THE CENTER AND CYCLICITY PROBLEMS IN A FAMILY OF THREE DIMENSIONAL POLYNOMIAL SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

by

Kokouvi Hounkanli

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Approved by:

Dr. Douglas S. Shafer

Dr. Joel Avrin

Dr. Thomas G. Lucas

Dr. Aidong Lu

ABSTRACT

KOKOUVI HOUNKANLI. The center and cyclicity problems in a family of three dimensional polynomial systems of ordinary differential equations. (Under the direction of DR. DOUGLAS S. SHAFER)

This dissertation is mainly a study of the center problem in the context of a family of three dimensional systems of ordinary differential equations of the form

$$\dot{u} = -v + P(u, v, w), \qquad \dot{v} = u + Q(u, v, w) \qquad \dot{w} = -\lambda w + R(u, v, w),$$

for which the right-hand sides are polynomials and $\lambda \neq 0$. Such systems are called polynomial systems.

There is a two dimensional local center manifold W^c_{loc} through the origin. It is invariant under the flow. The problem is to decide whether there is a focus or a center at the origin for the flow restricted to W^c_{loc}.

For two-dimensional systems a general method due to Poincaré and Lyapunov reduces the problem to that of solving an infinite system of polynomial equations whose variables are parameters of the system of differential equations. That is, the center-focus problem is reduced to the problem of finding the variety of the ideal generated by a collection of polynomials, called the focus quantities of the system ([12]). In this thesis we show how these ideas can be generalized to the setting of systems in $\mathbb{R}^3$ of the form above. This will involve generalizing to this setting the concepts of the complexification of real systems, normal forms and the center variety, described for two-dimensional systems by Valery G. Romanovski and Douglas S. Shafer in the Center and Cyclicity Problems: A Computational Algebra Approach. We then apply the ideas to the Moon-Rand family of systems that arise naturally. We will solve the center problem by providing sufficient conditions for the existence of a center, and otherwise determine the stability of the focus on W^c_{loc}.

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CHAPTER 1: INTRODUCTION

Consider the family of three dimensional polynomial system of ordinary differential equations of the form,

$$\begin{aligned}
\dot{u} &= -v + P(u, v, w) \\
\dot{v} &= u + Q(u, v, w) \\
\dot{w} &= -\lambda w + R(u, v, w),
\end{aligned} \tag{1.1}$$

for which, $P(x, y, z), Q(x, y, z)$ and $R(x, y, z)$ are polynomials without constant or linear part.

The system and its associated vector field $\mathfrak{X}$,

$$\begin{aligned}
\mathfrak{X}(u, v, w) =& (-v + P(u, v, w))\frac{\partial}{\partial u} + \\
& (u + Q(u, v, w))\frac{\partial}{\partial v} + \\
& (-\lambda w + R(u, v, w))\frac{\partial}{\partial w}
\end{aligned}$$

are analytic on a neighborhood of the origin, for which the eigenvalues of the associated linear part at the origin are $\pm i$ and $-\lambda$ with $\lambda \neq 0$.

There exists a two dimensional local center manifold W^c_{loc} through the origin, tangent to the $(u, v)-$plane and invariant under the flow ([4, 11, 13]). The center manifold need not be unique and also it need not be analytic. There is a C^r center manifold for every $r \in \mathbb{N}$. The local flows induced by $\mathfrak{X}$ on any two C^r center manifolds are C^{r-1} conjugate ([3]). It is known, however, that when the origin is a center for $\mathfrak{X} \mid W^c_{loc}$, then the local manifold is unique, and is analytic. The system is not linear and moreover the linear part at the singular point, the origin, has eigenvalues that are purely imaginary. Therefore, the topological type of the origin is not determined by the linear approximation. Although in general for a non-analytic system on the plane, an isolated singularity at which the eigenvalues are purely imaginary does not have to be either a center or a focus, in the situation of (1.1), for the flow induced by $\mathfrak{X}$ on any center manifold at the origin, the origin must be either a focus, in which case there is a neighborhood of the origin in which every orbit spirals towards or away from the origin, or center, in which case there is a neighborhood of the origin in which every orbit except the origin is periodic ([1]). Two problems naturally arise in this context. The first is the center problem, which is to determine whether the origin is a center or a focus for the flow restricted to W^c_{loc}. The second is the cyclicity problem, which is to determine the maximum number of limit cycles (isolated closed orbits) that can emerge from the focus or center when the right hand side of (1.1) is perturbed slightly.

Recent decades have seen a surge of interest in the center and cyclicity problems. Certainly an important reason for this is that the resolution of these problems involves extremely laborious computations, which nowadays can be carried out using powerful computational facilities. Applications of concepts that could not be utilized even 30 years ago are now feasible, often on a personal computer, because of advances in the mathematical theory, in the computer software of computational algebra, and in computer technology. This thesis explains and illustrates methods of computational algebra, as a means of approaching the center-focus and cyclicity problems in the context of system (1.1).

The methods we present can be most effectively exploited if the original real system of differential equations is properly complexified; hence, the idea of complexifying a real system, and more generally working in a complex setting, is one of the central ideas of the thesis. Although the idea of extracting information about a real system of ordinary differential equations from its complexification goes back to Lyapunov, it is still relatively scantily used ([12]).

Chapter 2 introduces the primary technical tools for this approach to the center and cyclicity problems for (1.1). We cover the complexification of real systems of ordinary differential equations and the basics of the theory of normal forms of ordinary differential equations, including examples for the normalization procedure. We next cover the generalization of the Lyapunov center theorem to our new setting, and other theorems that are aimed at the investigation of the stability of singularities (in this context termed equilibrium points) by means of Lyapunov functions. We then describe how the concept of a center can be generalized to complex systems, in order to take advantage of working over the algebraically closed field $\mathbb{C}$ in place of $\mathbb{R}$.

In Chapter 3, we explore the Lyapunov numbers, derive polynomials in the coefficients of the system whose vanishing is sufficient for existence of a center on the local center manifold. We prove that their vanishing is also necessary for existence of a center, and thus show that the set of parameter values for which there is a center on the center manifold is an affine variety. We present an efficient computational algorithm for computing the focus quantities, which are the polynomials that define the center variety. This program and its efficiency are demonstrated by applying the algorithm to compute the first few focus quantities for two particular examples of certain families of quadratic systems.

Chapter 4 is devoted to the application of the theory to the Moon-Rand family of systems that arose in the problem of modeling certain flexible structures.

CHAPTER 2: COMPLEXIFICATION AND NORMAL FORMS

2.1 Complexification of real systems

Our main concern is to work with the family of systems in the form (1.1), but it is more appealing to complexify the system because the eigenvalues of the linear part at the origin of every system in question are complex. We begin by considering the real space (x, y, z) as a complex plane cross a line.

$$x = u + iv, \qquad\qquad z = w. \tag{2.1}$$

Differentiating (2.1) and using (1.1) we obtain

$$\begin{aligned}\dot{x} &= \dot{u} + i\dot{v} \\ &= (-v + P(u, v, w)) + i(u + Q(u, v, w)) \\ &= i(u + iv) + P(u, v, w) + iQ(u, v, w).\end{aligned}$$

That is,

$$\dot{x} = ix + S\left(\tfrac{x+\bar{x}}{2}, \tfrac{x-\bar{x}}{2i}, z\right). \tag{2.2}$$

Equation (2.2) is a single complex equation that carries all the information in the first two equations in (1.1). Nothing changes if we adjoin to (2.2) its complex conjugate,

$$\dot{\bar{x}} = -i\bar{x} + \overline{S\left(\tfrac{x+\bar{x}}{2}, \tfrac{x-\bar{x}}{2i}, z\right)}.$$

If we let $U = R(\frac{x+\bar{x}}{2}, \frac{x-\bar{x}}{2i}, z)$ and replace every occurrence of $\bar{x}$ in $\dot{x}, \dot{\bar{x}}$, and $\dot{z}$ by a new variable y, and replace for any coefficient a_{jk} that arises in S its complex conjugate $\bar{a}_{jk}$ by b_{kj}, regarded as independent of a_{jk}, then we obtain

$$\begin{aligned}\dot{x} &= ix + S(x, y, z)\\ \dot{y} &= -iy + T(x, y, z)\\ \dot{z} &= -\lambda z + U(x, y, z)\end{aligned} \tag{2.3}$$

The system (2.3) in $\mathbb{C}^3$ is the complexification of the real system (1.1). This leads to the study of families of systems that are of the form,

$$\begin{aligned}\dot{x} &= \tilde{P}(x, y, z) = i(x - \sum_{(p,q,r)\in S} a_{pqr} x^{p+1} y^q z^r)\\ \dot{y} &= \tilde{Q}(x, y, z) = -i(y - \sum_{(p,q,r)\in S} b_{pqr} x^q y^{p+1} z^r)\\ \dot{z} &= \tilde{R}(x, y, z) = (-\lambda z - \sum_{(p_1,q_1,r_1)\in T} c_{p_1 q_1 r_1} x^{p_1} y^{q_1} z^{r_1+1})\end{aligned} \tag{2.4}$$

where the variables x, y, z are complex, the coefficients of $\tilde{P}, \tilde{Q}, \tilde{R}$ are also complex, where $S \subset (\{-1\} \cup \mathbb{N}_0) \times \mathbb{N}_0 \times \mathbb{N}_0$ is a finite set, every element (p, q, r) of which satisfies $p + q + r \geq 1$, where $b_{pqr} = \bar{a}_{qpr} \in S$, and where $T \subset \mathbb{N}_0 \times \mathbb{N}_0 \times (\{-1\} \cup \mathbb{N}_0)$. The set S specifies the collection of admissible nonzero coefficients, hence the family of systems under consideration.

The unusual indexing simplifies later expressions that will arise. Similarly, although system (2.4) is a system of the form (3.13) below, we will find that it is more convenient to completely factor out the i than it is to use the form (3.13). The complexification of any individual system of the form (1.1) can be written in the notation of (2.4) by choosing the set S and T and the individual coefficients a_{pqr}, b_{qpr} and $c_{p_1 q_1 r_1}$ suitably. The collection of admissible coefficients in (1.1) determines the collection of admissible coefficients in (2.4). For example, if P, Q, and R are arbitrary homogeneous polynomials of a fixed degree, then the nonlinearities in (2.4) correspond to all homogeneous nonlinearities of the same degree.

In general our interest is in families of systems of the form (2.4) when y is regarded as independent of x and the coefficients b_{pqr} do not necessarily satisfy $\bar{b}_{qpr} = a_{pqr}$, i.e., when they are not necessarily complexifications of real systems. We can then specialize the general results to the case of complexifications, as with the family of Moon-Rand systems that will be examined in Chapter 4. In any case the degree of the polynomials in (1.1) or the allowable non-zero coefficients must be restricted in order for computations to be feasible. In such a situation the collection of admissible coefficients is viewed as a parameter space; for (2.4) it will be denoted $E(a, b, c)$.

2.2 Normal Forms

Normal forms of differential equations are essential tools in the study of differential equations and in their applications. Given a relatively complicated differential system with a singularity, there exists, in many instances, a local change of coordinates accompanied by a possible rescaling in time after which the system takes a most simple form: a normal form. The idea is to modify the system by eliminating as many terms as possible. System (2.3) (hence (2.4)) may be written as

$$\dot{\underline{x}} = A\underline{x} + X_1(\underline{x}). \tag{2.5}$$

We will investigate normal forms of complex systems (2.5). Since we are working with systems whose right-hand sides are power series, we will also allow formal rather than convergent series as well.

We say that the original system (2.5) under consideration is formally equivalent to a like system

$$\dot{\underline{y}} = A\underline{y} + Y(\underline{y}) \tag{2.6}$$

if there is a change of variables

$$\underline{x} = H(\underline{y}) = \underline{y} + h(\underline{y}) \tag{2.7}$$

that transforms (2.5) into (2.6), where the coordinate functions of Y and h, Y_j and $h_j, j = 1, \cdots, n$, are formal power series. If all Y_j and h_j are convergent power series (and all X_j are as well), then by the Inverse Function Theorem the transformation (2.7) has an analytic inverse on a neighborhood of O and we say that (2.5) and (2.6) are analytically equivalent.

Consider the following example drawn from ([12]).

Example 2.2.1. Consider the linear system

$$\dot{x}_1 = 2x_1$$
$$\dot{x}_2 = x_2$$

which has a hyperbolic equilibrium at the origin, and the general quadratic system with the same linear part,

$$\dot{x}_1 = 2x_1 + ax_1^2 + bx_1x_2 + cx_2^2$$
$$\dot{x}_2 = x_2 + a^{'}x_1^2 + b^{'}x_1x_2 + c^{'}x_2^2.$$

We make the change of coordinates $x = y + h^{(2)}(y) + h^{(3)}(y) + \cdots$, where the linear terms are the identity because the linear part is already in canonical form. Then,

$$\dot{x} = \dot{y} + dh^{(2)}(y)\dot{y} + \cdots = (I + dh^{(2)}(y) + \cdots)\dot{y}.$$

Note that x and y could be either real or complex and that for y sufficiently close to

0 the linear transformation $(I + dh^{(2)}(y) + \cdots)$ is invertible. Hence, inserting,

$$\dot{y} = (I + dh^{(2)}(y) + \cdots)^{-1}\dot{x} = (I - dh^{(2)}(y) + \cdots)^{-1}\dot{x},$$

and writing

$$h^{(2)}(y) = \begin{pmatrix} a_{20}y_1^2 + a_{11}y_1y_2 + a_{02}y_2^2 + \cdots \\ b_{20}y_1^2 + b_{11}y_1y_2 + b_{02}y_2^2 + \cdots \end{pmatrix},$$

a computation gives

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 2y_1 + (a - 2a_{20})y_1^2 + (b - a_{11})y_1y_2 + cy_2^2 + \cdots \\ y_2 + (a' - 3b_{20})y_1^2 + (b' - 2b_{11})y_1y_2 + (c' + b_{02})y_2^2 + \cdots \end{pmatrix}.$$

From this last expression it is important to note that five of the six quadratic terms can be eliminated by a suitable choice of h, so that the normal form through order two is

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 2y_1 + cy_2^2 \\ y_2 \end{pmatrix}. \quad \square$$

In general, in $\mathbb{R}^n$, for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, let x^α denote $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, let $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and let $\mathcal{H}_s$ denote the vector space of functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ each of whose components is a homogeneous polynomial function of degree s; elements of $\mathcal{H}_s$ will be termed vector homogeneous functions. Then for $\dot{x} = Ax + \cdots$, working step by step attempting to eliminate homogeneous terms of higher and higher degree, it is well known that at each stage

(a) we can eliminate all terms of degree k iff the operator

$\mathcal{L} : \mathcal{H}_k \to \mathcal{H}_k : p(y) \mapsto dp(y)Ay - Ap(y)$ is onto,

(b) otherwise we can be certain to eliminate only those terms that lie in $Image(\mathcal{L})$ and there remain terms in any previously specified complementary subspace $\mathcal{K}_k$ of $\mathcal{H}_k$, such that $\mathcal{H}_k = Image(\mathcal{L}) \bigoplus \mathcal{K}_k$. (In general, for system (2.5) on $\mathbb{C}^n$ this is proved in

([12]) using Lemma 2.3.1.) Then, this means that higher order terms $X_m^{(\alpha)} x^\alpha$ in (2.24) corresponding to pairs (m, α) for which $(m, \alpha) - \kappa_m \neq 0$ for all $m \in \{1, 2, \cdots, n\}$ and for all $\alpha \in \mathbb{N}_0^n$ for which $|\alpha| > 2$ are the ones that can definitely be eliminated by a near-identity transformation (2.7). The remaining terms have the following special designation.

Definition 2.2.2. Let $\kappa_1, \cdots, \kappa_n$ be the eigenvalues of the matrix A in (2.5), ordered according to the choice of a Jordan normal J of A, and let $\kappa = (\kappa_1, \cdots, \kappa_n)$. Suppose $m \in \{1, \cdots, n\}$ and $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n \geq 2$, are such that $(\alpha, \kappa) - \kappa_m = 0$. Then m and α are called a resonant pair, the corresponding coefficient $X_m^{(\alpha)}$ of the monomial X^α in the mth component of X is called a resonant coefficient, and the corresponding term is called a resonant term of X. Index and multi-index pairs, terms, and coefficients that are not resonant are called nonresonant.

A "normal form" for system (2.5) should be a form that is as simple as possible.

The first step in the simplification process is to change the linear part A in (2.5) into a Jordan normal form. We will assume that the preliminary step has already been taken, so we begin with (2.5) in the form

$$\dot{x} = Jx + X(x), \tag{2.8}$$

where J is a lower-triangular Jordan matrix. (Note that the following definition is based on this supposition.) The simplest form that we are sure to be able to obtain is one in which all nonresonant terms are zero, so we will take this as the meaning of the term "normal form."

Definition 2.2.3. A normal form for system (2.5) is a system (2.8) in which every nonresonant coefficient is equal zero. A normalizing transformation for system (2.5) is any (possibly formal) change of variables (2.7) that transforms (2.5) into a normal form; it is called distinguished if for each resonant pair m and α, the corresponding coefficient $h_m^{(\alpha)}$ is zero, in which case the resulting normal form is likewise termed distinguished.

Example 2.2.4. Consider any C^∞ system (2.5) with an equilibrium at O that has the form:

$$\begin{aligned}
\dot{x}_1 &= x_1 + ax_1^2 + bx_1x_2 + cx_1x_3 + \cdots \\
\dot{x}_2 &= 2x_2 + a'x_1^2 + b'x_1x_2 + c'x_1x_3 + \cdots \\
\dot{x}_3 &= x_3 + a''x_1^2 + b''x_1x_2 + c''x_1x_3 + \cdots .
\end{aligned} \tag{2.9}$$

The resonant coefficients are determined by the equations

$$\begin{aligned}
(\alpha, \kappa) - 1 &= \alpha_1 + 2\alpha_2 + \alpha_3 - 1 = 0 \\
(\alpha, \kappa) - 2 &= \alpha_1 + 2\alpha_2 + \alpha_3 - 2 = 0 \\
(\alpha, \kappa) - 1 &= \alpha_1 + 2\alpha_2 + \alpha_3 - 1 = 0.
\end{aligned}$$

When $|\alpha| = 2$, the first and third equations do not have a solution and the second equation has solution $(\alpha_1, \alpha_2, \alpha_3) \in \{(2,0,0), (1,0,1), (0,0,2)\}$; for $|\alpha| \geq 3$, no equation has a solution. Thus by Definition 2.2.3, for any $k \in \mathbb{N}_0$, the normal form through order k is

$$\begin{aligned}
\dot{y}_1 &= y_1 + o(|y|^k) \\
\dot{y}_2 &= 2y_2 + Y_2^{(2,0,0)}y_1^2 + Y_2^{(1,0,1)}y_1y_3 + Y_2^{(0,0,2)}y_3^2 + o(|y|^k) \\
\dot{y}_3 &= y_3 + o(|y|^k). \quad \square
\end{aligned}$$

For families of systems (2.4), the eigenvalues of A are i,$-i$ and $-\lambda$. The resonant coefficients are determined by the equations

$$(\alpha, \kappa) - i = i\alpha_1 - i\alpha_2 - \lambda\alpha_3 - i = 0$$

$$(\alpha, \kappa) + i = i\alpha_1 - i\alpha_2 - \lambda\alpha_3 + i = 0$$

$$(\alpha, \kappa) + \lambda = i\alpha_1 - i\alpha_2 - \lambda\alpha_3 + \lambda = 0.$$

Solutions of the first equation that correspond to $|\alpha| \geq 2$ are the triplets $(n+1, n, 0), n \in \mathbb{N}_0, n \geq 1$; solutions of the second equation that correspond to $|\alpha| \geq 2$ are the triplets $(n, n+1, 0), n \in \mathbb{N}_0, n \geq 1$; solutions of the third equation that correspond to $|\alpha| \geq 2$ are the triplets $(n, n, 1), n \in \mathbb{N}_0, n \geq 1$. By Definition 2.2.3, the normal form of families of systems (2.4) is

$$\begin{aligned}
\dot{x}_1 &= ix_1 + \sum_{n=1}^{\infty} X^{(n+1,n,0)} x_1^{n+1} y_1^n \\
\dot{y}_1 &= -iy_1 + \sum_{n=1}^{\infty} Y^{(n,n+1,0)} x_1^n y_1^{n+1} \\
\dot{z}_1 &= -\lambda z_1 + \sum_{n=1}^{\infty} Z^{(n,n,1)} x_1^n y_1^n z_1,
\end{aligned}$$

which we will write as

$$\begin{aligned}
\dot{x}_1 &= ix_1 + x_1 \sum_{j=1}^{\infty} X^{(j+1,j,0)} (x_1 y_1)^j = ix_1 + x_1 X(x_1 y_1) \\
\dot{y}_1 &= -iy_1 + y_1 \sum_{j=1}^{\infty} Y^{(j,j+1,0)} (x_1 y_1)^j = -iy_1 + y_1 Y(x_1 y_1) \\
\dot{z}_1 &= -\lambda z_1 + z_1 \sum_{j=1}^{\infty} Z^{(j,j,1)} (x_1 y_1)^j = -\lambda z_1 + z_1 Z(x_1 y_1).
\end{aligned} \tag{2.10}$$

From now X and Y will specify the functions appearing in (2.10).

