BRAID INDICES IN A CLASS OF CLOSED BRAIDS

by

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ABSTRACT

KENNETH EDWARD HINSON. Braid Indices in a Class of Closed Braids.
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A long-standing problem in knot theory concerns the additivity of crossing numbers of links under the connected sum operation. It is conjectured that if L_1 and L_2 are links, then $Cr(L_1 \# L_2) = Cr(L_1) + Cr(L_2)$, but so far this has been proved only for certain classes of links. For example, in cases where both L_1 and L_2 are alternating or adequate links, the conjecture is known to be true. Another situation in which $Cr(L_1 \# L_2) = Cr(L_1) + Cr(L_2)$ is when both L_1 and L_2 are zero-deficiency links. Zero-deficiency links include some but not all of the links in the prior named classes, as well as some links that are not included in either of those. In addition, further results are known for situations in which only one of the links being connected has deficiency zero. In this paper we expand the known realm of zero-deficiency links to include some cases of links represented by alternating closed braids. The ultimate goal is to show that if D_k is any k-string, reduced, alternating, closed braid, then the braid index of D_k is k. Herein we show the result for a certain subset of these closed braids, those with at most two sequences of crossings between consecutive strings in the braid. This result is proved using a property of the HOMFLY polynomial, which provides a lower bound for the braid index of a link. In the process, a simplified formulation for computing the HOMFLY polynomial is implemented. It seems likely that this result can be extended to prove the result for more complex alternating closed braids.

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CHAPTER 1: INTRODUCTION TO KNOT THEORY

1.1 Basics of Knots and Links

The mathematical object known as a knot is almost what anyone would expect. Given a length of string, nearly everyone is familiar with the idea of tying a knot in the middle of the string. One could tie a relatively simple knot, or a very complicated one. Now to create a mathematical knot, the loose ends of the string would be connected together, to form a loop including the knot. Adding the fact that the string in a mathematical knot would typically be considered to have thickness zero, the concept can be formally defined as follows:

Definition 1.1. [1, 6] A *knot* K is a closed curve in $\mathbb{R}^3$ that is homeomorphic to a circle.

The simplest knot is a topological circle, and is sometimes called a trivial knot or the *unknot*. An infinite variety of non-trivial knots are possible, as one would probably guess from an exercise of tying progressively more complicated knots with a long string. However, the above definition is actually too general for most practical purposes, for it leaves open the possibility of a knot having a limit point where it is not differentiable (See [6], p.24 for an example). A knot with such a limit point is called *wild*, and a knot that is not wild is *tame*. Although most knots are wild [19], we are not concerned with those here — All knots considered in this paper will be assumed to be tame.

Expanding upon the idea of a knot, one can imagine several knotted loops of string, possibly linked together. Such a collection is called a link.

Definition 1.2. [1, 6] A link L is a finite disjoint union of knots. If K_1, K_2, ..., K_n are mutually non-intersecting knots, then $L = \bigcup_{i=1}^{n} K_i$ is an n-component link and the knots K_i are the components of L.

In particular, n can equal 1 so every knot is a link. In this paper, the term 'link' will be used to refer to both knots and links, unless only knots are intended. A link consisting of n trivial knots, none of which are linked together, is called an n-component trivial link. It is also an example of a *split link*.

Definition 1.3. [1,6] If L is a link and there exists a topological 2-sphere in $\mathbb{R}^3 \setminus L$ such that some components of a link are entirely on one side of the sphere and other components are entirely on the other side, then the link is a *split* (or *splittable*) link. If L_1 and L_2 represent the sets of components that lie on either side of the 2-sphere, then $L = L_1 \sqcup L_2$ indicates that L_1 and L_2 are the split components of L. A link that is not split is *connected*.

Links are often represented pictorially in two dimensions by a link diagram. The most important information contained in a link diagram involves the crossings, the points where the link crosses over or under itself. At a crossing in a diagram, the strand that lies on top is drawn uninterrupted, while the lower strand is drawn with a break where the upper strand passes over it. See figure 1.1 for some link diagrams. (An understanding of the notations 5_2^* and 4_1^2 is not crucial; 5_2 and 4_1^2 are simply names from standard knot tables, and 5_2^* is the mirror image of 5_2.)

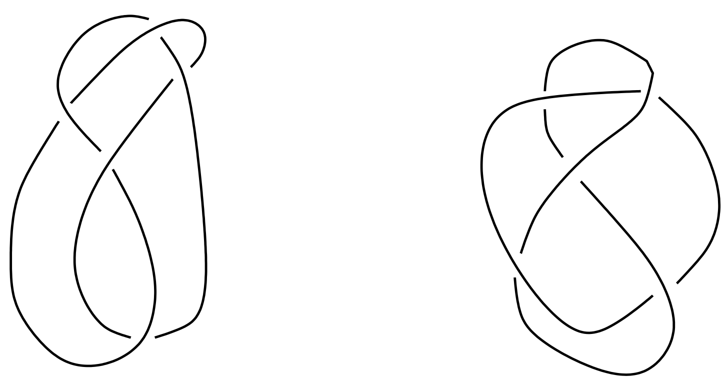

Figure 1.1: Diagrams of the knot 5_2^* (left) and the 2-component link 4_1^2 (right). Taken as a single diagram, these represent the split link $5_2^* \sqcup 4_1^2$.

It is important to distinguish between a link and its diagram. There are many different diagrams that can represent a given link, and some properties may have widely differing values in two diagrams of the same link. One could think of the underlying link type as an equivalence class, and its various diagrams as representing specific instances of the equivalence class. In this case the equivalence relation would be ambient isotopy [6]. If two links are ambient isotopic, that is if a link L_1 can be deformed without being cut or passing through itself to form another link L_2, then L_1 and L_2 are considered to be equivalent – They are two different representations of the same link type.

The act of deforming a link into an equivalent link can be simplified to three basic moves [24]. These are known as the Reidemeister moves, and are illustrated in Figure 1.2. A Reidemeister Type I move involves a single strand in the diagram. If there is no crossing in a section of the strand, then it can be twisted to create a small loop with a crossing; or if the diagram already has such a loop, it can be untwisted to remove the crossing. A Reidemeister Type II move involves two strands that lie alongside each other. If there is no crossing between them, then the strands can be moved together so that one lies over the other, creating two new crossings; or if there are two such crossings already, the strands can be moved apart to remove the crossings. A Reidemeister Type III move involves three strands that cross each other in a small area of the diagram. If one strand lies entirely above or entirely below the other two, then it can be moved across to the other side of the crossing between the other two strands.

The Reidemeister moves are widely used in knot theory due to the following result, proved by K. Reidemeister in 1926.

Theorem 1.1. *[24] Two links L_1 and L_2 are equivalent if and only if L_1 can be transformed into L_2 by performing some finite sequence of Reidemeister moves.*

Knots and links can be assigned an *orientation*, a direction to travel along the string. A diagram of an oriented link is drawn with arrows indicating the orientation. Figure 1.5 shows an example of an oriented knot diagram.

Definition 1.4. [1, 6] A crossing in an oriented link is said to be *positive* if, as the upper strand (following in its direction of orientation) crosses over the lower strand, the lower strand is oriented toward the left. If instead the lower strand is oriented toward the right, then the crossing is said to be *negative*.

Figure 1.3 illustrates positive and negative crossings between oriented strands. In a knot, the sign of a crossing is not dependent on the orientation, since a reversal of orientation affects the entire diagram. But if a link has two or more components, then there is a choice

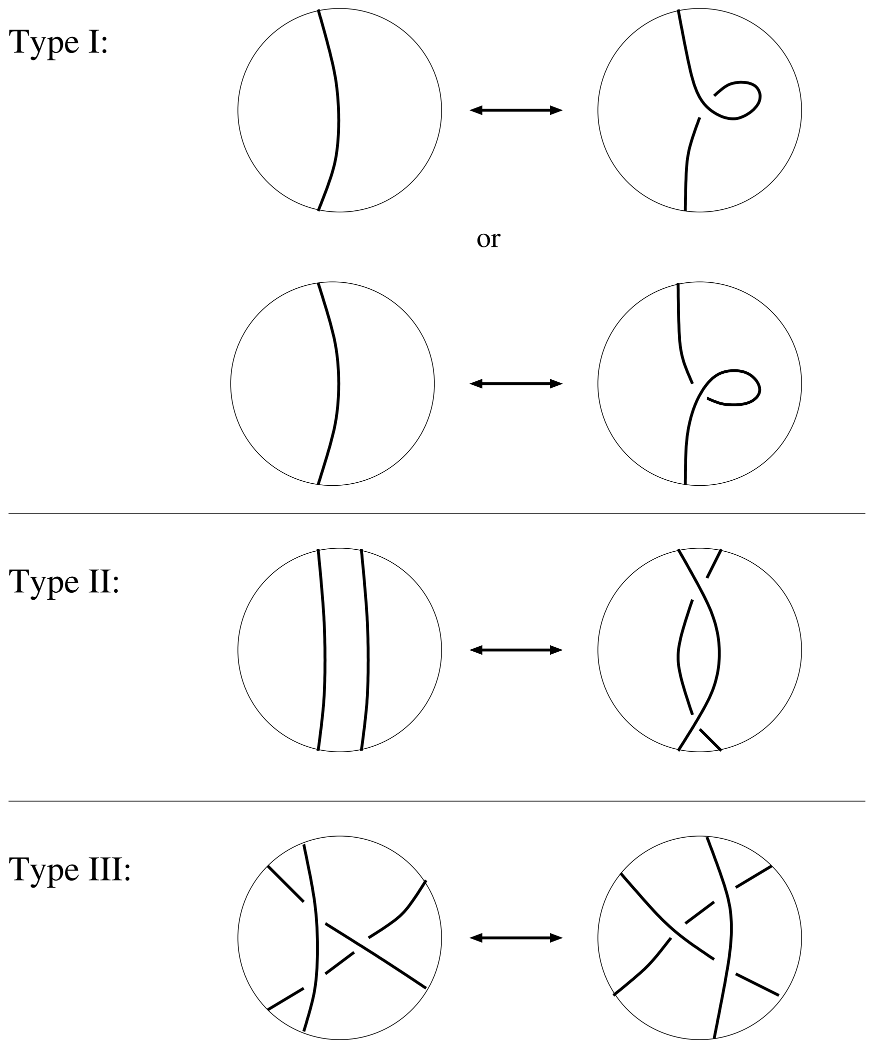

Figure 1.2: Reidemeister moves

of orientation for each component. Thus changing the orientation of one component may change the sign of some crossings involving that component.

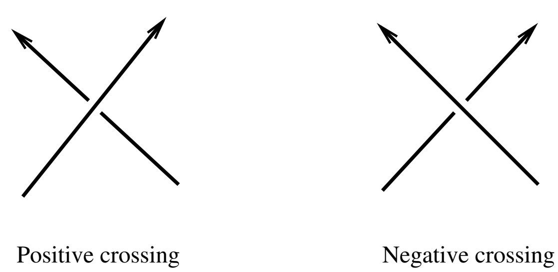

Figure 1.3: Positive and negative crossings

Finally, two links can be connected together to form a new composite link. The most common way of doing this is by performing a *connected sum.* A small arc that is not involved in any crossings is removed from each link, and the loose ends are then connected, as shown in Figure 1.4. The notation for a connected sum of links L_1 and L_2 is $L_1 \# L_2$. The connected sum operation is not well-defined [6], for $L_1 \# L_2$ could potentially be any of a variety of links. Perhaps the easiest way to see this is to suppose that L_1 is a split link. Then in forming $L_1 \# L_2$ there is a choice of which split component of L_1 to connect to L_2. The choices will most likely result in new links that are not equivalent (unless all of the split components of L_1 are equivalent to each other).

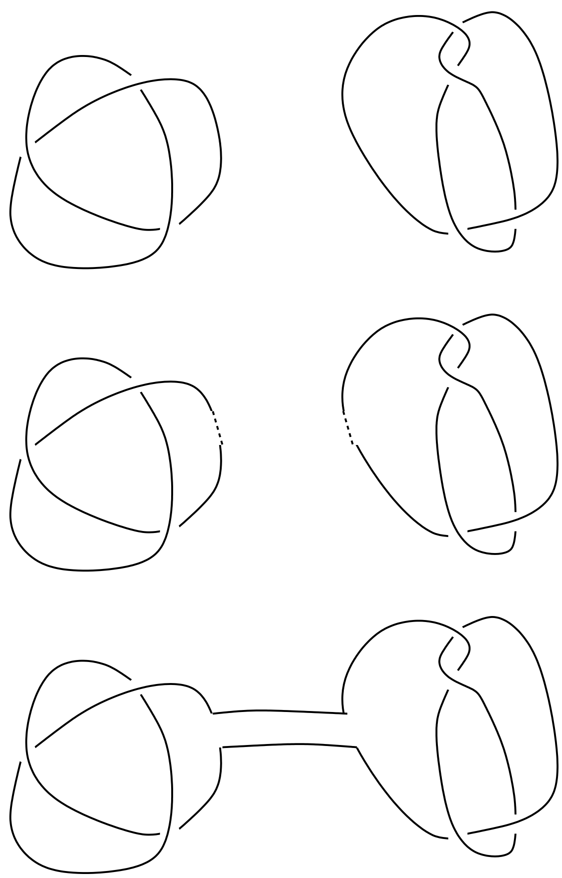

Figure 1.4: Creating a connected sum of two knots

1.2 Some Properties and Invariants of Links

A link has many properties that can be measured. Some, as noted above, can vary depending on the particular diagram being used. However, there are certain properties that remain the same regardless of the particular representation of the link.

Definition 1.5. [1, 6] A *link invariant* is a property of a link that has the same value regardless of which diagram of the link is being considered.

Invariants can be useful in distinguishing between links, and can sometimes provide other important insights. A very simple invariant is the number of components in a link [6]. Checking the Reidemeister moves, we see that each one moves the strands within a small area of the diagram concerned, but none would result in a change in the number of components in the link. Therefore if two links are equivalent, they must have the same number of components. The number of components in a link L is denoted by $\mu(L)$.

Definition 1.6. [6] Let D be a diagram representing an oriented link L. Let C be the set of crossings in D. For each crossing c, let $\varepsilon(c) = 1$ if c is a positive crossing, and $\varepsilon(c) = -1$ if c is a negative crossing. Then the sum $\sum_{c \in C} \varepsilon(c)$ is defined as the *writhe* of D, denoted $wr(D)$.

Definition 1.7. [1,6] Let D be a diagram representing an oriented link L. The number of crossings in D is denoted $Cr(D)$. The minimum number of crossings observed among all diagrams of L is defined as the *crossing number* of L, and is denoted $Cr(L)$.

$Cr(D)$ and $wr(D)$ are not invariants of L. However, $Cr(L)$ is a link invariant. Note that if L is the unknot or a trivial link with any number of components, then $Cr(L) = 0$. Otherwise, $Cr(L) \geq 2$. If there were a diagram with exactly one crossing, that crossing could be removed by a Reidemeister type I move. See Figure 1.5 for a diagram showing writhe and crossing number.

If a link diagram is of the form shown in Figure 1.6, then by flipping either section A or section B of the diagram in the appropriate direction, the crossing shown can simply be twisted out (similar to a Reidemeister type I move), thereby decreasing the crossing number of the diagram by 1.

Definition 1.8. [1,6] A link diagram that includes no easily-removed crossings such as seen in Figure 1.6 is called *reduced*.

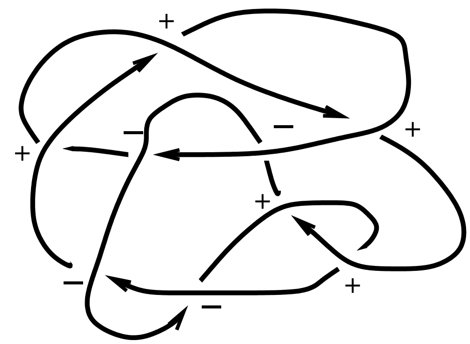

Figure 1.5: Oriented knot diagram D showing positive and negative crossings, $Cr(D) = 9$ and $wr(D) = 1$

Definition 1.9. [1, 6] A link diagram is called *alternating* if each strand passes alternately above and below the other strands that it crosses. A link is considered *alternating* if it has an alternating diagram.

The following theorem was proved independently by Kauffman [14], Murasugi [21], and Thistlethwaite [28] in 1986.

