier series in (13), we get

$$\begin{aligned} \frac{\partial \widehat{p}}{\partial t}(t, \sigma) &= \sum_{x \in Z^d} \frac{\partial p}{\partial t}(t, x) e^{-i(\sigma, x)} \\ &= \sum_{x \in Z^d} \left[\sum_{z \in Z^d} p(t, x - z) a(z) \right] e^{-i(\sigma, x)} - \sum_{x \in Z^d} p(t, x) e^{-i(\sigma, x)} \\ &= \sum_{z \in Z^d} \left[\sum_{x \in Z^d} p(t, x - z) e^{-i(\sigma, (x-z))} \right] a(z) e^{-i(\sigma, z)} - \widehat{p}(t, \sigma), \end{aligned}$$

i.e., $\widehat{p}(t, \sigma)$ is the solution of the following ordinary differential equation

$$\frac{\partial \widehat{p}}{\partial t}(t, \sigma) = \widehat{p}(t, \sigma) [\widehat{a}(\sigma) - 1] \quad (16)$$

with the condition that, $\widehat{p}(0, \sigma) = 1$.

This condition is true since we have $p(0, x) = \delta(x)$.

It is clear that the solution of the problem (16) above is:

$$\widehat{p}(t, \sigma) = e^{[\widehat{a}(\sigma) - 1]t}.$$

The Fourier coefficients $p(t, x)$ of the function $\widehat{p}(t, \sigma)$ are given by

$$p(t, x) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \widehat{p}(t, \sigma) e^{i(\sigma, x)} d\sigma$$

Thus

$$p(x, t) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{[\widehat{a}(\sigma)-1]t+i(\sigma, x)} d\sigma. \quad (17)$$

The equation (17) gives the integral representation of the function $p(t, x)$.

For the case when $\nu \neq 0$, then by passing to the Fourier series in (13) we obtain the following ordinary differential equation and initial condition

$$\frac{\partial \widehat{m}_1}{\partial t}(t, \sigma) = [\widehat{a}(\sigma) - 1] \widehat{m}_1(t, \sigma) + \nu \widehat{m}_1(t, \sigma)$$

$$\widehat{m}_1(0, \sigma) = 1.$$

The solution of this problem is

$$\widehat{m}_1(t, \sigma) = e^{[\widehat{a}(\sigma)-1+\nu]t} = e^{\nu t} e^{[\widehat{a}(\sigma)-1]t}.$$

Hence the integral representation of $m_1(t, x)$, is given by

$$m_1(t, x) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{\nu t} e^{[\widehat{a}(\sigma)-1]t+i(\sigma, x)} d\sigma, \quad (18)$$

It is clear that

$$m_1(t, x) = e^{\nu t} p(t, x).$$

2.3 Derivation of the Equation for m_2

For the derivation of equation (11) that is satisfied by m_2 , we consider again $m_2(t + \Delta t, x, y)$ on the time interval $(0, t + \Delta t)$. We split this interval in the backward way into two successive intervals $(0, \Delta t)$ and $(\Delta t, t + \Delta t)$. Then $m_2(t + \Delta t, x, y)$ can be represented on those two intervals in the following form

$$m_2(t + \Delta t, x, y) \sim \sum_{z \in Z^d} a(z) \Delta t m_2(t, x, y + z) + \nu \Delta t E(n_1 + n_2)^2 + (1 - \Delta t - \nu \Delta t) m_2(t, x, y). \quad (19)$$

Here the terms on the right hand side of (19) are similar to the terms on the right hand side of (12). In this case, $n_1 = n_1(t, x, y)$ is the number of particles at x that are descendant of the original particle, and $n_2 = n_2(t, x, y)$ is the number of particles at x that are descendant of the new born particle.

In (19) we use the fact that

$$E(n_1 + n_2)^2 = E(n_1)^2 + E(n_2)^2 + 2E(n_1)E(n_2) = 2m_2(t, x, y) + 2m_1^2(t, x, y).$$

We subtract $m_2(t, x, y)$ from both sides of (19), divide by Δt , and pass in both sides to the limit as $\Delta t \rightarrow 0$. After doing all these steps, the derivation of (11) is obtained.

2.4 Integral Representation of m_2

The solution of (11) gives the integral representation of m_2 . This solution can be found by using the following Duhamel formula:

$$m_2(t, x, y) = m_1(t, x, y) + \int_0^t u(t-s, x, y) ds. \quad (20)$$

This formula is used under the condition that $u(t, x)$ is the solution of the following problem:

$$\frac{\partial u}{\partial t}(t, x, y) = (\mathcal{L} + \nu)u, \quad t > s \quad (21)$$

such that

$$u(s, x, y) = 2\nu m_1^2(s, x, y).$$

Now since $m_1(t, x, y)$ is a fundamental solution of the problem (9) above, the solution of (21) on the lattice will be given by

$$u(t, x, y) = 2\nu \sum_{z \in Z^d} m_1(t, x-z, y) m_1^2(s, z, y). \quad (22)$$

Hence the substitution of (22) in the equation (20) implies the solution of (11).

Thus the integral representation of $m_2(t, x, y)$ is the following

$$m_2(t, x, y) = m_1(t, x, y) + 2\nu \int_0^t ds \sum_{z \in Z^d} m_1(t-s, x-z, y) m_1^2(s, z, y). \quad (23)$$

2.5 Starting From the Origin

Finally we note that the first and second moments m_1 and m_2 depend only on the difference $x - y$, (but not on x and y separately). Thus we can replace y in (20) and (22) by $y = 0$, and consider the two functions $m_1 = m_1(t, x)$ and $m_2 = m_2(t, x)$ as the

first and second moments respectively of the random variable $n(t, x)$ of the number of particles at each point $x \in Z^d$ under the condition that the process starts with one particle located at the origin.

Now we can consider the main equation for $m_1(t, x)$, $p(t, x)$ and $m_2(t, x)$ to be the following:

$m_1(t, x)$ is the solution of the equation

$$\frac{\partial m_1}{\partial t}(t, x) = (\mathcal{L} + \nu) m_1(t, x), \quad t \geq 0, \quad (24)$$

such that

$$m_1(0, x) = \delta(x).$$

Function $p(t, x)$ coincides with m_1 when $\nu = 0$. It is the solution of the equation

$$\frac{\partial p}{\partial t}(t, x) = \mathcal{L}p(t, x), \quad t \geq 0, \quad (25)$$

such that

$$p(0, x) = \delta(x).$$

Also from (23), the main integral representation equation for $m_2(t, x)$ has the form

$$m_2(t, x) = m_1(t, x) + 2\nu \int_0^t ds \sum_{z \in Z^d} m_1(t-s, x-z) m_1^2(s, z). \quad (26)$$

CHAPTER 3: ASYMPTOTIC BEHAVIOR OF THE FIRST MOMENT

3.1 Asymptotic Behavior of $\widehat{a}(\sigma)$ at 0

For $\widehat{a}(\sigma)$ defined in (15), the following Lemma 1 has been proved in [1]. It provides the asymptotic behavior of $\widehat{a}(\sigma)$ at 0. Namely, we assume that:

$$a(z) = \sum_{j=0}^{d+\epsilon} \frac{a_j(\dot{z})}{|z|^{d+\alpha+j}} + O\left(\frac{1}{|z|^{2d+\alpha+1+\epsilon}}\right), \quad |z| \rightarrow \infty, \quad \alpha \in (0, 2), \quad (27)$$

where

$$a_j \in C^{d+1-j+\epsilon}(S^{d-1}), \quad a_0(\dot{z}) > \delta > 0$$

and $\epsilon = 1$ if $\alpha = 1$, $\epsilon = 0$ otherwise.

Lemma 1. If (27) holds, then

$$\widehat{a}(\sigma) = 1 - \sum_{j=0}^d b_j(\dot{\sigma}) |\sigma|^{\alpha+j} + f(\sigma), \quad \sigma \in T^d = [-\pi, \pi]^d, \quad f(0) = 0, \quad (28)$$

where $b_j \in C^{d+[\alpha]+1}(S^{d-1})$ and function f , being extended periodically on R^d , belongs to $C^{d+[\alpha]+1}(R^d)$. Moreover, the homogeneous function $b_0(\dot{\sigma}) |\sigma|^\alpha$ in R^d is the Fourier transform of the homogeneous (of order $-d - \alpha$) distribution that is equal to $a_0(\dot{x}) |x|^{-d-\alpha}$ when $0 \neq x \in R^d$, and

$$b_0(\dot{\sigma}) = -\Gamma(-\alpha) \cos \frac{\alpha\pi}{2} \int_{S^{d-1}} a_0(\dot{x}) |(\dot{x}, \dot{\sigma})|^\alpha dS_{\dot{x}} > 0, \quad (29)$$

where Γ is the gamma-function.