It is important to note that the qualitative behavior of the system (1.1) is invariant under the nonlinear change of coordinates (2.7), which has an inverse in a small neighborhood of the origin since it is near-identity transformation; i.e., the two systems are topologically conjugate, and therefore they have the same qualitative behavior in a neighborhood of the origin. The method of reducing the system (1.1) to its normal form (system (2.3)) by means of a near-identity transformation of coordinates of the form (2.7) originated in the Ph.D. thesis of Poincaré ([11]).

Definition 3.0.5. A first integral on an open set Ω in $\mathbb{R}^n$ or $\mathbb{C}^n$ of a smooth or analytic system of differential equations

$$\dot{x}_1 = f_1(x), \cdots, \dot{x}_n = f_n(x) \tag{3.1}$$

defined everywhere on Ω is a differentiable function $\Psi : \Omega \longrightarrow \mathbb{C}$ that is not constant on any open subset of Ω but is constant on trajectories of (3.1) (that is, for any solution $x(t)$ of 3.1 in Ω the function $\psi(t) = \Psi(x(t))$ is constant). A formal first integral is a formal power series in x, not all of whose coefficients are zero, which under term-by-term differentiation satisfies $\frac{d}{dt}[\Psi(x(t))] \equiv 0$ in Ω [12].

Remark 3.0.6. (a) If Ψ is a first integral or formal first integral for system (3.1) on Ω, if $F : \mathbb{C} \to \mathbb{C}$ is any nonconstant differentiable function, and if λ is any constant, then $\Phi = F \circ \Psi$ is a first integral or formal first integral for the system $\dot{x}_1 = \lambda f_1(x), \cdots, \dot{x}_n = \lambda f_n(x)$ on Ω.

(b) If

$$\mathfrak{X}(x) = f_1(x)\frac{\partial}{\partial x_1} + \cdots + f_n(x)\frac{\partial}{\partial x_n} \tag{3.2}$$

is the smooth or analytic vector field on Ω associated to system (3.1), then a non-constant differentiable function (or formal powers series) Ψ on Ω is a first integral (or formal first integral) for (3.1) if and only if the function $\mathfrak{X}\Psi$ vanishes throughout Ω :

$$\mathfrak{X}\Psi = f_1\frac{\partial \Psi}{\partial x_1} + \cdots + f_n\frac{\partial \Psi}{\partial x_n} \equiv 0 \quad \text{on} \quad \Omega. \tag{3.3}$$

(c) Our concern is only with a neighborhood of the origin, so by "existence of a first integral" we will always mean "existence on a neighborhood of the origin".

Our first result is a generalization to (2.4) a result from the analogous two-dimensional case (system (2.4) without z). It connects existence of a first integral to properties of the normal form.

Theorem 3.0.7. System (2.4) admits a formal first integral of the form

$$\Psi(x, y, z) = xy + \cdots .$$

if and only if the functions X and Y in any normal form (2.10) satisfy

$$X + Y \equiv 0.$$

Proof. Suppose (2.4) has such a formal first integral of the form

$$\Psi(x, y, z) = xy + \cdots .$$

Let $\underline{x} = H(\underline{y})$ be the normalizing transformation that produces (2.10) from (2.4). Recalling from (2.7) the form of H and writing $F = \Psi \circ H$ according to our usual convention, $F(x_1, y_1, z_1)$ has the form

$$\begin{aligned} F(x_1, y_1, z_1) &= \sum_{(\alpha_1,\alpha_2,\alpha_3)} F^{(\alpha_1,\alpha_2,\alpha_3)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3} \\ &= x_1 y_1 + \cdots . \end{aligned} \tag{3.4}$$

Then F is the formal first integral for (2.10), hence

$$\begin{aligned}
&\frac{\partial F}{\partial x_1}(x_1,y_1,z_1)[ix_1+x_1X(x_1y_1)]+\\
&\frac{\partial F}{\partial y_1}(x_1,y_1,z_1)[-iy_1+y_1Y(x_1y_1)]+\\
&\frac{\partial F}{\partial z_1}(x_1,y_1,z_1)[-\lambda z_1+z_1Z(x_1y_1)]\equiv 0
\end{aligned}\tag{3.5}$$

which we arrange as

$$\begin{aligned}
&ix_1\frac{\partial F}{\partial x_1}(x_1,y_1,z_1)-iy_1\frac{\partial F}{\partial y_1}(x_1,y_1,z_1)-\lambda z_1\frac{\partial F}{\partial z_1}(x_1,y_1,z_1)=\\
&-x_1\frac{\partial F}{\partial x_1}(x_1,y_1,z_1)X(x_1y_1)-y_1\frac{\partial F}{\partial y_1}(x_1,y_1,z_1)Y(x_1y_1)\\
&-z_1\frac{\partial F}{\partial z_1}(x_1,y_1,z_1)Z(x_1y_1).
\end{aligned}\tag{3.6}$$

A simple computation on the left-hand side of (3.6) and inserting (2.10) into the right-hand side, yield

$$\begin{aligned}
&\sum_{|\alpha|\geq 2}(i\alpha_1-i\alpha_2-\lambda\alpha_3)F^{(\alpha_1,\alpha_2,\alpha_3)}x_1^{\alpha_1}y_1^{\alpha_2}z_1^{\alpha_3}=\\
&-\left[\textstyle\sum_{|\alpha|\geq 2}\alpha_1F^{(\alpha_1,\alpha_2,\alpha_3)}x_1^{\alpha_1}y_1^{\alpha_2}z_1^{\alpha_3}\right]\left[\textstyle\sum_{j=1}^{\infty}X^{(j+1,j,0)}(x_1y_1)^j\right]\\
&-\left[\textstyle\sum_{|\alpha|\geq 2}\alpha_2F^{(\alpha_1,\alpha_2,\alpha_3)}x_1^{\alpha_1}y_1^{\alpha_2}z_1^{\alpha_3}\right]\left[\textstyle\sum_{j=1}^{\infty}Y^{(j,j+1,0)}(x_1y_1)^j\right]\\
&-\left[\textstyle\sum_{|\alpha|\geq 2}\alpha_3F^{(\alpha_1,\alpha_2,\alpha_3)}x_1^{\alpha_1}y_1^{\alpha_2}z_1^{\alpha_3}\right]\left[\textstyle\sum_{j=1}^{\infty}Z^{(j,j,1)}(x_1y_1)^j\right].
\end{aligned}\tag{3.7}$$

We claim that $F(x_1,y_1,z_1)$ is a function of (x_1y_1) alone, so that it may be written as

$$F(x_1,y_1,z_1)=f_1(x_1y_1)+f_2(x_1y_1)^2+\cdots.$$

The claim is precisely the statement that for any term $F^{(\alpha_1,\alpha_2,\alpha_3)}x_1^{\alpha_1}y_1^{\alpha_2}z_1^{\alpha_3}$ of F,

$$(\alpha_1-\alpha_2,\alpha_3)\neq(0,0)\ \text{ implies }\ F^{(\alpha_1,\alpha_2,\alpha_3)}=0.\tag{3.8}$$

Equation (3.4) implies that for $|\alpha| \leq 2, F^{(\alpha_1,\alpha_2,\alpha_3)} = 0$ except for $F^{(1,1,0)} = 1$. That is,(3.4) shows (3.8) holds for $|\alpha| \leq 2$. This implies that the right-hand side of (3.7) has the form

$$\begin{aligned}
&-\left[x_1y_1 + \textstyle\sum_{|\alpha|\geq 3} \alpha_1 F^{(\alpha_1,\alpha_2,\alpha_3)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right]\left[\textstyle\sum_{j=1}^{\infty} X^{(j+1,j,0)}(x_1y_1)^j\right]\\
&-\left[x_1y_1 + \textstyle\sum_{|\alpha|\geq 3} \alpha_2 F^{(\alpha_1,\alpha_2,\alpha_3)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right]\left[\textstyle\sum_{j=1}^{\infty} Y^{(j,j+1,0)}(x_1y_1)^j\right]\\
&-\left[\textstyle\sum_{|\alpha|\geq 3} \alpha_3 F^{(\alpha_1,\alpha_2,\alpha_3)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right]\left[\textstyle\sum_{j=1}^{\infty} Z^{(j,j,1)}(x_1y_1)^j\right]\\
&= c_2(x_1y_1)^2 + \cdots
\end{aligned}$$

for $c_2 = -(X^{(2,1,0)} + Y^{(1,2,0)})$. Hence, the left-hand side of (3.7) has the same form. It has no terms of order 3, because $-\lambda\alpha_3 + (\alpha_1 - \alpha_2)i \neq 0$ if $\alpha_1 + \alpha_2 + \alpha_3$ is odd, $F^{(\alpha)} = 0$ for $|\alpha| = 3$. The terms of order 4 are $c_2(x_1y_1)^2$, so $F^{(2,2,0)} = c_2$ and $F^{(\alpha)} = 0$ for $|\alpha| = 4, \alpha \neq (2,2,0)$.

Now, for $k \in \mathbb{N}$, assume that implication (3.8) holds for $|\alpha| \leq 2k$. We want to show that it also holds for $|\alpha| \leq 2(k+1)$. For simplicity we will consider equation (3.7) for $|\alpha| \geq 2k$. That is

$$\begin{aligned}
\sum_{|\alpha|\geq 2k} (i\alpha_1 - i\alpha_2 - \lambda\alpha_3) F^{(\alpha_1,\alpha_2,\alpha_3)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3} =&\\
-\left[\textstyle\sum_{|\alpha|\geq 2k} \alpha_1 F^{(\alpha_1,\alpha_2,\alpha_3)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right]\left[\textstyle\sum_{j=1}^{\infty} X^{(j+1,j,0)}(x_1y_1)^j\right]&\\
-\left[\textstyle\sum_{|\alpha|\geq 2k} \alpha_2 F^{(\alpha_1,\alpha_2,\alpha_3)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right]\left[\textstyle\sum_{j=1}^{\infty} Y^{(j,j+1,0)}(x_1y_1)^j\right]&\\
-\left[\textstyle\sum_{|\alpha|\geq 2k} \alpha_3 F^{(\alpha_1,\alpha_2,\alpha_3)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right]\left[\textstyle\sum_{j=1}^{\infty} Z^{(j,j,1)}(x_1y_1)^j\right].&
\end{aligned} \tag{3.9}$$

If implication (3.8) holds for $|\alpha| \leq 2k$, then the right-hand side of (3.9) has the form

$$
\begin{aligned}
&- \left[kF^{(k,k,0)}(x_1y_1)^k + \textstyle\sum_{|\alpha|\geq 2k+1} \alpha_1 F^{(\alpha_1,\alpha_2,\alpha_3)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right] \left[\textstyle\sum_{j=1}^{\infty} X^{(j+1,j,0)}(x_1y_1)^j\right] \\
&- \left[kF^{(k,k,0)}(x_1y_1)^k + \textstyle\sum_{|\alpha|\geq 2k+1} \alpha_2 F^{(\alpha_1,\alpha_2,\alpha_3)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right] \left[\textstyle\sum_{j=1}^{\infty} Y^{(j,j+1,0)}(x_1y_1)^j\right] \\
&- \left[\textstyle\sum_{|\alpha|\geq 2k+1} \alpha_3 F^{(\alpha_1,\alpha_2,\alpha_3)} x_1^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3}\right] \left[\textstyle\sum_{j=1}^{\infty} Z^{(j,j,1)}(x_1y_1)^j\right] \\
&= c_{k+1}(x_1y_1)^{k+1} + \cdots
\end{aligned}
$$

for $c_{k+1} = -kF^{(k,k,0)}(X^{(2,1,0)} + Y^{(1,2,0)})$. Hence, the left-hand side of (3.9) has the same form. It has no terms of order $2k+1$, hence because $-\lambda\alpha_3 + (\alpha_1 - \alpha_2)i \neq 0$ if $\alpha_1 + \alpha_2 + \alpha_3$ is odd, $F^{(\alpha)} = 0$ for $|\alpha| = 2k+1$. The terms of order 2(k+1) are $c_{k+1}(x_1y_1)^{k+1}$, so $F^{(k+1,k+1,0)} = c_{k+1}$ and $F^{(\alpha)} = 0$ for $|\alpha| = 2(k+1), \alpha \neq (k+1, k+1, 0)$. Hence by (3.9) implication (3.8) holds for $|\alpha| \leq 2(k+1)$. Therefore, by mathematical induction, (3.8) must hold in general, establishing the claim.

But if $F(x_1, y_1, z_1) = f(x_1y_1)$, then

$$
\begin{aligned}
x_1 \frac{\partial F}{\partial x_1} &= x_1y_1 f'(x_1y_1), \\
y_1 \frac{\partial F}{\partial y_1} &= x_1y_1 f'(x_1y_1) \text{ and} \\
z_1 \frac{\partial F}{\partial x_1} &= 0.
\end{aligned}
$$

Letting $w = x_1y_1$, (3.6) becomes

$$0 \equiv -wf'(w)X(w) - wf'(w)Y(w) = -wf'(w)(X(w) + Y(w)).$$

But because F is a formal first integral, it is not a constant, so we immediately obtain

$$X + Y \equiv 0.$$

Direct calculations show that if $X + Y \equiv 0$, then

$$\widehat{\Psi}(x_1, y_1, z_1) = x_1 y_1 + \cdots$$

is the first integral of (2.10). The coordinate transformation that places (2.4)(hence (2.5)) in normal form has the form given in (2.7), hence has an inverse of the form $\underline{y} = \underline{x} + \widehat{h}(\underline{x})$. Therefore, system (2.4)(hence (2.5)) admits a formal first integral of the form $\Psi(x, y, z) = xy + \cdots$.$\square$

Theorem 3.0.8 (Lyapunov Center Theorem). An analytic system on a neighborhood of the origin in $\mathbb{R}^3$ of the form

$$\begin{aligned} \dot{u} &= -v + \cdots \\ \dot{v} &= u + \cdots \\ \dot{w} &= -\lambda w + \cdots \end{aligned} \tag{3.10}$$

has a center on some (hence every) local center manifold if and only if the system (in $\mathbb{R}^3$) admits a real analytic local first integral

$$\Phi(u, v, w) = u^2 + v^2 + \cdots \tag{3.11}$$

in a neighborhood of the origin in $\mathbb{R}^3$.

Proof. Section 3 of ([2]).

Now, rewriting (3.11) as

$$\Phi(u, v, w) = (u + iv)(u - iv) + \cdots ,$$

and by applying (2.1), $\Phi(u, v, w)$ in (3.11) is equal to $xy + \cdots$. This means that existence of a first integral or formal first integral of (1.1) on a neighborhood of the origin is equivalent to existence of a first integral of the form

$$\Psi(x,y,z) = xy + \sum_{j+k+n\geq 3} \nu_{j-1,k-1,n} x^j y^k z^n, \quad \text{where} \quad j,k,n \in \mathbb{N}_0 \tag{3.12}$$

for its complexification (2.4) in a neighborhood of the origin in $\mathbb{C}^3$. Here is the generalization of the concept of a center to the complex setting, based on the Lyapunov Center Theorem.

Definition 3.0.9. Consider the system

$$\begin{aligned} \dot{x}_1 &= ix_1 + X_1(x_1,x_2,x_3) \\ \dot{x}_2 &= -ix_2 + X_2(x_1,x_2,x_3) \\ \dot{x}_3 &= -\lambda x_3 + X_3(x_1,x_2,x_3), \end{aligned} \tag{3.13}$$

where x_1, x_2 and x_3 are complex variables and X_1, X_2 and X_3 are complex series without constant or linear terms that are convergent in a neighborhood of the origin. System (3.13) is said to have a center at the origin if it has a formal first integral of the form

$$\Psi(x_1,x_2,x_3) = x_1x_2 + \sum_{j+k+n\geq 3} w_{j,k,n} x_1^j x_2^k x_3^n. \tag{3.14}$$

Theorem 3.0.10. The following statements about real analytic system (1.1) are equivalent:

(1) The origin is a center for the system restricted to any local center manifold.

(2) The system (1.1) admits a formal first integral in the neighborhood of the origin in $\mathbb{R}^3$ of the form,

$$\Psi(u,v,w) = u^2 + v^2 + \cdots .$$

(3) The system (1.1) admits a real analytic local first integral in the neighborhood of

the origin in $\mathbb{R}^3$ of the form,

$$\Phi(u,v,w) = u^2 + v^2 + \cdots .$$

Proof.

(1) $\Leftrightarrow$ (3) is shown by the Lyapunov center theorem above.

(2) $\Rightarrow$ (3): In Section 5 of [2], it is shown that for the functions X and Y in the normal form (2.10), $X + Y \equiv 0$ implies that the distinguished normalizing transformation $\underline{x} = H(\underline{y}) = \underline{y} + h(\underline{y})$ that changes (2.5) to (2.6) is analytic. By the Inverse Function Theorem it has a local analytic inverse, hence by the last part of the proof of Theorem 3.0.7,the fact that $\widehat{\Psi}(x_1, y_1, z_1) = x_1 y_1 + \cdots$ is analytic implies that $\Psi(x,y,z) = xy + \cdots$ is a real analytic local first integral in the neighborhood of the origin in $\mathbb{R}^3$.

(3) $\Rightarrow$ (2) is automatic. $\square$

3.1 The Poincaré First Return Map and the Lyapunov Numbers

The object of this section is the real analytic system (1.1) that we write as $\dot{\underline{u}} = f(\underline{u})$ on a neighborhood of O in $\mathbb{R}^3$, where $f(0) = 0$ and the eigenvalues of the linear part of f at O are $\pm i$ and λ with $\lambda \neq 0$. The polynomials P, Q and R on the right-hand sides of (1.1) can be written as $P(u,v,w) = \sum_{k=2}^{N} P^{(k)}(u,v,w), Q(u,v,w) = \sum_{k=2}^{N} Q^{(k)}(u,v,w)$, and $R(u,v,w) = \sum_{k=2}^{N} R^{(k)}(u,v,w)$, where $P^k(u,v,w), Q^k(u,v,w)$ and $R^k(u,v,w)$ (if nonzero) are homogeneous polynomials of degree k. But we do not need to do this in $\mathbb{R}^3$ because we will be restricting to the two-dimensional center manifold and working in local coordinates, which means we are still in $\mathbb{R}^2$. We review the theory of the Poincaré first return map in a neighborhood of the origin in $\mathbb{R}^2$ for systems of the form

$$\begin{aligned} \dot{u} &= au - bv + P(u,v) \\ \dot{v} &= bu + av + Q(u,v) \end{aligned} . \tag{3.15}$$

In polar coordinates $u = r\cos\varphi, v = r\sin\varphi$, system (3.15) becomes

$$\begin{aligned}
\dot{r} &= ar + P(r\cos\varphi, r\sin\varphi)\cos\varphi + Q(r\cos\varphi, r\sin\varphi)\sin\varphi \\
&= ar + r^2[P^{(2)}(\cos\varphi, \sin\varphi)\cos\varphi + Q^{(2)}(\cos\varphi, \sin\varphi)\sin\varphi + \cdots] \\
\dot{\varphi} &= b - r^{-1}[P(r\cos\varphi, r\sin\varphi)\sin\varphi - Q(r\cos\varphi, r\sin\varphi)\cos\varphi] \\
&= b - r[P^{(2)}(\cos\varphi, \sin\varphi)\sin\varphi - Q^{(2)}(\cos\varphi, \sin\varphi)\cos\varphi + \cdots].
\end{aligned} \tag{3.16}$$

It is clear that for $|r|$ sufficiently small, if $b > 0$ then the polar angle φ increases as t increases, while if $b < 0$ then the angle decreases as t increases. It is convenient to consider, in place of system (3.16), the equation of its trajectories on the polar plane

$$\frac{dr}{d\varphi} = \frac{ar + r^2F(r, \sin\varphi, \cos\varphi)}{b + rG(r, \sin\varphi, \cos\varphi)} = R(r, \varphi). \tag{3.17}$$

The function $R(r, \varphi)$ is a 2π-periodic function of φ and is analytic for all φ and for all $|r| < r^*$, for some sufficiently small positive real number r^*. The fact that the origin is a singularity for (3.15) corresponds to the fact that $R(r, \varphi) \equiv 0$, so that $r = 0$ is a solution of (3.17). We can expand $R(r, \varphi)$ in a power series in r :

$$\frac{dr}{d\varphi} = R(r, \varphi) = rR_1(\varphi) + r^2R_2(\varphi) + r^3R_3(\varphi) + \cdots = \frac{a}{b}r + \cdots , \tag{3.18}$$

where $R_k(\varphi)$ are 2π-periodic functions of φ. The series is convergent for all φ and for all sufficiently small r. Denote by $r = f(\varphi, \varphi_0, r_0)$ the solution of system (3.18) with initial conditions $r = r_0$ and $\varphi = \varphi_0$. The function $f(\varphi, \varphi_0, r_0)$ is an analytic function of all three variables φ, φ_0 and r_0 and has the property that

$$f(\varphi, \varphi_0, 0) \equiv 0 \tag{3.19}$$

(because $r = 0$ is a solution of (3.18)). We can expand $f(\varphi, 0, r_0)$ in a power series in r_0,

$$r = f(\varphi, 0, r_0) = w_1(\varphi) r_0 + w_2(\varphi) r_0^2 + \cdots , \tag{3.20}$$

which is convergent for all $0 \leq \varphi \leq 2\pi$ and for $|r_0| < r^*$ for r^* sufficiently small. This function is a solution of (3.18), hence

$$\begin{aligned} & w_1' r_0 + w_2' r_0^2 + \cdots \equiv \\ & R_1(\varphi)(w_1(\varphi) r_0 + w_2(\varphi) r_0^2 + \cdots) + R_2(\varphi)(w_1(\varphi) r_0 + w_2(\varphi) r_0^2 + \cdots)^2 + \cdots , \end{aligned}$$

where the primes denote differentiation with respect to φ. Equating the coefficients of like powers of r_0 in this identity, we obtain recurrence differential equations for the functions $w_j(\varphi)$:

$$\begin{aligned} w_1' &= R_1(\varphi) w_1, \\ w_2' &= R_1(\varphi) w_2 + R_2(\varphi) w_1^2, \\ w_3' &= R_1(\varphi) w_3 + 2R_2(\varphi) w_1 w_2 + R_3(\varphi) w_1^3, \\ &\vdots \end{aligned} \tag{3.21}$$

The initial condition $r = f(0, 0, r_0) = r_0$ yields

$$w_1(0) = 1, \quad w_j(0) = 0 \text{ for } j > 1. \tag{3.22}$$

Using these conditions, we can consequently find the functions $w_j(\varphi)$ by integrating equations (3.21). In particular,

$$w_1(\varphi) = e^{\frac{a}{b}\varphi} \tag{3.23}$$

Definition 3.1.1. Fix a system of the form (3.15).