Theorem 1.2. *Let L be a link. If D is a reduced, alternating diagram of L, then D has the minimum number of crossings possible in a diagram of L. That is, $Cr(L) = Cr(D)$.*

Note that the diagram in Figure 1.5 is reduced but not alternating. So one could not necessarily conclude based on Theorem 1.2 that the represented knot has crossing number 9, the number of crossings in the diagram. (However, in this case it does – This is knot 9_{42} from a standard knot table.)

1.3 Link Polynomials

Link polynomials are invariants that have come into prominence relatively recently. The first polynomial invariant to be discovered was the Alexander polynomial in 1928 [3]. A Laurent polynomial in one variable, it was initially defined in terms of homology theory, using Seifert surfaces (see Chapter 2). The Alexander polynomial can provide some information about a link, for example its breadth (the difference between the greatest and least exponents of its variable) can be used to obtain a lower bound for the genus of a knot or link [16]. However, it cannot distinguish between many links. For example, a link L

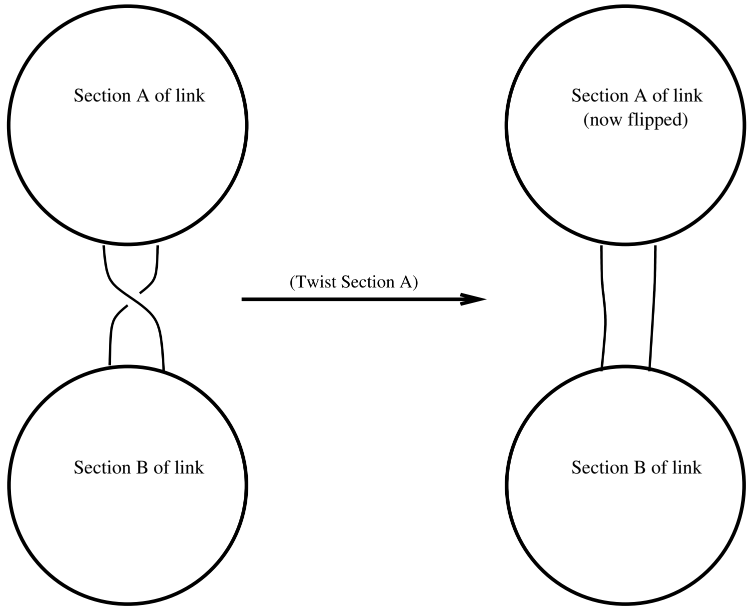

Figure 1.6: Reducing a non-reduced diagram

and its mirror image L^* will have the same Alexander polynomial, even though they may not be equivalent. In fact, there exist infinitely many non-equivalent knots having a given Alexander polynomial [6].

The Conway polynomial [5] is a true polynomial in one variable, which is found to be equivalent to the Alexander polynomial by a simple variable substitution. The main advancement associated with the Conway polynomial is in how it is computed, by a *skein relation*, using the link itself instead of a Seifert surface. A skein relation is an equation relating three variations of a link diagram that differ in only a small area containing one crossing. This discovery revealed that the Alexander polynomial can also be defined in terms of a simple skein relation.

A completely new polynomial for links was discovered by V. Jones in 1984 [13]. Working in von Neumann algebras, he realized that the algebras he was studying had applications in knot theory. The Jones polynomial is a Laurent polynomial in one variable, but is distinct from the Alexander polynomial. It can distinguish between many links that the Alexander polynomial cannot [6]. It too can be defined by a skein relation.

Definition 1.10. [13] Let L be an oriented link. The Jones polynomial $V(L)$ is defined by the following rules:

(i) $V(L) = 1$ if L is the unknot.

(ii) Let L_+, L_-, and L_0 be three oriented links whose diagrams differ in only a small region as shown in Figure 1.7. Then $t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0)$.

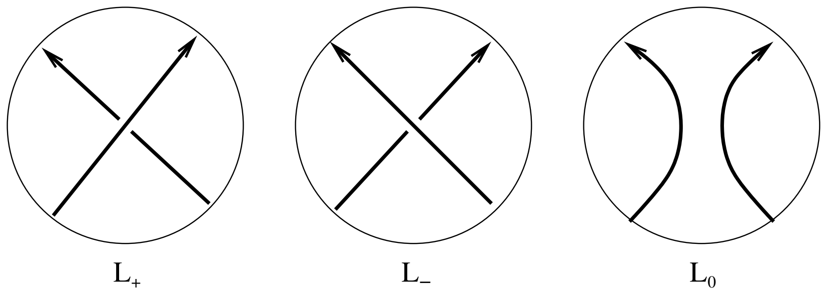

Figure 1.7: Diagrams of L_+, L_-, and L_0 are identical except for the region shown.

Examples using a skein relation will be seen in Chapters 3 and 5, with the HOMFLY polynomial.

Many results involving the Jones polynomial followed shortly after its discovery. For example, Theorem 1.2 had long been conjectured but never proved until it was discovered that the Jones polynomial provides a lower bound for the crossing number of a link.

Theorem 1.3. *[14, 22, 28] Let L be an oriented link with a connected n-crossing diagram D, and let $V(L)$ be the Jones polynomial of L. Then $B(V(L)) \leq n$, where $B(V(L))$ is the breadth of $V(L)$ (the difference between the highest and lowest degrees of t in $V(L)$). Also, if D is alternating and reduced, then $B(V(L)) = n$.*

In addition to the new results that could be proven using the Jones polynomial, more new polynomials were soon discovered. The HOMFLY polynomial (named as an acronym of its discoverers' names [11]) is a Laurent polynomial in two variables that generalizes both the Alexander and the Jones polynomials [16]. The HOMFLY polynomial is the main polynomial used in this paper, and will be discussed in much more detail later.

A useful fact about polynomial invariants is that they are multiplicative under the connected sum operation [6].

Theorem 1.4. *Let P be any of the Alexander, Conway, Jones or HOMFLY polynomials.*

If L_1 and L_2 are links with polynomials $P(L_1)$ and $P(L_2)$, then the polynomial of their connected sum is $P(L_1 \# L_2) = P(L_1)P(L_2)$.

1.4 Knot Theory Applications

Knot theory has been found to have applications in a variety of scientific areas, most notably in the study of molecular cell biology. DNA (deoxyribonucleic acid), the molecule in which an organism's genetic code is stored, exists in long, tangled strands inside of a cell. Certain enzymes, called topoisomerases, manipulate the DNA for cell processes such as replication and transcription. When the enzyme acts on the DNA molecule, it can make a number of different changes, such as breaking the molecule and reconnecting it in a different way. As such, knots can be introduced. If the DNA molecule is cyclic, then these knots are captured and can be detected experimentally [1].

For example, DNA of the P4 bacteriophage (a virus) is cyclic, and many DNA molecules extracted from P4 are found to be non-trivial knots [4]. The percentage of knot occurrence is much higher than that observed in experiments in which identical molecules are closed into circles in a free solution. Furthermore, those cyclic DNA molecules formed in solution are generally much less complicated than those taken from inside the virus [4]. The exact mechanism of knot formation inside P4 is not known, but it seems likely that the confinement of the DNA molecule in a small space is a factor in the higher incidence of complicated knots.

There are various ways of attempting to model DNA knotting in cells through random processes [4, 29]. For example, polygonal knots are often used. Instead of a smooth curve, a polygonal knot is composed of many line segments connected end-to-end, and eventually returning to the starting point. Random polygonal knots have the benefit of being fairly easy to generate. There are a variety of different generating techniques that have been tried, including methods to create random polygons inside a confined space, and more are being developed. Once generated, a random knot's degree of 'knottedness' can be evaluated using some basic knot properties such as the writhe, crossing number, number of Seifert circles (see Chapter 2) and braid index (see Chapter 4). There are also ways of allowing some randomness but also inducing a certain level of complexity. For example, a diagram can be forced to be alternating, which, by Theorem 1.2, will ensure a crossing number nearly as high as the number of crossings seen in the diagram (minus only the number that would need to be removed in order to make the diagram reduced).

Braids are another possible tool for generating random knots. Because of their simple structure, random braids would be easy to generate, and certain properties could be easily controlled. By using the results of this paper (and hopefully stronger results to come), for example, braids could be used to produce knots with a high crossing number and low braid index.

The ultimate goal of the various efforts to generate random knots is to create a model with results that closely match observed experimental data, such as those obtained from the P4 bacteriophage. Then perhaps the generating method can give some insight into how the DNA molecules become knotted inside the cell.

CHAPTER 2: A LONG-STANDING CONJECTURE

2.1 Seifert's Algorithm and Genus of a Link

In topology, the idea of the genus of a surface (or 2-manifold) without a boundary basically amounts to how many 'holes' the surface has [1]. For example, a sphere has no holes, and its genus is 0. A torus has a hole and its genus is 1. A surface with n holes has genus n. The genus of a link is closely related, but takes a bit more work to compute. In this section we look at the relationship between links and surfaces.

If a disc is removed from a surface, then it becomes a surface with a boundary. Several discs could be removed to create several boundary components. Each boundary component is a topological circle, and would remain a topological circle no matter how the surface is deformed within $\mathbb{R}^3$. However, the surface can be embedded in space in many different ways. In particular, there exist embeddings in which the boundary components are knotted and/or linked together. Seifert's algorithm is a method for constructing a surface with boundary from a given link diagram such that the boundary of the surface is the link itself [27]. Once such a surface is obtained, the genus of the corresponding surface without boundary can be calculated. The genus of the link diagram is defined as the genus of this surface, and the minimum genus among all diagrams of a link is defined as the genus of the link.

Seifert's algorithm is as follows [27]. Beginning with an oriented, connected diagram D of a link L, we first 'smooth' all of its crossings. That is, at each crossing, the two strands involved are cut and each incoming strand is reconnected with the outgoing strand to which it was not previously connected, thus removing the crossing and maintaining the original orientation for all link components. See Figure 2.1. After all crossings have been smoothed out, the remaining diagram consists of a number of unlinked topological circles. These are called Seifert circles, and their number is denoted $s(D)$.

Next, each Seifert circle is filled in to form a disc, and each disc is positioned at a different elevation. Finally, a small rectangular strip with a half twist in the appropriate

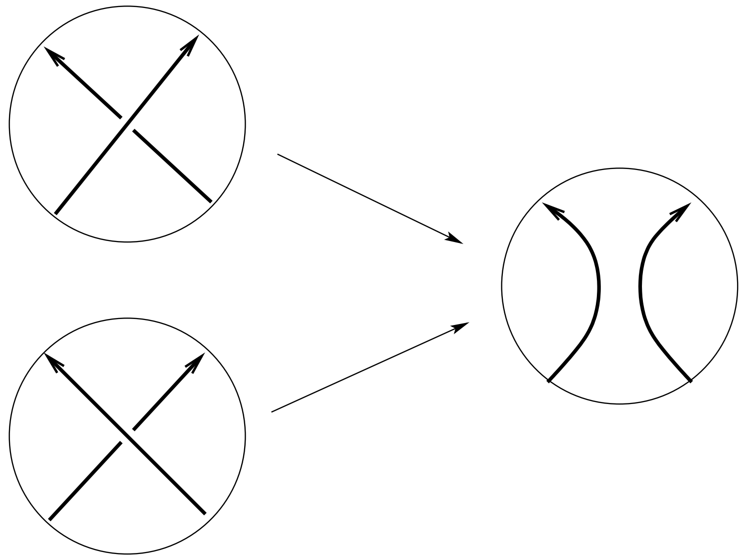

Figure 2.1: Smoothing a positive or negative crossing

direction is used to connect the discs at each point where there was a crossing in the original link diagram. The strip is twisted so that when projected downward, the two portions of its boundary that will not be attached to the discs cross in the same manner as the strands of the link in the original crossing at that position. The resulting surface is called a Seifert surface of D, and is an orientable (two-sided) surface whose boundary is D. See Figure 2.2 for an illustration of Seifert's algorithm.

Now that the Seifert surface has been found, its genus is to be computed. The genus g of a surface with boundary is defined by $g = \frac{2-\chi}{2}$, where χ is the Euler characteristic of the corresponding surface without boundary [1]. The Euler characteristic of a surface without boundary is defined by $\chi = f - e + v$, where f is the number of faces, e is the number of edges, and v is the number of vertices in any triangulation of the surface without boundary. Our Seifert surface with boundary is most easily triangulated by placing two vertices connected by an edge across each of the strips that were added to connect the discs, and then placing two more edges incident to each vertex that simply follow the boundary of the Seifert surface. Therefore the number of faces in this triangulation is equal to the number of Seifert circles obtained from the algorithm, $s(D)$, the number of vertices is 2 times the number of crossings, $2Cr(D)$, and the number of edges is 3 times the number of vertices divided by 2, $\frac{3(2Cr(D))}{2} = 3Cr(D)$. To obtain the Euler characteristic for the

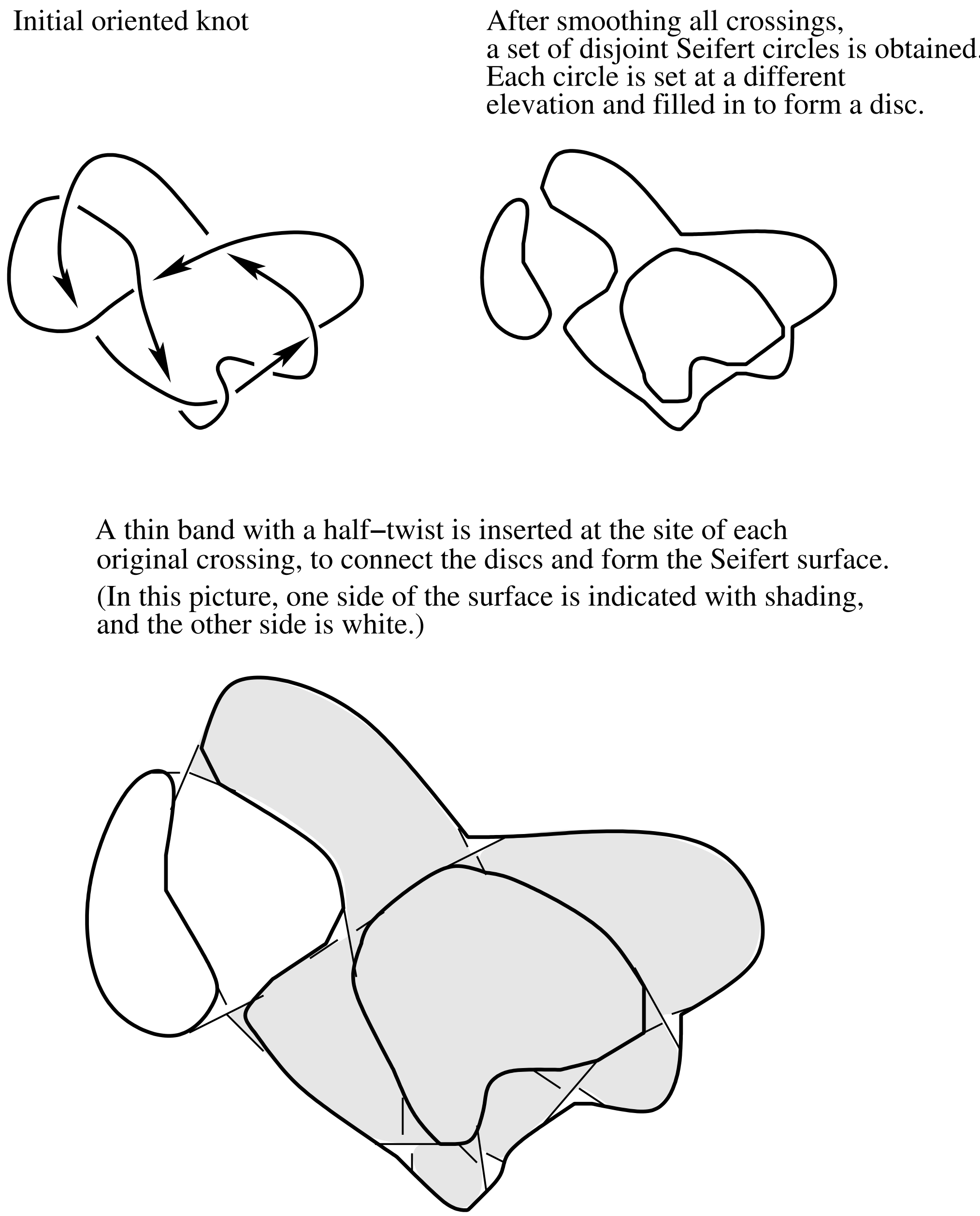

Figure 2.2: Creating a Seifert surface from an oriented knot

corresponding surface without boundary, we think of the surface as embedded in a higher-dimensional space. It is important to note that genus is an inherent property of a surface, not dependent on its particular embedding [6]. When embedded in a higher-dimensional space, each boundary component becomes a mere circle (see [1], p. 270-71), and as such we may 'cap off' each boundary component with a disc. This action adds one additional face per component to the triangulation, and leaves the numbers of vertices and edges unchanged. Thus we obtain $f = s(D) + \mu(L)$, $e = 3Cr(D)$, and $v = 2Cr(D)$. It follows that $g = \frac{2-\chi}{2} = \frac{2-f+e-v}{2} = \frac{2-s(D)+Cr(D)-\mu(L)}{2}$.