Remarks

1) Note that f can not be omitted, since a change in the values of $a(z)$ at several points does not perturb its asymptotic behavior at infinity, but changes $\widehat{a}(\sigma)$ by an analytic function.

2) The next two properties of $\widehat{a}(\sigma)$ follow immediately from the properties of $a_0(\dot{x})$:

$$\widehat{a}(-\sigma) = \widehat{a}(\sigma) \quad \text{and} \quad -1 < \widehat{a}(\sigma) < 1, \quad \text{when } 0 \neq \sigma \in T^d. \quad (30)$$

The second property in (30) follows from (15) and the relation $\sum_{x \in Z^d} a(x) = 1$, under the condition that, for each $\sigma \in T^d$, $\sigma \neq 0$, there is a point $z \in Z^d$, where $e^{-i(z,\sigma)} \neq 1$ and $a(z) \neq 0$. Such points z exist due to (27).

3.2 Asymptotic Behavior of $m_1(t, x)$

The asymptotic behavior of $m_1(t, x) = e^{\nu t} p(t, x)$ is a direct result of the uniform asymptotics of the function $p(t, x)$ given in the following theorem which is one of the main results of [1].

Theorem 2. Let the conditions (28)-(30) hold.

Then, the the following asymptotis holds for $p(t, x)$

$$(i) \quad p(t, x) = \frac{1}{t^{d/\alpha}} S\left(\frac{x}{t^{1/\alpha}}\right) (1 + o(1)), \quad \text{when } x \in Z^d, \quad |x| + t \rightarrow \infty. \quad (31)$$

Here, $S(y) = \frac{1}{(2\pi)^d} \int_{R^d} e^{i(\sigma,y) - b_0(\dot{\sigma})|\sigma|^\alpha} d\sigma > 0$, is the stable density $S = S_{\alpha, a_0}(y)$, which depends on $\alpha \in (0, 2)$, and coefficient a_0 , is defined in (27).

(ii) If $\frac{|x|}{t^{1/\alpha}} \rightarrow \infty$, $|x| \geq 1$, then the previous statement can be specified as follows:

$$p(t, x) = \frac{a_0(\dot{x})}{t^{d/\alpha}} \left(\frac{t^{1/\alpha}}{|x|} \right)^{d+\alpha} (1 + o(1)) = \frac{a_0(\dot{x})t}{|x|^{d+\alpha}} (1 + o(1)). \quad (32)$$

In the next section, a simplified version of this theorem will be proved. For this reason, we need to prove the following lemma:

Lemma 3. Function $p(t, x)$ given in (17), is strictly positive for all $x \in Z^d$, $t > 0$.

Proof. Denote by $a_n(x)$ the convolutions of n copies of $a(x)$:

$$a_n(x) := a(x) * a(x) * \dots * a(x), \quad (33)$$

where $a(x) * b(x) = \sum_{z \in Z^d} a(x - z) * b(z)$.

We multiply both sides of (33) by $e^{(-i\sigma x)}$, then we take the summation for all $x \in Z^d$. This implies that $\widehat{a}_n(\sigma) = [\widehat{a}(\sigma)]^n$. Further, since $\widehat{a}(0) = 1$, the second property in (30) implies that, $|\widehat{a}(\sigma)| \leq 1$, and therefore, $|\widehat{a}_n(\sigma)| \leq 1$. This allows us to write $p(t, x)$ as follows:

$$\begin{aligned} p(t, x) &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} e^{[\widehat{a}(\sigma)-1]t+i(\sigma, x)} d\sigma \\ &= \frac{e^{-t}}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left[1 + \sum_{n=1}^{\infty} \frac{[\widehat{a}(\sigma)]^n}{n!} t^n \right] e^{i(\sigma, x)} d\sigma. \end{aligned}$$

Thus

$$p(t, x) = e^{-t} [\delta(x) + \sum_{n=1}^{\infty} \frac{a_n(x)}{n!} t^n]. \quad (34)$$

Since $a(x) \geq 0$, all the convolutions $a_n(x)$ are non-negative. Hence (34) will imply

the statement of the lemma if we show that $a_2(x)$ is strictly positive for all $x \in Z^d$.

We have:

$$a_2(x) = \sum_{z \in Z^d} a(x-z)a(z). \quad (35)$$

Here $a(\cdot) \geq 0$, and from (27) it follows that the terms in (35) are positive for each fixed x if z is large enough. Thus $a_2(x) > 0$, and the proof of the lemma is completed.

3.3 Simplified Version Theorem of the Asymptotics of $p(t,x)$

As a consequence of the results above, the following theorem 4 has been proved, it can be considered as a simplified version of Theorem 2. First we need to introduce the notion of the equivalency between two functions.

Definition 1. Functions a and b are called equivalent, and it will be denoted by $a \sim b$, if two constants c_1 and c_2 exist, such that $c_1 b < a < c_2 b$.

Theorem 4. Let the condition given in (1) be hold.

For arbitrary $a_2 \geq a_1 > 0$, the following relations hold

$$(i) \quad |p(t, x)| \sim \frac{t}{|x|^{d+\alpha}} \quad \text{when } |x| > a_1 t^{\frac{1}{\alpha}}. \quad (36)$$

$$(ii) \quad |p(t, x)| \sim t^{-\frac{d}{\alpha}} \quad \text{when } 0 \neq |x| \leq a_2 t^{\frac{1}{\alpha}}, \quad t > \epsilon > 0. \quad (37)$$

$$(iii) \quad |p(t, x)| \sim C \quad \text{when } x = 0 \text{ and } t < 1, \quad C \neq 0 \text{ is a constant.} \quad (38)$$

Proof. The last statement is obvious since the boundedness of $p(t, x)$ follows immediately from (17).

In order to prove the first two statements, we need to split the lattice into three

regions, $U_1 = U_1(t)$, $U_2 = U_2(t)$, and $U_3 = U_3(t)$. Where U_1 is the region defined by the inequality $|x| > At^{\frac{1}{\alpha}}$ with A so large that the remainder term in (32) is less than $1/2$. If needed we increase the value of A to be sure that it is not smaller than the constant a_2 chosen in (37).

Note that $|x| > At^{\frac{1}{\alpha}}$ implies that $x \neq 0$. Thus, $|x| \geq 1$ in U_1 since x is a point on the lattice.

From (32) it follows that:

$$\left| p(t, x) - \frac{a_0(\dot{x})t}{|x|^{d+\alpha}} \right| < \frac{1}{2} \frac{a_0(\dot{x})t}{|x|^{d+\alpha}}, \text{ for all } x \in U_1.$$

and therefore,

$$\frac{1}{2} \frac{a_0(\dot{x})t}{|x|^{d+\alpha}} < p(t, x) < \frac{3}{2} \frac{a_0(\dot{x})t}{|x|^{d+\alpha}}, \text{ for all } x \in U_1.$$

Since $a_0(\dot{x})$ is continuous and does not vanish on the unite sphere $|\dot{x}| = 1$, it follows that:

$$|p(t, x)| \sim \frac{t}{|x|^{d+\alpha}} \text{ when } |x| > At^{\frac{1}{\alpha}}. \quad (39)$$

This justifies (36) for $a_1 = A$, but not for any arbitrary a_1 .

Now one can find a large enough constant B , such that the remainder term in (31) is less than $1/2$, (i.e. , $|o(1)| < 1/2$), whenever $|x| + t > B$.

Let U_2 be defined by the inequalities $|x| \leq At^{\frac{1}{\alpha}}$ and $|x| + t \geq B$, with the same A used in the previous step.

Since $\frac{|x|}{t^{\frac{1}{\alpha}}} \leq A$ is bounded in U_2 , and function $S(y)$ is positive and continuous, it follows that $S\left(\frac{|x|}{t^{\frac{1}{\alpha}}}\right)$ in (31) has upper and lower positive bounds in U_2 . Thus (31)

implies that $p(t, x) \sim \frac{1}{t^{\frac{d}{\alpha}}}$ on the region U_2 .