(a) The function

$$\Re(r_0) = f(2\pi, 0, r_0) = \widetilde{\eta_1} r_0 + \eta_2 r_0^2 + \eta_3 r_0^3 + \cdots \tag{3.24}$$

(defined for$|r_0| < r^*$), where $\widetilde{\eta_1} = w_1(2\pi)$ and $\eta_j = w_j(2\pi)$ for $j \geq 2$, is called the Poincaré first return map or just the return map.

(b) The function

$$\mathfrak{D}(r_0) = \Re(r_0) - r_0 = \eta_1 r_0 + \eta_2 r_0^2 + \eta_3 r_0^3 + \cdots \tag{3.25}$$

is called the difference function.

(c) The coefficient $\eta_j, j \in \mathbb{N}$, is called the j^{th} Poincaré-Lyapunov number.

Let $\Re(u)$ denote the Poincaré first return map on a sufficiently short segment of the positive $u-$axis and $\mathfrak{D}(u) = \Re(u) - u$. By the k^{th} Poincaré-Lyapunov quantity we mean the coefficient η_k in the expression $\mathfrak{D}(u) = \eta_1 u + \eta_2 u^2 + \cdots$. It is known that there always exists a sufficiently smooth function V from a neighborhood of the origin into $\mathbb{R}$ of the form $V(u, v) = \frac{1}{2}(u^2 + v^2) + \cdots$ such that if ς is the vector field associated to (3.15) then $\varsigma V = L_4(u^2+v^2)^2+L_6(u^2+v^2)^3+\cdots$. ([5]). The coefficient L_{2k} will be called the k^{th} Lyapunov quantity.

Theorem 3.1.2. System (3.15) has a center at the origin if and only if all the Poincaré-Lyapunov numbers are zero. Moreover, if $\eta_1 \neq 0$, or if for some $k \in \mathbb{N}$

$$\eta_1 = \eta_2 = \cdots = \eta_{2k} = 0, \eta_{2k+1} \neq 0, \tag{3.26}$$

then all trajectories in a neighborhood of the origin are spirals and the origin is a focus, which is stable if $\eta_1 < 0$ or (3.26) holds with $\eta_{2k+1} < 0$ and is unstable if $\eta_1 > 0$ or (3.26) holds with $\eta_{2k+1} > 0$.

Proof. Section 3 in ([12]).

3.2 Focus Quantities

Focus quantities are polynomials in the coefficients of certain polynomial systems of differential equations on $\mathbb{R}^2$ or $\mathbb{C}^2$ whose vanishing provides necessary and sufficient conditions for the existence of a center. In this section we derive analogous polynomials that determine existence of a center at the origin on the center manifold at the origin of system (1.1).

Beginning with a family (1.1) we form the complexification (2.4). We will actually work in more generality by not assuming that the coefficients b_{pqr} are restricted to satisfying the condition $b_{pqr} = \bar{a}_{qpr}$. To determine if a system of the form (2.4) has a center at the origin, by Definition 3.0.9 we must look for a formal first integral of the form (3.12). A function $\Psi(x, y, z)$ of the form (3.12) is a formal first integral of a system of the form (2.4) if and only if

$$\mathfrak{X}\Psi = \frac{\partial \Psi}{\partial x}\widetilde{P}(x, y, z) + \frac{\partial \Psi}{\partial y}\widetilde{Q}(x, y, z) + \frac{\partial \Psi}{\partial z}\widetilde{R}(x, y, z) \equiv 0,$$

which reads

$$\begin{aligned}
&i\left(y + \textstyle\sum_{j+k+n\geq 3} j \cdot \nu_{j-1,k-1,n} x^{j-1} y^k z^n\right)\left(x - \textstyle\sum_{(p,q,r)\in S} a_{pqr} x^{p+1} y^q z^r\right) + \\
&i\left(x + \textstyle\sum_{j+k+n\geq 3} k \cdot \nu_{j-1,k-1,n} x^{j} y^{k-1} z^n\right)\left(-y + \textstyle\sum_{(p,q,r)\in S} b_{qpr} x^{q} y^{p+1} z^r\right) + \\
&\left(\textstyle\sum_{j+k+n\geq 3} n \cdot \nu_{j-1,k-1,n} x^{j} y^{k} z^{n-1}\right)\left(-\lambda z - \textstyle\sum_{(p_1,q_1,r_1)\in T} c_{p_1 q_1 r_1} x^{p_1} y^{q_1} z^{r_1+1}\right) \equiv 0.
\end{aligned} \tag{3.27}$$

From (3.12), we must have $\nu_{0,0,0} = 1$ and $\nu_{1,-1,0} = \nu_{0,-1,1} = \nu_{-1,1,0} = \nu_{-1,0,1} = \nu_{-1,-1,2} = 0$, so that $\nu_{0,0,0}$ is the coefficient of xy in $\Psi(x, y, z)$. We set $a_{pqr} = b_{qpr} = 0$ for $(p, q, r) \notin S$ and $c_{p_1 q_1 r_1} = 0$ for $(p_1, q_1, r_1) \notin T$.

With these conventions, for k_1, k_2 in $\{-1\} \cup \mathbb{N}_0$ and k_3 in $\mathbb{N}_0$ the coefficients g_{k_1,k_2,k_3} of $x^{k_1+1}y^{k_2+1}z^{k_3}$ in (3.27) are zero for $k_1 + k_2 + k_3 < 1$ and for $k_1 + k_2 + k_3 \geq 1$ are

$$
\begin{aligned}
g_{k_1,k_2,k_3} = &- ia_{k_1,k_2,k_3} + ib_{k_1,k_2,k_3} + [-\lambda k_3 + (k_1 - k_2)i]\nu_{k_1,k_2,k_3} \\
&- i \sum_{\substack{j+k+n=2 \\ 1\leq j\leq k_1+2 \\ 0\leq k\leq k_2+1 \\ 0\leq n\leq k_3}}^{k_1+k_2+k_3+1} j a_{k_1-j+1,k_2-k+1,k_3-n}\nu_{j-1,k-1,n} \\
&+ i \sum_{\substack{j+k+n=2 \\ 0\leq j\leq k_1+1 \\ 1\leq k\leq k_2+2 \\ 0\leq n\leq k_3}}^{k_1+k_2+k_3+1} k b_{k_1-j+1,k_2-k+1,k_3-n}\nu_{j-1,k-1,n} \\
&- \sum_{\substack{j+k+n=2 \\ 0\leq j\leq k_1+1 \\ 0\leq k\leq k_2+1 \\ 1\leq n\leq k_3+1}}^{k_1+k_2+k_3+1} n c_{k_1-j+1,k_2-k+1,k_3-n}\nu_{j-1,k-1,n}.
\end{aligned}
\tag{3.28}
$$

Starting from (2.4), we wish to find $\nu_{j-1,k-1,n}$ so that $g_{k_1,k_2,k_3} = 0$ for all (k_1, k_2, k_3), thus yielding a formal first integral Ψ. If we proceed in a step-by-step process, at the first stage finding all suitable ν_{k_1,k_2,k_3} for which $k_1 + k_2 + k_3 = 1$, at the second stage finding all suitable ν_{k_1,k_2,k_3} for which $k_1 + k_2 + k_3 = 2$, and so on, then for a triplet k_1, k_2 and k_3, if $k_1 \neq k_2$ and $k_3 \neq 0$, and if all coefficients ν_{j_1,j_2,j_3} are already known for $j_1 + j_2 + j_3 < k_1 + k_2 + k_3$, then ν_{k_1,k_2,k_3} is uniquely determined by (3.28) and the conditions that g_{k_1,k_2,k_3} be zero, and the process is successful at this step.

By our specifications of $\nu_{1,-1,0}, \nu_{0,-1,1}, \nu_{0,0,0}, \nu_{-1,1,0}, \nu_{-1,0,1}$ and $\nu_{-1,-1,2}$, the procedure can be started. But at every second stage (in fact, at every even value of $k_1 + k_2 + k_3$), there is one triplet k_1, k_2 and k_3 such that $k_1 = k_2 = K > 0$ and $k_3 = 0$, for which (3.28) becomes

$$
\begin{aligned}
g_{K,K,0} = &- ia_{K,K,0} + ib_{K,K,0} \\
&- i \sum_{\substack{j+k=2 \\ 1\le j\le K+2 \\ 0\le k\le K+1}}^{2K+1} ja_{K-j+1,K-k+1,0}\nu_{j-1,k-1,0} \\
&+ i \sum_{\substack{j+k=2 \\ 0\le j\le K+1 \\ 1\le k\le K+2}}^{2K+1} kb_{K-j+1,K-k+1,0}\nu_{j-1,k-1,0} \\
&- \sum_{\substack{j+k+1=2 \\ 0\le j\le K+1 \\ 0\le k\le K+1}}^{2K+1} c_{K-j+1,K-k+1,-1}\nu_{j-1,k-1,1}.
\end{aligned}
\tag{3.29}
$$

The coefficient $\nu_{K,K,0}$ is now missing, so the process of constructing a formal first integral Ψ succeeds at this step only if the expression on the right-hand side of (3.29) is zero. The value of $\nu_{K,K,0}$ is not determined by the equation (3.28) and may be assigned arbitrarily. For a fixed choice of λ in $\mathbb{R} \setminus \{0\}$, it is evident from (3.28) that for all indices k_1, k_2 in $\{-1\} \cup \mathbb{N}_0$ and k_3 in $\mathbb{N}_0$, ν_{k_1,k_2,k_3} is a polynomial function of the coefficients of (2.4), that is, is a polynomial in elements of the set that we denote $E(a,b,c)$, hence by (3.29) so are the expressions $g_{K,K,0}$ for all K. The terms ν_{k_1,k_2,k_3} in (3.29) are obtained by setting $g_{k_1,k_2,k_3} = 0$ in (3.28) at the stage $2K-1$. Thus,

$$
g_{K,K,0} = \frac{\widetilde{g}_{K,K,0}}{d_K}, \quad \text{where} \quad d_K = f(\lambda) = \prod_{k_1+k_2+k_3=2K-1} [-\lambda k_3 + (k_1 - k_2)i].
$$

Then for the values of $\lambda = iz$, where z is a nonzero integer, $d_K = 0$, that is $g_{K,K,0}$ is not defined. Therefore, if λ is allowed (like the a, b, c) to be complex, we do not have a center variety, and when λ is restricted to only real values there is a center variety.

The polynomial $g_{1,1,0}$ is unique, but for $K \geq 2$ the polynomial $g_{K,K,0}$ depends on the arbitrary choices made for $\nu_{j,j,0}$ for $1 \leq < K$. So while it is clear that if for the system $(a^*, b^*, c^*), g_{K,K,0}(a^*, b^*, c^*) = 0$ for all $K \in \mathbb{N}$, then there is a center at the origin, since the process of constructing the formal first integral ψ succeeds at every step, the truth of the converse is not immediately apparent. For even if for some $K \geq 2$ we obtained $g_{K,K,0}(a^*, b^*, c^*) \neq 0$, it is conceivable that if we had made different choices for the polynomials $\nu_{j,j,0}$ for $1 \leq K$, we might have gotten $g_{K,K,0}(a^*, b^*, c^*) = 0$. We will show below (Theorem 3.2.3) that in fact whether or not $g_{K,K,0}$ vanishes at any particular $(a^*, b^*, c^*) \in E[a, b, c]$ is independent of the choices of the $\nu_{j,j,0}$. Thus, the polynomial $g_{K,K,0}$ may be thought of the K^{th} " obstacle " to the existence of a first integral (3.4). If at a point (a^*, b^*, c^*) of our parameter space $E[a, b, c]$, $g_{K,K,0}(a^*, b^*, c^*) \neq 0$, then the construction process fails at that step, no formal first integral of the form (3.12) exists for the corresponding system (2.4), and by Theorem 3.0.8, the system does not have a center at the origin. Only if all the polynomials $g_{K,K,0}$ vanish, $g_{K,K,0}(a^*, b^*, c^*) = 0$ for all $K > 0$, does the corresponding system (2.4) have a formal first integral of the form (3.12), hence have a center at the origin of $\mathbb{C}^3$. Although it is not generally true that a first integral of the form (3.12) exists, the construction process always yields a series of the form (3.12) for which $\mathfrak{X}\Psi = \mathfrak{X}_X \widetilde{P} + \mathfrak{X}_Y \widetilde{Q} + \mathfrak{X}_Z \widetilde{R}$ reduces to

$$\mathfrak{X}\Psi = g_{1,1,0}(xy)^2 + g_{2,2,0}(xy)^3 + g_{3,3,0}(xy)^4 + \cdots . \qquad (3.30)$$

A pseudocode algorithm for applying this idea for computing the coefficients in Ψ and the focus quantities is given in Table 4.1.

Example 3.2.1. Let us consider the set of all systems of the form (2.4) whose sets of admissible indices ordered from greatest to least under degree lexicographic order are

S and T, $S = \{(1,0,0),(-1,1,1)\}$ and $T = \{(1,1,-1),(0,0,1)\}$. Then (2.4) reads

$$\begin{aligned}
\dot{x} &= i(x - a_{1,0,0}x^2 - a_{-1,1,1}yz) \\
\dot{y} &= -i(y - b_{1,-1,1}xz - b_{0,1,0}y^2) \\
\dot{z} &= -\lambda z - c_{1,1,-1}xy - c_{0,0,1}z^2
\end{aligned} \tag{3.31}$$

We will use (3.28) to compute ν_{k_1,k_2,k_3} through the first stage, and use (3.29) to compute $g_{1,1,0}$.

Stage 0: $k_1 + k_2 + k_3 = 0$:

$(k_1, k_2, k_3) \in \{(-1,-1,2),(-1,0,1),(-1,1,0),(0,-1,1),(0,0,0),(1,-1,0)\}$.

By definition, $\nu_{0,0,0} = 1$ and $\nu_{-1,-1,2} = \nu_{-1,0,1} = \nu_{-1,1,0} = \nu_{0,-1,1} = \nu_{1,-1,0} = 0$.

Stage 1: $k_1 + k_2 + k_3 = 1$:

$$\begin{aligned}
(k_1, k_2, k_3) \in \{&(-1,-1,3),(-1,0,2),(-1,1,1),(-1,2,0),(0,-1,2),(0,0,1), \\
&(0,1,0),(1,-1,1),(1,0,0),(2,-1,0)\}.
\end{aligned}$$

In (3.28), $j + k + n$ runs from 2 to 2.

If $(k_1, k_2, k_3) = (-1,-1,3), (j,k,n) \in \{(1,0,1)\}$ for the first sum, $(j,k,n) \in \{(0,1,1)\}$ for the second sum, and $(j,k,n) \in \{(0,0,2)\}$ for the third sum. Inserting the values of ν_{k_1,k_2,k_3} from stage 0 into (3.28) yields

$$\begin{aligned}
g_{-1,-1,3} =& - ia_{-1,-1,3} + ib_{-1,-1,3} - 3\lambda\nu_{-1,-1,3} - ia_{-1,0,2}\nu_{0,-1,1} + ib_{0,1,2}\nu_{-1,0,1} \\
&- 2c_{0,0,1}\nu_{-1,-1,2} = -3\lambda\nu_{-1,-1,3}.
\end{aligned}$$

Setting $g_{-1,-1,3} = 0$ yields $\nu_{-1,-1,3} = 0$.

If $(k_1, k_2, k_3) = (-1,0,2), (j,k,n) \in \{(1,0,1),(1,1,0)\}$ for the first sum, $(j,k,n) \in \{(0,1,1),(0,2,0)\}$ for the second sum, and $(j,k,n) \in \{(0,0,2),(0,1,1)\}$ for the third sum. Inserting the values of ν_{k_1,k_2,k_3} from stage 0 into (3.28) yields

$$\begin{aligned} g_{-1,0,2} =& -ia_{-1,0,2} + ib_{-1,0,2} + (-2\lambda - i)\nu_{-1,0,2} - ia_{-1,0,2}\nu_{0,-1,1} \\ & - ia_{1,0,2}\nu_{0,0,0} + ib_{0,1,2}\nu_{-1,0,1} + ib_{0,-1,2}\nu_{-1,1,0} - 2c_{0,0,1}\nu_{-1,-1,2} \\ & - c_{0,0,1}\nu_{-1,0,1} = (-2\lambda - i)\nu_{-1,0,2}. \end{aligned}$$

Setting $g_{-1,0,2} = 0$ yields $\nu_{-1,0,2} = 0$.

If $(k_1, k_2, k_3) = (-1, 1, 1)$, $(j, k, n) \in \{(1, 0, 1), (1, 1, 0)\}$ for the first sum, $(j, k, n) \in \{(0, 1, 1), (0, 2, 0)\}$ for the second sum, and $(j, k, n) \in \{(0, 0, 2), (0, 1, 1)\}$ for the third sum. Inserting the values of ν_{k_1,k_2,k_3} from stage 0 into (3.28) yields

$$\begin{aligned} g_{-1,1,1} =& -ia_{-1,1,1} + ib_{-1,1,1} + (-\lambda - 2i)\nu_{-1,1,1} - ia_{-1,0,0}\nu_{0,-1,1} - ia_{1,1,1}\nu_{0,0,0} \\ & + ib_{0,1,1}\nu_{-1,0,1} + ib_{0,0,1}\nu_{-1,1,0} - c_{0,1,0}\nu_{-1,0,1} - 2c_{0,1,-1}\nu_{-1,-1,2} \\ =& -2ia_{-1,1,1} + (\lambda - 2i)\nu_{-1,1,1}. \end{aligned}$$

Setting $g_{-1,1,1} = 0$ yields $\nu_{-1,1,1} = \frac{-2ia_{-1,1,1}}{\lambda+2i}$.

Applying the same procedure for all the remaining choices of (k_1, k_2, k_3), we obtain

$\nu_{-1,2,0} = 0,$

$\nu_{0,-1,2} = 0,$

$\nu_{0,0,1} = 0,$

$\nu_{0,1,0} = 2b_{0,1,0},$

$\nu_{-1,1,1} = \frac{2ib_{1,-1,1}}{\lambda-2i},$

$\nu_{1,0,0} = 2a_{1,0,0},$

$\nu_{2,-1,0} = 0.$

Notice that we have applied the conventions $a_{p,q,r} = b_{q,p,r} = 0$ if $(p, q, r) \notin S$ and $c_{p_1,q_1,r_1} = 0$ if $(p_1, q_1, r_1) \notin T$.

We now use (3.29) with $K = 1$ to compute $g_{1,1,0}$. In (3.29) $j + k$ runs from 2 to 3. The sums are over the terms (j, k) in the index sets $\{(1, 1), (2, 0), (1, 2), (3, 0)\}$,

$\{(0,2),(1,1),(0,3),(1,2),(2,1)\}$ and $\{(0,1),(1,0),(0,2),(1,1),(2,0)\}$ respectively for the first, the second and the third sums. Inserting the values of ν_{k_1,k_2,k_3} from stages 0 and 1 into (3.29), we obtain

$$\begin{aligned}
g_{1,1,0} =& -ia_{1,1,0} + ib_{1,1,0} - ia_{1,1,0}\nu_{0,0,0} - 2ia_{0,2,0}\nu_{1,-1,0} - ia_{1,0,0}\nu_{0,1,0} \\
& - 2ia_{0,1,0}\nu_{1,0,0} - 3ia_{-1,2,0}\nu_{2,-1,0} + 2ib_{2,0,0}\nu_{-1,1,0} + ib_{1,1,0}\nu_{0,0,0} \\
& + 3ib_{2,-1,0}\nu_{-1,2,0} + 2ib_{1,0,0}\nu_{0,1,0} + ib_{0,1,0}\nu_{1,0,0} - c_{2,1,-1}\nu_{-1,0,1} \\
& - c_{1,2,-1}\nu_{0,-1,1} - c_{2,0,-1}\nu_{-1,1,1} - c_{1,1,-1}\nu_{0,0,1} - c_{0,2,-1}\nu_{1,-1,1} \\
=& 0.
\end{aligned}$$

This is the first focus quantity for an element of family (3.31).
Similarly, we compute ν_{k_1,k_2,k_3} at stages $2,3,4,\cdots$, for all admissible (k_1,k_2,k_3) except when $k_1 - k_2 = k_3 = 0 (k_1 = k_2 = K)$ for which we set $\nu_{K,K,0} = 0$. Then all $g_{K,K,0}$ must vanish in order for an element of family (3.31) to have a center at the origin. □

The computations quickly become too large to be feasible for hand computation. A Mathematica procedure derived from the algorithm presented in Table 4.1 is shown in Table 4.2. It is used in the following example.