Definition 2.1. [16] If D is a diagram of a link L, the genus of D is denoted $g(D)$ and is equal to the genus of the Seifert surface of D. The genus of L is denoted $g(L)$, and is defined as the minimum genus among all diagrams of L.

Genera of links are not always easy to compute, as the above process demonstrates. The following two results are sometimes useful.

Theorem 2.1. *[26] Let L_1 and L_2 be links. Then $g(L_1 \# L_2) = g(L_1) + g(L_2)$.*

Theorem 2.2. *[7] Let L be an alternating link represented by an alternating, reduced diagram D. Then $g(L) = g(D)$.*

2.2 The Additivity of Crossing Numbers

It is an open question whether the crossing number of a connected sum of two links is equal to the sum of the crossing numbers of the two individual links. That is, if L_1 and L_2 are links, is it true that $Cr(L_1 \# L_2) = Cr(L_1) + Cr(L_2)$? Clearly $Cr(L_1 \# L_2) \leq Cr(L_1) + Cr(L_2)$ just by looking at a diagram of $L_1 \# L_2$. But it remains unknown whether that diagram can be manipulated to produce a diagram with fewer crossings. It has not even been proven whether in general $Cr(L_1 \# L_2) \geq Cr(L_1)$ or $Cr(L_1 \# L_2) \geq Cr(L_2)$.

It is known that crossing numbers are additive for some classes of links, but results pertaining to all links remain, for the most part, elusive. If L_1 and L_2 are both alternating links, then it has been established that $Cr(L_1 \# L_2) = Cr(L_1) + Cr(L_2)$. It has also been shown to be true for *adequate* links [17] and for links with *zero deficiency* [10]. In a recent paper [15], M. Lackenby presents the first known result of a non-trivial lower bound on $Cr(K_1 \# K_2)$ for any two knots K_1 and K_2. He demonstrates that $\frac{1}{152}(Cr(K_1) + Cr(K_2)) \leq Cr(K_1 \# K_2) \leq Cr(K_1) + Cr(K_2)$.

This paper endeavors to expand the known realm of the zero deficiency links. As explained in [10], the zero deficiency links include many (but not all) alternating links, all torus knots, and some Montesinos links. Using the results of this paper, a class of alternating closed braids will also be seen to have deficiency zero.

Definition 2.2. [10] Let L be a link. The deficiency of L, denoted $d(L)$, is defined by

$$d(L) = Cr(L) - b(L) - 2g(L) - \mu(L) + 2$$

where $Cr(L)$ is the crossing number of L, $b(L)$ is the braid index of L, $g(L)$ is the genus of L, and $\mu(L)$ is the number of components in L.

The braid index will be defined in Chapter 4. In the meantime, we can make use of the following theorem, due to S. Yamada [30].

Theorem 2.3. *Let L be a link, let $b(L)$ be the braid index of L, and let $s(L)$ be the minimum number of Seifert circles among all diagrams of L. Then $b(L) = s(L)$.*

In Chapter 6, a certain type of reduced, alternating closed braid diagram D will be shown to have both the minimum number of Seifert circles and the minimum crossing number possible for its link type. If L is the link represented by D, then this will mean that $s(D) = s(L)$ and $Cr(L) = Cr(D)$. Since we also have $b(L) = s(L)$ by Theorem 2.3, the deficiency of L will thus be

$$\begin{aligned} d(L) &= Cr(L) - b(L) - 2g(L) - \mu(L) + 2 \\ &= Cr(L) - b(L) - 2\left(\frac{2 - s(D) + Cr(D) - \mu(L)}{2}\right) - \mu(L) + 2 \\ &= Cr(L) - b(L) - 2 + s(L) - Cr(L) + \mu(L) - \mu(L) + 2 \\ &= 0 \end{aligned}$$

Then by the following theorem and corollary we can conclude that the crossing numbers of links represented by these closed braid diagrams are additive under the connected sum operation.

Theorem 2.4. *[10] Let L_1 and L_2 be links such that $d(L_1) = 0$ and $d(L_2) = 0$. Then $Cr(L_1 \# L_2) = Cr(L_1) + Cr(L_2)$ and $d(L_1 \# L_2) = 0$.*

Corollary 2.5. *[10] Let $n \geq 2$ and let $L_1, L_2, \ldots, L_n$ be links with $d(L_1) = d(L_2) = \cdots = d(L_n) = 0$. Then $Cr(L_1 \# L_2 \# \ldots \# L_n) = Cr(L_1) + Cr(L_2) + \cdots + Cr(L_n)$ and $d(L_1 \# L_2 \# \ldots \# L_n) = 0$.*

Even though the diagrams that will be considered are alternating and thus already known to have the property of their crossing numbers being additive, the fact that they are of deficiency zero is still valuable information since not all zero-deficiency links are alternating. For instance, if L_1 is one of this class of closed braids and L_2 is a torus knot, the result for alternating knots alone would not tell us whether $Cr(L_1 \# L_2) = Cr(L_1) + Cr(L_2)$ (since torus knots are not generally alternating).

With adequate links, not only is there the fact that $Cr(L_1 \# L_2) = Cr(L_1) + Cr(L_2)$, but if only one of the links is adequate, say L_1, then it is also known that $Cr(L_1 \# L_2) \geq Cr(L_1) + B(V(L_2))$, where $B(V(L_2))$ is the breadth of the Jones polynomial of L_2 [17]. There is an analogous result for zero-deficiency links:

Theorem 2.6. *[10] Let L_1 and L_2 be links such that $d(L_1) = 0$. Then $Cr(L_1 \# L_2) \geq Cr(L_1)$. If L_2 is a non-trivial knot, then $Cr(L_1 \# L_2) \geq Cr(L_1) + 3$, and if L_2 is a non-trivial link with $\mu(L_2)$ components then $Cr(L_1 \# L_2) \geq Cr(L_1) + 2\mu(L_2) - 2$.*

CHAPTER 3: THE HOMFLY POLYNOMIAL

The HOMFLY polynomial is a Laurent polynomial in two variables, m and ℓ. If L is an oriented link, $P(L)$ will denote the HOMFLY polynomial of L. As with the polynomials discussed in Chapter 1, it is defined in terms of a skein relation.

Definition 3.1. [11] Let L be an oriented link. The HOMFLY polynomial $P(L)$ is defined by the following rules:

(i) $P(L) = 1$ if L is the unknot.

(ii) Let L_+, L_-, and L_0 be three oriented links whose diagrams differ in only a small region as shown in Figure 1.7. Then $\ell P(L_+) + \ell^{-1}P(L_-) + mP(L_0) = 0$.

The procedure for computing the HOMFLY polynomial is to split the diagram at various crossings and apply the skein relation from part (ii) of the above definition. When splitting a positive crossing, we have $P(L_+) = -\ell^{-2}P(L_-) - m\ell^{-1}P(L_0)$, and when splitting a negative crossing we have $P(L_-) = -\ell^2 P(L_+) - m\ell P(L_0)$. The goal is to choose crossings to split so that the resulting secondary diagrams will be simpler than the initial diagram. Eventually, after applying the skein relation some finite number of times, each remaining diagram should be reduced to either the unknot (whose HOMFLY polynomial is 1) or to a trivial link of more than one component, i.e. a set of disconnected circles.

Regarding the latter situation, the HOMFLY polynomial of an n-component trivial link can be calculated in the following way. Suppose two of the components are placed beside each other such that in the region where they are closest to each other they are oriented in the same direction (One component can be flipped over if necessary). Then this region will be a case of the L_0 diagram in Figure 1.7. According to the skein relation, $P(L_0) = -m^{-1}(\ell^{-1}P(L_-) + \ell P(L_+))$. But L_- and L_+ are each the result of adding a crossing between the two components, thereby connecting them into one component. Moreover, this new component is simply the unknot with a twist in the middle. The newly-added crossing can be removed by twisting it out. Therefore, $P(L_-) = 1$ and $P(L_+) = 1$, and $P(L_0) = -m^{-1}(\ell^{-1}+\ell)$. Now this 2-component trivial link can be placed beside another component, and the same action repeated, obtaining a second factor of $-m^{-1}(\ell^{-1}+\ell)$. Repeating the process as many times as needed, if L is an n-component trivial link, then $P(L) = \left[-m^{-1}(\ell^{-1}+\ell)\right]^{n-1}$.

A split link need not be trivial in order to use the process described above. The following example will illustrate that the HOMFLY polynomial of an n-component split link is the product of the the quantity found above, $\left[-m^{-1}(\ell^{-1}+\ell)\right]^{n-1}$, and the HOMFLY polynomials of each of its split components.

Example 3.1. Let L_1, L_2, ..., L_n be links and let $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_n$ be a split link. We place two of the split components, say L_1 and L_2, beside each other as described above so that the region where they are closest to each other is a case of the L_0 diagram in Figure 1.7, and then apply the relation $P(L_0) = -m^{-1}(\ell^{-1}P(L_-) + \ell P(L_+))$. In this situation each of L_- and L_+ is a connected sum of L_1 and L_2, with a crossing between them that can be removed by twisting either L_1 or L_2 in the appropriate direction. By Theorem 1.4, the HOMFLY polynomial of $L_1 \# L_2$ is simply the product of the polynomials $P(L_1)$ and $P(L_2)$. Thus we find that

$$\begin{aligned} P(L_1 \sqcup L_2) &= -m^{-1}(\ell^{-1}P(L_1)P(L_2) + \ell P(L_1)P(L_2)) \\ &= -m^{-1}(\ell^{-1}+\ell)P(L_1)P(L_2) \end{aligned}$$

Now if we place L_3 beside $L_1 \sqcup L_2$ and repeat the process, we obtain

$$\begin{aligned} P(L_1 \sqcup L_2 \sqcup L_3) &= -m^{-1}(\ell^{-1}P(L_1 \sqcup L_2)P(L_3) + \ell P(L_1 \sqcup L_2)P(L_3)) \\ &= -m^{-1}(\ell^{-1}+\ell)P(L_1 \sqcup L_2)P(L_3) \\ &= \left[-m^{-1}(\ell^{-1}+\ell)\right]^2 P(L_1)P(L_2)P(L_3) \end{aligned}$$

Continuing in this manner, we eventually obtain the HOMFLY polynomial of the entire split link,

$$P(L) = P(L_1 \sqcup L_2 \sqcup \cdots \sqcup L_n) = \left[-m^{-1}(\ell^{-1}+\ell)\right]^{n-1} \prod_{i=1}^{n} P(L_i)$$

Finally, we calculate the HOMFLY polynomial of a simple knot. When calculating link polynomials using skein relations, a resolving tree such as in Figure 3.1 is often used.

Example 3.2. Let L be the trefoil knot shown at the top of Figure 3.1. Suppose the diagram is split at the top crossing, which is negative. Then we think of this diagram as L_- and use the equation $P(L_-) = -\ell^2 P(L_+) - m\ell P(L_0)$. Using the labels in the figure, L_1 is the result of changing the negative crossing into a positive crossing, and L_2 is the result of removing the crossing while keeping the orientation intact, so $P(L) = -\ell^2 P(L_1) - m\ell P(L_2)$. Now if untwisted L_1 can be seen seen to be equivalent to the unknot, so $P(L_1) = 1$. The skein relation needs to be applied once more on L_2, since it is a non-trivial link. The result of the skein relation on the rightmost crossing of L_2 is shown in the third line in the figure. $L_{2,1}$ is the result of changing the negative crossing into a positive crossing. The two components can now be separated, so $L_{2,1}$ is the trivial 2-component link. $L_{2,2}$ is the result of removing the rightmost crossing from L_2, and it can be untwisted to reveal the unknot. So we have

$$\begin{aligned}
P(L_2) &= -\ell^2 P(L_{2,1}) - m\ell P(L_{2,2}) \\
&= -\ell^2(-m^{-1}(\ell^{-1} + \ell)) - m\ell(1) \\
&= m^{-1}(\ell + \ell^3)) - m\ell
\end{aligned}$$

and then

$$\begin{aligned}
P(L) &= -\ell^2 P(L_1) - m\ell P(L_2) \\
&= -\ell^2(1) - m\ell(m^{-1}(\ell + \ell^3)) - m\ell) \\
&= -2\ell^2 - \ell^4 + m^2\ell^2
\end{aligned}$$

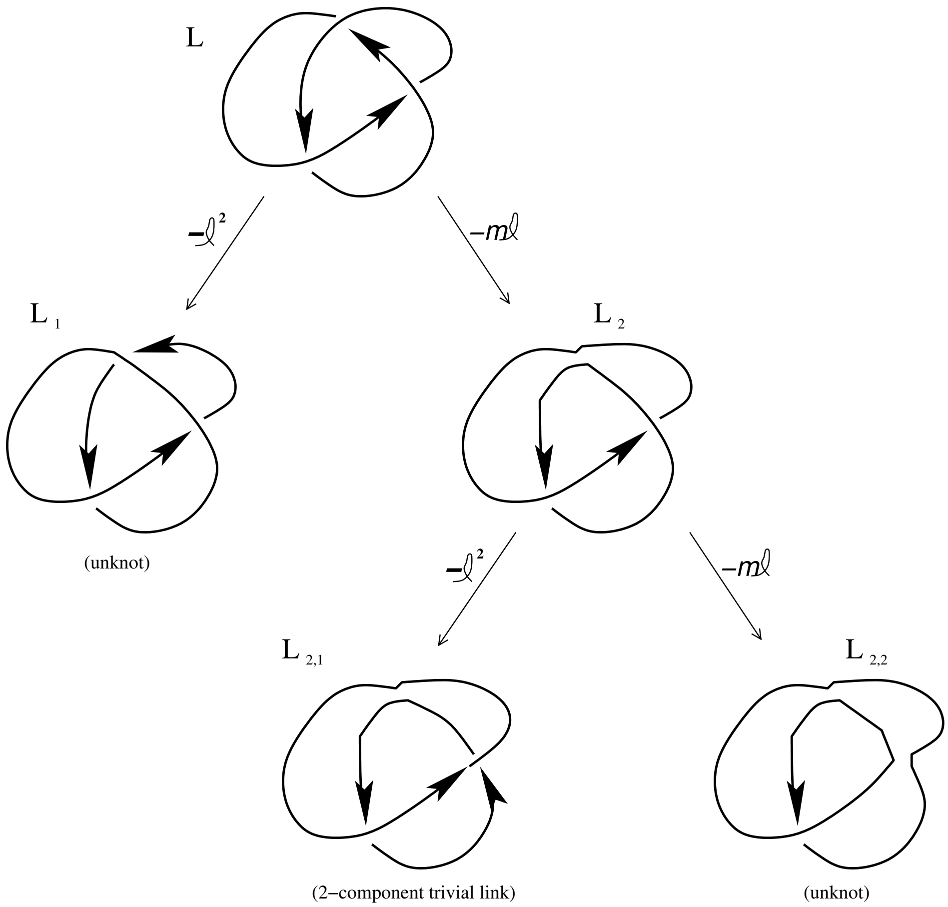

Figure 3.1: Resolving tree for the trefoil knot 3_1

CHAPTER 4: BRAIDS AND CLOSED BRAIDS

Braids are important tools in knot theory, both because they have a very simple structure and because they can be used to represent any knot or link [2].

Definition 4.1. [1] A *braid* (or *open braid*) is a set of k strings arranged vertically alongside one another, with the ends of each string fixed at the top and bottom, as if attached to two bars. The strings may pass over or under one another as they traverse the space between the bars, but any horizontal cross-section can only be intersected by each string once.