Now consider the region U_3 defined as follows

$$U_3 = \{|x| \leq At^{\frac{1}{\alpha}}, \epsilon < |x| + t \leq B\} \setminus \{(t, x) : x = 0, t \leq 1\}.$$

We take into account that region U_3 is bounded with $t \geq \epsilon > 0$. Since $p(t, x) > 0$ for $t > 0$, (see Lemma 3), and p is continuous on U_3 , it follows that p has lower and upper bounds on U_3 , i.e., there are two constants C_1 and C_2 such that

$$0 < C_1 < p(t, x) < C_2 \text{ on } U_3.$$

Similar estimates are valid for the function $\frac{1}{t^{\frac{d}{\alpha}}}$.

Thus, $p(t, x) \sim \frac{1}{t^{\frac{d}{\alpha}}}$ on $U_2 \cup U_3$, and the second statement of the theorem is proved with $a_2 = A$. Therefore it is valid for arbitrary a_2 , since A was chosen to be such that $A \geq a_2$.

To complete the proof of the theorem, it remains to show that the equivalency relation (39) can be extended to the region $At^{\frac{1}{\alpha}} \geq |x| > a_1t^{\frac{1}{\alpha}}$. Thus the proof of the theorem will be completed as soon as the following lemma is proved.

Lemma 5. In the intermediate region $\epsilon < a_1t^{\frac{1}{\alpha}} < |x| \leq At^{\frac{1}{\alpha}}$, the following relations hold

$$p(t, x) \sim \frac{1}{t^{\frac{d}{\alpha}}} \sim \frac{t}{|x|^{d+\alpha}}.$$

Proof. Relation (37) holds in the region $a_1t^{\frac{1}{\alpha}} < |x| \leq At^{\frac{1}{\alpha}}$, described in the lemma. So it is enough to show that $\frac{1}{t^{\frac{d}{\alpha}}} \sim \frac{t}{|x|^{d+\alpha}}$ in this region.

The ratio of the latter two functions is equal to

$$\frac{\frac{1}{t^{\frac{d}{\alpha}}}}{\frac{t}{|x|^{d+\alpha}}} = \frac{|x|^{d+\alpha}}{t^{1+\frac{d}{\alpha}}} = \frac{|x|^{d+\alpha}}{t^{\frac{d+\alpha}{\alpha}}} = \left(\frac{|x|}{t^{\frac{1}{\alpha}}}\right)^{d+\alpha}.$$

This ratio is bounded from below by $a_1^{d+\alpha}$ and bounded from above by $A^{d+\alpha}$. Hence the proofs of Lemma 5 and Theorem 4 are complete.

Lemma 5 allows us to use any of the equivalency results (36) or (37) for $p(t, x)$ on the region where $t^{\frac{d}{\alpha}} < |x| < At^{\frac{1}{\alpha}}$.

We also can combine the estimates (37) and (38) in the region $|x| \leq t^{\frac{1}{\alpha}}$ by the following form, given in the next lemma, with no need to mention that $t \geq 1$.

Lemma 6. The relation, $|p(t, x)| \sim \frac{1}{(t+1)^{\frac{d}{\alpha}}}$, holds in the region where $|x| \leq a_2 t^{\frac{1}{\alpha}}$.

3.4 Finding the Front

Recall that the front of propagation is the region where, $m_1(t, x) \sim 1$ as $t \rightarrow \infty$.

The following theorem states that, the front propagates exponentially in time.

Theorem 7. The front of propagation is located in the region where,

$$|x| = (a_0(\dot{x})t)^{\frac{1}{d+\alpha}} e^{\frac{\nu t}{d+\alpha}} (1 + o(1)) , \quad t \rightarrow \infty.$$

Proof. There is no front in the region where $|x| \leq t^{\frac{1}{\alpha}}$. This follows from (37) since in this region, we have

$$m_1(t, x) = p(t, x) e^{\nu t} \sim \frac{1}{t^{\frac{d}{\alpha}}} e^{\nu t} \rightarrow \infty. \quad (40)$$

From (40), it is clear that $m_1(t, x)$ is unbounded and grows exponentially as $t \rightarrow \infty$.

Now consider the region $|x| > t^{\frac{1}{\alpha}}$, as $t \rightarrow \infty$.

From (18) and (32), we have:

$$m_1(t, x) = p(t, x)e^{\nu t} = \frac{a_0(\dot{x})t}{|x|^{d+\alpha}} e^{\nu t}(1 + o(1)). \quad (41)$$

The statement of the theorem follows immediately from here.

CHAPTER 4: INTERMITTENCY

4.1 Intermittency On and Outside the Front:

Recall that our goal is to investigate the intermittent regions in the domain of our problem. They are the regions in which

$$\frac{m_2(t, x)}{m_1^2(t, x)} \rightarrow \infty, \text{ as } t \rightarrow \infty.$$

For this reason, we will estimate $\frac{m_2}{m_1^2}$ separately, on, inside and outside of the front of propagation (see Theorem 7 for the definition of the front).

In the following theorem, we will prove that $\frac{m_2(t, x)}{m_1^2(t, x)} \rightarrow \infty$, as $t \rightarrow \infty$ in the regions on and outside the front. For the regions inside the front, the intermittency will be discussed in much details in the sections that follow this section.

Theorem 8. $\frac{m_2(t, x)}{m_1^2(t, x)} \rightarrow \infty$ on and outside of the front, where $|x| \geq t^{\frac{1}{d+4\alpha}} e^{\frac{\nu t}{d+4\alpha}}$, as $t \rightarrow \infty$.

Proof. For the proof we are going to directly estimate $\frac{m_2(t, x)}{m_1^2(t, x)}$ in the region given above.

Since always $m_1(t, x) \geq 0$, the equality (26) implies that

$$m_2(t, x) > \int_0^t \sum_{z \in Z} m_1(t-s, x-z) m_1^2(s, z) ds. \tag{42}$$

We need to find an estimation for the integral on the right hand side of (42). For

this reason, we use that $m_1(s, z) = p(s, z) \cdot e^{\nu s}$, and replace $m_1(t-s, x-z)$ and $m_1(s, z)$ by their corresponding formulas from (36) and (37).

For estimating this integral, we divide the set $\{(s, z) : s > 0, z \in Z^d\}$ into several sub-regions by the following two parboloids, each of them in $R^1 \times Z^d$, namely:

$$|z| = s^{\frac{1}{\alpha}} \text{ and } |x - z| = (t - s)^{\frac{1}{\alpha}} .$$

It is clear that this division depends on the values of t and x .

We will need to estimate $m_2(t, x)$ in each of those sub-regions. In fact it is enough to find a small sub-region, ω , in $R^1 \times Z^d$, where the expression in the right hand side of (42) restricted to ω is unbounded as $t \rightarrow \infty$, because in this case, the contribution to the right side of (42) from the other parts of $R^1 \times Z^d$ will only increase the low bound to the value of $m_2(t, x)$.

Let us consider the small sub-region, $\omega : z = 0, t - 1 \leq s \leq t$. It is located inside the region $|z| \leq s^{\frac{1}{\alpha}}$. In this case (42) implies that

$$m_2(t, x) > \int_0^t \sum_{z \in Z} m_1(t-s, x-z) m_1^2(s, z) ds \geq \int_{t-1}^t m_1(t-s, x-0) m_1^2(s, 0) ds. \quad (43)$$

Function $m_1(t-s, x-0)$ in (43), can be estimated from below using (36) as follows:

since we have

$$m_1(t-s, x-z) \geq \frac{C(t-s)}{|x-z|^{\alpha+d}} e^{\nu(t-s)} \text{ as } \frac{|x-z|}{(t-s)^{\frac{1}{\alpha}}} \rightarrow \infty .$$

So when, $z = 0, t - 1 \leq s \leq t$, then

$$m_1(t-s, x-0) \geq \frac{C(t-s)}{|x|^{d+\alpha}} e^{\nu(t-s)} \text{ as } \frac{|x|}{(t-s)^{\frac{1}{\alpha}}} \rightarrow \infty .$$

Also, $m_1(s, 0)$ in (43), can be estimated from below using (37) by

$$m_1(s, 0) \geq \frac{C}{s^{\frac{d}{\alpha}}} e^{\nu s}.$$

Hence $m_2(t, x)$ in (43) is estimated by the following

$$m_2(t, x) \geq C \int_{t-1}^t \frac{(t-s)}{|x|^{d+\alpha}} e^{\nu(t-s)} \frac{1}{s^{\frac{2d}{\alpha}}} e^{2\nu s} ds. \quad (44)$$

For further estimation of $m_2(t, x)$ in (44), we can use that

$$e^{\nu(t-s)} \cdot e^{2\nu s} = e^{\nu(t+s)} \geq ce^{2\nu t} \text{ since } s \geq t-1 \text{ and } \frac{1}{s^{\frac{2d}{\alpha}}} > \frac{1}{t^{\frac{2d}{\alpha}}} \text{ (as we have } s < t \text{).}$$

We use that $\int_{t-1}^t (t-s)ds = 1/2$. Hence (44) can be read as follows

$$m_2(t, x) \geq \frac{Ce^{2\nu t}}{|x|^{d+\alpha}} \cdot \frac{1}{t^{\frac{2d}{\alpha}}}. \quad (45)$$

So if x is located on the front (see Theorem 7, where $|x| \sim t^{\frac{1}{d+\alpha}} e^{\frac{\nu t}{d+\alpha}}$, t is large, and $|x|$ is exponentially large) or outside of it, then (41) and (45), together with the relation $m_1 < C$, imply that

$$\frac{m_2(t, x)}{m_1^2(t, x)} \Big|_{|x| \geq t^{\frac{1}{d+\alpha}} e^{\frac{\nu t}{d+\alpha}}} \geq \frac{Ce^{\nu t}}{t^{1+2d/\alpha}} \rightarrow \infty.$$

In the above result, $\frac{m_2}{m_1^2}$ is unbounded since it has an exponential growth as $t \rightarrow \infty$.