Example 3.2.2. Let us consider the set of all systems of the form (2.4) that have some quadratic nonlinearities, so that the sets of admissible indices ordered from greatest to least under degree lexicographic order, are S and T,
$S = \{(1,0,0),(0,0,1),(-1,1,1),(-1,0,2)\}$ and $T = \{(2,0,-1),(0,0,1)\}$. Then (2.4) reads

$$\begin{aligned}
\dot{x} &= i(x - a_{1,0,0}x^2 - a_{0,0,1}xz - a_{-1,1,1}yz - a_{-1,0,2}z^2) \\
\dot{y} &= -i(y - b_{1,-1,1}xz - b_{0,1,0}y^2 - b_{0,0,1}yz - b_{0,-1,2}z^2) \\
\dot{z} &= -\lambda z - c_{2,0,-1}x^2 - c_{0,0,1}z^2
\end{aligned} \tag{3.32}$$

We will use the algorithm in Table 4.1 and the associated Mathematica code for

Example 3.2.2 in Table 4.2 to compute ν_{k_1,k_2,k_3} through the second stage and also compute $g_{1,1,0}$. We obtain,

at stage=0, by definition $\nu_{0,0,0} = 1$ and $\nu_{-1,-1,2} = \nu_{-1,0,1} = \nu_{-1,1,0} = \nu_{0,-1,1} = \nu_{1,-1,0}$,

at stage=1,

$\nu_{-1,-1,3} = 0$,

$\nu_{-1,0,2} = \frac{-2ia_{-1,0,2}}{i+2\lambda}$,

$\nu_{-1,1,1} = \frac{-2ia_{-1,1,1}}{2i+\lambda}$,

$\nu_{-1,2,0} = 0$,

$\nu_{0,-1,2} = \frac{-2ib_{0,-1,2}}{-i+2\lambda}$,

$\nu_{0,0,1} = \frac{-2i(a_{0,0,1}-b_{0,0,1})}{\lambda}$,

$\nu_{0,1,0} = 2b_{0,1,0}$,

$\nu_{1,-1,1} = \frac{2ib_{1,-1,1}}{-2i+\lambda}$,

$\nu_{1,0,0} = 2a_{1,0,0}$,

$\nu_{2,-1,0} = 0$.

When $(k_1, k_2, k_3) = (1, 1, 0)$, (3.29) becomes $g_{1,1,0} = \frac{2ia_{-1,1,1}c_{2,0,-1}}{2i+\lambda}$. This is the first focus quantity, which must be zero in order for an element of family (3.32) to have a center at the origin. □

We now show that for fixed $K \in \mathbb{N}$, the variety $V(g_{1,1,0}, g_{2,2,0}, \cdots)$ is the same for all choices of the polynomials $\nu_{j,j,0}, j < K$, which determine $g_{K,K,0}$, and thus that the center variety $V_{\mathcal{C}}$ is well-defined.

Theorem 3.2.3. Fix sets S and T and consider family (2.4).

1. Let Ψ be a formal series of the form (3.12) and let $g_{1,1,0}(a, b, c)$, $g_{2,2,0}(a, b, c), \cdots$ be polynomials satisfying (3.30) with respect to system (2.4). Then system (2.4) with parameters $(a^\star,\, b^\star,\, c^\star)$ has a center at the origin if and only if $g_{k,k,0}(a^\star, b^\star, c^\star) = 0$ for all $k \in \mathbb{N}$.

2. Let Ψ and $g_{k,k,0}$ be as in (1) and suppose there exist another function Ψ' of the form (3.12) and polynomials $g'_{1,1,0}(a, b, c), g'_{2,2,0}(a, b, c), \cdots$ that satisfy (3.30) with respect

to family (2.4). Then $V_{\mathcal{C}} = V'_{\mathcal{C}}$,

where $V_{\mathcal{C}} = V(g_{1,1,0}(a,b,c), g_{2,2,0}(a,b,c), \cdots)$ and where

$V'_{\mathcal{C}} = V(g'_{1,1,0}(a,b,c), g'_{2,2,0}(a,b,c), \cdots)$.

Proof. 1. Suppose that family (2.4) is as in the statement of the theorem. Let Ψ be a formal series of the form (3.12) and let $\{g_{k,k,0}(a,b,c) : k \in \mathbb{N}\}$ be polynomials in (a,b,c) that satisfy (3.30). If for $(a^\star, b^\star, c^\star) \in E(a,b,c)$, $g_{k,k,0}(a^\star, b^\star, c^\star) = 0$ for all $k \in \mathbb{N}$, then Ψ is a formal first integral for the corresponding family in (2.4). By Definition 3.0.9 the system has a center at the origin of $\mathbb{C}^3$.

To prove the converse, we first make the following observations. Suppose that there exist $k \in \mathbb{N}$ and a choice $(a^\star, b^\star, c^\star)$ of the parameters such that $g_{j,j,0}(a^\star, b^\star, c^\star) = 0$ for $1 \le j \le k-1$ but $g_{k,k,0}(a^\star, b^\star, c^\star) \ne 0$. Let $H(x_1, y_1, z_1)$ be the distinguished normalizing transformation (2.7), producing the distinguished normal form (2.10), and consider the function $F = \Psi \circ H$. By construction

$$
\begin{aligned}
&\frac{\partial F}{\partial x_1}(x_1,y_1,z_1)[ix_1 + x_1 X(x_1y_1)] + \frac{\partial F}{\partial y_1}(x_1,y_1,z_1)[-iy_1 + y_1 Y(x_1y_1)] \\
&\quad + \frac{\partial F}{\partial z_1}(x_1,y_1,z_1)[-\lambda z_1 + z_1 Z(x_1y_1)] \\
&= g_{k,k,0}(a^\star, b^\star, c^\star)[x_1 + h_1(x_1,y_1,z_1)]^{k+1}[y_1 + h_2(x_1,y_1,z_1)]^{k+1} + \cdots \\
&= g_{k,k,0}(a^\star, b^\star, c^\star) x_1^{k+1} y_1^{k+1} + \cdots .
\end{aligned}
\tag{3.33}
$$

Through order $2k+1$ this is almost precisely equation (3.5), so if we repeat verbatim the argument that follows (3.5), we obtain identity (3.8), through order $2k+2$. Therefore,

$$
F(x_1,y_1,z_1) = f_1(x_1y_1) + \cdots + f_{k+1}(x_1y_1)^{k+1} + U(_1, y_1) = f(_1y_1) + U(_1, y_1),
$$

where $f_1 = 1$ and $U(_1, y_1)$ begins with terms of order at least $2k+3$. Thus,

$$x_1 \frac{\partial F}{\partial x_1} = x_1 y_1 f'(_1 y_1) + \alpha(_1, y_1) \text{ and } y_1 \frac{\partial F}{\partial y_1} = x_1 y_1 f'(_1 y_1) + \beta(_1, y_1),$$

where $\alpha(_1, y_1)$ and $\beta(_1, y_1)$ begin with terms of order at least $2k+3$, and so the left-hand side of (3.33) is

$$i[\alpha(_1, y_1) - \beta(_1, y_1)] + (X(x_1 y_1) + Y(x_1 y_1)) x_1 y_1 f'(_1 y_1) +$$
$$X(x_1 y_1)\alpha(_1, y_1) + Y(x_1 y_1)\beta(_1, y_1).$$

Hence if we subtract from both sides of (3.33)

$$i[\alpha(_1, y_1) - \beta(_1, y_1)] + X(x_1 y_1)\alpha(_1, y_1) + Y(x_1 y_1)\beta(_1, y_1),$$

which begins with terms of order at least $2k+3$, we obtain

$$(X(x_1 y_1) + Y(x_1 y_1)) f'(_1 y_1) = g_{k,k,0}(a^\star, b^\star, c^\star)(x_1 y_1)^k + \cdots, \tag{3.34}$$

where X and Y are functions in (2.10). Thus supposing, contrary to what we wish to show, that system (2.10) for the choice of $(a, b, c) = (a^\star, b^\star, c^\star)$ has a center at the origin of $\mathbb{C}^3$, so that it admits a first integral $\Phi(x, y, z) = xy + \cdots$. Then by Theorem 3.0.8, the function $X + Y$ vanishes identically, hence the left-hand side of (3.34) is identically zero, whereas the right-hand side is not, a contradiction.

2. If $V_{\mathcal{C}} \neq V'_{\mathcal{C}}$, then there exists $(a^\star, b^\star, c^\star)$ that belongs to one of the varieties $V_{\mathcal{C}}$ and $V'_{\mathcal{C}}$ but not to the other, say $(a^\star, b^\star, c^\star) \in V_{\mathcal{C}}$ and $(a^\star, b^\star, c^\star) \notin V'_{\mathcal{C}}$. The inclusion $(a^\star, b^\star, c^\star) \in V_{\mathcal{C}}$ means that the system corresponding to $(a^\star, b^\star, c^\star)$ has a center at the origin. Therefore by part (1) $g'_{k,k,0}(a^\star, b^\star, c^\star) = 0$ for all $k \in \mathbb{N}$. This contradicts our assumption that $(a^\star, b^\star, c^\star) \notin V'_{\mathcal{C}}$. $\square$

3.3 The Center Variety

Definition 3.3.1. Fix sets S and T of admissible indices of (2.4). The polynomial $g_{K,K,0}$ defined by (3.29) is called the Kth focus quantity for the singularity at the origin of system (2.4). The ideal of focus quantities, $\mathcal{B} = \langle g_{1,1,0}, g_{2,2,0}, \cdots, g_{j,j,0}, \cdots \rangle \subset \mathbb{C}[a,b,c]$, is called the Bautin ideal, and the affine variety $V_{\mathcal{C}} = V(\mathcal{B})$ is called the center variety for the singularity at the origin of system (2.4), or more simply, of system (2.4). $\mathcal{B}_K$ will denote the ideal generated by the first K focus quantities, $\mathcal{B}_K = \langle g_{1,1,0}, g_{2,2,0}, \cdots, g_{K,K,0} \rangle$.

Remark 3.3.2. Let $G(w) = \sum_{k=1}^{\infty} G_{2k+1} w^k$ be the function of complex variable w defined by $G \equiv X + Y$. Note that it is a consequence of (3.34) that if, for a particular $(a^\star, b^\star, c^\star) \in E(a,b,c)$, $g_{k,k,0}(a^\star, b^\star, c^\star)$ is the first nonzero focus quantity, then the first nonzero coefficient of $G(a^\star, b^\star, c^\star)$ is $G_{2k+1}(a^\star, b^\star, c^\star)$ and $G_{2k+1}(a^\star, b^\star, c^\star) = g_{k,k,0}(a^\star, b^\star, c^\star)$.

Thus points of $V_{\mathcal{C}}$ correspond precisely to systems in family (2.4) that have a center at the origin of $\mathbb{C}^3$, in the sense that there exists a first integral of the form (3.12). If $(a,b,c) \in V_{\mathcal{C}}$ and $a_p qr = \bar{b}_q pr$ for all $(p,q,r) \in S$, which we denote $b = \bar{a}$, then such point corresponds to a system that is the complexification of the real system expressed in complex coordinates as (2.4), which then has a topological center at the origin for the system restricted to the center variety. More generally, we can consider the intersection of the center variety $V_{\mathcal{C}}$ with the set $\Pi := \{(a,b,c) : b = \bar{a}\}$ whose elements correspond to complexifications of real systems; we call this the real center variety $V_{\mathcal{C}}^R := V_{\mathcal{C}} \cap \Pi$. To the set Π there corresponds a family of real systems of differential equations on $\mathbb{R}^3$ expressed in complex form as (2.4) or in real form as (1.1), and for this family there is a space E^R of real parameters. Within E^R, there is a variety V corresponding to systems that have a center at the origin. These ideas are the generalization to (1.1) of similar concepts for analogous two-dimensional systems (essentially (1.1) without z).

By Theorems 3.0.8, and 3.2.3, in order to find either $V_{\mathcal{C}}$ or $V_{\mathcal{C}}^{R}$, one can compute either the coefficients G_{2k+1} of the function G defined in Remark 3.3.2 or the focus quantities $g_{k,k,0}$, all of which are polynomial functions of the parameters.

From the point of view of applications the most interesting systems are the real systems. The trouble is, of course, that the field $\mathbb{R}$ is not algebraically closed, making it far more difficult to study real varieties than complex varieties. This is why we will primarily investigate the center problem for complex systems (2.4).

CHAPTER 4: THE MOON-RAND SYSTEM

4.1 The Polynomial Moon-Rand System

The polynomial Moon-Rand family of systems introduced in ([9]) to model certain flexible structures is the family of three dimensional polynomial systems of ordinary differential equations of the form

$$\begin{aligned}\dot{u} &= v\\ \dot{v} &= -u - uw\\ \dot{w} &= -\lambda w + c_{20}u^2 + c_{11}uv + c_{02}v^2,\end{aligned} \tag{4.1}$$

where λ, c_{20}, c_{11} and c_{02} are real parameters and $\lambda \neq 0$. There is an isolated equilibrium at the origin at which the associated linear part has two eigenvalues that are purely imaginary ($\pm i$) and one eigenvalue that is real $(-\lambda.)$ We analyze the local flows induced by $\mathfrak{X}$ on a neighborhood of the origin in any center manifold of the polynomial Moon-Rand systems. We then solve the center problem on the center manifold W^c_{loc}, find the Lyapunov numbers to determine the stability of the focus on W^c_{loc}, and find the cyclicity of the foci.

4.1.1 Complexification of the Polynomial Moon-Rand System

We start by applying the following change of coordinates to put system (4.1) into a system of the form (1.1):

$$U = -u \qquad V = v \qquad W = w.$$

We obtain

$$\begin{aligned}
\dot{U} &= -V \\
\dot{V} &= U + UW \\
\dot{W} &= -\lambda W + c_{20}U^2 - c_{11}UV + c_{02}V^2.
\end{aligned} \tag{4.2}$$

We now consider the real space (U, V, W) as a complex plane cross a line, $x = U + iV, w = W$, for the complexification of the system (4.2).

Simple computations yield

$$\begin{aligned}
\dot{x} =& ix + \frac{i}{2}xz + \frac{i}{2}yz \\
\dot{y} =& -iy - \frac{i}{2}xz - \frac{i}{2}yz \\
\dot{z} =& -\lambda z + \frac{1}{4}(c_{20} - c_{02} + ic_{11})x^2 + \frac{1}{2}(c_{20} + c_{02})xy \\
& + \frac{1}{4}(c_{20} - c_{02} - ic_{11})y^2.
\end{aligned} \tag{4.3}$$

We then factor out i from the first two equations to put system (4.3) in the form of system (2.4), hence, to use the results of the theory previously presented in this dissertation. We obtain

$$\begin{aligned}
\dot{x} =& i[x - (-\frac{1}{2}xz - \frac{1}{2}yz)] \\
\dot{y} =& -i[y - (-\frac{1}{2}xz - \frac{1}{2}yz)] \\
\dot{z} =& [-\lambda z - (-\frac{1}{4}(c_{20} - c_{02} + ic_{11})x^2 - \frac{1}{2}(c_{20} + c_{02})xy \\
& - \frac{1}{4}(c_{20} - c_{02} - ic_{11})y^2)].
\end{aligned} \tag{4.4}$$

Therefore, the index sets S and T are given by

$S = \{(0, 0, 1), (-1, 1, 1)\}$ and $T = \{(2, 0, -1), (1, 1, -1), (0, 2, -1)\}$, and the corresponding nonzero coefficients are

$a_{0,0,1} = a_{-1,1,1} = b_{1,-1,1} = b_{0,0,1} = -\frac{1}{2}$,

$c_{2,0,-1} = -\frac{1}{4}(c_{20} - c_{02} + ic_{11}),$

$c_{1,1,-1} = -\frac{1}{2}(c_{20} + c_{02}),$

$c_{0,2,-1} = -\frac{1}{4}(c_{20} - c_{02} - ic_{11}).$

4.1.2 Focus Quantities for the Polynomial Moon-Rand System

Let $\mathfrak{X}$ denote the associated vector field to system (4.2). By the Lyapunov center theorem, the real system (4.2) has a center at the origin of $\mathbb{R}^3$ on a local center manifold W^c_{loc} if and only if it admits a first integral of the form

$$\Phi(U, V, W) = U^2 + V^2 + \cdots .$$

We have shown that it is equivalent to the existence of the first integral Ψ of the form (3.12) for its complexification (4.4). From the discussion after the formula for $g_{K,K,0}$ in Section 3.2, a series of the form (3.12) for which

$$\mathfrak{X}\Psi = \mathfrak{X}_X \widetilde{P} + \mathfrak{X}_Y \widetilde{Q} + \mathfrak{X}_Z \widetilde{R},$$

reduces to

$$\mathfrak{X}\Psi = g_{1,1,0}(xy)^2 + g_{2,2,0}(xy)^3 + g_{3,3,0}(xy)^4 + \cdots .$$

The first focus quantity $g_{1,1,0}$ is uniquely determined, but for $K \in \mathbb{N}, K \geq 2, g_{K,K,0}$ depend on the choices made for $\nu_{j,j,0}, j \in \mathbb{N}, j < K$. It is natural that we assign 0 to all $\nu_{j,j,0}, j \in \mathbb{N}$. The focus quantities $g_{K,K,0}$ are polynomials in the coefficients c_{20}, c_{02}, c_{11}, that have parameter $\lambda, \lambda \neq 0$, and are computed using the algorithm in Table 4.1 with the associated Mathematica code in Table 4.3. The first five nonzero focus quantities are computed, but we only show the results for the first two since the others have too many terms (more than 300) to display here.

$$g_{1,1,0} = \frac{2c_{20} - 2c_{02} - \lambda c_{11}}{8 + 2\lambda^2}$$
$$g_{2,2,0} = \frac{(c_{20} + c_{02})[2c_{02}(-4 + \lambda^2) + 2c_{20}(12 + \lambda^2) - \lambda c_{11}(12 + \lambda^2)]}{8\lambda(4 + \lambda^2)^2}$$

Recall that vanishing of all the focus quantities is sufficient for the existence of the formal first integral. We also know that by Theorem 3.2.3, for any fixed $K \in \mathbb{N}$, the variety $V(g_{K,K,0})$ is the same for all choices of the polynomials $\nu_{j,j,0}, j < K$, which determine $g_{K,K,0}$, and thus that the center variety $V_{\mathcal{C}}$ is well-defined.

4.1.3 The Center Variety for the Polynomial Moon-Rand System

We now introduce a concept, the radical of an ideal I, that will be of fundamental importance in the procedure for identifying the variety $V(I)$ of I.

Definition 4.1.1. Let $I \subset k[x_1, \cdots, x_n]$ be an ideal. The radical of I, denoted $\sqrt{I}$, is the set

$$\sqrt{I} = \{f \in k[x_1, \cdots, x_n] : \text{there exists } p \in \mathbb{N} \text{ such that } f^p \in I\}.$$

An ideal $J \in k[x_1, \cdots, x_n]$ is called a radical ideal if $J = \sqrt{I}$. ([12]).

Example 4.1.2. [Example 1.3.13 on page 31 of ([12]).] Consider the set of ideals $I^{(p)} = \langle (x - y)^p \rangle, p \in \mathbb{N}$. All these ideals define the same variety V, which is the line $y = x$ in the plane k^2. It is easy to see that $I^{(1)} \supsetneqq I^{(2)} \supsetneqq I^{(3)} \supsetneqq \cdots$ and that $\sqrt{I^{(p)}} = I^{(1)}$ for every index p, so the only radical ideal among the $I^{(p)}$ is $I^{(1)} = \langle x-y \rangle$.

This example indicates that it is the radical of an ideal that is fundamental in picking out the variety that it and other ideals may determine.

Part of the importance for us of the concept of radical ideal is that when the field k in question is $\mathbb{C}$, it completely characterizes when two ideals determine the same affine variety:

Proposition 4.1.3. Let I and J be ideals in $\mathbb{C}[\mathrm{x}_1, \cdots, \mathrm{x}_\mathrm{n}]$. Then,
$\mathrm{V(I)} = \mathrm{V(J)}$ if and only if $\sqrt{\mathrm{I}} = \sqrt{\mathrm{J}}$.

Proof. Section 1.3 of ([12]).

It is a difficult computational problem to compute the radical of a given ideal (unless the ideal is particulary simple, as was the case in Example 4.1.2). However, we obtain in Section 1.3 of ([12]), the following method for checking whether or not a given polynomial belongs to the radical of a given ideal.

Theorem 4.1.4. Let k be an arbitrary field and let $\mathrm{I} = \langle \mathrm{f}_1, \cdots, \mathrm{f}_\mathrm{s} \rangle$ be an ideal in $\mathrm{k}[\mathrm{x}_1, \cdots, \mathrm{x}_\mathrm{n}]$. Then $\mathrm{f} \in \sqrt{\mathrm{I}}$ if and only if
$1 \in \mathrm{J} := \langle \mathrm{f}_1, \cdots, \mathrm{f}_\mathrm{s}, 1 - \mathrm{wf} \rangle \subseteq \mathrm{k}[\mathrm{x}_1, \cdots, \mathrm{x}_\mathrm{n}, \mathrm{w}]$.