The last requirement means that as one follows the course of a string from top to bottom, the string cannot at any point turn back upward. Obviously this also precludes any string extending above the top bar or below the bottom.

The usual manner of describing a braid is by a *braid word* consisting of symbols such as σ_1, σ_3^5, and σ_2^{-1}. If we think of the strings as traveling downward, σ_i indicates a crossing in which the string in position i (counting from left to right) passes below the string in position $i+1$. The symbol σ_i^{-1} indicates a crossing in which the string in position i passes above the string in position $i+1$. (Note: Some texts use notation that is exactly the reverse of that just stated. However, the choice is arbitrary and this notation is more convenient for the purposes of this paper, for a positive crossing has a positive exponent and a negative crossing has a negative exponent.) An exponent with absolute value greater than 1 simply indicates that there are the corresponding number of consecutive σ_i or σ_i^{-1} crossings. The braid word then is a listing of the symbols describing all of the crossings in the braid, in the order they are encountered as one moves downward from the top of the braid until one reaches the bottom. Occasionally in this paper the notation σ_i^0 may be used to indicate that there are no crossings between the strings in positions i and $i+1$ at a particular location.

There is a natural product operation on braids. The product of two braids A and B can be defined as the braid represented by the concatenation of the braid word of A with the braid word of B (in that order, for clearly this operation is not commutative). If n is a positive integer and $\mathcal{B}_n$ is the set of all braids with n strings, then $\mathcal{B}_n$ is a group under this product operation [1,8].

Definition 4.2. [1] Let k be the number of strings in a braid. A *closed braid* is a link formed from the braid, by connecting the string in position i at the top with the string in position i at the bottom, for $1 \leq i \leq k$.

A braid and its corresponding closed braid are illustrated in Figure 4.1. In this paper, the notation $D_k[\ldots]$ or D_k will be used to indicate the closure of a k-string braid, where the braid word (or the relevant parts thereof) will be specified in the square brackets. The square brackets may be omitted for ease of reading, if such omission will not cause confusion.

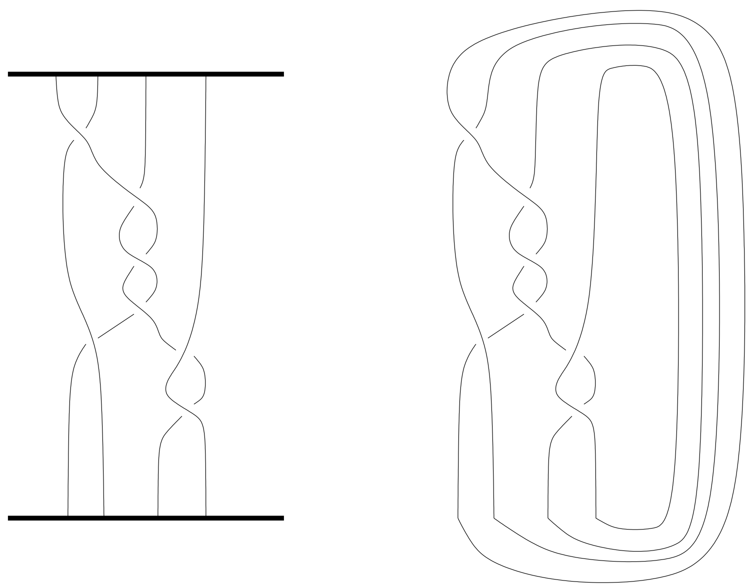

Figure 4.1: The braid $\sigma_1^{-1}\sigma_2^{-3}\sigma_1^{-1}\sigma_3\sigma_3^{-1}$ and its corresponding closed braid

Two braids are equivalent if one can be transformed into the other by moving the strings around without moving the bars, cutting the strings or detaching any strings from the bars. The ways of moving the strings around can be simplified to three basic moves. A braid B_1 is equivalent to a braid B_2 if B_1 can be transformed into B_2 by applying some sequence of the following three types of moves [1]:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ and } \sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1} = \sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} \tag{4.1}$$

$$\sigma_i \sigma_i^{-1} = \sigma_i^0 \text{ and } \sigma_i^{-1} \sigma_i = \sigma_i^0 \tag{4.2}$$

$$\text{If } |i-j| > 1 \text{ and } \alpha,\, \beta \in \{-1, 1\} \text{ , then } \sigma_i^{\alpha} \sigma_j^{\beta} = \sigma_j^{\beta} \sigma_i^{\alpha} \tag{4.3}$$

Clearly, if two braids are equivalent and each one is closed to form a link, then the two links will be equivalent. However, it is also possible for two non-equivalent braids, upon closure, to produce equivalent links. So when considering closed braids, two additional rules, known as the Markov moves, are required. Introduced in 1935 [18], these two moves combined with the three original rules above are sufficient to demonstrate equivalence of two closed braids [8].

The first Markov move is *conjugation*: A closed braid representation is equivalent to its conjugate. If B_k is a braid with k strings, $C(B_k)$ is the closure of B_k, and $1 \leq i \leq k-1$, then $C(\sigma_i^{-1} B_k \sigma_i) = C(B_k) = C(\sigma_i B_k \sigma_i^{-1})$. In practice, this rule provides the means to move a crossing from the bottom of a braid all the way around (along the strings that are added when the braid is closed) to the top, or vice versa.

The second Markov move is known as *stabilization* and has the effect of increasing or decreasing the number of strings in the braid by 1. If B_k is defined as above, and one more string and one crossing $\sigma_k^{\pm 1}$ are added, then $B_{k+1} = B_k \sigma_k^{\pm 1}$ is a $(k+1)$-string braid such that $C(B_{k+1}) = C(B_k)$. Similarly, if $C(B_k)$ is a closed braid with only one crossing between strings in positions $k-1$ and k, then that crossing can be "twisted out" to reduce the braid to $k-1$ strings.

It was shown by J. W. Alexander in 1923 that every link has a closed braid represention [2]. Knowing this, and applying the above rules, it is clear that there is an infinite variety of braids whose closures could represent a given link. However, a link has closed braid representations that are minimal in the sense that they use the smallest possible number of strings.

Definition 4.3. [1,6,9] The *braid index* of a link L is the minimum number of strings used among all braids whose closure is L.

If L has a closed braid diagram D_k, then clearly $b(L) \leq k$. The only link with braid index of 1 is the unknot, and an n-component trivial link has braid index n. The braid index is a link invariant and, as noted in Chapter 2, the braid index is equal to the minimum number of Seifert circles in a diagram of a link [30].

In addition to the above five rules, the following fact is useful in the discussion that follows.

Theorem 4.1. *Let* $n \in \mathbb{Z}$. *Then* $\sigma_i \sigma_{i+1}^n \sigma_i^{-1} = \sigma_{i+1}^{-1} \sigma_i^n \sigma_{i+1}$.

Proof. The proof is straightforward using the rules outlined above. If $n = 0$, then clearly $\sigma_i \sigma_i^{-1} = \sigma_i^0 = \sigma_i^{-1} \sigma_i$. If $n > 0$,

$$
\begin{aligned}
\sigma_i \sigma_{i+1}^n \sigma_i^{-1} &= (\sigma_{i+1}^{-1} \sigma_{i+1}) \sigma_i \sigma_{i+1}^n \sigma_i^{-1} \\
&= \sigma_{i+1}^{-1} (\sigma_{i+1} \sigma_i \sigma_{i+1}) \sigma_{i+1}^{n-1} \sigma_i^{-1} \\
&= \sigma_{i+1}^{-1} (\sigma_i \sigma_{i+1} \sigma_i) \sigma_{i+1}^{n-1} \sigma_i^{-1} \\
&= \sigma_{i+1}^{-1} \sigma_i (\sigma_{i+1} \sigma_i \sigma_{i+1}) \sigma_{i+1}^{n-2} \sigma_i^{-1} \\
&= \sigma_{i+1}^{-1} \sigma_i (\sigma_i \sigma_{i+1} \sigma_i) \sigma_{i+1}^{n-2} \sigma_i^{-1} \\
&= \sigma_{i+1}^{-1} \sigma_i^2 (\sigma_{i+1} \sigma_i \sigma_{i+1}) \sigma_{i+1}^{n-3} \sigma_i^{-1} \\
&= \sigma_{i+1}^{-1} \sigma_i^2 (\sigma_i \sigma_{i+1} \sigma_i) \sigma_{i+1}^{n-3} \sigma_i^{-1} \\
&= \dots \\
&= \sigma_{i+1}^{-1} \sigma_i^{n-2} (\sigma_{i+1} \sigma_i \sigma_{i+1}) \sigma_{i+1} \sigma_i^{-1} \\
&= \sigma_{i+1}^{-1} \sigma_i^{n-2} (\sigma_i \sigma_{i+1} \sigma_i) \sigma_{i+1} \sigma_i^{-1} \\
&= \sigma_{i+1}^{-1} \sigma_i^{n-1} (\sigma_{i+1} \sigma_i \sigma_{i+1}) \sigma_i^{-1} \\
&= \sigma_{i+1}^{-1} \sigma_i^{n-1} (\sigma_i \sigma_{i+1} \sigma_i) \sigma_i^{-1} \\
&= \sigma_{i+1}^{-1} \sigma_i^{n} \sigma_{i+1} (\sigma_i \sigma_i^{-1}) \\
&= \sigma_{i+1}^{-1} \sigma_i^{n} \sigma_{i+1}
\end{aligned}
$$

A similar process gives the proof for the $n < 0$ case. □

CHAPTER 5: COMPUTING HOMFLY POLYNOMIALS OF CLOSED BRAIDS

Now we consider a few examples of calculating HOMFLY polynomials of closed braids. First note that in the final example in Chapter 3 the knot is equivalent to the closed braid $D_2[\sigma_1^{-3}]$. So $P(D_2[\sigma_1^{-3}]) = -2\ell^2 - \ell^4 + m^2\ell^2$ and we do not repeat that calculation here. Also recall that $P(D_2[\sigma_1]) = P(D_2[\sigma_1^{-1}]) = P(D_1) = 1$ by the Markov move of stabilization, and $D_2[\sigma_1^0]$ is simply a two-component trivial link so $P(D_2[\sigma_1^0]) = -m^{-1}(\ell^{-1} + \ell)$.

Example 5.1. Let L be the link represented by the closed braid $D_2[\sigma_1^{-4}]$. Note that at each step where there remain two or more consecutive negative crossings we may apply the skein relation to any of the consecutive negative crossings and the result will be the same. Without loss of generality we assume the skein relation is applied to the last crossing in the sequence. See Figure 5.1 for a resolving tree for this calculation. The computation of the HOMFLY polynomial of L is then

$$
\begin{aligned}
P(L) &= P(D_2[\sigma_1^{-4}]) \\
&= -\ell^2 P(D_2[\sigma_1^{-3}\sigma_1]) - m\ell P(D_2[\sigma_1^{-3}]) \\
&= -\ell^2 P(D_2[\sigma_1^{-2}]) - m\ell P(D_2[\sigma_1^{-3}]) \\
&= -\ell^2 \left(-\ell^2 P(D_2[\sigma_1^{-1}\sigma_1]) - m\ell P(D_2[\sigma_1^{-1}])\right) \\
&\qquad -m\ell \left(-\ell^2 P(D_2[\sigma_1^{-2}\sigma_1]) - m\ell P(D_2[\sigma_1^{-2}])\right) \\
&= \ell^4 P(D_2[\sigma_1^{0}]) + m\ell^3 P(D_2[\sigma_1^{-1}]) + m\ell^3 P(D_2[\sigma_1^{-1}]) + m^2\ell^2 P(D_2[\sigma_1^{-2}]) \\
&= \ell^4(-m^{-1}(\ell^{-1} + \ell)) + 2m\ell^3 + m^2\ell^2 \left(-\ell^2 P(D_2[\sigma_1^{-1}\sigma_1]) - m\ell P(D_2[\sigma_1^{-1}])\right) \\
&= -m^{-1}(\ell^3 + \ell^5) + 2m\ell^3 - m^2\ell^4 P(D_2[\sigma_1^{0}]) - m^3\ell^3 \\
&= -m^{-1}(\ell^3 + \ell^5) + 2m\ell^3 - m^2\ell^4(-m^{-1}(\ell^{-1} + \ell)) - m^3\ell^3 \\
&= -m^{-1}(\ell^3 + \ell^5) + m(3\ell^3 + \ell^5) - m^3\ell^3
\end{aligned}
$$

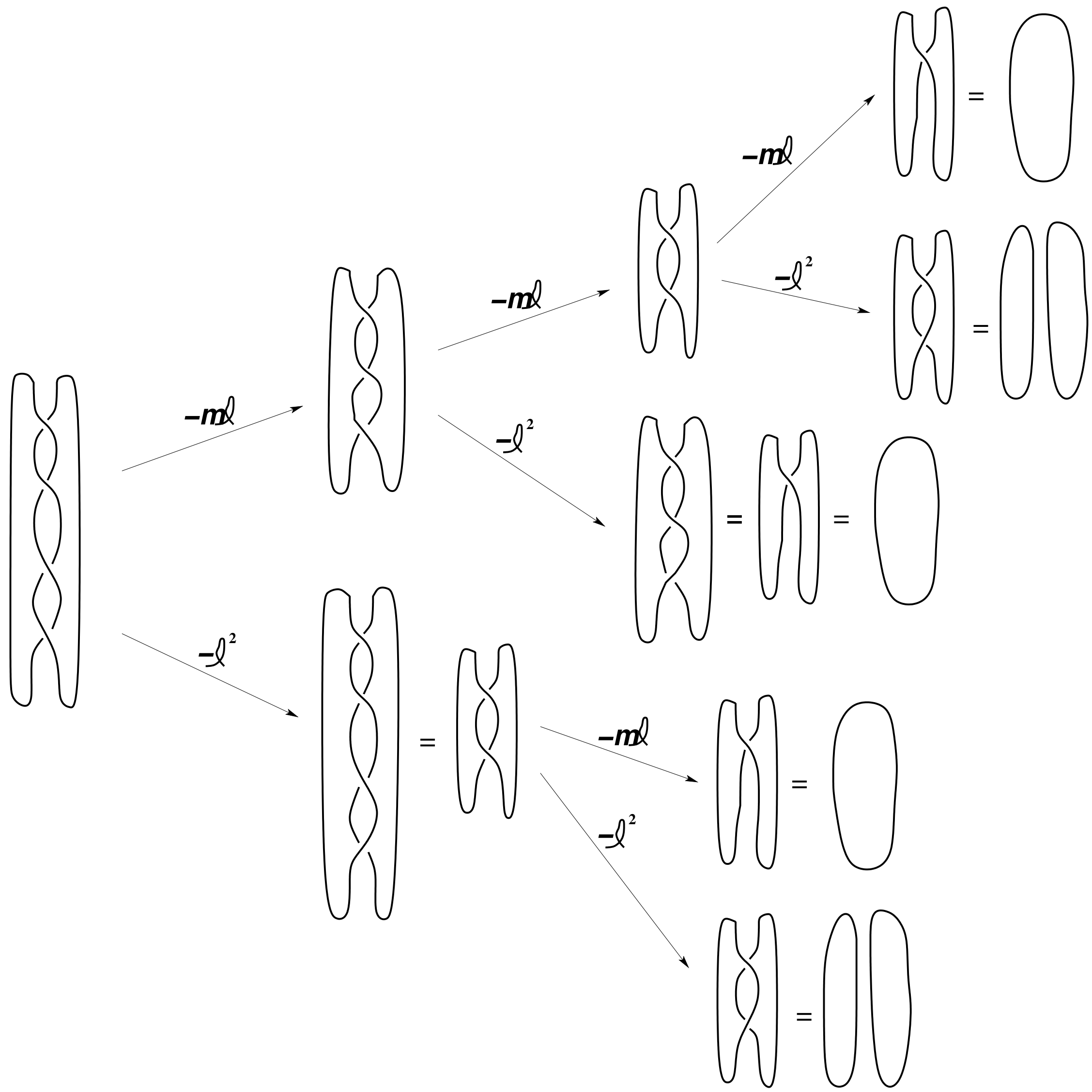

Figure 5.1: Resolving tree for $P(D_2[\sigma_1^{-4}])$

Let us list (without calculations) HOMFLY polynomials for closed braids of the form $P(D_2[\sigma_1^{-n}])$, for the first few $n \geq 0$.

$$\begin{aligned}
P(D_2[\sigma_1^{0}]) &= -m^{-1}(\ell^{-1} + \ell) \\
P(D_2[\sigma_1^{-1}]) &= 1 \\
P(D_2[\sigma_1^{-2}]) &= m^{-1}(\ell + \ell^3) - m\ell \\
P(D_2[\sigma_1^{-3}]) &= -2\ell^2 - \ell^4 + m^2\ell^2 \\
P(D_2[\sigma_1^{-4}]) &= -m^{-1}(\ell^3 + \ell^5) + m(3\ell^3 + \ell^5) - m^3\ell^3 \\
P(D_2[\sigma_1^{-5}]) &= 3\ell^4 + 2\ell^6 - m^2(4\ell^4 + \ell^6) + m^4\ell^4
\end{aligned}$$

Remark. A convenient property of the HOMFLY polynomial is that if L^* is the mirror-image of link L, then $P(L^*)$ is of the same form as $P(L)$ except that all exponents of ℓ are negated [6]. So for example, since $D_2[\sigma_1^4]$ is the mirror image of $D_2[\sigma_1^{-4}]$, we have $P(D_2[\sigma_1^4]) = -m^{-1}(\ell^{-5} + \ell^{-3}) + m(\ell^{-5} + 3\ell^{-3}) - m^3\ell^{-3}$.