Hence the theorem is proved.

In fact, in Theorem 9, given in the next section, we will show that intermittency takes place inside the front for some extent of the form $|x| = t^\gamma$.

4.2 Intermittency Inside the Front

Now we consider the region where $|x| \gg t^{\frac{1}{\alpha}}$ but still inside the front, i.e., we consider the region where,

$$t^{\frac{1}{\alpha}} < |x| < t^{\frac{1}{d+\alpha}} e^{\frac{vt}{d+\alpha}}.$$

The following theorem is essential in our work. In this theorem we are going to prove that, there are intermittent regions inside the front to the extent where $|x| > t^{\gamma+\epsilon}$, such that $\gamma = \frac{2\alpha + d}{\alpha(\alpha + d)}$. In other word $x = t^\gamma$ is the boundary between the intermittent and the non-intermittent regions inside the front.

Theorem 9. Let the heavy tail condition (1) holds. Then

1) The ratio $\frac{m_2(t, x)}{m_1^2(t, x)}$ is uniformly bounded in each ball $|x| < Bt^\gamma$, such that $\gamma = \frac{2\alpha+d}{\alpha(\alpha+d)}$, i.e., the random variable n is non-intermittent in this ball.

2) For each domain $\Omega_\varepsilon(t) = \{x : |x| > t^{\gamma+\varepsilon}\}$, $\varepsilon > 0$, we have $\frac{m_2(t, x)}{m_1^2(t, x)} \rightarrow \infty$ uniformly in $x \in \Omega_\varepsilon(t)$, as $t \rightarrow \infty$, i.e., n is intermittent in $\Omega_\varepsilon(t)$.

Proof: The proof of the first part of this theorem will be given in the following sections of this chapter starting from the next one.

The proof of the second part is given here.

From the estimation of $m_1(t, x)$ given in (41), when $t \rightarrow \infty$, we have:

$$m_1^2(t, x) \leq \frac{ct^2}{|x|^{2(d+\alpha)}} e^{2vt}. \quad (46)$$

We will consider the contribution to the lower bound of $m_2(t, x)$ given in (42) from the region where, $t - 1 < s < t$, $|z| \leq t^{\frac{1}{\alpha}}/2$ (this region is clear in Figure 1 when

$d=1$).

We replace $m_1(t-s, x-z)$ and $m_1(s, z)$ in (42) by their estimates given in (36) and (37) respectively. Then

$$m_2(t, x) \geq C \int_{t-1}^t \sum_{\substack{|z| \leq \frac{t^{\frac{1}{\alpha}}}{2}, \\ z \in Z^d}} \frac{(t-s)}{|x-z|^{d+\alpha}} e^{\nu(t-s)} \frac{1}{s^{\frac{2d}{\alpha}}} e^{2\nu s} ds. \quad (47)$$

We are going to estimate the right hand side of (47) from below as follows:

Since $|x| \geq t^{\gamma+\epsilon}$, and since $|z| \leq \frac{t^{\frac{1}{\alpha}}}{2} < \frac{t^{\gamma+\epsilon}}{2} < \frac{|x|}{2}$, these imply that $|x-z| < \frac{|x|}{2}$. Hence we can replace $\frac{1}{|x-z|}$ in (47) by $\frac{C_1}{|x|}$. We also use that $\frac{1}{s^{\frac{2d}{\alpha}}} \geq \frac{1}{t^{\frac{2d}{\alpha}}}$ since $t-1 \leq s \leq t$.

Hence, (47) can be read as follows:

$$m_2(t, x) \geq \frac{ce^{\nu t}}{t^{\frac{2d}{\alpha}}} \int_{t-1}^t (t-s) e^{\nu s} ds \sum_{\substack{|z| \leq \frac{t^{\frac{1}{\alpha}}}{2}, \\ z \in Z^d}} \frac{1}{|x|^{\alpha+d}}. \quad (48)$$

The Summation in (48) is taken over all the integer points z in the ball

$|z| \leq t^{\frac{1}{\alpha}}/2$. The number of these points has an order of $O(t^{\frac{1}{\alpha}})^d$, as $t \rightarrow \infty$. Hence this summation can be replaced by $t^{\frac{d}{\alpha}}$.

After performing the integration with respect to s in (48), $m_2(t, x)$ is estimated for $x \in \Omega_\epsilon(t)$ as $t \rightarrow \infty$, by

$$m_2(t, x) \geq \frac{ce^{2\nu t}}{t^{\frac{d}{\alpha}}} \frac{1}{|x|^{d+\alpha}}, \quad (49)$$

Hence if $|x| > t^\gamma$, for any $\gamma \geq 0$, then from (49) and (46), we have

$$\frac{m_2(t, x)}{m_1^2(t, x)} \geq \frac{C|x|^{(d+\alpha)}}{t^{2+d/\alpha}} = \frac{C|x|^{(d+\alpha)}}{t^{\frac{2\alpha+d}{\alpha}}} = C\left(\frac{|x|}{t^{\frac{2\alpha+d}{\alpha(\alpha+d)}}}\right)^{d+\alpha} = C\left(\frac{|x|}{t^\gamma}\right)^{d+\alpha}.$$

The expression above is unbounded when $|x| \geq t^{\gamma+\epsilon}$, which means that there is an intermittency in the region Ω_ϵ , where $|x| \geq t^{\gamma+\epsilon}$, with $\gamma = \frac{2\alpha+d}{\alpha(d+\alpha)}$. This proves the second part of the theorem.

Notice that for $\gamma = \frac{2\alpha+d}{\alpha(d+\alpha)}$ then $t^\gamma < t^{\frac{1}{d+\alpha}} e^{\frac{\nu t}{d+\alpha}}$. This means that the region Ω_ϵ is located inside the front where $|x|$ is given as in Theorem 7.

4.3 No Intermittency in the Region $|x| \leq Bt^\gamma$

For the proof of the first part of the theorem, we need to prove that, for any $B > 0$, $\frac{m_2}{m_1^2}$ is bounded in the ball where $|x| \leq Bt^\gamma$, $\gamma = \frac{2\alpha+d}{\alpha(\alpha+d)}$.

Using (26) and the fact that $m_1 \geq 1$ inside the front, $\frac{m_2}{m_1^2}$ satisfies the following relation

$$\frac{m_2(t, x)}{m_1^2(t, x)} \leq 1 + \frac{2\nu}{m_1^2(t, x)} \int_0^t \sum_{z \in \mathbb{Z}^d} m_1(t-s, x-z) m_1^2(s, z) ds, \quad |x| \leq Bt^\gamma, \quad t \rightarrow \infty. \quad (50)$$

We are going to find an estimation from above, for the right hand-side of (50). It is as follows :

For fixed s , such that $0 \leq s \leq t$, we split the region \mathbb{Z}^d in (50) into four regions, separated by the two spheres which are, $|z| < s^{\frac{1}{\alpha}}$ and $|x-z| < (t-s)^{\frac{1}{\alpha}}$.