Proof. Section 1.3 of ([12]).

This theorem provides the simple algorithm presented in Table 1.4 [The Radical Membership Test] in ([12]) for deciding whether or not a polynomial f lies in $\sqrt{\langle f_1, \cdots, f_s \rangle}$.

Definition 4.1.5. Consider systems

$$\dot{x} = \widetilde{P}(x, y, z), \qquad \dot{y} = \widetilde{Q}(x, y, z), \qquad \dot{z} = \widetilde{R}(x, y, z), \tag{4.5}$$

where $x, y, z \in \mathbb{C}, \widetilde{P}, \widetilde{Q}$ and $\widetilde{R}$ are polynomials without constant terms that have no nonconstant common factor,and
$m = \max(deg(\widetilde{P}), deg(\widetilde{Q}), deg(\widetilde{R}))$.
A nonconstant polynomial $f(x, y, z) \in \mathbb{C}[x, y, z]$ is called an algebraic partial integral of system (4.5) if there exists a polynomial $K(x, y, z) \in \mathbb{C}[x, y, z]$, such that

$$\mathfrak{X}f = \frac{\partial f}{\partial x}\widetilde{P} + \frac{\partial f}{\partial y}\widetilde{Q} + \frac{\partial f}{\partial z}\widetilde{R} = Kf. \tag{4.6}$$

The polynomial K is termed a cofactor of f; it has degree at most $m-1$.[12].

Theorem 4.1.6 (The center conditions). Consider the polynomial Moon-Rand system (4.1). Let $\mathfrak{X}$ denote its associated vector field. Then for the flow induced by $\mathfrak{X}$ on the neighborhood of the origin of $\mathbb{R}^3$ in any center manifold W^c of (4.1), the origin is a center if and only if $c_{02} = -\frac{1}{2}\lambda c_{11} + c_{20} = 0$.

Proof. Let $\widetilde{g}_{K,K,0}$ denote the numerator of $g_{K,K,0}$ and
$\mathcal{B}_K = \langle \widetilde{g}_{1,1,0}, \widetilde{g}_{2,2,0}, \cdots, \widetilde{g}_{K,K,0} \rangle$.
Using Singular, a highly efficient computer algebra system for polynomial computations, we decompose the radicals of $\mathcal{B}_K, 1 \leq K \leq 5$ into a unique intersection of prime ideals. We replace for simplicity c_{20}, c_{11}, c_{02} and λ by x, y, z and L respectively. The code (with only parts of $\widetilde{g}_{3,3,0}, \widetilde{g}_{4,4,0}$ and $\widetilde{g}_{5,5,0}$ displayed) for carrying out the decompositions and the outputs are as follows.

```
> LIB "primdec.lib";

> ring r=0,(x,y,z,L),dp;

> poly g1=(2*x - L*y - 2*z);

> poly g2=((x + z)*(-2*(12 + L^2)*x + 12*L*y + L^3*y + 8*z

          - 2*L^2*z));

> poly g3=(-2*(-98304 - 188672*L^2 - 53744*L^4 - 1360*L^6

          + 947*L^8 +...+2*L^7*(903*y^2*z + 1471*z^3) +

          32*L^3*(61*y^2*z + 3725*z^3)));

> poly g4=-(-2*(-33554432 - 536346624*L^2 -2400976896*L^4

          -...+ 2048*L^8*(499118*y^2*z + 3438707*z^3)));

> poly g5=(-2*(18029791672270848 + 244935868857450496*L^2

           + ...- 16777216*L^8*(1925486675*y^4 +

           105107739599*y^2*z^2 + 250632146354*z^4)));

> ideal j1=g1;
```

```
> ideal j2=g1,g2;
> ideal j3=g1,g2,g3;
> ideal j4=g1,g2,g3,g4;
> ideal j5=g1,g2,g3,g4,g5;
> minAssGTZ(j1);
[1]:
   _[1]=-1/2yL+x-z
> minAssGTZ(j2);
[1]:
   _[1]=L2+4
   _[2]=-1/2yL+x-z
[2]:
   _[1]=yL+4z
   _[2]=-1/2yL+x-z
[3]:
   _[1]=z
   _[2]=-1/2yL+x-z
> minAssGTZ(j3);
[1]:
   _[1]=L2+4
   _[2]=-1/2yL+x-z
[2]:
   _[1]=L2+10
   _[2]=-2zL+5y
   _[3]=-1/2yL+x-z
[3]:
   _[1]=L2+1
```

```
   _[2]=-4zL+y
   _[3]=-1/2yL+x-z
[4]:
   _[1]=z
   _[2]=-1/2yL+x-z
> minAssGTZ(j4);
[1]:
   _[1]=L2+4
   _[2]=-1/2yL+x-z
[2]:
   _[1]=L2+1
   _[2]=-4zL+y
   _[3]=-1/2yL+x-z
[3]:
   _[1]=z
   _[2]=-1/2yL+x-z
> minAssGTZ(j5);
[1]:
   _[1]=L2+4
   _[2]=-1/2yL+x-z
[2]:
   _[1]=L2+1
   _[2]=-4zL+y
   _[3]=-1/2yL+x-z
[3]:
   _[1]=z
   _[2]=-1/2yL+x-z
```

The first command downloads a Singular library that enables computation of primary and prime decompositions. The second command declares that the polynomial ring involved has characteristic zero, that the variables are x, y, z, and L is also treated as variable, and in the order L > x > y > z, and that the term order to be used (as specified by the parameter dp) is degree lexicographic order. The next five command lines specify the polynomials $g_{1,1,0}$ through $g_{5,5,0}$. The next five lines declare successively that the ideals under consideration are $\mathcal{B}_K = \langle g_{1,1,0}, \cdots, g_{K,K,0}\rangle$ from $K = 1$ to $K = 5$. Finally, minAssGTZ(jK) commands the computation of a primary decomposition of jK using the minimal associated primes via Gianni,Trager,Zacharias (with modifications by Laplagne)([6]). In the output the symbol L2 is Singular's short notation for L^2. Each output to the exception of the first one, is a list of ideals, where each ideal is, of course specified by a list of generators.

The outputs indicate that $\sqrt{\mathcal{B}_1} \subsetneqq \sqrt{\mathcal{B}_2} \subsetneqq \sqrt{\mathcal{B}_3} \subsetneqq \sqrt{\mathcal{B}_4} = \sqrt{\mathcal{B}_5}$. Hence, we suspect that $V_{\mathcal{C}} = V(\mathcal{B}_4)$. Moreover, we obtain the prime decomposition of the variety $V(\mathcal{B}_4)$ as the union of three varieties $V(J_i)$ which are:

$$\begin{aligned}
J_1 &= \langle \lambda^2 + 4, -\frac{1}{2}\lambda c_{11} + c_{20} - c_{02}\rangle \\
J_2 &= \langle \lambda^2 + 1, -4\lambda c_{02} + c_{11}, -\frac{1}{2}\lambda c_{11} + c_{20} - c_{02}\rangle \\
J_3 &= \langle c_{02}, -\frac{1}{2}\lambda c_{11} + c_{20} - c_{02}\rangle
\end{aligned}$$

Notice that setting the generators of either J_1 or J_2 to zero yields complex numbers, which contradicts the fact that system (4.1) in question is real. Therefore, the necessary conditions given by setting the polynomial generators of J_3 equal to zero are:

$$c_{02} = -\frac{1}{2}\lambda c_{11} + c_{20} = 0 \tag{4.7}$$

for the origin to be a center for $\mathfrak{X} \mid W^c_{loc}$.

Conversely, suppose that conditions (4.7) hold. We must show that there exists, on a neighborhood of the origin, a first integral Φ of (4.1) of the form $\Phi(u,v,w) = u^2+v^2+\cdots$. By the Darboux theory (see [12]), for a polynomial vector field $\mathfrak{X}$ of degree m on $\mathbb{R}^3$, if $F_1,\cdots,F_k$ are polynomial functions on $\mathbb{R}^3$ for which there are polynomials $K_1,\cdots,K_k$ and constants $\alpha_1,\cdots,\alpha_k$ such that $\mathfrak{X}(F_j) = K_jF_j$ for $1 \leq j \leq k$ and $\sum \alpha_j K_j = 0$, then a first integral is $F = F_1^{\alpha_1}\cdots F_k^{\alpha_k}$. (Each surface $f_j = 0$ is an invariant surface for $\mathfrak{X}$; each F_j is called an algebraic partial integral with cofactor K_j, which can have degree at most $deg(\mathfrak{X})-1$.) To apply this theory we let $\mathfrak{X}$ denote the vector field associated to system (4.1) when conditions (4.7) hold and search for polynomials F (which could have any degree) and K (of degree at most 1) satisfying

$$\mathfrak{X}(F) = KF.$$

So let $F(u,v,w) = [F_{200}u^2 + F_{110}uv + F_{101}uw + F_{020}v^2 + F_{011}vw + F_{002}w^2 + \cdots]$, and $K(u,v,w) = [K_{000} + K_{100}u + K_{010}v + K_{001}w]$. Using the Mathematica code (where \[Lambda] represents λ) below, we are able to construct the polynomials $F(u,v,w)$ and $K(u,v,w)$.

```
F[u_, v_, w_] := a*u^2 + b*u*v + c*u*w + d*v^2 + e*v*w +
                 f*w^2 +g*u + h*v + i*w + j;
K[u_, v_, w_] := k + l*u + m*v + n*w;
F1[u_, v_, w_] := Expand[D[F[u, v, w], u]*v +
                  D[F[u, v, w], v]*(-u - u*v) +
                  D[F[u, v, w], w]*(-\[Lambda]*w +
                  x*(u^2) + y*u*v + z*(v^2))];
F2[u_, v_, w_] := Expand[K[u, v, w]*F[u, v, w]];
F3[u_, v_, w_] := Expand[F2[u, v, w] - F1[u, v, w]];
z=0;
Coefficient[F3[u, 0, 0], u]
```

```
Coefficient[F3[0, v, 0], v]
Coefficient[F3[0, 0, w], w]
Coefficient[F3[u, 0, 0], u^2]
Coefficient[F3[u, 0, 0], u^3]
Coefficient[F3[u, v, 0], u*v]
Coefficient[F3[u, v, 0], u*u*v]
Coefficient[F3[u, v, 0], u*v*v]
Coefficient[F3[u, 0, w], u*w]
Coefficient[F3[u, 0, w], u*u*w]
Coefficient[F3[u, 0, w], u*w*w]
Coefficient[F3[0, v, 0], v*v]
Coefficient[F3[0, v, 0], v*v*v]
Coefficient[F3[0, v, w], v*v*w]
Coefficient[F3[0, v, w], v*w]
Coefficient[F3[0, v, w], v*w*w]
Coefficient[F3[u, v, w], u*v*w]
Coefficient[F3[0, 0, w], w*w]
Coefficient[F3[0, 0, w], w*w*w]
```

The first two lines define the polynomial $F(u, v, w)$ where its coefficients F_{200}, F_{110}, F_{101}, F_{020}, F_{011}, F_{002}, F_{100}, F_{010}, F_{001} and F_{000} are replaced by a, b, c, d, e, f, g, h, i and j respectively. The third line defines the cofactor polynomial $K(u, v, w)$ where its coefficients K_{100}, K_{010}, K_{001} and K_{000} are replaced by k, l, m and n respectively. The next three lines define the expression $\mathfrak{X}F$ where the coefficients c_{20}, c_{11} and c_{02} are replaced x, y, and z respectively, and the following line defines the expression KF, both expressions are clearly polynomials in coefficients of F and K. The next line defines the polynomial $\mathfrak{X}F - KF$ that need to be a zero polynomial. Line nine set the coefficient $z = c_{02}$ to zero to meet the first of the two center conditions (4.7). The last nineteen lines extract the coefficients of the zero polynomial $\mathfrak{X}F - KF$ that we set each equal to zero. The output (where \[Lambda] represents λ) is the following.

```
h + g k + j l

-g + h k + j m

i k + j n + i \[Lambda]

b + a k + g l - i x

a l - c x

-2 a + 2 d + h + b k + h l + g m - i y

b + b l + a m - e x - c y

2 d + d l + b m - e y

e + c k + i l + g n + c \[Lambda]

c l + a n - 2 f x

f l + c n

-b + d k + h m

d m

e m + d n

-c + e k + i m + h n + e \[Lambda]

f m + e n

e + e l + c m + b n - 2 f y

f k + i n + 2 f \[Lambda]

f n
```

We then solve the equations obtained that have fewer coefficient variables and run the code again, but this time with the values of the coefficients found. For instance, we solve $dm = fn = 0$ to set $d = f = m = n = 0$ when we run the code again, so to create zeroes and to leave us with fewer equations with less coefficient variables to solve for. We obtain

```
h + g k + j l
```

```
-g + h k

i k + i \[Lambda]

b + a k + g l - i x

a l - c x

-2 a + h + b k + h l - i y

b + b l - e x - c y

-e y

e + c k + i l + c \[Lambda]

c l

0

-b

0

0

-c + e k + e \[Lambda]

0

e + e l

0

0
```

Next we set $b = c = e = g = h = l = 0$, and after running the code again with these values, we obtain all zeroes except for the following: $ik + i\lambda$, $ak - ix$ and $-2a - iy$. Therefore, substituting back the coefficients of $F(u, v, w)$ and $K(u, v, w)$, yield the following first three equations to which we add a forth equation that is just the second of the center conditions (4.7):

$$F_{001}(K_{000}+\lambda)=0$$
$$F_{200}K_{000}-F_{001}c_{20}=0$$
$$-2F_{200}-F_{001}c_{11}=0$$
$$c_{20}-\frac{1}{2}\lambda c_{11}=0$$

The above four equations yield

$$K_{000}=-\lambda$$
$$F_{200}=-c_{20}$$
$$F_{001}=\lambda$$

Here we find that

$$F(u,v,w)=-c_{20}u^2+\lambda w=0, \tag{4.8}$$

with cofactor $K(u,v,w)=-\lambda$, is an invariant surface that satisfies our conditions. This result is not sufficient for applying the Darboux theory, but we notice that the surface given by (4.8) is tangent to the $(u,v)-$plane, the center eigenspace, hence because it is invariant it must be a center manifold. Solving (4.8) for w and inserting this expression into the first two equations in (4.1) gives $\mathfrak{X} \mid W^c_{loc}$ in local coordinates about the origin:

$$\begin{aligned}\dot{u}&=v\\ \dot{v}&=-u-\frac{c_{20}}{\lambda}u^3\end{aligned} \tag{4.9}$$

on W^c_{loc}.

In Section 2.14 of ([11]), this system is one particular type of Hamiltonian system, the Newtonian system with the Hamiltonian function $H(u,v)=T(v)+U(u)$ where

$T(v) = \frac{v^2}{2}$ is the kinetic energy and

$$U(u) = -\int_{u_0}^{u} (-s - \frac{c_{20}}{\lambda} s^3) ds = \frac{u^2}{2} + \frac{c_{20}}{4\lambda} u^4$$

is the potential energy. Therefore, by Theorem 3 in Section 2.14 of ([11]), since the origin is a strict local minimum of the analytic function $U(u)$, it is a center for the polynomial Moon-Rand system (4.1). $\square$

4.1.4 The Lyapunov numbers and the stability of the focus

In this section we turn our attention to the case that the center conditions (4.7) are not met. That is if

$$c_{02} \neq 0 \qquad \text{or} \qquad -\frac{1}{2}\lambda c_{11} + c_{20} \neq 0 \qquad \text{or} \qquad \text{both.} \tag{4.10}$$

We investigate the stability of the focus at the origin, with respect to $\mathfrak{X} \mid W^c$. To do this, we implement a code in Mathematica to compute the first few Lyapunov numbers by using the Poincaré first return map in the Definition 3.1.1. Then we appeal to Theorem 3.1.2 to decide whether or not the focus at the origin is stable or not.

Definition 4.1.7. When (3.26) in Theorem 3.1.2 holds for $k > 0$ the focus at the origin is called a fine focus of order k.

Theorem 4.1.8 (The stability of the focus). Consider the polynomial Moon-Rand system (4.1) with its associated vector field $\mathfrak{X}$. Suppose that system (4.1) meets conditions (4.10). For any flow induced by $\mathfrak{X}$ on any center manifold W^c of system

(4.1) at the origin of $\mathbb{R}^3$ the following hold:

1. if $2c_{20} - 2c_{02} - \lambda c_{11} \neq 0$, then $\eta_3 = -\dfrac{\pi(2c_{20} - 2c_{02} - \lambda c_{11})}{4(4+\lambda^2)}$ and the origin is the first order fine focus which is stable when $\eta_3 < 0$ and unstable when $\eta_3 > 0$;
2. if $c_{02} = -\dfrac{\lambda c_{11}}{2} + c_{20}$ and $\lambda c_{11} - 4c_{20} \neq 0$, then $\eta_5 = -\dfrac{\pi(\lambda c_{11} - 4c_{20})(\lambda c_{11} - 2c_{20})}{16\lambda(4+\lambda^2)}$ and the origin is the second order fine focus which is stable when $\eta_5 < 0$ and unstable when $\eta_5 > 0$;
3. if $c_{02} = -\dfrac{\lambda c_{11}}{2} + c_{20}$, $\lambda c_{11} - 4c_{20} = 0$ and $c_{11} \neq 0$ (to agree with our assumption), then $\eta_7 = -\dfrac{\pi\lambda(10+\lambda^2)c_{11}^3}{512(4+\lambda^2)(16+\lambda^2)}$ and the origin is the third order fine focus which is stable when $-\lambda c_{11} < 0$ and unstable when $-\lambda c_{11} > 0$.

Proof. Any center manifold at the origin for system (4.1) (whether the center conditions are met or not) can be expressed as the graph of a function

$$w = h(u, v)$$

such that $h(0,0) = 0$ (it passes through the origin) and $h_u(0,0) = h_v(0,0) = 0$ (it is tangent to the $(u, v)-$plane, which is the center eigenspace, at the origin). Since the center manifold is arbitrary smooth h can be written in the form

$$h(u, v) = h_{20}u^2 + h_{11}uv + h_{02}v^2 + \cdots$$

with as many terms as are needed for a particular computation.

We obtain an expression for $\mathfrak{X} \mid W^c_{loc}$ in local coordinates about the origin by inserting an initial segment of the series for h into $\dot{u}$ and $\dot{v}$ (since $\mathfrak{X} \mid W^c_{loc}$ is the graph of the function $h(u,v)$.) Thus

$$\mathfrak{X} \mid W^c_{loc} : \begin{aligned} \dot{u} &= v \\ \dot{v} &= -u - uh(u,v) = -u - h_{20}u^3 - h_{11}u^2v - h_{02}uv^2 + \cdots \end{aligned} \tag{4.11}$$

To find the h_{jk} (that are in terms of λ and the c_{jk}) we equate coefficient of like powers of u and v in the expression that we get by differentiating the formula for the center manifold, $w = h(u,v)$ and using the expression for $\dot{w}$ with w replaced by h, namely,

$$h_u\dot{u} + h_v\dot{v} = -\lambda h + c_{20}u^2 + c_{11}uv + c_{02}v^2.$$

We can manually compute the first few terms in h. Using just the quadratic terms in h, this is already pretty large when written out:

$$\begin{aligned} &v(2h_{20}u + h_{11}v + \cdots)+ \\ &(-u - u(h_{20}u^2 + h_{11}uv + h_{02}v^2 + \cdots))(h_{11}u + 2h_{02}v + \cdots) \\ &= -\lambda(h_{20}u^2 + h_{11}uv + h_{02}v^2 + \cdots) + c_{20}u^2 + c_{11}uv + c_{02}v^2, \end{aligned}$$

we obtain

$$\begin{aligned} h_{20} &= \frac{\lambda c_{11} + (2+\lambda^2)c_{20} + 2c_{02}}{\lambda(4+\lambda^2)}, \\ h_{11} &= \frac{\lambda c_{11} - 2c_{20} + 2c_{02}}{4+\lambda^2}, \\ h_{02} &= \frac{-\lambda c_{11} + 2c_{20} + (2+\lambda^2)c_{02}}{\lambda(4+\lambda^2)}. \end{aligned}$$

These coefficients are sufficient for finding the first few Lyapunov quantities with the approach to stability described in section 3.1 (with the direction of time reversed, hence the Lyapunov quantities computed are the negatives of the correct values.)