Some patterns are apparent in this short list. First, it seems that the highest power of ℓ in $P(D_2[\sigma_1^{-n}])$ is ℓ^{n+1} and the lowest power of ℓ is ℓ^{n-1}, except in the case $n = 1$. Not coincidentally, the diagram $D_2[\sigma_1^{-1}]$ is the only one in the list that is not a reduced diagram; it can be reduced to a 1-string braid by the Markov move of stabilization. Also, whenever n is even there are terms containing m^{-1} in the polynomial, and whenever n is odd the lowest power of m is m^0. This is due to the following fact [16]:

Theorem 5.1. *Let L be a link with $\mu(L)$ components, and let $P(L)$ be the HOMFLY polynomial of L. Then the lowest degree of m in $P(L)$ is $1 - \mu(L)$.*

Though interesting, this fact has no great bearing on the results to come. The observation about the powers of ℓ is more significant, but not immediately obvious why it would be true. Perhaps more instructive for the purpose at hand would be to look not at the final result of the calculation, but at the process. Note that at a certain point we could have simply substituted the previously computed value of $P(D_2[\sigma_1^{-3}])$. In fact, for any $n \geq 2$ we could define $P(D_2[\sigma_1^{\pm n}])$ recursively by $P(D_2[\sigma_1^{-n}]) = -\ell^2 P(D_2[\sigma_1^{2-n}]) - m\ell P(D_2[\sigma_1^{1-n}])$ and $P(D_2[\sigma_1^{n}]) = -\ell^{-2} P(D_2[\sigma_1^{n-2}]) - m\ell^{-1} P(D_2[\sigma_1^{n-1}])$. However, we followed the process all the way through in order to observe the following. Each time the skein relation is applied to one of the sequences of crossings that remains, the sequence is shortened by either one or two crossings. If we continue, then we eventually reach a state in which every term remaining contains a factor of $P(D_2[\sigma_1^{-1}]) = 1$ or $P(D_2[\sigma_1^{0}]) = -m^{-1}(\ell^{-1} + \ell)$. If we look back at the example in those terms we obtain

$$\begin{aligned}
P(D_2[\sigma_1^{-4}]) &= \ell^4 P(D_2[\sigma_1^0]) + 2m\ell^3 P(D_2[\sigma_1^{-1}]) + m^2\ell^2 P(D_2[\sigma_1^{-2}]) \\
&= \ell^4 P(D_2[\sigma_1^0]) + 2m\ell^3 P(D_2[\sigma_1^{-1}]) \\
&\quad +m^2\ell^2\left(-\ell^2 P(D_2[\sigma_1^0]) - m\ell P(D_2[\sigma_1^{-1}])\right) \\
&= (\ell^4 - m^2\ell^4) P(D_2[\sigma_1^0]) + (2m\ell^3 - m^3\ell^3) P(D_2[\sigma_1^{-1}])
\end{aligned}$$

A similar formulation will be used to simplify most of the calculations needed later.

Example 5.2. In this example we look at a slightly more complex braid of three strings. Let L be the link represented by the closed braid $D_3[\sigma_1\sigma_2^{-4}\sigma_1\sigma_2^{-1}]$. We will apply the skein relation to the sequence σ_2^{-4} first.

$$\begin{aligned}
P(L) &= P(D_3[\sigma_1\sigma_2^{-4}\sigma_1\sigma_2^{-1}]) \\
&= -\ell^2 P(D_3[\sigma_1\sigma_2^{-2}\sigma_1\sigma_2^{-1}]) - m\ell P(D_3[\sigma_1\sigma_2^{-3}\sigma_1\sigma_2^{-1}]) \\
&= -\ell^2 \left(-\ell^2 P(D_3[\sigma_1\sigma_2^{0}\sigma_1\sigma_2^{-1}]) - m\ell P(D_3[\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}])\right) \\
&\quad -m\ell \left(-\ell^2 P(D_3[\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}]) - m\ell P(D_3[\sigma_1\sigma_2^{-2}\sigma_1\sigma_2^{-1}])\right) \\
&= -\ell^4 P(D_3[\sigma_1^2\sigma_2^{-1}]) + 2m\ell^3 P(D_3[\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}]) \\
&\quad +m^2\ell^2 \left(-\ell^2 P(D_3[\sigma_1\sigma_2^{0}\sigma_1\sigma_2^{-1}]) - m\ell P(D_3[\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}])\right) \\
&= (\ell^4 - m^2\ell^4) P(D_3[\sigma_1^2\sigma_2^{-1}]) + (2m\ell^3 - m^3\ell^3) P(D_3[\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}])
\end{aligned}$$

We pause here to point out that the polynomials multiplied with the $P(D_3[\ldots])$ expressions are exactly the same as those seen above from the previous example multiplied with $P(D_2[\sigma_1^{-1}])$ and $P(D_2[\sigma_1^0])$. The same steps have been performed in each example to reduce a sequence of four consecutive negative crossings to either one or zero negative crossings. As these reductions of the original sequence leave the rest of the diagram unchanged, it is clear that any sequence of four consecutive negative crossings between the same two strings can be reduced in the same way, obtaining the same two polynomials and essentially performing several applications of the HOMFLY skein relation at one time.

All that remains then is to finish evaluating $P(D_3[\sigma_1^2\sigma_2^{-1}])$ and $P(D_3[\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}])$. Note that $P(D_3[\sigma_1^2\sigma_2^{-1}])$ can be reduced to $P(D_2[\sigma_1^2])$ by stabilization, and then

$$\begin{aligned}
P(D_2[\sigma_1^2]) &= -\ell^{-2} P(D_2[\sigma_1^0]) - m\ell^{-1} P(D_2[\sigma_1]) \\
&= -\ell^{-2}\left(-m^{-1}(\ell^{-1} + \ell)\right) - m\ell^{-1}(1) \\
&= m^{-1}(\ell^{-3} + \ell^{-1}) - m\ell^{-1}
\end{aligned}$$

$P(D_3[\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}])$ cannot be simplified initially, so the skein relation can be applied to any crossing. Choosing the first σ_2^{-1} crossing,

$$\begin{aligned}
P(D_3[\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}]) &= -\ell^2 P(D_3[\sigma_1\sigma_2\sigma_1\sigma_2^{-1}]) - m\ell P(D_3[\sigma_1^2\sigma_2^{-1}]) \\
&= -\ell^2 P(D_3[\sigma_2\sigma_1\sigma_2\sigma_2^{-1}]) - m\ell P(D_2[\sigma_1^2]) \\
&= -\ell^2 P(D_3[\sigma_2\sigma_1]) - m\ell P(D_2[\sigma_1^2]) \\
&= -\ell^2 P(D_2[\sigma_1]) - m\ell \left(m^{-1}(\ell^{-3}+\ell^{-1}) - m\ell^{-1}\right) \\
&= -\ell^2 P(D_1) - \left((\ell^{-2}+1) - m^2\right) \\
&= -\ell^{-2} - 1 - \ell^2 + m^2
\end{aligned}$$

Finally, substituting these back into the earlier equation,

$$\begin{aligned}
P(L) &= (\ell^4 - m^2\ell^4)P(D_3[\sigma_1^2\sigma_2^{-1}]) + (2m\ell^3 - m^3\ell^3)P(D_3[\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}]) \\
&= (\ell^4 - m^2\ell^4)\left(m^{-1}(\ell^{-3}+\ell^{-1}) - m\ell^{-1}\right) \\
&\quad +(2m\ell^3 - m^3\ell^3)\left(-\ell^{-2} - 1 - \ell^2 + m^2\right) \\
&= m^{-1}(\ell+\ell^3) - m(3\ell + 4\ell^3 + 2\ell^5) + m^3(\ell + 4\ell^3 + \ell^5) - m^5\ell^3
\end{aligned}$$

The following lemma formalizes the observations of the last two examples, and simplifies the calculations needed in Chapter 6.

Lemma 5.2. *Let $D_k[\ldots\sigma_i^n\ldots]$ be a k-string closed braid containing a sequence of crossings σ_i^n with $1 \le i \le k-1$ and $n \ge 2$. Let $D_k\left[\ldots\sigma_i^0\ldots\right]$ be the closed braid diagram obtained by removing the crossings σ_i^n from $D_k[\ldots\sigma_i^n\ldots]$ (and leaving the rest of the diagram unchanged). Let $D_k[\ldots\sigma_i\ldots]$ be the diagram obtained by replacing the crossings σ_i^n with the single crossing σ_i. Then $P(D_k[\ldots\sigma_i^n\ldots]) = Q_0(n)P(D_k\left[\ldots\sigma_i^0\ldots\right]) + Q_1(n)P(D_k[\ldots\sigma_i\ldots])$, where*

$$Q_0(n) = \begin{cases} \ell^{-n}\displaystyle\sum_{j=0}^{\frac{n-2}{2}}(-1)^{n-1-j}\binom{n-2-j}{j}m^{n-2-2j} & \textit{if } n \textit{ is even} \\ \ell^{-n}\displaystyle\sum_{j=0}^{\frac{n-3}{2}}(-1)^{n-1-j}\binom{n-2-j}{j}m^{n-2-2j} & \textit{if } n \textit{ is odd} \end{cases}$$

and

$$Q_1(n) = \begin{cases} \ell^{1-n} \sum_{j=0}^{\frac{n-2}{2}} (-1)^{n-1-j} \binom{n-1-j}{j} m^{n-1-2j} & \textit{if } n \textit{ is even} \\ \ell^{1-n} \sum_{j=0}^{\frac{n-1}{2}} (-1)^{n-1-j} \binom{n-1-j}{j} m^{n-1-2j} & \textit{if } n \textit{ is odd} \end{cases}$$

If $D_k\left[\ldots\sigma_i^{-n}\ldots\right]$ *is a closed braid containing a sequence of crossings* σ_i^{-n}*, and* $D_k\left[\ldots\sigma_i^{0}\ldots\right]$ *and* $D_k\left[\ldots\sigma_i^{-1}\ldots\right]$ *are defined similarly as above, then* $P(D_k\left[\ldots\sigma_i^{-n}\ldots\right]) = Q_0(-n)P(D_k\left[\ldots\sigma_i^{0}\ldots\right]) + Q_1(-n)P(D_k\left[\ldots\sigma_i^{-1}\ldots\right])$*, where*

$$Q_0(-n) = \begin{cases} \ell^{n} \sum_{j=0}^{\frac{n-2}{2}} (-1)^{n-1-j} \binom{n-2-j}{j} m^{n-2-2j} & \textit{if } n \textit{ is even} \\ \ell^{n} \sum_{j=0}^{\frac{n-3}{2}} (-1)^{n-1-j} \binom{n-2-j}{j} m^{n-2-2j} & \textit{if } n \textit{ is odd} \end{cases}$$

and

$$Q_1(-n) = \begin{cases} \ell^{n-1} \sum_{j=0}^{\frac{n-2}{2}} (-1)^{n-1-j} \binom{n-1-j}{j} m^{n-1-2j} & \textit{if } n \textit{ is even} \\ \ell^{n-1} \sum_{j=0}^{\frac{n-1}{2}} (-1)^{n-1-j} \binom{n-1-j}{j} m^{n-1-2j} & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. Upon applying the HOMFLY skein relation to one of the (positive) crossings in the sequence σ_i^n, we obtain two terms. For one term a factor of $-m\ell^{-1}$ is introduced and the crossing is simply deleted from the diagram, so $D_k[\ldots\sigma_i^n\ldots]$ is reduced to $D_k\left[\ldots\sigma_i^{n-1}\ldots\right]$. For the other term a factor of $-\ell^{-2}$ is introduced and the positive crossing is changed to a negative crossing. This negative crossing can then cancel with an adjacent positive crossing, so $D_k[\ldots\sigma_i^n\ldots]$ is reduced to $D_k\left[\ldots\sigma_i^{n-2}\ldots\right]$. Thus we obtain $P(D_k[\ldots\sigma_i^n\ldots]) = -\ell^{-2}P(D_k\left[\ldots\sigma_i^{n-2}\ldots\right]) - m\ell^{-1}P(D_k\left[\ldots\sigma_i^{n-1}\ldots\right])$. By repeatedly applying the skein relation on the new shorter sequences as long as the exponent of σ_i is at least 2, we eventually obtain an expression for $P(D_k[\ldots\sigma_i^n\ldots])$ in which every term contains a factor of either $P(D_k\left[\ldots\sigma_i^0\ldots\right])$ or $P(D_k\left[\ldots\sigma_i^1\ldots\right])$. Grouping terms together according to which of these factors they contain, we obtain the polynomials $Q_0(n)$ and $Q_1(n)$.

To calculate $Q_1(n)$, we must enumerate the ways of reducing $D_k[\ldots\sigma_i^n\ldots]$ to

$D_k\left[\ldots\sigma_i^1\ldots\right]$ by the HOMFLY skein relation. Each time the skein relation is applied, it reduces the exponent of σ_i by either 1 or 2, so finding the ways that $D_k\left[\ldots\sigma_i^n\ldots\right]$ can be reduced to $D_k\left[\ldots\sigma_i^1\ldots\right]$ is equivalent to finding the compositions of the integer $n-1$ in which each summand is either 1 or 2. One such composition is $\underbrace{1+1+\cdots+1}_{(n-1\text{ times})}$.

Combining two of the 1's, and noting that the resulting 2 can be placed at any position in a composition, we see that there are $\binom{n-2}{1}=n-2$ possible compositions consisting of exactly one 2 and $(n-3)$ 1's. Combining two more 1's, there are $\binom{n-3}{2}$ possible compositions containing exactly two 2's and $(n-5)$ 1's. In general, there are $\binom{n-1-j}{j}$ possible compositions containing exactly j 2's and $(n-1-2j)$ 1's.

Now since a 1 in a composition corresponds to a factor of $-m\ell^{-1}$ in the skein relation and a 2 corresponds to a factor of $-\ell^{-2}$, we see that the composition of $(n-1)$ 1's gives us the term $(-1)^{n-1}m^{n-1}\ell^{1-n}$, the compositions of one 2 and $(n-3)$ 1's give us $(-1)^{n-2}\binom{n-2}{1}m^{n-3}\ell^{1-n}$, and so on. In general the compositions of j 2's and $(n-1-2j)$ 1's give us $(-1)^{n-1-j}\binom{n-1-j}{j}m^{n-1-2j}\ell^{1-n}$.