Let $P_1 = P_1(s)$, and $P_2 = P_2(x, t-s)$, be two (bounded) sets of points $z \in \mathbb{Z}^d$. P_1 is located inside or at the boundary of the first sphere and P_2 is located inside or at the boundary of the second sphere, i.e.,

$$P_1 = \{z \in \mathbb{Z}^d : |z| \leq s^{\frac{1}{\alpha}}\}, \text{ and } P_2 = \{z \in \mathbb{Z}^d : |x - z| \leq (t - s)^{\frac{1}{\alpha}}\}.$$

For $i, j = 1, 2$, we denote by each $D_{i,j} = D_{i,j}(s, t, x)$, the set of points $z \in \mathbb{Z}^d$ such that $0 \leq s \leq t$, $x \in \mathbb{Z}^d$ and satisfy the following

D_{11} is the the set of points $z \in \mathbb{Z}^d$ located inside or at the boundary of both spheres, i.e., $D_{11} = P_1 \cap P_2$.

D_{22} is the the set of points $z \in \mathbb{Z}^d$, located outside of both spheres, i.e.,

$$D_{22} = \mathbb{Z}^d \setminus (P_1 \cup P_2).$$

D_{12} is the set of points $z \in \mathbb{Z}^d$ located outside of the first sphere, but inside of the second one or on its boundary, i.e., $D_{12} = (\mathbb{Z}^d \setminus P_1) \cup P_2$.

D_{21} is the the set of points $z \in \mathbb{Z}^d$ located inside of the first sphere or on its boundary, but outside of the second one, i.e., $D_{21} = (\mathbb{Z}^d \setminus P_2) \cup P_1$.

It is convenient to visualize the domains D_{ij} by drawing the two paraboloids, $|z| < s^{\frac{1}{\alpha}}$ and $|x - z| < (t - s)^{\frac{1}{\alpha}}$, in the $(1 + d)$ dimensional space $R^1 \times Z^d$. The graph given above is for $d = 1$. Then D_{ij} is defined by the intersection of the region inside or outside of the corresponding paraboloid with the space $s = \text{constant}$.

Now under the condition that $|x| \leq Bt^\gamma$, $t \rightarrow \infty$, we are going to estimate the contribution to the lower bound of $\frac{m_2}{m_1^2}$, given in the right hand side of (50), that is made by integration/summation over each of the above regions D_{ij} instead of the integral for all Z^d .

Thus

$$\frac{m_2(t, x)}{m_1^2(t, x)} \leq 1 + 2\nu \sum_{2 \geq i, j \geq 1} I_{ij}, \quad (51)$$

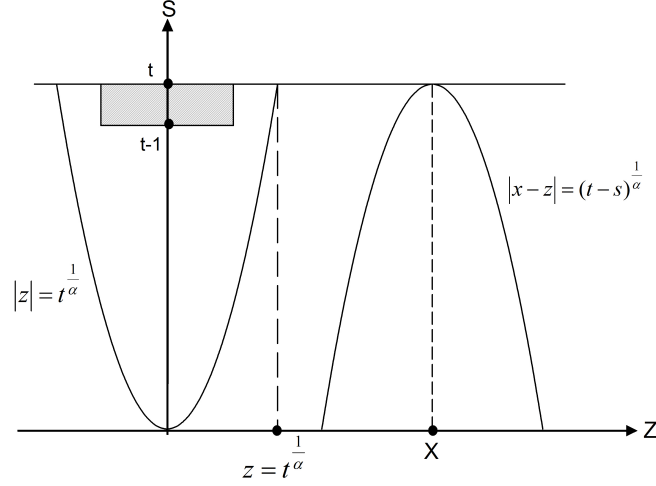


Figure 1: This graph is for $d=1$.

such that for $i, j = 1, 2$, I_{ij} , is given as follows

$$I_{ij} := \frac{1}{m_1^2(t, x)} \int_0^t \sum_{z \in Z^d \subset D_{ij}} m_1(t-s, x-z) \cdot m_1^2(s, z) ds. \quad (52)$$

In the following sections, we are going to estimate the upper bound of the right hand side of (52), for each I_{ij} , $i, j = 1, 2$. We will prove that they are bounded as $t \rightarrow \infty$.

4.4 Contribution to $\frac{m_2}{m_1^2}$ From the Region D_{12}

Let us start with the region D_{12} , and discuss two cases: first when $2t^{\frac{1}{\alpha}} < |x| \leq \beta t^\gamma$, and second when $|x| \leq 2t^{\frac{1}{\alpha}}$. Note that, from the definition of the region D_{12} , we have $|z| > s^{\frac{1}{\alpha}}$ and $|x-z| \leq (t-s)^{\frac{1}{\alpha}}$, so the two balls P_1 and P_2 , are separated in the first case (when $0 \leq s \leq t$) (see figure 1), and they may intersect each other in the second case.

4.4.1 Case 1 : Region D_{12} When $2t^{\frac{1}{\alpha}} < |x| \leq Bt^\gamma$

In this case, since $|x| > 2t^{\frac{1}{\alpha}}$, estimate (36) implies that

$$m_1^2(t, x) \sim \frac{t^2}{|x|^{2(d+\alpha)}} e^{2tv}.$$

In D_{12} , because we have $|z| > s^{\frac{1}{\alpha}}$ and $|x - z| \leq (t - s)^{\frac{1}{\alpha}}$, estimate (36) and Lemma 6 imply that

$$m_1(s, z) \sim \frac{s}{|z|^{(d+\alpha)}} e^{s\nu}, \text{ and } m_1(t - s, x - z) \sim \frac{1}{(t - s + 1)^{\frac{1}{\alpha}}} e^{(t-s)\nu}.$$

Hence for I_{12} , defined in (52), we have

$$I_{12} < \frac{C|x|^{2(d+\alpha)}}{t^2 e^{2t\nu}} \int_0^t \sum_{z \in D_{12}} \frac{s^2 e^{\nu(t+s)}}{(t - s + 1)^{\frac{d}{\alpha}} |z|^{2(d+\alpha)}} ds, \quad 2t^{\frac{1}{\alpha}} < |x| \leq \beta t^\gamma. \quad (53)$$

For the estimation of the right hand side of (53), we can replace z by x . This is true because, in this region we have, $|x| > 2t^{\frac{1}{\alpha}}$, hence, $|x - z| \leq (t - s)^{\frac{1}{\alpha}} \leq t^{\frac{1}{\alpha}} \leq \frac{1}{2}|x|$.

This implies that $|z| \geq \frac{1}{2}|x|$ in D_{12} , and accordingly, $\frac{1}{|z|^{2(d+\alpha)}} \leq \frac{C}{|x|^{2(d+\alpha)}}$.

The summation in (53), is applied to the z -independent terms and therefore the summation sign can be replaced by the factor, K , that estimates the number of terms in this sum from above. This factor K can be estimated by the volume of the ball P_2 that results from the second paraboloid $|x - z| \leq (t - s)^{\frac{1}{\alpha}}$ (when s is fixed). It is clear that

$$K \leq C[(t - s)^{\frac{1}{\alpha}} + 1]^d \sim C[(t - s)^{\frac{d}{\alpha}} + 1].$$

Thus

$$I_{12} < \frac{C}{t^2 e^{2\nu t}} \int_0^t \frac{[(t - s)^{\frac{d}{\alpha}} + 1] s^2}{(t - s + 1)^{\frac{d}{\alpha}}} e^{\nu(t+s)} ds.$$

By using that, $\frac{[(t-s)^{\frac{d}{\alpha}} + 1]}{(t-s+1)^{\frac{d}{\alpha}}} \leq 1$, we have

$$I_{21} < \frac{C_1}{t^2 e^{\nu t}} \int_0^t s^2 e^{\nu s} ds. \quad (54)$$

After performing the integral above, it is clear that the right side of (54), is bounded when $t \rightarrow \infty$.

Thus we have proved that, I_{12} is bounded in the first case when, $2t^{\frac{1}{\alpha}} < |x| \leq Bt^\gamma$, as $t \rightarrow \infty$.

4.4.2 Case 2 : Region D_{12} , When $|x| \leq 2t^{\frac{1}{\alpha}}$

In case $|x| \leq 2t^{\frac{1}{\alpha}}$, due to (37), then, $|m_1^2(t, x)| \sim \frac{1}{t^{\frac{2d}{\alpha}}} e^{2t\nu}$, for $t \geq 1$.

Further, due to (36) and Lemma 4, relation (52) implies that

$$I_{12} < \frac{Ct^{\frac{2d}{\alpha}}}{e^{2\nu t}} \int_0^t \sum_{z \in D_{12}} \frac{s^2 e^{\nu(t+s)}}{[(t-s+1)^{\frac{d}{\alpha}}] |z|^{2(d+\alpha)}} ds, \quad |x| \leq 2t^{\frac{1}{\alpha}}, \quad t \rightarrow \infty. \quad (55)$$

For the estimation of the right hand side of (55), we split the integral above so that

$$\int_0^t = \int_0^{t/2} + \int_{t/2}^t.$$

For estimating the integral $\int_0^{t/2}$, we replace $|z|^{2(d+\alpha)}$ by 1 since by the definition of D_{12} we have $|z| \geq 1$ on the lattice.