Using the notations in section 3.1, for system (4.11), (3.17) reads

$$\frac{dr}{d\varphi} = \frac{r\cos\varphi\sin\varphi h(r\cos\varphi, r\sin\varphi)}{1+\cos^2\varphi h(r\cos\varphi, r\sin\varphi)} = R(r,\varphi), \tag{4.12}$$

and (3.18) reads

$$\begin{aligned}\frac{dr}{d\varphi} =& R(r,\varphi) = rR_1(\varphi) + r^2R_2(\varphi) + r^3R_3(\varphi) + \cdots \\ =&(r\cos\varphi\sin\varphi h(r\cos\varphi, r\sin\varphi))[1-(\cos^2\varphi h(r\cos\varphi, r\sin\varphi)) \\ &+ (\cos^2\varphi h(r\cos\varphi, r\sin\varphi))^2 + \cdots] \\ =& r^3\cos\varphi\sin\varphi(h_{20}\cos^2\varphi + h_{11}\cos\varphi\sin\varphi + h_{02}\sin^2\varphi + \cdots)[1 \\ &- r^2\cos^2\varphi(h_{20}\cos^2\varphi + h_{11}\cos\varphi\sin\varphi + h_{02}\sin^2\varphi + \cdots) + \cdots]\end{aligned}$$

where $R_1(\varphi) = R_2(\varphi) = 0$,

$R_3(\varphi) = cos\varphi\sin\varphi(h_{20}\cos^2\varphi + h_{11}\cos\varphi\sin\varphi + h_{02}\sin^2\varphi), \text{and} R_j(\varphi)$ for $3 < j < 9$ (that are material) will be computed using Mathematica. With initial conditions (3.22), the first three equations in (3.21) give

1. $w_1' = R_1(\varphi)w_1 = 0,\ w_1(0) = 1$ yields $w_1(\varphi) = 1,$
2. $w_2' = R_1(\varphi)w_2 + R_2(\varphi)w_1^2 = 0,\ w_2(0) = 0$ yields $w_2(\varphi) = 0, \text{and}$
3. $w_3' = R_1(\varphi)w_3 + 2R_2(\varphi)w_1w_2 + R_3(\varphi)w_1^3 = R_3(\varphi),\ w_3(0) = 0$ yields
$$w_3(\varphi) = \frac{1-\cos^4\varphi}{4}h_{20} + \frac{4\varphi - \sin 4\varphi}{32}h_{11} + \frac{\sin^4\varphi}{4}h_{02}.$$

Therefore, by Definition 3.1.1, $\eta_1 = \eta_2 = 0$ and

$\eta_3 = -w_3(2\pi) = -\frac{\pi}{4}h_{11} = -\frac{\pi(\lambda c_{11} - 2c_{20} + 2c_{02})}{4(4+\lambda^2)}$. This together with Theorem 3.1.2 establish part 1. Proceeding on the assumption that $\eta_3 = 0$ allows us to eliminate a parameter ($c_{02} = -\frac{\lambda}{2}c_{11} + c_{20},$) thereby simplifying the computations. At this point hand computations are too long to perform. We implement a code in Mathematica (see Table 4.4) that computes the next few terms in h, R_4 and R_5, and η_4 and η_5

(whose results are displayed below.)

$\eta_4 = 0$ and $\eta_5 = -\frac{\pi(\lambda c_{11} - 4c_{20})(\lambda c_{11} - 2c_{20})}{16\lambda(4+\lambda^2)}$.

Next, if $\eta_5 = 0$ then $\lambda c_{11} - 4c_{20} = 0$ or $\lambda c_{11} - 2c_{20} = 0$. We must exclude the last condition ($\lambda c_{11} - 2c_{20} = 0$) since then the origin is a center by Theorem 4.1.6. Therefore, we proceed on the assumption that $\lambda c_{11} - 2c_{20} \neq 0$ and that $\eta_3 = 0$ by setting $c_{20} = \frac{\lambda}{4} c_{11}$ with $c_{11} \neq 0$ (otherwise it would imply $c_{02} = 0$). The Mathematica code in Table 4.4 gives $\eta_6 = 0$ and $\eta_7 = -\frac{\pi\lambda(10+\lambda^2)c_{11}^3}{512(4+\lambda^2)(16+\lambda^2)}$ that has the same sign as $-\lambda c_{11}$. These together with Theorem 3.1.2 establish parts 2 and 3. $\square$

Remark 4.1.9. If we want to get $\eta_1 = \eta_3 = \eta_5 = \eta_7 = 0$, we must have $c_{20} = c_{11} = c_{02} = 0$. Then the origin is a center for the flow restricted to W^c_{loc}. This means for system (4.1), if the origin is a focus, it can only be of at most third order.

In this next section we will investigate the possible number of limit cycles (isolated periodic orbits) occurring in the phase portrait of system (4.1) in a neighborhood of the origin.

4.1.5 Bifurcation of the Limit Cycles

We consider the problem of the cyclicity of a simple singularity of system (4.1) (that is, one at which the determinant of the linear part is nonzero.)

Let Ξ be a subset of $\mathbb{R}^3$ and let $\mathfrak{P} : \mathbb{R} \times \Xi \to \mathbb{R} : (r, \xi) \mapsto \mathfrak{P}(r, \xi)$ be an analytic function, which represents the difference function, and which we will write in a neighborhood of the origin of $r = 0$ in the form

$$\mathfrak{P}(r, \xi) = \Re(r, \xi) - r = \eta_1(\xi) r + \eta_2(\xi) r^2 + \eta_3(\xi) r^3 + \cdots . \tag{4.13}$$

Definition 4.1.10. We consider a family of systems (4.1) with coefficients drawn from the parameter space Ξ equipped with a topology. A limit periodic set is a point set Γ in the phase portrait of the system (4.1) that corresponds to some choice ξ_0 of the parameters that has the property that a limit cycle can be made to bifurcate from Γ under a suitable but arbitrary small change in the parameters. That is, for any neighborhood M of Γ in $\mathbb{R}^3$ and any neighborhood N of ξ_0 in Ξ there exists $\xi_1 \in N$ such that the system corresponding to parameter choice ξ_1 has a limit cycle lying wholly within M. The limit periodic Γ has cyclicity c with respect to Ξ if and only if for any choice ξ of parameters in the neighborhood of ξ_0 in Ξ the corresponding system (4.1) has at most c limit cycles wholly contained in a neighborhood of Γ, and c is the smallest number with this property ([12]).

Examples of limit periodic sets under consideration are singularities of focus or center type. It is worth to recall that zeros of (4.13) regarded as an equation in r alone, correspond to cycles (closed orbits, that is, orbits that are ovals) of system (4.1); isolated zeros correspond to limit cycles (isolated closed orbits).

We now treat the cyclicity of the polynomial Moon-Rand system restricted to W^c_{loc}. Even though W^c_{loc} is not unique, existence of a topological conjugacy between the flows on any two center manifolds insures that the cyclicity is well-defined.

Theorem 4.1.11. Consider the polynomial Moon-Rand system (4.1) with its associated vector field $\mathfrak{X}$.

1. If the origin is a third order focus, then the origin on the local center manifold has cyclicity two: the origin can be made to bifurcate two, but no more than two, limit cycles on a neighborhood of the origin in W^c_{loc}.

2. If the origin is a second order focus, then the origin on the local center manifold has cyclicity one: the origin can be made to bifurcate one, but at most one limit cycle on a neighborhood of the origin in W^c_{loc}.

3. If the origin is a first order focus, then the origin on the local center manifold has cyclicity zero on a neighborhood of the origin in W^c_{loc}.

4. If the origin is a center, then the origin on the local center manifold on a neighborhood of the origin in W^c_{loc} can be made to bifurcate two limit cycles if

$c_{20} = c_{11} = c_{02} = 0$ and one limit cycle if $c_{20}, c_{11} \neq 0$.

Proof. 1. If the origin is a third order focus, then (4.13) reads

$$\begin{aligned}\mathfrak{P}(r,\xi) &= \eta_1(\xi)r + \eta_3(\xi)r^3 + \eta_5(\xi)r^5 + \eta_7(\xi)r^7 + o(r,\xi)\\ &= Z(r,\xi) + \eta_7(\xi)r^7 + o(r,\xi),\end{aligned} \tag{4.14}$$

on $\{(r,\xi) : |r| < \epsilon \text{ and } |\xi - \xi^*| < \delta\}$, for some positive real numbers δ and ϵ, and $\eta_j(0,\xi^*) = 0$ for $j = 1,3,5$, and $\eta_7(0,\xi^*) \neq 0$ and $o(0,\xi) = 0$. To finish the proof we will imitate, but in a much simpler way, the proof of Proposition 6.1.2 of ([12]).

Let δ_1 and ϵ_1 be such that $0 < \delta_1 < \delta$ and $0 < \epsilon_1 < \epsilon$ and $| \eta_j(r,\xi) | < 1$ if $|r| \leq \epsilon_1$ and $|\xi - \xi^*| \leq \delta_1$ for $j = 1,3,5$. let $B(\xi^*, \delta_1)$ denote the closed ball in $\mathbb{R}^3$ of radius δ_1 centered at ξ^*. For each $j \in \{1,3,5\}, \eta_j$ is not the zero function, else the corresponding term is not present in (4.14). We begin by defining the set V_0 by $V_0 := \{\xi \in B(\xi^*,\delta_1) : \eta_j(\xi) = 0 \text{ for all } j = 1,3,5\}$, a closed, proper subset of $B(\xi^*,\delta_1)$. For $\xi_0 \in V_0$, as a function of $r, Z(r,\xi_0)$ vanishes identically on $(0,\epsilon_0)$, so the origin is indeed a second order focus since $\eta_7(0,\xi_0) \neq 0$.

For any $\xi_0 \in B(\xi^*,\delta_1) \setminus V_0$, let $l \in \{1,3,5\}$ be the least index for which $\eta_l(\xi_0) \neq 0$. Then $Z(r,\xi_0) = \eta_l(\xi_0)r^{k_l} + r^{k_l+1}v(r,\xi_0)$, where $v(r,\xi_0)$ is a real analytic function on $[-\epsilon_1,\epsilon_1]$. Thus the k_l^{th} derivative of $Z(r,\xi_0)$ is nonzero at $r = 0$, so $Z(r,\xi_0)$ is not identically zero, hence has a finite number $S_0(\xi_0)$ of zeros in $(0,\epsilon_1)$. Since, for all values of (c_{20}, c_{11}, c_{02}) in Ξ, and for all admissible values of $\lambda, \eta_1 = 0$, then $l \in \{3,5\}$.

Let $V_1 = \{\xi \in B(\xi^*,\delta_1) : \eta_5(\xi) = 0\}; V_1 \supset V_0$. For $\xi_0 \in V_1$, if $\eta_3(\xi_0) = 0$, then $Z(r,\xi_0)$ vanishes identically on $(0,\epsilon_1)$; if $\eta_3(\xi_0) \neq 0$, then, as a function of $r, Z(r,\xi_0) = \eta_3(\xi_0)r^3(1 + v(r,\xi_0))$ has no zeros in $(0,\epsilon_1)$.

For $\xi \in B(\xi^*,\delta_1) \setminus V_1$, we divide $Z(r,\xi)$ by $r^3(1+\eta_3(\xi))$ to form a real analytic function $\widetilde{Z}^{(1)}(r,\xi)$ of r on $[-\epsilon_1,\epsilon_1]$ that can be written in the form $\widetilde{Z}^{(1)}(r,\xi) = \frac{\eta_3(\xi)}{1+\eta_3(\xi)} + \frac{\eta_5(\xi)r^2}{1+\eta_3(\xi)}$. Then we differentiate with respect to r to obtain a real analytic function $Z^{(1)}(r,\xi)$ of r on $[-\epsilon_1,\epsilon_1]$ that can be written in the form $Z^{(1)}(r,\xi) = 2\frac{\eta_5(\xi)r}{1+\eta_3(\xi)}$. As a function of $r, \widetilde{Z}^{(1)}(r,\xi)$ has the same number $S_0(\xi)$ of zeros in $(0,\epsilon_1)$ as does $Z(r,\xi)$. As a function of $r, Z^{(1)}(r,\xi)$ is not identically zero, and moreover has no zeros in $(0,\epsilon_1)$. By Rolle's Theorem, $\widetilde{Z}^{(1)}(r,\xi)$ has at most one more zero in $(0,\epsilon_1)$ than does $Z^{(1)}(r,\xi)$, so $S_0(\xi) \leq 1$.

Let $V_2 = \{\xi \in B(\xi^*,\delta_1) : \eta_3(\xi) = 0\}; V_2 \supset V_1 \supset V_0$. For $\xi_0 \in V_2$, if $\eta_5(\xi_0) = 0$, then $Z(r,\xi_0)$ vanishes identically on $(0,\epsilon_1)$; if $\eta_5(\xi_0) \neq 0$, then, as a function of $r, Z(r,\xi_0) = \eta_5(\xi_0)r^5$ has no zeros in $(0,\epsilon_1)$.

For $\xi \in B(\xi^*,\delta_1) \setminus V_2$, we divide $Z(r,\xi)$ by $r^3(1+\eta_3(\xi))$ to form a real analytic function $\widetilde{Z}^{(2)}(r,\xi)$ of r on $[-\epsilon_1,\epsilon_1]$ that can be written in the form $\widetilde{Z}^{(2)}(r,\xi) = \frac{\eta_3(\xi)}{1+\eta_3(\xi)} + \frac{\eta_5(\xi)r^2}{1+\eta_3(\xi)}$. Then we differentiate with respect to r to obtain a real analytic function $Z^{(2)}(r,\xi)$ of r on $[-\epsilon_1,\epsilon_1]$ that can be written in the form $Z^{(2)}(r,\xi) = 2\frac{\eta_5(\xi)r}{1+\eta_3(\xi)}$. As a function of $r, \widetilde{Z}^{(2)}(r,\xi)$ has the same number $S_0(\xi)$ of zeros in $(0,\epsilon_1)$ as does $\widetilde{Z}^{(1)}(r,\xi)$. As a function of $r, Z^{(2)}(r,\xi)$ is not identically zero, and moreover has no zeros in $(0,\epsilon_1)$. By Rolle's Theorem, $\widetilde{Z}^{(2)}(r,\xi)$ has at most one more zero in $(0,\epsilon_1)$ than does $Z^{(2)}(r,\xi)$, so $S_0(\xi) \leq 2$. Therefore the upper bound in point 1 holds for all $\xi \in B(\xi^*,\delta_1)$.

Similarly, if the origin is a second order focus, then the origin on the local center manifold can be made to bifurcate at most one limit cycle on a neighborhood of the origin in W^c_{loc}.

3. For all values of (c_{20}, c_{11}, c_{02}) in Ξ, and for all admissible values of $\lambda, \eta_1(\xi) = 0$. Therefore, we will not be able to create any isolated zeros with a change in the parameters if the origin is a first order focus. This proves 3.

4. If $c_{20} = c_{11} = c_{02} = 0$, then we will be able to perturb the system to create a third order fine focus near the origin. Thus statement (1) applies. If $c_{20}, c_{11} \neq 0$, then there is no third order fine focus near the center at the origin. We will then be able to perturb the system to create a second order fine focus near the origin. Thus statement (2) applies. This proves 4.

The bounds in points 1 and 2 are sharp because the Lyapunov numbers can be adjusted independently. (The bounds in point 4 might not be sharp because of possible lack of analyticity of the center manifold under perturbation.) □

4.2 The Generalized Rational Moon-Rand System

In this section we analyze the generalized rational Moon-Rand family of systems which are three dimensional systems of ordinary differential equations of the form,

$$\begin{aligned}\dot{u} &= v\\ \dot{v} &= -u - uw\\ \dot{w} &= -\lambda w + \frac{c_{20}u^2 + c_{11}uv + c_{02}v^2}{1+\eta u^2},\end{aligned} \tag{4.15}$$

where $\lambda, c_{20}, c_{11}, c_{02}$ and η are real parameters with $\eta \neq 0$.

We first perform a time rescaling by $1+\eta u^2$ on system (4.15) to obtain

$$\begin{aligned}\dot{u} &= (1+\eta u^2)v\\ \dot{v} &= (1+\eta u^2)(-u-uw)\\ \dot{w} &= -\lambda(1+\eta u^2)w + c_{20}u^2 + c_{11}uv + c_{02}v^2\end{aligned}$$

or

$$\begin{aligned}\dot{u} &= v + \eta u^2 v\\ \dot{v} &= -u - uw - \eta u^3 - \eta u^3 w\\ \dot{w} &= -\lambda w + c_{20}u^2 + c_{11}uv + c_{02}v^2 - \lambda\eta u^2 w\end{aligned} \tag{4.16}$$

that has the same structures in consideration as (4.15).

Theorem 4.2.1. Consider the generalized rational Moon-Rand system (4.15). Let $\mathfrak{X}$ denote the associated vector field to system (4.16) obtained by performing a time rescaling by $1+\eta u^2$ on system (4.15). Then for the flow induced by $\mathfrak{X}$ on the neighborhood of the origin of $\mathbb{R}^3$ in any center manifold W^c of (4.16), the origin is a center if and only if $c_{02}+\lambda\eta = c_{20}-\frac{\lambda}{2}c_{11}+\lambda\eta = 0$ or $c_{20}=c_{11}=c_{02}=0$.

Proof. As for the Polynomial Moon-Rand system in section 4.1, we complexify system (4.16) and hand computations give

$$\begin{aligned}\dot{x} =& i[x-(-\frac{1}{2}xz-\frac{1}{2}yz-\frac{\eta}{4}x^3-\frac{\eta}{2}x^2y-\frac{\eta}{4}xy^2\\&-\frac{\eta}{8}x^3z-\frac{3\eta}{8}x^2yz-\frac{3\eta}{8}xy^2z-\frac{\eta}{8}y^3z)]\\\dot{y} =& -i[y-(-\frac{1}{2}xz-\frac{1}{2}yz-\frac{\eta}{4}x^2y-\frac{\eta}{2}xy^2-\frac{\eta}{4}y^3\\&-\frac{\eta}{8}x^3z-\frac{3\eta}{8}x^2yz-\frac{3\eta}{8}xy^2z-\frac{\eta}{8}y^3z)]\\\dot{z} =& [-\lambda z-(-\frac{1}{4}(c_{20}-c_{02}+ic_{11})x^2-\frac{1}{2}(c_{20}+c_{02})xy\\&-\frac{1}{4}(c_{20}-c_{02}-ic_{11})y^2)+\frac{\lambda\eta}{4}x^2z+\frac{\lambda\eta}{2}xyz+\frac{\lambda\eta}{4}y^2z].\end{aligned} \tag{4.17}$$

Therefore, the index sets S and T are given by

$$\begin{aligned}S =& \{(0,0,1),(-1,1,1),(2,0,0),(1,1,0),(0,2,0),(2,0,1),(1,1,1),\\&(0,2,1),(-1,3,1)\}\\T =& \{(2,0,-1),(1,1,-1),(0,2,-1),(2,0,0),(1,1,0),(0,2,0)\},\end{aligned}$$

and the corresponding nonzero coefficients are

$a_{0,0,1}=a_{-1,1,1}=b_{1,-1,1}=b_{0,0,1}=-\frac{1}{2}$,

$a_{2,0,0}=a_{0,2,0}=b_{2,0,0}=b_{0,2,0}=-\frac{\eta}{4}$,

$a_{1,1,0}=b_{1,1,0}=-\frac{\eta}{2}$,

$a_{2,0,1}=a_{-1,3,1}=b_{3,-1,1}=b_{0,2,1}=-\frac{\eta}{8}$,

$a_{1,1,1}=a_{0,2,1}=b_{1,1,1}=b_{2,0,1}=-\frac{3\eta}{8}$,

$c_{2,0,-1}=-\frac{1}{4}(c_{20}-c_{02}+ic_{11})$,

$c_{1,1,-1}=-\frac{1}{2}(c_{20}+c_{02})$,

$c_{0,2,-1}=-\frac{1}{4}(c_{20}-c_{02}-ic_{11})$,

$c_{2,0,0}=c_{0,2,0}=\frac{\lambda\eta}{4}$,

$c_{1,1,0} = \frac{\lambda\eta}{2}$.

The first five nonzero focus quantities $g_{1,1,0}, \cdots, g_{5,5,0}$ in the coefficients c_{20}, c_{02}, c_{11}, with parameters $\lambda, \eta, \neq 0$, are computed using the algorithm in Table 4.1 with the associated Mathematica code in Table 4.5. We only show below the results for the first two since the others have too many terms to display here.

$$\begin{aligned}
g_{1,1,0} =& \frac{2c_{20} - \lambda c_{11} - 2c_{02}}{2(4+\lambda^2)} \\
g_{2,2,0} =& \frac{(c_{20}+c_{02})}{8\lambda(4+\lambda^2)^2}[2(12+\lambda^2)c_{20} - \lambda(12+\lambda^2)c_{11} + 2(-4+\lambda^2)c_{02} + \\
& 4\lambda\eta(4+\lambda^2)]
\end{aligned}$$

As in the proof of Theorem 4.1.6, we use Singular to decompose the radicals of $\mathcal{B}_K, 1 \leq j \leq 5$ into a unique intersection of prime ideals. We replace for simplicity $c_{20}, c_{11}, c_{02}, \lambda$ and η by x, y, z, L and M respectively. The code (with only parts of $\widetilde{g}_{3,3,0}, \widetilde{g}_{4,4,0}$ and $\widetilde{g}_{5,5,0}$ displayed) for carrying out the decompositions and the outputs are displayed in Table 4.5. We also obtain the prime decomposition of the variety $V(\mathcal{B}_5)$ as the union of five varieties $V(J_i), 1 \leq i \leq 5$ which are:

$$\begin{aligned}
J_1 =& \langle \lambda\eta + c_{02}, c_{20} - \frac{\lambda}{2}c_{11} - c_{02} \rangle \\
J_2 =& \langle \lambda^2 + 4, c_{20} - \frac{\lambda}{2}c_{11} - c_{02} \rangle \\
J_3 =& \langle 9\lambda^2 + 4, -9\lambda^{19}\eta + \cdots + 162502848c_{02}, 9\lambda^{18}\eta + \cdots + 260025846\eta, \\
& c_{20} - \frac{\lambda}{2}c_{11} - c_{02} \rangle \\
J_4 =& \langle \lambda^2 + 1, -4\lambda c_{02} + c_{11}, c_{20} - \frac{\lambda}{2}c_{11} - c_{02} \rangle \\
J_5 =& \langle c_{02}, c_{11}, c_{20} - \frac{\lambda}{2}c_{11} - c_{02} \rangle
\end{aligned}$$

Notice that setting the generators of either J_2, J_3 or J_4 to zero yields complex numbers, which contradicts the fact that system (4.15) in question is real. Therefore, the necessary conditions are given by setting the polynomial generators of J_1 or J_5 to zero. We get

$$c_{02} + \lambda\eta = c_{20} - \frac{\lambda}{2}c_{11} + \lambda\eta = 0 \text{ or } \quad c_{20} = c_{11} = c_{02} = 0 \tag{4.18}$$

for the origin to be a center for $\mathfrak{X} \mid W^c_{loc}$.