If n is odd (so $n-1$ is even), the last composition of $n-1$ will consist of $\frac{n-1}{2}$ 2's and no 1's, so

$$Q_1(n)=\ell^{1-n}\sum_{j=0}^{\frac{n-1}{2}}(-1)^{n-1-j}\binom{n-1-j}{j}m^{n-1-2j}$$

If n is even (so $n-1$ is odd), the last set of compositions of $n-1$ will consist of $\frac{n-2}{2}$ 2's and one 1, so

$$Q_1(n)=\ell^{1-n}\sum_{j=0}^{\frac{n-2}{2}}(-1)^{n-1-j}\binom{n-1-j}{j}m^{n-1-2j}.$$

To calculate $Q_0(n)$, we must enumerate the ways of reducing $D_k\left[\ldots\sigma_i^n\ldots\right]$ to $D_k\left[\ldots\sigma_i^0\ldots\right]$ by the HOMFLY skein relation. The procedure differs slightly from that of $Q_1(n)$, in that in our reductions we never apply the skein relation to $D_k\left[\ldots\sigma_i^1\ldots\right]$. Therefore whenever $D_k\left[\ldots\sigma_i^n\ldots\right]$ is reduced to $D_k\left[\ldots\sigma_i^0\ldots\right]$, the last step must be a reduction from $D_k\left[\ldots\sigma_i^2\ldots\right]$ to $D_k\left[\ldots\sigma_i^0\ldots\right]$. This will be as if we are listing the compositions of the integer n in which each summand is 1 or 2 and the final summand is always 2. One such composition is $\underbrace{1+1+\cdots+1}_{(n-2 \text{ times})} + 2$, which corresponds to the term $(-1)^{n-1}m^{n-2}\ell^{-n}$. Combining two of the 1's, there are $\binom{n-3}{1}$ possible compositions of exactly two 2's and $(n-4)$ 1's, giving the term $(-1)^{n-2}\binom{n-3}{1}m^{n-4}\ell^{-n}$. Combining two more 1's, there are $\binom{n-4}{2}$ possible compositions containing exactly three 2's and $(n-6)$ 1's, giving the term $(-1)^{n-3}\binom{n-4}{2}m^{n-6}\ell^{-n}$. In general, there are $\binom{n-2-j}{j}$ possible compositions containing of exactly $(j+1)$ 2's and $(n-2-2j)$ 1's, giving the general term $(-1)^{n-1-j}\binom{n-2-j}{j}m^{n-2-2j}\ell^{-n}$.

If n is even, the final composition of n will consist of $\frac{n}{2}$ 2's and no 1's, so

$$Q_0(n) = \ell^{-n}\sum_{j=0}^{\frac{n-2}{2}}(-1)^{n-1-j}\binom{n-2-j}{j}m^{n-2-2j}$$

If n is odd, the last compositions of $n-1$ will each consist of $\frac{n-1}{2}$ 2's and one 1, so

$$Q_0(n) = \ell^{-n}\sum_{j=0}^{\frac{n-3}{2}}(-1)^{n-1-j}\binom{n-2-j}{j}m^{n-2-2j}$$

The derivations of $Q_0(-n)$ and $Q_1(-n)$ are similar. $\square$

CHAPTER 6: MAIN RESULTS

In this chapter the braid index for a certain class of closed braids is discussed. Recall that the braid index of a link L is the minimum number of strings needed in a braid whose closure is L. The main new result will show that a reduced, alternating k-string closed braid with at most two sequences of crossings between each pair of adjacent strings has braid index equal to the number of strings in the braid, k. The proof makes use of properties of the HOMFLY polynomial. However, it is worth noting that a stronger result is easily proven for some of the simplest cases, without resorting to polynomials at all.

Theorem 6.1. *Let $1 \leq k \leq 3$ and let D_k be a reduced, alternating diagram which is the closure of a k-string braid. Let L be the link type represented by D_k. Then the braid index of L is k.*

Proof. Note that if L is represented by a diagram D_k then clearly $b(L) \leq k$, so it suffices in each case to show that $b(L) \geq k$.

For $k = 1$ there is only one 1-string braid, and its closure is the unknot. Since there is no braid with fewer strings than 1, the braid index of the unknot is 1.

For $k = 2$, note that in order for its closure to be a reduced diagram, the underlying braid must be of the form $\sigma_1^{\pm n}$ with $n \geq 2$ or $n = 0$. If $n = 0$ then L is the trivial 2-component link. Otherwise, since the diagram is alternating, we know that $Cr(L) = Cr(D_2[\sigma_1^{\pm n}]) = n \geq 2$. In either case, L is not the unknot, so $b(L) \geq 2$. Thus $b(L) = 2$.

For $k = 3$, suppose L can be expressed as the closure of a 2-string braid. Then that diagram (in its reduced form) would be of the form $D_2[\sigma_1^{\pm n}]$, where $n = Cr(D_3)$, since D_3 and $D_2[\sigma_1^{\pm n}]$ are both alternating and reduced. If n is even, then $\mu(D_3)$ is odd[1], but $\mu(D_2[\sigma_1^{\pm n}]) = 2$. And if n is odd then $\mu(D_3)$ is even, but $\mu(D_2[\sigma_1^{\pm n}]) = 1$. Since $\mu(L)$ is a link invariant, we obtain a contradiction in either case. Therefore, $b(L) \geq 3$, and then $b(L) = 3$. □

Note that the specific class of closed braids that we will be concerned with here excludes situations where there is only one crossing between two adjacent strings, because in such cases the diagram would not be reduced. The crossing in question could simply be twisted out, as in Figure 1.6. There could however be no crossings between consecutive string positions. In such a case the link would be a split link. This special case will be considered separately from other cases. In a reduced, alternating closed braid that does not represent a split link it is observed that not only is the diagram alternating in the manner defined previously, but the columns of crossings in their entirety also alternate in sign. For instance, all of the crossings between string positions 1 and 2 might be positive, and then all of the crossings between positions 2 and 3 would be negative, all crossings between positions 3 and 4 would be positive, and so on. (In a split link this would not necessarily be the case.) Because of this, it is sometimes convenient to use a slight abuse of terminology and notation by referring to σ_i as being 'positive' or 'negative'. This will simply mean that all of the crossings between strings i and $i+1$ are positive crossings, or negative crossings, respectively.

The main result will make use of the following theorem, first proved by Morton [20] and Franks and Williams [12]:

Theorem 6.2. *Let L be a link and let $P(L)$ be the HOMFLY polynomial of L. If E_ℓ and e_ℓ are the maximum and minimum exponents of ℓ, respectively, in $P(L)$, then $E_\ell - e_\ell \leq 2(S(D)-1)$, where $S(D)$ is the number of Seifert circles in a diagram of L.*

That is, this theorem gives $\frac{1}{2}(E_\ell - e_\ell) + 1$ as a lower bound for the number of Seifert circles. Important to the purpose here, it has also been shown [30] that the minimum number of Seifert circles for a link is equal to the link's braid index.

Theorem 6.2 gives a lower bound on the braid index for a link L. If it is known that there is a closed braid representation for L using a certain number of strings, then

[1] Suppose that D_3 has no crossings. Then D_3 is simply a 3-component trivial link. Now imagine that crossings are added to D_3, one at a time. Each crossing that is added either combines two components (if the strings involved were previously in distinct components) or breaks one component into two (if the strings were previously in the same component). So the effect of inserting one crossing is to change $\mu(D_3)$ by ± 1. The number of crossings inserted will thus be congruent modulo 2 to the change in $\mu(D_3)$. Since $\mu(D_3) = 3$ when $Cr(D_3) = 0$, $\mu(D_3) + Cr(D_3) \equiv 1 \pmod 2$. Therefore if $Cr(D_3)$ is even, $\mu(D_3)$ is odd, and if $Cr(D_3)$ is odd, $\mu(D_3)$ is even.

that number is an upper bound for the braid index. Therefore, if there is a closed braid representation of L using exactly $\frac{1}{2}(E_\ell - e_\ell) + 1$ string positions, then $\frac{1}{2}(E_\ell - e_\ell) + 1$ is the braid index for L.

The strategy here will be to consider the special class of k-string closed braids specified above, calculate the HOMFLY polynomial to obtain the values of E_ℓ and e_ℓ, and then show that $\frac{1}{2}(E_\ell - e_\ell) + 1 = k$ in all cases.

The calculation of the HOMFLY polynomial can become lengthy, as seen in the relatively small examples in Chapter 5. Fortunately, since the only parts of the polynomial that are relevant to the present purpose are the highest and lowest powers of ℓ, we do not necessarily need to calculate the entire polynomial. However, we cannot simply apply the skein relation and look only for the powers of ℓ, because generally there could be several branches of the resolving tree that could yield terms containing the highest or lowest power of ℓ. Without determining more precisely what those terms are, there is no way of knowing whether they might cancel each other out when all of the terms are added together. Lemma 5.2 gives us a means to calculate the HOMFLY polynomial more efficiently, and eliminate some of the uncertainty about whether some terms might cancel. Instead of using the basic skein relation to reduce one crossing at a time, we will use Lemma 5.2 to reduce whole sequences of crossings. Instead of a potentially very large and messy resolving tree, we could draw an *enhanced resolving tree* which will result in at most four expressions from which to draw high and low powers of ℓ.

Example 6.1. Figure 6.1 illustrates an example of an enhanced resolving tree for the closed braid $P(D_3[\sigma_1^2\sigma_2^{-4}\sigma_1\sigma_2^{-3}])$. (For clarity only the braids are shown in each step, rather than the entire closed braid diagram.) The first split of the tree reduces the sequence σ_2^{-4}, and then each of those branches is split to reduce the sequence σ_2^{-3}. Of the four diagrams that remain at the right, the middle two now can be reduced to the two-string closed braid $D_2[\sigma_1^3]$ by stabilization, and the top one is a connected sum of $D_2[\sigma_1^3]$ with $D_2[\sigma_1^0]$. The bottom diagram will require some more work using the regular skein relation. The

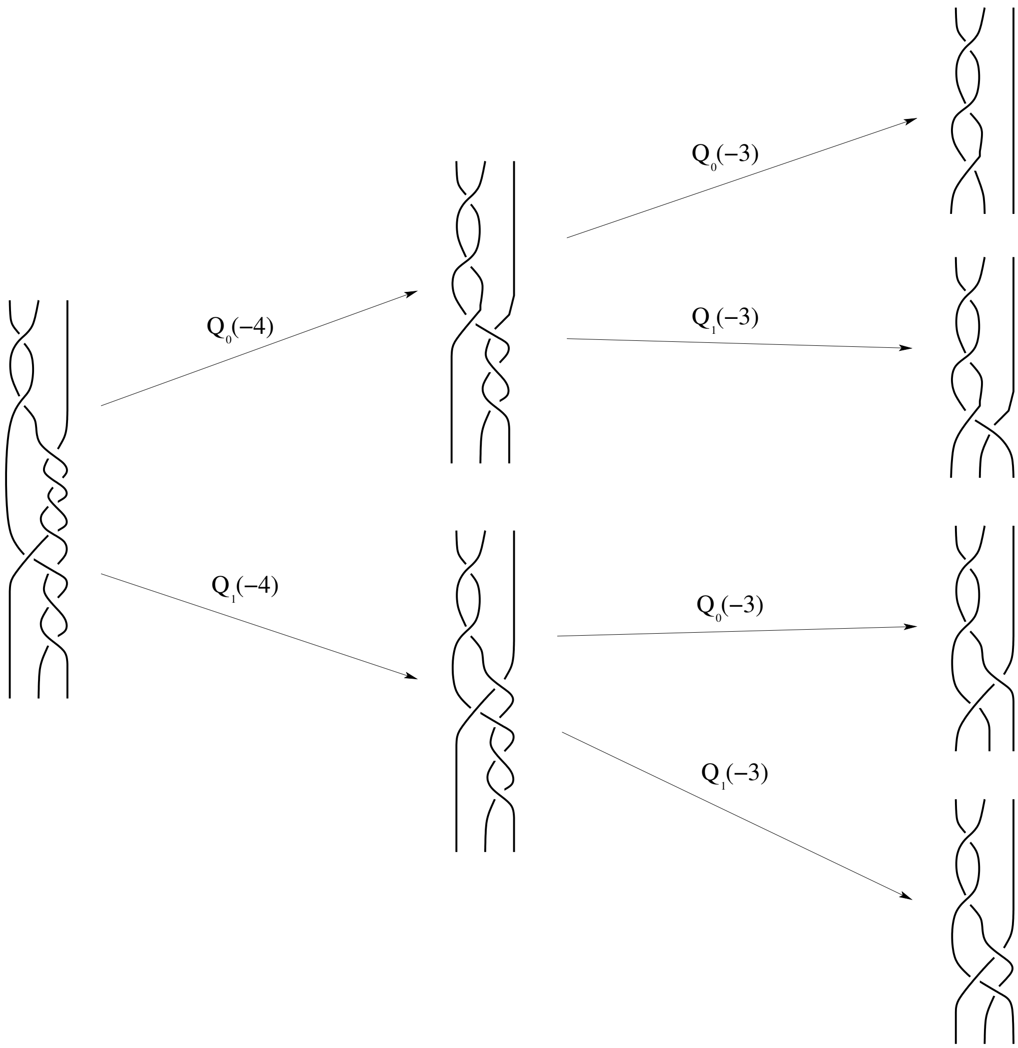

Figure 6.1: Enhanced resolving tree for $P(D_3[\sigma_1^2\sigma_2^{-4}\sigma_1\sigma_2^{-3}])$

HOMFLY polynomial for $D_3[\sigma_1^2\sigma_2^{-4}\sigma_1\sigma_2^{-3}]$ is thus

$$\begin{aligned} P(D_3[\sigma_1^2\sigma_2^{-4}\sigma_1\sigma_2^{-3}]) &= Q_0(-4)Q_0(-3)P(D_2[\sigma_1^0])P(D_2[\sigma_1^3]) \\ &\quad + [Q_0(-4)Q_1(-3) + Q_1(-4)Q_0(-3)]\, P(D_2[\sigma_1^3]) \\ &\quad + Q_1(-4)Q_1(-3)P(D_3[\sigma_1^2\sigma_2^{-1}\sigma_1\sigma_2^{-1}]) \end{aligned}$$

It will be seen in the proof of the next theorem that it is relatively easy to pick out the terms with the highest and lowest powers of ℓ from the expression on the right.

Theorem 6.3. *Let L be a link and let D_k be a reduced alternating k-string closed braid* *representation of L, with at least one and at most two sequences of consecutive crossings between each pair of adjacent strings. If $P(L)$ is the HOMFLY polynomial of L, then the minimum exponent of ℓ in $P(L)$ is $1-k-w$, and the maximum exponent of ℓ in $P(L)$ is $k-1-w$, where w is the writhe of D_k.*

Furthermore, if k is odd, then the maximum exponent of m multiplied with ℓ^{1-k-w} is $c-2k+2$, and the maximum exponent of m multiplied with ℓ^{k-1-w} is $c-2k+2$, where c is the number of crossings in D_k (also the crossing number of L). If k is even, we have two subcases: If σ_1 is positive, then the maximum exponent of m multiplied with ℓ^{1-k-w} is $c-2k+1$ and the maximum exponent of m multiplied with ℓ^{k-1-w} is $c-2k+3$. If instead σ_1 is negative, then the maximum exponent of m multiplied with ℓ^{1-k-w} is $c-2k+3$ and the maximum exponent of m multiplied with ℓ^{k-1-w} is $c-2k+1$.

Finally, the coefficient of each of the terms thus described is ± 1.

Proof. We will prove the theorem by induction on the number of strings k in the braid diagram. The basis step for the induction will include the cases $k=1$ and $k=2$, since the induction step uses the two previous values of k.

For $k=1$, only one 1-string braid exists, with no crossings and writhe 0. Its closure is the unknot, of which the HOMFLY polynomial is 1. Since $1-k-w=0$, $k-1-w=0$, and $c-2k+2=0$, all conditions hold.

For $k=2$, a reduced 2-string braid is in one of two forms, either $D_2[\sigma_1^n]$ or $D_2[\sigma_1^{-n}]$, with $n \geq 2$ (since if $n=1$ or $n=-1$ then the closure is not a reduced diagram, and if $n=0$ there are no crossings between the two adjacent strings). Note however, for use in the following calculations that $P(D_2[\sigma_1^0]) = -m^{-1}(\ell^{-1}+\ell)$ and $P(D_2[\sigma_1]) = P(D_2[\sigma_1^{-1}]) = 1$.

Let us consider $P(D_2[\sigma_1^n])$ first. Since the writhe and number of crossings are each n, we need to find $1-k-w=-1-n$ and $k-1-w=1-n$ for the smallest and largest exponents of ℓ, respectively, and we need the largest exponent of m multiplied with ℓ^{-1-n} to be $c-2k+1=n-3$, and the largest exponent of m multiplied with ℓ^{1-n} to be $c-2k+3=n-1$.