Similar to the previous case, the summation along z can be replaced by the factor $K \leq C[(t-s)^{\frac{d}{\alpha}} + 1]$.

Hence

$$\frac{Ct^{\frac{2d}{\alpha}}}{e^{2\nu t}} \int_0^{t/2} \frac{[(t-s)^{\frac{d}{\alpha}} + 1] s^2 e^{\nu(t+s)}}{(t-s+1)^{\frac{d}{\alpha}}} ds \leq \frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \int_0^{t/2} s^2 e^{\nu s} ds \leq C < \infty. \quad (56)$$

The right hand side of (56) is bounded since it decays exponentially as $t \rightarrow \infty$.

Now consider the case when the integral in (55) is $\int_{t/2}^t$. In this case we can replace $|z|$ by $(t/2)^{\frac{1}{\alpha}}$. The reason for that is, $t/2 < s < t$ implies that $|z| \geq s^{\frac{1}{\alpha}} \geq (t/2)^{\frac{1}{\alpha}}$. After that, we replace the summation in z by the factor $K \leq C[(t-s)^{\frac{d}{\alpha}} + 1]$ as before. This leads to the following estimate,

$$\frac{Ct^{\frac{2d}{\alpha}}}{e^{2\nu t} t^{\frac{2(d+\alpha)}{\alpha}}} \int_{t/2}^t s^2 e^{\nu(t+s)} ds \leq C < \infty, \quad t \rightarrow \infty. \quad (57)$$

After implementing the integral, the expression (57) above is bounded as $t \rightarrow \infty$.

Now (56) and (57) imply that, I_{12} is bounded in the second case when $|x| \leq 2t^{\frac{1}{\alpha}}$.

The two estimates of I_{12} , obtained in the previous two cases, prove that I_{12} , given in (55), is bounded in the region D_{12} , under the condition that $|x| < t^\gamma$.

4.5 Contribution to $\frac{m_2}{m_1^2}$ From the Region D_{11} (Inside of Both Paraboloids)

In this region we will estimate I_{11} , given in (52) under the assumption that

$$|x| \leq 2t^{\frac{1}{\alpha}}.$$

The reason for that is due to the definition of D_{11} , given in section 4.3.

Since in D_{11} we have

$$|z| < s^{\frac{1}{\alpha}} \text{ and } |x - z| < (t - s)^{\frac{1}{\alpha}}.$$

The two balls P_1 and P_2 , defined in Section 4.3, do not intersect each other when $|x| > 2t^{\frac{1}{\alpha}}$. Hence the region D_{11} exists only when $|x| \leq 2t^{\frac{1}{\alpha}}$.

For I_{11} , given in (52), we use that, $m_1^2(t, x) \sim \frac{C}{t^{\frac{2d}{\alpha}}} e^{2\nu t}$ for $t \rightarrow \infty$, and replace $m_1(t - s, x - z)$ and $m_1(t, x)$ by their corresponding formulas in D_{11} using (37) and

lemma 6.

Now the following inequality is obtained

$$I_{11} < \frac{Ct^{\frac{2d}{\alpha}}}{e^{2\nu t}} \int_0^t \sum_{z \in D_{11}} \frac{1}{(t-s+1)^{\frac{d}{\alpha}} (s+1)^{\frac{2d}{\alpha}}} e^{\nu(t+s)} ds, \text{ as } t \rightarrow \infty. \quad (58)$$

The estimation of the right hand side of (58), can be done as follows:

The summation, $\sum_{z \in D_{11}}$, can be replaced by the factor K which has been defined in subsection 4.4.1. And by the same reason, we use that

$$\sum_{z \in D_{11}} \leq C[(t-s)^{\frac{d}{\alpha}} + 1].$$

After that, we use in (58) the fact that $\frac{C[(t-s)^{\frac{d}{\alpha}} + 1]}{(t-s+1)^{\frac{d}{\alpha}}} < C_1$.

Hence, the following relation is true:

$$I_{11} < \frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \int_0^t \frac{e^{\nu s}}{(s+1)^{\frac{2d}{\alpha}}} ds, \text{ as } t \rightarrow \infty. \quad (59)$$

For further estimation of I_{11} in (59), the integral in the right hand side is split so that

$$\int_0^t = \int_0^{t/2} + \int_{t/2}^t .$$

When the integral is $\int_0^{t/2}$, we drop the bottom of the integrand in (59) since

$(s+1)^{\frac{2d}{\alpha}} > 1$. Hence

$$\frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \int_0^{t/2} e^{\nu s} ds \leq \frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} [e^{\nu t/2} - e^{\nu}]. \quad (60)$$

The right side of (60) is bounded, as it decays exponentially when $t \rightarrow \infty$.

For the second integral $\int_{t/2}^t$, we drop 1 from the bottom of the integrand of (59), and make the substitution $s = t\tau$, hence

$$\int_{t/2}^t \frac{e^{\nu s}}{s^{\frac{2d}{\alpha}}} ds = \int_{1/2}^1 \frac{e^{\nu t\tau}}{(t\tau)^{\frac{2d}{\alpha}}} t d\tau = \int_{1/2}^1 \frac{e^{\nu t\tau}}{t^{\frac{2d}{\alpha}-1} \tau^{\frac{2d}{\alpha}}} d\tau \quad (61)$$

The integral in the right hand side of (61) can be estimated by using the Laplace method, as $t \rightarrow \infty$. It is less than, $\frac{1}{t^{\frac{2d}{\alpha}-1}} \frac{e^{\nu t}}{t\nu}$.

Hence,

$$\frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \int_{t/2}^t \frac{e^{\nu s}}{s^{\frac{2d}{\alpha}}} ds < \frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \frac{1}{t^{\frac{2d}{\alpha}-1}} \frac{e^{\nu t}}{t\nu}. \quad (62)$$

It is clear that the right hand side of (62) is bounded by a constant as $t \rightarrow \infty$.

The boundedness of both (60) and (62), prove that, (59) is bounded as $t \rightarrow \infty$.

Hence I_{11} is bounded in D_{11} .

4.6 Contribution to $\frac{m_2}{m_1^2}$ From Region D_{22} (Outside of Both Paraboloids)

From the definition of the region D_{22} , we have $|z| > s^{\frac{1}{\alpha}}$ and $|x - z| > (t - s)^{\frac{1}{\alpha}}$.

Then due to (36), we can replace the factors under the summation in (52) by

$$m_1(s, z) \sim \frac{se^{\nu s}}{|z|^{d+\alpha}} \quad \text{and} \quad m_1(t - s, x - z) \sim \frac{(t - s)e^{\nu(t-s)}}{|x - z|^{d+\alpha}}.$$

For this region, we will consider the two cases, first when $\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$, and second when $|x| < \frac{1}{2}t^{1/\alpha}$. These two cases will be given in following two subsections.

4.6.1 Case 1: Region D_{22} When $\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$

In this case, from (36), we have $m_1^2(t, x) \sim \frac{t^2}{|x|^{2(d+\alpha)}} e^{2\nu t}$.

Hence I_{22} , defined in (52) satisfies the following relation

$$I_{22} < \frac{C|x|^{2(d+\alpha)}}{t^2 e^{2\nu t}} \int_0^t \sum_{z \in D_{22}} \frac{(t-s)s^2}{|x-z|^{d+\alpha}|z|^{2(d+\alpha)}} e^{\nu(t+s)} ds. \quad (63)$$

For an estimation of the right hand side of (63), we divide the region D_{22} into two sub-regions, namely,

$$D_{22}^{(1)} = D_{22} \cap \{z : |z| > \frac{x}{2}\} \quad \text{and} \quad D_{22}^{(2)} = D_{22} \cap \{z : |z| \leq \frac{x}{2}\}.$$

Let I_1 and I_2 be the right-hand side of (63), with D_{22} is replaced by $D_{22}^{(1)}$ and $D_{22}^{(2)}$ respectively. It is clear that $I_{22} \leq I_1 + I_2$.

In order to find an estimation for I_1 from above, we use in (63) the following:

$|x-z|$ is strictly greater than $(t-s)^{\frac{1}{\alpha}}$ on the lattice in D_{22} . This implies that,

$|x-z| \geq 1$, for all $z \in D_{22}$.