Conversely, when conditions (4.18) hold, we search for polynomials F (which could have any degree) and K (of degree at most 3) satisfying

$$\mathfrak{X}(F) = KF. \tag{4.19}$$

So let $F(u,v,w) = F_{300}u^3 + F_{210}u^2v + F_{201}u^2w + F_{120}uv^2 + \cdots$ (this time we were unsuccessful to find a quadratic F that satisfies (4.19)) and $K(u,v,w) = K_{300}u^3 + K_{210}u^2v + K_{201}u^2w + K_{120}uv^2 + \cdots$.

Using the Mathematica code in Table 4.6 (where, \[Lambda] and \[Eta] represent λ and η respectively), we were able to construct the polynomials $F(u,v,w)$ and $K(u,v,w)$. In the process, we first obtained a system of eighty four equations with forty variables that we were unable to solve with Mathematica. Then we manually solved for a few equations by successively setting

$$\begin{aligned}
&F_{000} = 0,\\
&F_{010} = F_{100} = 0; K_{000} = -\lambda,\\
&F_{003} = F_{012} = F_{102} = F_{111} = F_{120} = 0 \quad \text{and}\\
&F_{021} = F_{030} = 0.
\end{aligned}$$

This reduced the system of equations to sixty-three equations with fewer unknown coefficient variables that we were able to solve with the Mathematica code in Table 4.6. The output yields a material solution which is the following.

$\{F_{300} \to 0, F_{210} \to 0, F_{200} \to \frac{(2\eta - c_{11})F_{201}}{2\eta}, F_{110} \to 0, F_{101} \to 0,$

$F_{020} \to F_{201}, F_{011} \to 0, F_{002} \to 0, F_{001} \to \frac{F_{201}}{\eta}, K_{100} \to 0, K_{010} \to 0,$

$K_{001} \to 0, K_{200} \to -\eta\lambda, K_{110} \to 0, K_{101} \to 0, K_{020} \to 0, K_{011} \to 0,$

$K_{002} \to 0, K_{300} \to 0, K_{210} \to 0, K_{201} \to 0, K_{120} \to 0, K_{111} \to 0,$

$K_{102} \to 0, K_{030} \to 0, K_{021} \to 0, K_{012} \to 0, K_{003} \to 0\}.$

For convenience, we choose $F_{201} = 2\eta$, and obtain

$$\begin{aligned} F(u,v,w) &= 2\eta u^2 w + (2\eta - c_{11})u^2 + 2\eta v^2 + 2w, \quad \text{and} \\ K(u,v,w) &= -\eta\lambda u^2 - \lambda \end{aligned}$$

that satisfy conditions (4.19). We have found an invariant algebraic surface

$$F(u,v,w) = 2\eta u^2 w + (2\eta - c_{11})u^2 + 2\eta v^2 + 2w = 0 \tag{4.20}$$

that is tangent to the $(u,v)-$plane, hence it must be the center manifold. Solving (4.20) for w and inserting this expression into the first two equations in (4.16) gives $\mathfrak{X} \mid W^c_{loc}$ in local coordinates about the origin:

$$\begin{aligned} \dot{u} &= v + \eta u^2 v \\ \dot{v} &= -u - \frac{c_{11}}{2}u^3 + \eta u v^2. \end{aligned} \tag{4.21}$$

If we set

$$\dot{u} = \widetilde{U}(u,v) = v + \eta u^2 v \quad \text{and} \quad \dot{v} = \widetilde{V}(u,v) = -u - \frac{c_{11}}{2}u^3 + \eta u v^2$$

then,

$$\widetilde{U}(u,-v) = -\widetilde{U}(u,v) \quad \text{and} \quad \widetilde{V}(u,-v) = \widetilde{V}(u,v),$$

so system (4.21) possesses a time-reversible symmetry with respect to the $u-$axis. Therefore, the origin is a center by Theorem 3.5.5. in ([12]). $\square$

Theorem 4.2.2. Consider the generalized rational Moon-Rand system (4.16) with its associated vector field $\mathfrak{X}$. Suppose that system (4.16) does not meet conditions (4.18). For any flow induced by $\mathfrak{X}$ on any center manifold W^c of system (4.16) at the origin of $\mathbb{R}^3$ the following hold:

1. If $\eta_3 = \dfrac{\pi(2c_{20} - \lambda c_{11} - 2c_{02})}{4(4+\lambda^2)} \neq 0$ then the origin is a first order fine focus which is asymptotically stable if and only if $2c_{20} - \lambda c_{11} - 2c_{02} < 0$ (or unstable iff $2c_{20} - \lambda c_{11} - 2c_{02} > 0$.)
2. If $\eta_3 = 0$ and $\eta_5 = \dfrac{\pi(\lambda c_{11} - 4c_{20})(2c_{20} - \lambda c_{11} + 2\lambda\eta)}{16\lambda(4+\lambda^2)} \neq 0$ [that is if $c_{02} = -\dfrac{\lambda c_{11}}{2} + c_{20} \neq -\lambda\eta$ and $c_{20} \neq \dfrac{\lambda c_{11}}{4}$] then the origin is a second order fine focus whose stability is determined by the sign of $\lambda(\lambda c_{11} - 4c_{20})(2c_{20} - \lambda c_{11} + 2\lambda\eta)$.
3. If $\eta_3 = \eta_5 = 0$ and
$$\eta_7 = \frac{\pi\lambda c_{11}(4\eta - c_{11})(64\eta + 4\eta\lambda^2 + 10c_{11} + \lambda^2 c_{11})}{512(4+\lambda^2)(16+\lambda^2)} \neq 0$$
[that is if $c_{02} = -\dfrac{\lambda c_{11}}{2} + c_{20} \neq -\lambda\eta$, $c_{20} = \dfrac{\lambda c_{11}}{4} \neq \lambda\eta$ $(c_{11} \neq 0)$ and $c_{11} \neq -\dfrac{64\eta + 4\eta\lambda^2}{10+\lambda^2}$] then the origin is a third order fine focus whose stability is determined by the sign of
$$\lambda c_{11}(4\eta - c_{11})(64\eta + 4\eta\lambda^2 + 10c_{11} + \lambda^2 c_{11}).$$
4. If $\eta_3 = \eta_5 = \eta_7 = 0$ [that is if $c_{02} = -\dfrac{\lambda c_{11}}{2} + c_{20} \neq -\lambda\eta$, $c_{20} = \dfrac{\lambda c_{11}}{4} \neq \lambda\eta$ $(c_{11} \neq 0)$ and $c_{11} = -\dfrac{64\eta + 4\eta\lambda^2}{10+\lambda^2}$] then $\eta_9 = \dfrac{5\pi\eta^4\lambda(13+\lambda^2)(16+\lambda^2)}{8(10+\lambda^2)^4} \neq 0$, and the origin is a fourth order fine focus whose stability is determined by the sign of λ.

Proof. As we did in Theorem 4.1.8 we compute the Lyapunov quantities for

$\mathfrak{X} \mid W^c_{loc}$ using the same ideas in the proof of the polynomial Moon-Rand system, for the generalized rational Moon-Rand system. Using the same notations we obtained $\eta_1 = \eta_{2k} = 0$ for $1 \leq k \leq 4$, and first

$$\eta_3 = \frac{\pi(2c_{20} - \lambda c_{11} - 2c_{02})}{4(4+\lambda^2)}$$

which has the same sign as $2c_{20} - \lambda c_{11} - 2c_{02}$, this proves 1.

Setting $\eta_3 = 0$ yields $c_{02} = -\frac{\lambda c_{11}}{2} + c_{20} \neq -\lambda\eta$, where the inequality is necessary for the system not to meet conditions (4.18). With the substitution of c_{02}, we obtain

$$\eta_5 = \frac{\pi(\lambda c_{11} - 4c_{20})(2c_{20} - \lambda c_{11} + 2\lambda\eta)}{16\lambda(4+\lambda^2)}$$

which has the same sign as $\lambda(\lambda c_{11} - 4c_{20})(2c_{20} - \lambda c_{11} + 2\lambda\eta)$. The assumption $-\frac{\lambda c_{11}}{2} + c_{20} \neq -\lambda\eta$ implies $2c_{20} - \lambda c_{11} + 2\lambda\eta \neq 0$ hence $\eta_5 \neq 0$ if and only if $c_{20} \neq \frac{\lambda c_{11}}{4}$, this proves 2.

Next setting $\eta_3 = \eta_5 = 0$ yields [$c_{02} = -\frac{\lambda c_{11}}{2} + c_{20} \neq -\lambda\eta,\ \ c_{20} = \frac{\lambda c_{11}}{4} \neq \lambda\eta\ \ (c_{11} \neq 0)$], where the last two inequalities are necessary for the system not to meet conditions (4.18). Substituting $c_{20} = \frac{\lambda c_{11}}{4}$, gives

$$\eta_7 = \frac{\pi\lambda c_{11}(4\eta - c_{11})(64\eta + 4\eta\lambda^2 + 10c_{11} + \lambda^2 c_{11})}{512(4+\lambda^2)(16+\lambda^2)}$$

which has the same sign as $\lambda c_{11}(4\eta - c_{11})(64\eta + 4\eta\lambda^2 + 10c_{11} + \lambda^2 c_{11})$. The assumption $c_{20} = \frac{\lambda c_{11}}{4} \neq \lambda\eta\ \ (c_{11} \neq 0)$ implies $\eta_7 \neq 0$ if and only if $64\eta + 4\eta\lambda^2 + 10c_{11} + \lambda^2 c_{11} \neq 0$ (or $c_{11} \neq -\frac{64\eta + 4\eta\lambda^2}{10+\lambda^2}$), this proves 3.

Finally, we substitute $c_{11} = -\frac{64\eta + 4\eta\lambda^2}{10+\lambda^2}$ so for $\eta_3 = \eta_5 = \eta_7 = 0$ and obtain

$$\eta_9 = \frac{5\pi\eta^4\lambda(13+\lambda^2)(16+\lambda^2)}{8(10+\lambda^2)^4}$$

that is a nonzero number that has the same sign as λ, this proves 4. $\square$

Theorem 4.2.3. Consider the generalized rational Moon-Rand system (4.16) with its associated vector field $\mathfrak{X}$.

1. If the origin for $2 \leq k \leq 4$ is a kth order fine focus, then the origin on the local center manifold has cyclicity k : the origin can be made to bifurcate $k-1$ limit cycles on a neighborhood of the origin in W^c_{loc}.

2. If the origin is a first order focus, then the origin on the local center manifold has cyclicity zero: no limit cycles bifurcate from the origin on a neighborhood of the origin in W^c_{loc}.

3. If the origin is a center, then the origin on the local center manifold on a neighborhood of the origin in W^c_{loc}.

can be made to bifurcate:

a. one limit cycle if $c_{02} = -\frac{\lambda}{2}c_{11} + c_{20} = -\lambda\eta$ and $\lambda c_{11} - 4c_{20} \neq 0$.

b. two limit cycles if $c_{02} = -\frac{\lambda}{2}c_{11} + c_{20} = -\lambda\eta$ and $\lambda c_{11} - 4c_{20} = 0$.

c. two limit cycles if $c_{20} = c_{11} = c_{02} = 0$.

Proof. The proof is similar to the one of Theorem 4.1.11 for the polynomial Moon-Rand system since the computation of the Lyapunov quantities give analogous results, so we use the same technique to create zeros that correspond to limit cycles under small perturbation of the parameter system $(c_{20}, c_{11}, c_{02}, \lambda, \eta)$ within system (4.16). $\square$

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APPENDIX: TABLES

Table 1: The Algorithm for computing $g_{1,1,0}, g_{2,2,0}, \cdots, g_{k,k,0}$.

The Algorithm for computing ν_{k_1,k_2,k_3} and $g_{k,k,0}$.

Input:
$S = \{(p,q,r)|(p,q,r) \in S\}$
$T = \{(p_1,q_1,r_1)|(p_1,q_1,r_1) \in T\}$
Stage = 2k.

Output:
$\nu_{k_1,k_2,k_3}(k_1, k_2 \in \{-1\} \cup \mathbb{N}_0 \text{ and } k_3 \in \mathbb{N}_0), k_1 + k_2 + k_3 \leq 2k$
$g_{1,1,0}, g_{2,2,0}, \cdots, g_{k,k,0}$.

Procedure:
g:=Array[g,{stage,stage,1},{1,1,0}]

ν:=Array[ν,{stage+3,stage+3,stage+3},{-1,-1,0}]

a:=Array[a,{stage+3,stage+3,stage+3},{-1,-1,0}]

b:=Array[b,{stage+3,stage+3,stage+3},{-1,-1,0}]

c:=Array[c,{stage+3,stage+3,stage+3},{0,0,-1}]

For i=-1 To stage+3 Do
For j=-1 To stage+3 Do
For k=-1 To stage+3 Do
ν [i,j,k]=0
If$(i,j,k) \in S$ Then $a[i,j,k] = a_{i,j,k}; b[j,i,k] = b_{j,i,k}$ Otherwise $a[i,j,k] = b[j,i,k] = 0$
If $(i,j,k) \in T$ Then $c[i,j,k] = c_{i,j,k}$ Otherwise $c[i,j,k] = b[i,j,k]$

ν[0,0,0]=1

For i=1 To stage Do
For $k_1 = -1$ To i+1 Do
For $k_2 = -1$ To i-k_1 Do
$k_3 = i-(k_1 - k_2)$
Compute A=the first sum in (3.28)
Compute B=the second sum in (3.28)
Compute A=the third sum in (3.28)

If $k_3 = k_2 - k_1 = 0$ Then Compute $g_{k_1,k_1,0}$ using (3.29); Set $\nu_{k_1,k_1,0} = 0$
Otherwise Compute ν_{k_1,k_2,k_3} by solving for it in (3.28)

Table 2: The Mathematica code for Example 3.2.2

```
S = {(1, 0, 0), (0, 0, 1), (−1, 1, 1), (−1, 0, 2)}; T = {(2, 0, −1), (0, 0, 1)};
Stage=2;
g:=Array[g,{stage,stage,1},{1,1,0}];
ν:=Array[ν,{stage+3,stage+3,stage+3},{-1,-1,0}];
a:=Array[a,{stage+3,stage+3,stage+3},{-1,-1,0}];
b:=Array[b,{stage+3,stage+3,stage+3},{-1,-1,0}];
c:=Array[c,{stage+3,stage+3,stage+3},{0,0,-1}];
For[i = −1, i ≤ stage + 3, i + +,
For[j = −1, j ≤ stage + 3, j + +,
For[k = −1, k ≤ stage + 3, k + +,
ν [i,j,k]=0;
If[MemberQ[S,{i,j,k}], a[i,j,k]=Subscript[a,i,j,k];b[j,i,k]=Subscript[b,j,i,k],
a[i,j,k]=0;b[j,i,k]=0;
If[MemberQ[T,{i,j,k}], c[i,j,k]=Subscript[c,i,j,k],c[i,j,k]=0]]]];
ν [0,0,0]=1;
For[i = 1, i ≤ stage, i + +,
For[k1 = −1, k1 ≤ i + 1, k1 + +,
For[k2 = −1, k2 ≤ i − k1, k2 + +,
k3 = i − (k1 + k2); A = 0; B = 0; Z = 0;
For[j = 1, j ≤ k1 + 2, j + +,
For[k = 0, k ≤ k2 + 1, k + +,
For[n = 0, n ≤ k3, n + +,
If[2 ≤ j + k + n ≤ k1 + k2 + k3 + 1,
A = A + j ⋆ a[k1 − j + 1, k2 − k + 1, k3 − n] ⋆ ν[j − 1, k − 1, n]]]]];
For[j = 0, j ≤ k1 + 1, j + +,
For[k = 1, k ≤ k2 + 2, k + +,
For[n = 0, n ≤ k3, n + +,
If[2 ≤ j + k + n ≤ k1 + k2 + k3 + 1,
B = B + k ⋆ b[k1 − j + 1, k2 − k + 1, k3 − n] ⋆ ν[j − 1, k − 1, n]]]]];
For[j = 0, j ≤ k1 + 1, j + +,
For[k = 0, k ≤ k2 + 1, k + +,
For[n = 1, n ≤ k3 + 1, n + +,
If[2 ≤ j + k + n ≤ k1 + k2 + k3 + 1,
Z = Z + n ⋆ c[k1 − j + 1, k2 − k + 1, k3 − n] ⋆ ν[j − 1, k − 1, n]]]]];
If k_3 == k_2 − k_1 == 0, g_{k1,k1,0} = i ⋆ a[k1, k2, k3] + i ⋆ b[k1, k2, k3] − i ⋆ A + i ⋆ B − Z;
ν[k1, k2, k3] = 0,
ν[k1, k2, k3] = (1/((− ⋆ λ) ⋆ (k3) + (k1 − k2) ⋆ i)) ⋆ (i ⋆ a[k1, k2, k3] − i ⋆ b[k1, k2, k3]
+i ⋆ A − i ⋆ B + Z)]; ]]];
For[i = 1, i ≤ stage, i + +,
For[k1 = −1, k1 ≤ i + 1, k1 + +,
For[k2 = −1, k2 ≤ i − k1, k2 + +,
k3 = i − (k1 + k2); A = 0; B = 0; Z = 0;
Print[Subscript[ν, k1, k2, k3], " = ", Simplify[ν[k1, k2, k3]]]; ]]];
For[i = 1, i ≤ stage/2, i + +,
Print[Subscript[g, i, i, 0], " = ", Simplify[g[i, i, 0]]]];
```

Table 3: $g_{1,1,0}, \cdots, g_{5,5,0}$. for Polynomial Moon-Rand system

$S = \{(0,0,1),(-1,1,1)\}$; $T = \{(2,0,-1),(1,1,-1),(0,2,-1)\}$;
Stage=10;
g:=Array[g,{stage,stage,1},{1,1,0}];
ν:=Array[ν,{stage+3,stage+3,stage+3},{-1,-1,0}];
a:=Array[a,{stage+3,stage+3,stage+3},{-1,-1,0}];
b:=Array[b,{stage+3,stage+3,stage+3},{-1,-1,0}];
c:=Array[c,{stage+3,stage+3,stage+3},{0,0,-1}];
a[0,0,1]=a[-1,1,1]=b[0,0,1]=b[1,-1,1]=$-\frac{1}{2}$;
c[2,0,-1]=$-\frac{1}{4}(c_{20}-c_{02}+ic_{11})$;
c[1,1-1]=$-\frac{1}{2}(c_{20}+c_{02})$;
c[0,2,-1]=$-\frac{1}{4}(c_{20}-c_{02}-ic_{11})$;
ν [0,0,0]=1;
$\mathrm{For}[i=1, i\le \mathrm{stage}, i++,$
$\mathrm{For}[k1=-1, k1\le i+1, k1++,$
$\mathrm{For}[k2=-1, k2\le i-k1, k2++,$
$k3=i-(k1+k2); A=0; B=0; Z=0;$
$\mathrm{For}[j=1, j\le k1+2, j++,$
$\mathrm{For}[k=0, k\le k2+1, k++,$
$\mathrm{For}[n=0, n\le k3, n++,$
$\mathrm{If}[2\le j+k+n\le k1+k2+k3+1,$
$A=A+j\star a[k1-j+1, k2-k+1, k3-n]\star\nu[j-1, k-1, n]]]]];$
$\mathrm{For}[j=0, j\le k1+1, j++,$
$\mathrm{For}[k=1, k\le k2+2, k++,$
$\mathrm{For}[n=0, n\le k3, n++,$
$\mathrm{If}[2\le j+k+n\le k1+k2+k3+1,$
$B=B+k\star b[k1-j+1, k2-k+1, k3-n]\star\nu[j-1, k-1, n]]]]];$
$\mathrm{For}[j=0, j\le k1+1, j++,$
$\mathrm{For}[k=0, k\le k2+1, k++,$
$\mathrm{For}[n=1, n\le k3+1, n++,$
$\mathrm{If}[2\le j+k+n\le k1+k2+k3+1,$
$Z=Z+n\star c[k1-j+1, k2-k+1, k3-n]\star\nu[j-1, k-1, n]]]]];$
$\mathrm{If}\ k_3==k_2-k_1==0, g_{k_1,k_1,0}=i\star a[k1,k2,k3]+i\star b[k1,k2,k3]-i\star A+i\star B-Z;$
$\nu[k1,k2,k3]=0,$
$\nu[k1,k2,k3]=(1/((-\star\lambda)\star(k3)+(k1-k2)\star i))\star(i\star a[k1,k2,k3]-i\star b[k1,k2,k3]$
$+i\star A-i\star B+Z)];]]];$
$\mathrm{For}[i=1, i\le \mathrm{stage}, i++,$
$\mathrm{For}[k1=-1, k1\le i+1, k1++,$
$\mathrm{For}[k2=-1, k2\le i-k1, k2++,$
$k3=i-(k1+k2); A=0; B=0; Z=0;$
$\mathrm{For}[i=1, i\le \mathrm{stage}/2, i++,$
$\mathrm{Print}[\mathrm{Subscript}[g,i,i,0], "=", \mathrm{Simplify}[g[i,i,0]]]]];$