Using Lemma 5.2, if n is odd then

$$\begin{aligned}
P(D_2[\sigma_1^n]) &= Q_0(n)P(D_2[\sigma_1^0]) + Q_1(n)P(D_2[\sigma_1]) \\
&= -m^{-1}(\ell^{-1}+\ell)Q_0(n) + Q_1(n) \\
&= -m^{-1}(\ell^{-1}+\ell)\ell^{-n}\sum_{j=0}^{\frac{n-3}{2}}(-1)^j\binom{n-2-j}{j}m^{n-2-2j} \\
&\quad +\ell^{1-n}\sum_{j=0}^{\frac{n-1}{2}}(-1)^j\binom{n-1-j}{j}m^{n-1-2j} \\
&= \ell^{-1-n}\sum_{j=0}^{\frac{n-3}{2}}(-1)^{j+1}\binom{n-2-j}{j}m^{n-3-2j} \\
&\quad +\ell^{1-n}\sum_{j=0}^{\frac{n-3}{2}}(-1)^{j+1}\binom{n-2-j}{j}m^{n-3-2j} \\
&\quad +\ell^{1-n}\sum_{j=0}^{\frac{n-1}{2}}(-1)^j\binom{n-1-j}{j}m^{n-1-2j} \\
&= \ell^{-1-n}\sum_{j=0}^{\frac{n-3}{2}}(-1)^{j+1}\binom{n-2-j}{j}m^{n-3-2j} \\
&\quad +\ell^{1-n}\left(m^{n-1}+\sum_{j=0}^{\frac{n-3}{2}}(-1)^{j+1}\left[\binom{n-2-j}{j}+\binom{n-2-j}{j+1}\right]m^{n-3-2j}\right)
\end{aligned}$$

The relevant terms from this polynomial are $-\ell^{-1-n}m^{n-3}$ and $\ell^{1-n}m^{n-1}$, so the conditions hold.

If n is even then

$$\begin{aligned}
P(D_2[\sigma_1^n]) &= Q_0(n)P(D_2[\sigma_1^0]) + Q_1(n)P(D_2[\sigma_1]) \\
&= -m^{-1}(\ell^{-1}+\ell)Q_0(n) + Q_1(n) \\
&= -m^{-1}(\ell^{-1}+\ell)\ell^{-n}\sum_{j=0}^{\frac{n-2}{2}}(-1)^{j+1}\binom{n-2-j}{j}m^{n-2-2j} \\
&\quad +\ell^{1-n}\sum_{j=0}^{\frac{n-2}{2}}(-1)^{j+1}\binom{n-1-j}{j}m^{n-1-2j} \\
&= \ell^{-1-n}\sum_{j=0}^{\frac{n-2}{2}}(-1)^{j}\binom{n-2-j}{j}m^{n-3-2j} \\
&\quad +\ell^{1-n}\sum_{j=0}^{\frac{n-2}{2}}(-1)^{j}\binom{n-2-j}{j}m^{n-3-2j} \\
&\quad +\ell^{1-n}\sum_{j=0}^{\frac{n-2}{2}}(-1)^{j+1}\binom{n-1-j}{j}m^{n-1-2j} \\
&= \ell^{-1-n}\sum_{j=0}^{\frac{n-2}{2}}(-1)^{j}\binom{n-2-j}{j}m^{n-3-2j} \\
&\quad +\ell^{1-n}\left(-m^{n-1}+(-1)^{\frac{n+2}{2}}m^{-1}\right) \\
&\quad +\ell^{1-n}\sum_{j=0}^{\frac{n-4}{2}}(-1)^{j}\left[\binom{n-2-j}{j}+\binom{n-2-j}{j+1}\right]m^{n-3-2j}
\end{aligned}$$

The relevant terms here are $\ell^{-1-n}m^{n-3}$ and $-\ell^{1-n}m^{n-1}$, so the conditions hold.

Continuing with the $D_2[\sigma_1^{-n}]$ cases, since the number of crossings and writhe are each $-n$, we need to find the greatest and least exponents of ℓ to be $k-1-w=n+1$ and $1-k-w=n-1$, respectively. The greatest exponent of m multiplied with ℓ^{n+1} should be $c-2k+1=n-3$, and the greatest exponent of m multiplied with ℓ^{n-1} should be $c-2k+3=n-1$.

Applying Lemma 5.2, if n is odd then

$$\begin{aligned}
P(D_2[\sigma_1^{-n}]) &= Q_0(-n)P(D_2[\sigma_1^0]) + Q_1(-n)P(D_2[\sigma_1^{-1}]) \\
&= -m^{-1}(\ell^{-1}+\ell)Q_0(-n) + Q_1(-n) \\
&= -m^{-1}(\ell^{-1}+\ell)\ell^n \sum_{j=0}^{\frac{n-3}{2}}(-1)^j \binom{n-2-j}{j} m^{n-2-2j} \\
&\quad +\ell^{n-1}\sum_{j=0}^{\frac{n-1}{2}}(-1)^j \binom{n-1-j}{j} m^{n-1-2j} \\
&= \ell^{n-1}\sum_{j=0}^{\frac{n-3}{2}}(-1)^{j+1} \binom{n-2-j}{j} m^{n-3-2j} \\
&\quad +\ell^{n+1}\sum_{j=0}^{\frac{n-3}{2}}(-1)^{j+1} \binom{n-2-j}{j} m^{n-3-2j} \\
&\quad +\ell^{n-1}\sum_{j=0}^{\frac{n-1}{2}}(-1)^j \binom{n-1-j}{j} m^{n-1-2j} \\
&= \ell^{n-1}\left(m^{n-1} + \sum_{j=0}^{\frac{n-3}{2}}(-1)^{j+1}\left[\binom{n-2-j}{j} + \binom{n-2-j}{j+1}\right] m^{n-3-2j}\right) \\
&\quad +\ell^{n+1}\sum_{j=0}^{\frac{n-3}{2}}(-1)^{j+1} \binom{n-2-j}{j} m^{n-3-2j}
\end{aligned}$$

The relevant terms from this polynomial are $-\ell^{n+1}m^{n-3}$ and $\ell^{n-1}m^{n-1}$, so the conditions hold.

If n is even then

$$\begin{aligned}
P(D_2[\sigma_1^{-n}]) &= Q_0(-n)P(D_2[\sigma_1^0]) + Q_1(-n)P(D_2[\sigma_1^{-1}]) \\
&= -m^{-1}(\ell^{-1}+\ell)Q_0(-n) + Q_1(-n) \\
&= -m^{-1}(\ell^{-1}+\ell)\ell^n \sum_{j=0}^{\frac{n-2}{2}} (-1)^{j+1}\binom{n-2-j}{j} m^{n-2-2j} \\
&\quad +\ell^{n-1}\sum_{j=0}^{\frac{n-2}{2}} (-1)^{j+1}\binom{n-1-j}{j} m^{n-1-2j} \\
&= \ell^{n-1}\sum_{j=0}^{\frac{n-2}{2}} (-1)^{j}\binom{n-2-j}{j} m^{n-3-2j} \\
&\quad +\ell^{n+1}\sum_{j=0}^{\frac{n-2}{2}} (-1)^{j}\binom{n-2-j}{j} m^{n-3-2j} \\
&\quad +\ell^{n-1}\sum_{j=0}^{\frac{n-2}{2}} (-1)^{j+1}\binom{n-1-j}{j} m^{n-1-2j} \\
&= \ell^{n-1}\left(-m^{n-1} + (-1)^{\frac{n+2}{2}} m^{-1}\right) \\
&\quad +\ell^{n-1}\sum_{j=0}^{\frac{n-4}{2}} (-1)^{j}\left[\binom{n-2-j}{j} + \binom{n-2-j}{j+1}\right] m^{n-3-2j} \\
&\quad +\ell^{n+1}\sum_{j=0}^{\frac{n-2}{2}} (-1)^{j}\binom{n-2-j}{j} m^{n-3-2j}
\end{aligned}$$

The relevant terms here are $\ell^{n+1}m^{n-3}$ and $-\ell^{n-1}m^{n-1}$, so the conditions hold. Thus concludes the basis step of the induction.

Let us now suppose that for some $k \geq 3$ the proposition holds for all D_i with $1 \leq i \leq k-1$. For the induction step, there are many cases to consider. For all cases, the closed braid D_k will be assumed to have writhe w and number of crossings c. Also, the abbreviations HℓHm and LℓHm are introduced for purposes of brevity. We are concerned with the highest and lowest powers of ℓ, and the highest power of m multiplied with each of those powers of ℓ. The HℓHm term of a polynomial will indicate the term containing the highest m power among all terms containing the highest ℓ power. The LℓHm term of a polynomial will indicate the term containing the highest m power among all terms containing the lowest ℓ power.

Case 1, subcases 1 and 2. First let us consider the cases where a reduced alternating closed braid $D_k[\ldots\sigma_{k-1}^a\ldots]$ could be thought to have been formed from a reduced alternating closed braid D_{k-1} by adding one string to the braid and a single sequence of consecutive crossings σ_{k-1}^a, with $a > 1$ (If $a = 1$ then the closed braid diagram could be reduced back to $k-1$ strings by a Reidemeister type I move). If we apply Lemma 5.2 we obtain

$$\begin{aligned} P(D_k[\ldots\sigma_{k-1}^a\ldots]) &= Q_0(a)P(D_k[\ldots\sigma_{k-1}^0\ldots]) + Q_1(a)P(D_k[\ldots\sigma_{k-1}^1\ldots]) \\ &= Q_0(a)P(D_2[\sigma_1^0])P(D_{k-1}) + Q_1(a)P(D_{k-1}) \\ &= \left[Q_0(a)P(D_2[\sigma_1^0]) + Q_1(a)\right] P(D_{k-1}) \\ &= P(D_2[\sigma_1^a])P(D_{k-1}) \end{aligned}$$

This is to be expected due to the fact that $D_k[\ldots\sigma_{k-1}^a\ldots]$ is simply a connected sum of D_{k-1} and $D_2[\sigma_1^a]$. See Figure 6.2. $P(D_2[\sigma_1^a])$ has HℓHm term $\pm\ell^{1-a}m^{a-1}$ and LℓHm term $\pm\ell^{-1-a}m^{a-3}$, as seen in the $k = 2$ case. Note that the diagram D_{k-1} has $c-a$ crossings and writhe $w-a$. If k is even, then $k-1$ is odd so the HℓHm term of D_{k-1} is $\pm\ell^{(k-1)-1-(w-a)}m^{(c-a)-2(k-1)+2} = \pm\ell^{k-2-w+a}m^{c-a-2k+4}$ and the LℓHm term of D_{k-1} is $\pm\ell^{1-(k-1)-(w-a)}m^{(c-a)-2(k-1)+2} = \pm\ell^{2-k-w+a}m^{c-a-2k+4}$. Therefore the H$\ell$H$m$ term of the product $P(D_2[\sigma_1^a])P(D_{k-1})$ is $\pm\ell^{k-1-w}m^{c-2k+3}$ and the LℓHm term is $\pm\ell^{1-k-w}m^{c-2k+1}$.

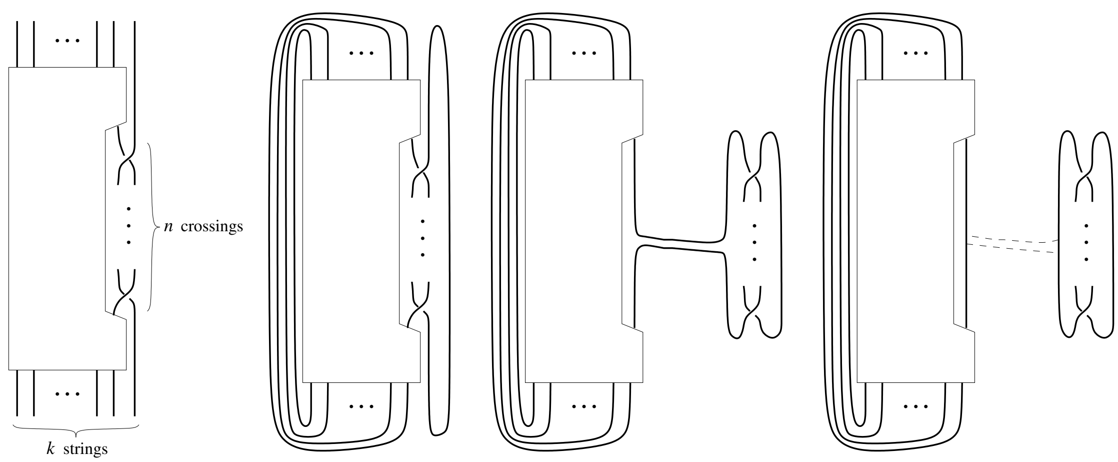

Figure 6.2: The closed braid $D_k[\ldots\sigma_{k-1}^n\ldots]$ is a connected sum of D_{k-1} and $D_2[\sigma_1^n]$

If k is odd, then $k-1$ is even and σ_1 is negative so the HℓHm term of D_{k-1} is $\pm\ell^{(k-1)-1-(w-a)}m^{(c-a)-2(k-1)+1} = \pm\ell^{k-2-w+a}m^{c-a-2k+3}$ and the LℓHm term of D_{k-1} is $\pm\ell^{1-(k-1)-(w-a)}m^{(c-a)-2(k-1)+3} = \pm\ell^{2-k-w+a}m^{c-a-2k+5}$. Therefore the H$\ell$H$m$ term of the product $P(D_2[\sigma_1^a])P(D_{k-1})$ is $\pm\ell^{k-1-w}m^{c-2k+2}$ and the LℓHm term is $\pm\ell^{1-k-w}m^{c-2k+2}$.

Case 1, subcases 3 and 4. Next we consider the cases in which a reduced alternating closed braid $D_k[\ldots\sigma_{k-1}^{-a}\ldots]$ is formed from a reduced alternating closed braid D_{k-1} by adding one string to the braid and a single sequence of consecutive crossings σ_{k-1}^{-a}, with $a > 1$. Applying the lemma we obtain

$$\begin{aligned}
P(D_k[\ldots\sigma_{k-1}^{-a}\ldots]) &= Q_0(-a)P(D_k[\ldots\sigma_{k-1}^{0}\ldots]) + Q_1(-a)P(D_k[\ldots\sigma_{k-1}^{-1}\ldots]) \\
&= Q_0(-a)P(D_2[\sigma_1^0])P(D_{k-1}) + Q_1(-a)P(D_{k-1}) \\
&= \left[Q_0(-a)P(D_2[\sigma_1^0]) + Q_1(-a)\right]P(D_{k-1}) \\
&= P(D_2[\sigma_1^{-a}])P(D_{k-1})
\end{aligned}$$

$P(D_2[\sigma_1^{-a}])$ has HℓHm term $\pm\ell^{a+1}m^{a-3}$ and LℓHm term $\pm\ell^{a-1}m^{a-1}$. Note that the diagram D_{k-1} has $c-a$ crossings and writhe $w+a$. If k is even, then $k-1$ is odd so the HℓHm term of D_{k-1} is $\pm\ell^{(k-1)-1-(w+a)}m^{(c-a)-2(k-1)+2} = \pm\ell^{k-2-w-a}m^{c-a-2k+4}$ and the LℓHm term of D_{k-1} is $\pm\ell^{1-(k-1)-(w+a)}m^{(c-a)-2(k-1)+2} = \pm\ell^{2-k-w-a}m^{c-a-2k+4}$. Therefore the H$\ell$H$m$ term of the product $P(D_2[\sigma_1^{-a}])P(D_{k-1})$ is $\pm\ell^{k-1-w}m^{c-2k+1}$ and the LℓHm term is $\pm\ell^{1-k-w}m^{c-2k+3}$.

If k is odd, then $k-1$ is even and σ_1 is positive so the HℓHm term of D_{k-1} is $\pm\ell^{(k-1)-1-(w+a)}m^{(c-a)-2(k-1)+3} = \pm\ell^{k-2-w-a}m^{c-a-2k+5}$ and the LℓHm term of D_{k-1} is $\pm\ell^{1-(k-1)-(w+a)}m^{(c-a)-2(k-1)+1} = \pm\ell^{2-k-w-a}m^{c-a-2k+3}$. Therefore the H$\ell$H$m$ term of the product $P(D_2[\sigma_1^{-a}])P(D_{k-1})$ is $\pm\ell^{k-1-w}m^{c-2k+2}$ and the LℓHm term is $\pm\ell^{1-k-w}m^{c-2k+2}$.