Now since we have, $\sum_{z \in D_{22}} \frac{1}{|x-z|^{d+\alpha}} < \sum_{z \in \mathbb{Z}^d \setminus \{x\}} \frac{1}{|x-z|^{d+\alpha}}$, and the later series converges and does not depend on x . This implies that,

$\sum_{z \in D_{22}} \frac{1}{|x-z|^{d+\alpha}}$ is bounded by a constant.

We also use that $\frac{1}{|z|^{2(d+\alpha)}} < \frac{1}{(|x|/2)^{2(d+\alpha)}}$. This is true since $|z| > |x|/2$. Because

those two reasons, the following relation is true

$$\sum_{z \in D_{22}^{(1)}} \frac{1}{|x-z|^{d+\alpha}|z|^{2(d+\alpha)}} \leq \left(\frac{2}{|x|}\right)^{2(d+\alpha)} \sum_{z: |x-z| \geq 1} \frac{1}{|x-z|^{d+\alpha}} \leq \frac{C}{|x|^{2(d+\alpha)}}. \quad (64)$$

Using (64) in (63), we find that

$$I_1 \leq \frac{c}{t^2 e^{\nu t}} \int_0^t (t-s)s^2 e^{\nu s} ds. \quad (65)$$

For further estimation from above of I_1 , given in (65), as $t \rightarrow \infty$, in the case when

$\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$, we use the following Lemma, which is a consequence of the

Laplace method, see ([10]). This Lemma will be used to show that I_1 is bounded from above as $t \rightarrow \infty$.

Lemma 10. The asymptotics of the integral $I = \int_0^1 (1-s)^\alpha g(s) e^{\phi(s)t} ds$, for $\alpha > -1$, when $\phi(s)$ has a maximum at $s = 1$ and $\phi'(1) \neq 0$, is

$$I = C_o \frac{g(1)}{[\phi'(1)]^{\alpha+1}} \cdot \frac{e^{\phi(1)t}}{t^{\alpha+1}} (1 + o(1)), \quad t \rightarrow \infty. \quad (66)$$

In order to use the lemma 10 above, we change the variable s in (65) by, $s = \tau t$.

This implies that

$$I_1 \leq \frac{Ct^2}{e^{\nu t}} \int_0^1 (1-\tau)\tau^2 e^{\nu\tau t} d\tau. \quad (67)$$

Using Lemma 10, the integral in (67) is estimated by $\frac{e^{\nu t}}{\nu^2 t^2}$ as $t \rightarrow \infty$. Hence it is easy to see that, I_1 is bounded from above by a constant when $t \rightarrow \infty$.

Now let us estimate I_2 .

The inequality $|z| \leq \frac{|x|}{2}$, implies that $|x-z| \geq \frac{|x|}{2}$. Hence $\frac{1}{|x-z|^{(d+\alpha)}} \leq \left(\frac{2}{|x|}\right)^{(d+\alpha)}$.

From here and (63), it follows that, for $\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$, $t \rightarrow \infty$, we have

$$I_2 \leq \frac{C|x|^{(d+\alpha)}}{t^2 e^{\nu t}} \int_0^t (t-s)s^2 e^{\nu s} \left(\sum_{z \in D_{22}^{(2)}} \frac{1}{|z|^{2(d+\alpha)}} \right) ds. \quad (68)$$

Since we have, $|z| > s^{\frac{1}{\alpha}}$ in D_{22} , and $|z| \geq 1$ on the lattice when $z \neq 0$. This implies that the summation in (68) does not exceed $\frac{C}{(1+s)^{\frac{2\alpha+d}{\alpha}}}$.

Thus

$$I_2 \leq \frac{C|x|^{d+\alpha}}{t^2 e^{\nu t}} \int_0^t \frac{(t-s)s^2}{(1+s)^{\frac{2\alpha+d}{\alpha}}} e^{\nu s} ds. \quad (69)$$

In (69), we can also use that $\frac{|x|^{(d+\alpha)}}{t^2} < C_1 t^{d/\alpha}$. This is due to reason that $|x| \leq Bt^\gamma$,

with $\gamma = \frac{2\alpha+d}{\alpha(\alpha+d)}$). Hence, for $\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$, $t \rightarrow \infty$, we have

$$I_2 \leq \frac{Ct^{d/\alpha}}{e^{\nu t}} \int_0^t \frac{(t-s)s^2}{(1+s)^{\frac{2\alpha+d}{\alpha}}} e^{\nu s} ds, \quad (70)$$

For further estimation of I_2 , given in (70), we split the integral above as follows:

$$\int_0^t = \int_0^{t/2} + \int_{t/2}^t .$$

In the case when the integral in (70) is $\int_0^{t/2}$, we replace s^2 in the integrand by t^2 since $s \leq t/2$, we replace $(t-s)$ by t since $(t-s) \leq t$, and we drop the term $\frac{1}{(1+s)^{\frac{2\alpha+d}{\alpha}}}$ since it is less than 1. Then

$$\frac{Ct^{d/\alpha}}{e^{\nu t}} \int_0^{t/2} \frac{(t-s)s^2}{(1+s)^{\frac{2\alpha+d}{\alpha}}} e^{\nu s} ds < \frac{Ct^{3+\frac{d}{\alpha}}}{e^{\nu t/2}}. \quad (71)$$

The right hand side of (71) is bounded, since it decays exponentially when,

$$\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma, \text{ and } t \rightarrow \infty .$$

Now Consider (70) when the integral is $\int_{t/2}^t$. In this case we drop 1 from the denominator of the term $\frac{1}{(1+s)^{\frac{2\alpha+d}{\alpha}}}$, and make the change of variable $s = \tau t$.

Now the right side of (70) satisfies that

$$\frac{Ct^{d/\alpha}}{e^{\nu t}} \int_{t/2}^t \frac{(t-s)}{s^{\frac{d}{\alpha}}} e^{\nu s} ds \leq \frac{Ct^2}{e^{\nu t}} \int_{1/2}^1 \frac{(1-\tau)}{\tau^{\frac{d}{\alpha}}} e^{\nu \tau t} d\tau. \quad (72)$$

Using Lemma 10 to estimate the integral in the right hand side of (72), it is easy to see that this side is bounded, as $t \rightarrow \infty$.

The boundedness of each of (71) and (72) implies that, I_2 given in (70) is bounded

when, $|x| \leq Bt^\gamma$, with $\gamma = \frac{2\alpha + d}{\alpha(\alpha + d)}$.

The boundedness of both I_1 and I_2 prove that, I_{22} is bounded in the first case when $\frac{1}{2}t^{1/\alpha} \leq |x| \leq Bt^\gamma$, $t \rightarrow \infty$.

4.6.2 Case 2: Region D_{22} When $|x| < \frac{1}{2}t^{1/\alpha}$

In this case, $|m_1^2(t, x)| \sim \frac{1}{t^{2d/\alpha}} e^{2\nu t}$ for $t \rightarrow \infty$. Hence, I_{22} defined in (52), satisfies that

$$I_{22} < \frac{Ct^{2d/\alpha}}{e^{2\nu t}} \int_0^t \sum_{z \in D_{22}} \frac{(t-s)s^2}{|x-z|^{d+\alpha} |z|^{2(d+\alpha)}} e^{\nu(t+s)} ds. \quad (73)$$

From the definition of D_{22} , we have

$$|z| > s^{\frac{1}{\alpha}}, \quad |x-z| > (t-s)^{\frac{1}{\alpha}}, \quad 0 < s < t, \quad x, z \in \mathbb{R}^d. \quad (74)$$

This implies that, there exists $\beta > 0$, such that, $|z| \geq \beta > 0$, when $t = 1$.

Indeed, if we assume that $|z|$ can be as small as we pleased, i.e., $|z| = \epsilon$, then the first inequality in (74) implies that, $|s| < \epsilon^\alpha$, and when $t = 1$, the second inequality in (74) implies that $|x| > 1/2$. The latter inequality contradicts our assumption that $|x| < \frac{1}{2}t^{\frac{1}{\alpha}}$, $t = 1$.