Table 4: Poincaré-Lyapunov numbers for the Polynomial Moon-Rand system

```
Print["POINCARE − LYAPUNOV NUMBERS / STABILITY OF THE FOCI"];
Clear[u]; Clear[v]; Clear[s]; Clear[h]; Clear[λ]; Clear[W]; Clear[R];
Remove["Global`*"];
(*Defining the center manifold
w = h(u, v) = h20 u^2 + h11 uv + h02 v^2 + h40 u^4 + h31 u^3 v + ···*)
h[u_, v_] := (Subscript[h, 20]) * u^2 + (Subscript[h, 11]) * u * v+
(Subscript[h, "02"]) * v^2 + (Subscript[h, 40]) * u^4+
(Subscript[h, 31]) * (u^3) * v + (Subscript[h, 22]) * (u^2) * (v^2)+
(Subscript[h, 13]) * u * (v^3) + (Subscript[h, "04"]) * v^4+
(Subscript[h, 60]) * u^6 + (Subscript[h, 51]) * (u^5) * v+
(Subscript[h, 42]) * (u^4) * (v^2) + (Subscript[h, 33]) * (u^3) * (v^3)+
(Subscript[h, 24]) * (u^2) * (v^4) + (Subscript[h, 15]) * u * (v^5)+
(Subscript[h, "06"]) * v^6;
(*Defining the zero function
s(u, v) = h_u u̇ + h_v v̇ + λh(u, v) − c20 u^2 − c11 uv − c02 v^2*)
s[u_, v_] = D[h[u, v], u] * v + D[h[u, v], v] * (−u − u * h[u, v]) + λ * h[u, v]−
(Subscript[c, 20]) * u^2 − (Subscript[c, 11]) * u * v − (Subscript[c, "02"]) * v^2;
(*Solving for h20, h11, h02, and assigning results*)
sols = Solve [ Coefficient[s[u, 0], u^2] == 0, Coefficient[s[u, v], u * v] == 0,
Coefficient[s[0, v], v^2] == 0, Subscript[h, 20], Subscript[h, 11], Subscript[h, "02"]; ]
Subscript[h, 20] = Simplify[Subscript[h, 20]/.sols[[1]]];
Subscript[h, 11] = Simplify[Subscript[h, 11]/.sols[[1]]];
Subscript[h, "02"] = Simplify[Subscript[h, "02"]/.sols[[1]]];
Print["h20", "=", Subscript[h, 20]];
Print["h11", "=", Subscript[h, 11]];
Print["h02", "=", Subscript[h, "02"]];
(*Defining dr/dφ = r cos[φ] sin[φ] h(r cos[φ], r sin[φ]) / (1 − cos^2[φ] h(r cos[φ], r sin[φ])) =
r cos[φ] sin[φ] h(r cos[φ], r sin[φ]) [1 − cos^2[φ] h(r cos[φ], r sin[φ]) + ···]*)
R[r_, φ_] := (r * cos[φ] * sin[φ] * h[r * cos[φ], r * sin[φ]]) * (1 + (−(cos[φ])^2)*)
h[r * cos[φ], r * sin[φ]] + (−((cos[φ])^2) * h[r * cos[φ], r * sin[φ]])^2+
(−((cos[φ])^2) * h[r * cos[φ], r * sin[φ]])^3;)
Print[Subscript[R, 1], "=", Coefficient[R[r, φ], r]];
Print[Subscript[R, 2], "=", Coefficient[R[r, φ], r^2]];
Print[Subscript[R, 3], "=", Coefficient[R[r, φ], r^3]];
Print[Subscript[R, 4], "=", Coefficient[R[r, φ], r^4]];
Print[Subscript[R, 5], "=", Coefficient[R[r, φ], r^5]];
Print[Subscript[R, 6], "=", Coefficient[R[r, φ], r^6]];
Print[Subscript[R, 7], "=", Coefficient[R[r, φ], r^7]];
Print[Subscript[R, 8], "=", Coefficient[R[r, φ], r^8]];
(*Computing w3(φ) by integrating w3'(φ) = R3(φ) and
using the initial condition w3(0) = 0. η3 = w3(2π) = N3(2π). *)
w3[φ_] = Integrate[Coefficient[R[r, φ], r^3], φ];
N3[φ_] = w3[φ] − w3[0];
Print["η3 =", Simplify[N3[2 * Pi]]];
(*Conditions for η3 = 0*)
Print["With", Subscript[c, "02"], "=", Subscript[c, 20] − λ * Subscript[c, 11]/2];
Subscript[c, "02"] = Subscript[c, 20] − λ * Subscript[c, 11]/2;
```

Table 4.4 : (continued)

```
(*Solving for h_40, h_31, h_22, h_13, h_04, and assigning corresponding results.*)
sols4 = Solve [{Coefficient[s[u, 0], u^4] == 0,
Coefficient[s[u, v], (u^3) * v] == 0,
Coefficient[s[u, v], (u^2) * (v^2)] == 0,
Coefficient[s[u, v], u * (v^3)] == 0,
Coefficient[s[0, v], v^4] == 0 },
{Subscript[h, 40], Subscript[h, 31], Subscript[h, 22], Subscript[h, 13],
Subscript[h, "04"]}];
Subscript[h, 40] = Simplify[Subscript[h, 40]/.sols4[[1]]];
Subscript[h, 31] = Simplify[Subscript[h, 31]/.sols4[[1]]];
Subscript[h, 22] = Simplify[Subscript[h, 22]/.sols4[[1]]];
Subscript[h, 13] = Simplify[Subscript[h, 13]/.sols4[[1]]];
Subscript[h, "04"] = Simplify[Subscript[h, "04"]/.sols4[[1]]];
(*Computing w_5(φ) by integrating w'_5(φ) = 3R_3(φ)w_3(φ) + R_5(φ) and
using the initial condition w_5(0) = 0. η_5 = w_5(2π) = N5(2π). *)
w_5[φ_] = Integrate [ 3 * Coefficient[R[r, φ], r^3] * w_3[φ] + Coefficient[R[r, φ], r^5], φ];
N5[φ_] = w_5[φ] − w_5[0];
Print["η_3 =", Simplify[N3[2 * Pi]], " andη_5 =", Factor[Simplify[N5[2 * Pi]]]];
(*Conditions for η_5 = 0*)
Print["With", Subscript[c, 20], " =", λ * Subscript[c, 11]/4];
Subscript[c, 20] = λ * Subscript[c, 11]/4;
(*Solving for h_60, h_51, h_42, h_33, h_24, h_15, h_06, and assigning corresponding results .*)
sols6 = Solve [{ Coefficient[s[u, 0], u^6] == 0,
Coefficient[s[u, v], (u^5) * v] == 0,
Coefficient[s[u, v], (u^4) * (v^2)] == 0,
Coefficient[s[u, v], (u^3) * (v^3)] == 0,
Coefficient[s[u, v], (u^2) * (v^4)] == 0,
Coefficient[s[u, v], u * (v^5)] == 0,
Coefficient[s[0, v], v^6] == 0 },
{ Subscript[h, 60], Subscript[h, 51], Subscript[h, 42], Subscript[h, 33],
Subscript[h, 24], Subscript[h, 15], Subscript[h, 06]}];
Subscript[h, 60] = Simplify[Subscript[h, 60]/.sols6[[1]]];
Subscript[h, 51] = Simplify[Subscript[h, 51]/.sols6[[1]]];
Subscript[h, 42] = Simplify[Subscript[h, 42]/.sols6[[1]]];
Subscript[h, 33] = Simplify[Subscript[h, 33]/.sols6[[1]]];
Subscript[h, 24] = Simplify[Subscript[h, 24]/.sols6[[1]]];
Subscript[h, 15] = Simplify[Subscript[h, 15]/.sols6[[1]]];
Subscript[h, "06"] = Simplify[Subscript[h, "06"]/.sols6[[1]]];
(*Computing η_7(φ) by integrating
w'_7(φ) = 3R_3(φ)(w_3^2(φ) + w_5(φ)) + 5R_5(φ)w_3(φ) + R_7(φ) and
using the intial condition w_7(0) = 0. η_7 = w_7(2π) = N7(2π). *)
w_7[φ_] = Simplify [ Integrate [ Coefficient[R[r, φ], r^3] * ((w_3[φ])^2 + w_5[φ])+
5 * Coefficient[R[r, φ], r^5] * w_3[φ] + Coefficient[R[r, φ], r^7], φ ]];
N7[φ_] = w_7[φ] − w_7[0];
Print["η_3 = η_5 =", Simplify[N5[2 * Pi]], " andη_7 =", Factor[Simplify[N7[2 * Pi]]]];
```

Table 5: Prime decomposition of $\mathcal{B}$ for the Generalized Rational Moon-Rand

```
> LIB "primdec.lib";
> ring r=0,(x,y,z,L,M),dp;
> poly g1=2*x - L*y - 2*z;
> poly g2=-((x + z)*(24*x + 4*L*(4*M - 3*y) +
           L^3*(4*M - y) - 8*z + 2*L^2*(x + z)));
> poly g3=16*L^13*M^2*y + 196608*x*(x + z)^2 +...
          +y*(-501*x^2 - 138*y^2 + 1222*x*z + 1171*z^2));
> poly g4=-1152*L^24*M^3*y -...
          -x*(76*y^2 + 45*z^2) - 61*(2*y^2*z + z^3)));
> poly g5=31352832*L^41*M^4*y - ...
         -85878606571852*y^2*z^2 + 60687397590549*z^4)));
> ideal j1=g1;
> ideal j2=g1,g2;
> ideal j3=g1,g2,g3;
> ideal j4=g1,g2,g3,g4;
> ideal j5=g1,g2,g3,g4,g5;
> minAssGTZ(j1);
[1]:
   _[1]=-1/2yL+x-z
> minAssGTZ(j2);
[1]:
   _[1]=LM+z
   _[2]=-1/2yL+x-z
[2]:
   _[1]=L2+4
   _[2]=-1/2yL+x-z
[3]:
   _[1]=yL+4z
   _[2]=-1/2yL+x-z
> minAssGTZ(j3);
[1]:
   _[1]=LM+z
   _[2]=-1/2yL+x-z
[2]:
   _[1]=L2+4
   _[2]=-1/2yL+x-z
[3]:
   _[1]=-L3M+zL2-16LM+10z
   _[2]=2L2M-2zL+5y+32M
   _[3]=-1/2yL+x-z
[4]:
   _[1]=L2+1
   _[2]=-4zL+y
   _[3]=-1/2yL+x-z
```

Table 4.5 : (continued)

```
[5]:
   _[1]=z
   _[2]=y
   _[3]=-1/2yL+x-z

> minAssGTZ(j4);
[1]:
   _[1]=LM+z
   _[2]=-1/2yL+x-z
[2]:
   _[1]=L2+4
   _[2]=-1/2yL+x-z
[3]:
   _[1]=9L2+4
   _[2]=-9L19M-265L17M-3149L15M-20887L13M-81782L11M-
         214324L9M-211640L7M-1245136L5M+9628192L3M-
         260025856LM+162502848z
   _[3]=9L18M+265L16M+3149L14M+20887L12M+81782L10M+
        214324L8M+211640L6M+1245136L4M-9628192L2M+
        40625712y+260025856M
   _[4]=-1/2yL+x-z
[4]:
   _[1]=L2+1
   _[2]=-4zL+y
   _[3]=-1/2yL+x-z
[5]:
   _[1]=z
   _[2]=y
   _[3]=-1/2yL+x-z
> minAssGTZ(j5);
[1]:
   _[1]=LM+z
   _[2]=-1/2yL+x-z
[2]:
   _[1]=L2+4
   _[2]=-1/2yL+x-z
```

Table 4.5 : (continued)

```
[3]:
   _[1]=9L2+4
   _[2]=-9L19M-265L17M-3149L15M-20887L13M-81782L11M-
         214324L9M-211640L7M-1245136L5M+9628192L3M-
         260025856LM+162502848z
   _[3]=9L18M+265L16M+3149L14M+20887L12M+81782L10M+
        214324L8M+211640L6M+1245136L4M-9628192L2M+
        40625712y+260025856M
   _[4]=-1/2yL+x-z
[4]:
   _[1]=L2+1
   _[2]=-4zL+y
   _[3]=-1/2yL+x-z
[5]:
   _[1]=z
   _[2]=y
   _[3]=-1/2yL+x-z
>
```

Table 6: Coefficients of F for the Generalized Rational Moon-Rand

```
Clear[\[Lambda]]; Clear[j]; Clear[k];
F[u_, v_, w_] := Subscript[F, 300]*u^3 +
   Subscript[F, 210]*(u^2)*v +
   Subscript[F, 201]*(u^2)*w + Subscript[F, 120]*u*(v^2) +
   Subscript[F, 111]*u*v*w + Subscript[F, 102]*u*(w^2) +
   Subscript[F, "030"]*v^3 + Subscript[F, "021"]*(v^2)*w +
   Subscript[F, "012"]*v*(w^2) + Subscript[F, "003"]*w^3 +
   Subscript[F, 200]*u^2 + Subscript[F, 110]*u*v +
   Subscript[F, 101]*u*w + Subscript[F, "020"]*v^2 +
   Subscript[F, "011"]*v*w + Subscript[F, "002"]*w^2 +
   Subscript[F, 100]*u + Subscript[F, "010"]*v +
   Subscript[F, "001"]*w + Subscript[F, "000"];

K[u_, v_, w_] := Subscript[K, "000"] + Subscript[K, 100]*u +
   Subscript[K, "010"]*v + Subscript[K, "001"]*w +
   Subscript[K, 200]*u^2 + Subscript[K, 110]*u*v +
   Subscript[K, 101]*u*w + Subscript[K, "020"]*v^2 +
   Subscript[K, "011"]*v*w + Subscript[K, "002"]*w^2 +
   Subscript[K, 300]*u^3 + Subscript[K, 210]*(u^2)*v +
   Subscript[K, 201]*(u^2)*w + Subscript[K, 120]*u*(v^2) +
   Subscript[K, 111]*u*v*w + Subscript[K, 102]*u*(w^2) +
   Subscript[K, "030"]*v^3 + Subscript[K, "021"]*(v^2)*w +
   Subscript[K, "012"]*v*(w^2) + Subscript[K, "003"]*w^3 ;

Subscript[C, "02"] = -\[Lambda]*\[Eta];
Subscript[C, 20] = (\[Lambda]/2)*Subscript[C, 11] - \[Lambda]*\[Eta];

F1[u_, v_, w_] = Expand[D[F[u, v, w], u]*v*(1 + \[Eta]*u^2) +
    D[F[u, v, w], v]*(-u - u*w)*(1 + \[Eta]*u^2) +
    D[F[u, v, w], w]*(-\[Lambda]*w*(1 + \[Eta]*u^2) +
       Subscript[C, 20]*(u^2) + Subscript[C, 11]*u*v +
       Subscript[C, "02"]*(v^2))];

F2[u_, v_, w_] := Expand[K[u, v, w]*F[u, v, w]];

F3[u_, v_, w_] := Expand[F2[u, v, w] - F1[u, v, w]];

Subscript[F, "000"] = Subscript[F, "010"] = Subscript[F, 100] = 0;
Subscript[K, "000"] = -\[Lambda];
Subscript[F, "003"] = Subscript[F, "012"] = Subscript[F, 102] =
    Subscript[F, 111] = Subscript[F, 120] = 0;
Subscript[F, "021"] = Subscript[F, "030"] = 0;
```

Table 4.6 : (continued)

```
sols = Solve[{
   Coefficient[F3[u, 0, 0], u^2] == 0,
   Coefficient[F3[u, v, 0], u*v] == 0,
   Coefficient[F3[u, 0, w], u*w] == 0,
   Coefficient[F3[0, v, 0], v*v] == 0,
   Coefficient[F3[0, v, w], v*w] == 0,
   Coefficient[F3[0, 0, w], w^2] == 0,
   Coefficient[F3[u, 0, 0], u^3] == 0,
   Coefficient[F3[u, v, 0], (u^2)*v] == 0,
   Coefficient[F3[u, 0, w], (u^2)*w] == 0,
   Coefficient[F3[u, v, 0], u*(v^2)] == 0,
   Coefficient[F3[u, v, w], u*v*w] == 0,
   Coefficient[F3[u, 0, w], u*(w^2)] == 0,
   Coefficient[F3[0, v, 0], v^3] == 0,
   Coefficient[F3[0, v, w], (v^2)*w] == 0,
   Coefficient[F3[0, v, w], v*(w^2)] == 0,
   Coefficient[F3[0, 0, w], w^3] == 0,
   Coefficient[F3[u, 0, 0], u^4] == 0,
   Coefficient[F3[u, v, 0], u^3*v] == 0,
   Coefficient[F3[u, 0, w], u^3*w] == 0,
   Coefficient[F3[u, v, 0], u^2*v^2] == 0,
   Coefficient[F3[u, v, w], u^2*v*w] == 0,
   Coefficient[F3[u, 0, w], u^2*w^2] == 0,
   Coefficient[F3[u, v, 0], u*v^3] == 0,
   Coefficient[F3[u, v, w], u*(v^2)*w] == 0,
   Coefficient[F3[u, v, w], u*v*(w^2)] == 0,
   Coefficient[F3[u, 0, w], u*(w^3)] == 0,
   Coefficient[F3[0, v, 0], v^4] == 0,
   Coefficient[F3[0, v, w], (v^3)*w] == 0,
   Coefficient[F3[0, v, w], (v^2)*(w^2)] == 0,
   Coefficient[F3[0, v, w], v*(w^3)] == 0,
   Coefficient[F3[0, 0, w], w^4] == 0,
   Coefficient[F3[u, 0, 0], u^5] == 0,
   Coefficient[F3[u, v, 0], u^4*v] == 0,
   Coefficient[F3[u, 0, w], u^4*w] == 0,
   Coefficient[F3[u, v, 0], u^3*v^2] == 0,
   Coefficient[F3[u, v, w], u^3*v*w] == 0,
   Coefficient[F3[u, 0, w], u^3*w^2] == 0,
   Coefficient[F3[u, v, 0], u^2*v^3] == 0,
   Coefficient[F3[u, v, w], u^2*v^2*w] == 0,
   Coefficient[F3[u, v, w], u^2*v*w^2] == 0,
   Coefficient[F3[u, 0, w], u^2*w^3] == 0,
   Coefficient[F3[u, v, 0], u*v^4] == 0,
   Coefficient[F3[u, v, w], u*v^3*w] == 0,
   Coefficient[F3[u, v, w], u*v^2*w^2] == 0,
```

Table 4.6 : (continued)

```
  Coefficient[F3[u, v, w], u*v*w^3] == 0,
Coefficient[F3[u, 0, w], u*w^4] == 0,
   Coefficient[F3[0, v, 0], v^5] == 0,
   Coefficient[F3[0, v, w], (v^4)*w] == 0,
   Coefficient[F3[0, v, w], (v^3)*(w^2)] == 0,
   Coefficient[F3[0, v, w], (v^2)*(w^3)] == 0,
   Coefficient[F3[0, v, w], v*(w^4)] == 0,
   Coefficient[F3[0, 0, w], w^5] == 0,
Coefficient[F3[u, 0, 0], u^6] == 0,
  Coefficient[F3[u, v, 0], u^5*v] == 0,
  Coefficient[F3[u, 0, w], u^5*w] == 0,
  Coefficient[F3[u, v, 0], u^4*v^2] == 0,
  Coefficient[F3[u, v, w], u^4*v*w] == 0,
  Coefficient[F3[u, 0, w], u^4*w^2] == 0,
  Coefficient[F3[u, v, 0], u^3*v^3] == 0,
  Coefficient[F3[u, v, w], u^3*v^2*w] == 0,
  Coefficient[F3[u, v, w], u^3*v*w^2] == 0,
  Coefficient[F3[u, 0, w], u^3*w^3] == 0,
  Coefficient[F3[u, v, 0], u^2*v^4] == 0,
  Coefficient[F3[u, v, w], u^2*v^3*w] == 0,
  Coefficient[F3[u, v, w], u^2*v^2*w^2] == 0,
  Coefficient[F3[u, v, w], u^2*v*w^3] == 0,
  Coefficient[F3[u, 0, w], u^2*w^4] == 0},
  {Subscript[F, 300],Subscript[F, 210], Subscript[F, 201],
  Subscript[F, 200],Subscript[F, 110], Subscript[F, 101],
  Subscript[F, "020"],Subscript[F, "011"],Subscript[F, "002"],
  Subscript[F, "001"],Subscript[K, 100], Subscript[K, "010"],
  Subscript[K, "001"],Subscript[K, 200], Subscript[K, 110],
  Subscript[K, 101],Subscript[K, "020"], Subscript[K, "011"],
  Subscript[K, "002"],Subscript[K, 300], Subscript[K, 210],
  Subscript[K, 201],Subscript[K, 120], Subscript[K, 111],
  Subscript[K, 102],Subscript[K, "030"], Subscript[K, "021"],
  Subscript[K, "012"], Subscript[K, "003"] }]
```