Now we can use the homogeneity argument as follows:

Assume that $s_1 = \frac{s}{t^{1/\alpha}}$, $z_1 = \frac{z}{t^{1/\alpha}}$, $x_1 = \frac{x}{t^{1/\alpha}}$, $t_1 = \frac{t}{t^{1/\alpha}}$. Then one can easily see that the inequalities in (74) imply that, $|z_1| > \frac{1}{2}$ for $t_1 = 1$. Thus for an arbitrary $t > 0$, we have

$$|z| \geq \beta t^{1/\alpha}.$$

Now replacing $|z|$ in (73) by $\beta t^{1/\alpha}$, we find that

$$I_{22} \leq \frac{Ct^{2d/\alpha}}{e^{2\nu t}t^{2(d+\alpha)/\alpha}} \int_0^t \sum_{z \in D_{22}} \frac{(t-s)s^2 e^{\nu(t+s)} ds}{|x-z|^{d+\alpha}}, \quad \text{when } |x| < \frac{1}{2}t^{1/\alpha}, \quad t \rightarrow \infty.$$

Note that for all $z \in D_{22}$, then $|x-z| \geq 1$ on the lattice, and consequently, $\sum_{z \in D_{22}} \frac{1}{|x-z|^{d+\alpha}}$ is a convergent series since it can be estimated from above by the convergent series $\sum_{|x-z| \geq 1} \frac{1}{|x-z|^{d+\alpha}}$. The latter series does not depend on x . Hence

$$I_{22} \leq \frac{C}{t^2 e^{\nu t}} \int_0^t (t-s)s^2 e^{\nu s} ds. \quad (75)$$

By changing the variable $s = \tau t$ in (75), then

$$I_{22} \leq \frac{Ct^2}{e^{\nu t}} \int_0^1 (1-\tau)\tau^2 e^{\nu \tau t} d\tau. \quad (76)$$

Estimating the integral in (76) by using Lemma 10, it can easily be seen that I_{22} is bounded when $t \rightarrow \infty$. Thus we have proved that I_{22} is bounded in the second case when $|x| < \frac{1}{2}t^{1/\alpha}$, $t \rightarrow \infty$.

Together with the boundedness of I_{22} in the first case given in section 4.6.1, when $2t^{\frac{1}{\alpha}} \leq |x| \leq Bt^\gamma$, we have proved that I_{22} is bounded.

4.7 Contribution to $\frac{m_2}{m_1^2}$ From the Region D_{21}

In the region D_{21} , we have, $|z| \leq s^{\frac{1}{\alpha}}$ and $|x-z| > (t-s)^{\frac{1}{\alpha}}$. We will consider the two cases: first, when $2t^{\frac{1}{\alpha}} < |x| \leq \beta t^\gamma$ and second, when $|x| \leq 2t^{\frac{1}{\alpha}}$. These two cases has been considered in section 4.4 when we discussed the contribution to $\frac{m_2}{m_1^2}$ from the region D_{12} . The reason is that, the two balls P_1 and P_2 , defined in section 4.3,

are separated in the first case, and they may be intersect each other in the second case.

4.7.1 Case 1: Region D_{21} When $2t^{\frac{1}{\alpha}} < |x| \leq Bt^\gamma$.

From (36), it follows that $m_1^2(t, x) \sim \frac{t^2}{|x|^{2(d+\alpha)}} e^{2\nu t}$, for $t > 1$.

Formula (52) implies that

$$I_{21} \leq \frac{C|x|^{2(d+\alpha)}}{t^2 e^{2\nu t}} \int_0^t \sum_{z \in D_{21}} \frac{(t-s)}{|x-z|^{d+\alpha} (s+1)^{\frac{2d}{\alpha}}} e^{\nu(t+s)} ds. \quad (77)$$

Since, $|z| \leq s^{\frac{1}{\alpha}} \leq t^{\frac{1}{\alpha}} < |x|/2$, it follows that $|x-z| \geq |x|/2$.

So in order to estimate the right side of (77), we replace $|x-z|$ in (77) by $|x|/2$.

After that, the summation sign, $\sum_{z \in D_{21}}$, in (77) can be replaced by the number of terms in this sum which is κ_1 . Obviously, $\kappa_1 \leq C(A_1^d + 1)$, where A_1 is the radius of the first ball P_1 , i.e., $\kappa_1 \leq C[s^{\frac{d}{\alpha}} + 1]$.

Then

$$I_{21} \leq \frac{C|x|^{d+\alpha}}{t^2 e^{\nu t}} \int_0^t \frac{(t-s)}{(s+1)^{\frac{d}{\alpha}}} e^{\nu s} ds.$$

Since we have $|x| \leq Bt^\gamma$, $\gamma = \frac{2\alpha + d}{\alpha(\alpha + d)}$, we can replace $|x|^{d+\alpha}$ by $Bt^{\gamma(d+\alpha)}$.

Hence

$$I_{21} \leq \frac{Ct^{d/\alpha}}{e^{\nu t}} \int_0^t \frac{(t-s)}{(s+1)^{\frac{d}{\alpha}}} e^{\nu s} ds. \quad (78)$$

For the estimation of (78), we split the integral in its right hand side so that:

$$\int_0^t = \int_0^{t/2} + \int_{t/2}^t.$$

Let us denote by, $I_{21}^{(1)}$ and $I_{21}^{(2)}$, the right hand side in (78), with the integral over $[0, t/2]$ and $[t/2, t]$ respectively.

is replaced by the integrals over $[0, t/2]$ and $[t/2, t]$ respectively.

In order to estimate $I_{21}^{(1)}$, we replace $(t-s)$ by t since $(t-s) \leq t$, and we drop the term $\frac{1}{(1+s)^{\frac{d}{\alpha}}}$ since it is less than 1. Then

$$I_{21}^{(1)} = \frac{Ct^{d/\alpha}}{e^{\nu t}} \int_0^{t/2} \frac{(t-s)}{(s+1)^{\frac{d}{\alpha}}} e^{\nu s} ds < \frac{Ct^{\frac{d}{\alpha}+1}}{e^{\nu t}} \int_0^{t/2} e^{\nu s} ds. \quad (79)$$

The right hand side of (79), decays exponentially as $t \rightarrow \infty$, so it is bounded.

In order to estimate $I_{21}^{(2)}$, we drop the 1 in $(s+1)^{\frac{d}{\alpha}}$, and make the change of variable, $s = \tau t$. Then

$$I_{21}^{(2)} = \frac{Ct^{d/\alpha}}{e^{\nu t}} \int_{t/2}^t \frac{(t-s)}{s^{\frac{d}{\alpha}}} e^{\nu s} ds \leq \frac{Ct^2}{e^{\nu t}} \int_{1/2}^1 \frac{(1-\tau)}{\tau^{\frac{d}{\alpha}}} e^{\nu \tau t} d\tau. \quad (80)$$

From Lemma 10, it follows that, the right hand side of (80) is bounded as $t \rightarrow \infty$.

The boundedness of both (79) and (80) prove that I_{21} , given in (78), is bounded in the first case when, $2t^{\frac{1}{\alpha}} < |x| \leq Bt^\gamma$ and $t \rightarrow \infty$.

4.7.2 Case 2: Region D_{21} When $|x| \leq 2t^{\frac{1}{\alpha}}$

For this case, we have that, $m_1^2(t, x) \sim \frac{1}{t^{\frac{2d}{\alpha}}} e^{2t\nu}$, as $t \rightarrow \infty$.

From (52), I_{21} satisfies that

$$I_{21} < \frac{Ct^{\frac{2d}{\alpha}}}{e^{2\nu t}} \int_0^t \sum_{z \in D_{21}} \frac{(t-s)}{|x-z|^{d+\alpha} (s+1)^{\frac{2d}{\alpha}}} e^{\nu(t+s)} ds. \quad (81)$$

For the estimation of the right side of (81), we use that $\sum_{z \in D_{21}} \frac{1}{|x-z|^{d+\alpha}} < C$ for some x -independent constant $C < \infty$ (since $|x-z| \geq 1$, see the details in the subsection on D_{22}).

Thus for $|x| \leq 2t^{\frac{1}{\alpha}}$ and $t \rightarrow \infty$, we have

$$I_{21} \leq \frac{Ct^{\frac{2d}{\alpha}}}{e^{\nu t}} \int_0^t \frac{(t-s)}{(s+1)^{\frac{2d}{\alpha}}} e^{\nu s} ds \leq C < \infty. \quad (82)$$

The boundedness of the right hand side of (82) can be proved exactly in the same way that was used to prove that (78) is bounded.

Results obtained in sections 4.7.1 and 4.7.2 imply that I_{21} is bounded as $t \rightarrow \infty$.

Now the boundedness of all I_{ij} , for $i, j = 1, 2$ has been proved. This proves that $\frac{m_2}{m_1^2}$, given in (51) is bounded when $|x| \leq Bt^\gamma, t \rightarrow \infty$.

This completes the proof of the 2nd part of Theorem 9.

□

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