Case 2, subcases 1 and 2. The next set of cases to consider are those in which D_k is formed by adding one string to D_{k-1} and exactly two non-consecutive crossings σ_{k-1}. In order to be truly non-consecutive, there must be crossings σ_{k-2}^{-a} and σ_{k-2}^{-b}, with $a \geq 1$ and $b \geq 1$, separating the two σ_{k-1} crossings. That is, the braid word must be of a form such as

$$\ldots\sigma_{k-1}\ldots\sigma_{k-2}^{-a}\ldots\sigma_{k-1}\ldots\sigma_{k-2}^{-b}\ldots$$

for otherwise the σ_{k-1} crossings could commute with other crossings and combine into a single sequence of crossings σ_{k-1}^2. We shall use the notation $D_k[\dots\sigma_{k-1}\dots\sigma_{k-2}^{-a}\dots\sigma_{k-1}\dots\sigma_{k-2}^{-b}\dots]$ to indicate this diagram, and changes made to these crossings while leaving the rest of the diagram unchanged.

Since σ_{k-1} commutes with all symbols in the braid word except the σ_{k-2}'s, we can rearrange the braid word so that both crossings σ_{k-1} are adjacent to one of the sequences σ_{k-2}^{-a} or σ_{k-2}^{-b}. For example, $D_k[\dots\sigma_{k-1}\dots\sigma_{k-2}^{-a}\dots\sigma_{k-1}\dots\sigma_{k-2}^{-b}\dots]$ could be written as $D_k[\dots\sigma_{k-1}\sigma_{k-2}^{-a}\sigma_{k-1}\dots\sigma_{k-2}^{-b}\dots]$.

Now applying the HOMFLY skein relation to one of the σ_{k-1} crossings, we obtain

$$\begin{aligned} P(D_k[\dots\sigma_{k-1}\sigma_{k-2}^{-a}\sigma_{k-1}\dots\sigma_{k-2}^{-b}\dots]) &= -\ell^{-2}P(D_k[\dots\sigma_{k-1}^{-1}\sigma_{k-2}^{-a}\sigma_{k-1}\dots\sigma_{k-2}^{-b}\dots]) \\ &\quad -m\ell^{-1}P(D_k[\dots\sigma_{k-2}^{-a}\sigma_{k-1}\dots\sigma_{k-2}^{-b}]) \end{aligned}$$

In the far right term, the remaining σ_{k-1} crossing can be removed by a Reidemeister type I move, so that diagram is reduced to D_{k-1}. In the first term on the right hand side, we can apply Theorem 1.1:

$$\dots\sigma_{k-1}^{-1}\sigma_{k-2}^{-a}\sigma_{k-1}\dots\sigma_{k-2}^{-b}\dots = \dots\sigma_{k-2}\sigma_{k-1}^{-a}\sigma_{k-2}^{-1}\dots\sigma_{k-2}^{-b}\dots$$

This closed braid is now clearly a connected sum of $D_{k-1}[\dots\sigma_{k-2}\sigma_{k-2}^{-1}\dots\sigma_{k-2}^{-b}\dots]$ and $D_2[\sigma_1^{-a}]$. Indeed, we can go further: $D_{k-1}[\dots\sigma_{k-2}\sigma_{k-2}^{-1}\dots\sigma_{k-2}^{-b}\dots] = D_{k-1}[\dots\sigma_{k-2}^{-b}\dots]$ is the connected sum of $D_2[\sigma_1^{-b}]$ and the reduced alternating closed braid D_{k-2}. Therefore we find that

$$\begin{aligned} P(D_k[\dots\sigma_{k-1}\sigma_{k-2}^{-a}\sigma_{k-1}\dots\sigma_{k-2}^{-b}\dots]) &= -\ell^{-2}P(D_2[\sigma_1^{-a}])P(D_2[\sigma_1^{-b}])P(D_{k-2}) \\ &\quad -m\ell^{-1}P(D_{k-1}) \end{aligned}$$

Note that D_{k-1} has $c-2$ crossings and writhe $w-2$, and that D_{k-2} has $c-2-a-b$ crossings and writhe $w-2+a+b$.

Suppose that k is odd. Then the HℓHm term of $-\ell^{-2}P(D_2[\sigma_1^{-a}])P(D_2[\sigma_1^{-b}])P(D_{k-2})$

is $-\ell^{-2}(\pm\ell^{a+1}m^{a-3})(\pm\ell^{b+1}m^{b-3})(\pm\ell^{(k-2)-1-(w-2+a+b)}m^{(c-2-a-b)-2(k-2)+2}) =$ $\pm\, \ell^{k-1-w}m^{c-2k-2}$, and the H$\ell$H$m$ term of $-m\ell^{-1}P(D_{k-1})$ is $-m\ell^{-1}(\pm\ell^{(k-1)-1-(w-2)}m^{(c-2)-2(k-1)+1}) = \pm\ell^{k-1-w}m^{c-2k+2}$. Thus the H$\ell$H$m$ term for the entire expression is $\pm\ell^{k-1-w}m^{c-2k+2}$.

Likewise, the LℓHm term of $-\ell^{-2}P(D_2[\sigma_1^{-a}])P(D_2[\sigma_1^{-b}])P(D_{k-2})$ is $-\ell^{-2}(\pm\ell^{a-1}m^{a-1})(\pm\ell^{b-1}m^{b-1})(\pm\ell^{1-(k-2)-(w-2+a+b)}m^{(c-2-a-b)-2(k-2)+2}) =$ $\pm\, \ell^{1-k-w}m^{c-2k+2}$, and the L$\ell$H$m$ term of $-m\ell^{-1}P(D_{k-1})$ is $-m\ell^{-1}(\pm\ell^{1-(k-1)-(w-2)}m^{(c-2)-2(k-1)+3}) = \pm\ell^{3-k-w}m^{c-2k+4}$. Thus the L$\ell$H$m$ term for the entire expression is $\pm\ell^{1-k-w}m^{c-2k+2}$.

Now suppose that k is even. In this case the HℓHm term of $-\ell^{-2}P(D_2[\sigma_1^{-a}])P(D_2[\sigma_1^{-b}])P(D_{k-2})$ is $-\ell^{-2}(\pm\ell^{a+1}m^{a-3})(\pm\ell^{b+1}m^{b-3})(\pm\ell^{(k-2)-1-(w-2+a+b)}m^{(c-2-a-b)-2(k-2)+3}) =$ $\pm\, \ell^{k-1-w}m^{c-2k-1}$, and the H$\ell$H$m$ term of $-m\ell^{-1}P(D_{k-1})$ is $-m\ell^{-1}(\pm\ell^{(k-1)-1-(w-2)}m^{(c-2)-2(k-1)+2}) = \pm\ell^{k-1-w}m^{c-2k+3}$. Thus the H$\ell$H$m$ term for the entire expression is $\pm\ell^{k-1-w}m^{c-2k+3}$.

Likewise, the LℓHm term of $-\ell^{-2}P(D_2[\sigma_1^{-a}])P(D_2[\sigma_1^{-b}])P(D_{k-2})$ is $-\ell^{-2}(\pm\ell^{a-1}m^{a-1})(\pm\ell^{b-1}m^{b-1})(\pm\ell^{1-(k-2)-(w-2+a+b)}m^{(c-2-a-b)-2(k-2)+1}) =$ $\pm\, \ell^{1-k-w}m^{c-2k+1}$, and the L$\ell$H$m$ term of $-m\ell^{-1}P(D_{k-1})$ is $-m\ell^{-1}(\pm\ell^{1-(k-1)-(w-2)}m^{(c-2)-2(k-1)+2}) = \pm\ell^{3-k-w}m^{c-2k+3}$. Thus the L$\ell$H$m$ term for the entire expression is $\pm\ell^{1-k-w}m^{c-2k+1}$.

Case 2, subcases 3 and 4. We follow a similar procedure if D_k is formed by adding one string to D_{k-1} and exactly two non-consecutive crossings σ_{k-1}^{-1}. Here again the braid word must be of a form such as

$$\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-2}^{a}\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-2}^{b}\ldots$$

with $a \geq 1$ and $b \geq 1$, and we will use the notation $D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-2}^{a}\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-2}^{b}\ldots]$ to indicate changes made to the crossings shown while leaving the rest of the diagram unchanged.

As before we will rearrange the braid word so that both crossings σ_{k-1}^{-1} are adjacent to one of the sequences σ_{k-2}^a or σ_{k-2}^b. For example, $D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-2}^a\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-2}^b\ldots]$ could be written as $D_k[\ldots\sigma_{k-1}^{-1}\sigma_{k-2}^a\sigma_{k-1}^{-1}\ldots\sigma_{k-2}^b\ldots]$.

Now applying the HOMFLY skein relation to one of the σ_{k-1}^{-1} crossings, we obtain

$$\begin{aligned} P(D_k[\ldots\sigma_{k-1}^{-1}\sigma_{k-2}^a\sigma_{k-1}^{-1}\ldots\sigma_{k-2}^b\ldots]) &= -\ell^2 P(D_k[\ldots\sigma_{k-1}^{-1}\sigma_{k-2}^a\sigma_{k-1}\ldots\sigma_{k-2}^b\ldots]) \\ &\quad -m\ell P(D_k[\ldots\sigma_{k-1}^{-1}\sigma_{k-2}^a\ldots\sigma_{k-2}^b\ldots]) \end{aligned}$$

In the far right term, the remaining σ_{k-1}^{-1} crossing can be removed by a Reidemeister type I move, so that diagram is reduced to D_{k-1}. In the first term on the right hand side, we apply Theorem 1.1:

$$\ldots\sigma_{k-1}^{-1}\sigma_{k-2}^a\sigma_{k-1}\ldots\sigma_{k-2}^b\cdots = \ldots\sigma_{k-2}\sigma_{k-1}^a\sigma_{k-2}^{-1}\ldots\sigma_{k-2}^b\ldots$$

This closed braid is a connected sum of $D_{k-1}[\ldots\sigma_{k-2}\sigma_{k-2}^{-1}\ldots\sigma_{k-2}^b\ldots]$ and $D_2[\sigma_1^a]$, and $D_{k-1}[\ldots\sigma_{k-2}\sigma_{k-2}^{-1}\ldots\sigma_{k-2}^b\ldots] = D_{k-1}[\ldots\sigma_{k-2}^b\ldots]$ is the connected sum of $D_2[\sigma_1^b]$ and the reduced alternating closed braid D_{k-2}. Therefore we find that

$$P(D_k[\ldots\sigma_{k-1}^{-1}\sigma_{k-2}^a\sigma_{k-1}^{-1}\ldots\sigma_{k-2}^b\ldots]) = -\ell^2 P(D_2[\sigma_1^a])P(D_2[\sigma_1^b])P(D_{k-2}) - m\ell P(D_{k-1})$$

Note that D_{k-1} has $c-2$ crossings and writhe $w+2$, and that D_{k-2} has $c-2-a-b$ crossings and writhe $w+2-a-b$.

Suppose that k is odd. Then the HℓHm term of $-\ell^2 P(D_2[\sigma_1^a])P(D_2[\sigma_1^b])P(D_{k-2})$ is $-\ell^2(\pm\ell^{1-a}m^{a-1})(\pm\ell^{1-b}m^{b-1})(\pm\ell^{(k-2)-1-(w+2-a-b)}m^{(c-2-a-b)-2(k-2)+2}) =$ $\pm\,\ell^{k-1-w}m^{c-2k+2}$, and the H$\ell$H$m$ term of $-m\ell P(D_{k-1})$ is $-m\ell(\pm\ell^{(k-1)-1-(w+2)}m^{(c-2)-2(k-1)+3}) = \pm\ell^{k-3-w}m^{c-2k+4}$. Thus the H$\ell$H$m$ term for the entire expression is $\pm\ell^{k-1-w}m^{c-2k+2}$.

Likewise, the LℓHm term of $-\ell^2 P(D_2[\sigma_1^a])P(D_2[\sigma_1^b])P(D_{k-2})$ is $-\ell^2(\pm\ell^{-1-a}m^{a-3})(\pm\ell^{-1-b}m^{b-3})(\pm\ell^{1-(k-2)-(w+2-a-b)}m^{(c-2-a-b)-2(k-2)+2}) =$ $\pm\,\ell^{1-k-w}m^{c-2k-2}$, and the L$\ell$H$m$ term of $-m\ell P(D_{k-1})$ is $-m\ell(\pm\ell^{1-(k-1)-(w+2)}m^{(c-2)-2(k-1)+1}) = \pm\ell^{1-k-w}m^{c-2k+2}$. Thus the L$\ell$H$m$ term for the entire expression is $\pm\ell^{1-k-w}m^{c-2k+2}$.

Now suppose that k is even. In this case the HℓHm term of $-\ell^2 P(D_2[\sigma_1^a])P(D_2[\sigma_1^b])P(D_{k-2})$ is $-\ell^2(\pm\ell^{1-a}m^{a-1})(\pm\ell^{1-b}m^{b-1})(\pm\ell^{(k-2)-1-(w+2-a-b)}m^{(c-2-a-b)-2(k-2)+1}) = \pm\,\ell^{k-1-w}m^{c-2k+1}$, and the H$\ell$H$m$ term of $-m\ell P(D_{k-1})$ is $-m\ell(\pm\ell^{(k-1)-1-(w+2)}m^{(c-2)-2(k-1)+2}) = \pm\ell^{k-3-w}m^{c-2k+3}$. Thus the H$\ell$H$m$ term for the entire expression is $\pm\ell^{k-1-w}m^{c-2k+1}$.

Likewise, the LℓHm term of $-\ell^2 P(D_2[\sigma_1^a])P(D_2[\sigma_1^b])P(D_{k-2})$ is $-\ell^2(\pm\ell^{-1-a}m^{a-3})(\pm\ell^{-1-b}m^{b-3})(\pm\ell^{1-(k-2)-(w+2-a-b)}m^{(c-2-a-b)-2(k-2)+3}) = \pm\,\ell^{1-k-w}m^{c-2k-1}$, and the L$\ell$H$m$ term of $-m\ell P(D_{k-1})$ is $-m\ell(\pm\ell^{1-(k-1)-(w+2)}m^{(c-2)-2(k-1)+2}) = \pm\ell^{1-k-w}m^{c-2k+3}$ Thus the LℓHm term for the entire expression is $\pm\ell^{1-k-w}m^{c-2k+3}$.

Case 3, subcase 1. Here k is odd and and D_k is formed from D_{k-1} by adding one string to the braid, one sequence σ_{k-1}^a with $a > 1$ and one crossing σ_{k-1} that is separate from the sequence σ_{k-1}^a. Applying the lemma to the sequence σ_{k-1}^a, we have

$$P(D_k[\ldots\sigma_{k-1}^a\ldots\sigma_{k-1}\ldots]) = Q_0(a)P(D_{k-1}) + Q_1(a)P(D_k[\ldots\sigma_{k-1}\ldots\sigma_{k-1}\ldots])$$

Notice that the polynomial $P(D_k[\ldots\sigma_{k-1}\ldots\sigma_{k-1}\ldots])$ is exactly the situation considered in Case 2. Note also that with k odd, $k-1$ is even and σ_1 is negative, and the writhes and crossing numbers of the relevant diagrams are as follows:

$$wr(D_{k-1}) = w - a - 1$$

$$Cr(D_{k-1}) = c - a - 1$$

$$wr(D_k[\ldots\sigma_{k-1}\ldots\sigma_{k-1}\ldots]) = w - a + 1$$

$$Cr(D_k[\ldots\sigma_{k-1}\ldots\sigma_{k-1}\ldots]) = c - a + 1$$

We are then able to calculate the necessary terms of $P(D_k)$. The HℓHm term of

$Q_0(a)P(D_{k-1})$ is

$$(\pm\ell^{-a}m^{a-2})(\pm\ell^{(k-1)-1-(w-a-1)}m^{(c-a-1)-2(k-1)+1}) = \pm\ell^{k-1-w}m^{c-2k}$$

and the HℓHm term of $Q_1(a)P(D_k[\ldots\sigma_{k-1}\ldots\sigma_{k-1}\ldots])$ is

$$(\pm\ell^{1-a}m^{a-1})(\pm\ell^{k-1-(w-a+1)}m^{(c-a+1)-2k+2}) = \pm\ell^{k-1-w}m^{c-2k+2}$$

Similarly, the LℓHm term of $Q_0(a)P(D_{k-1})$ is

$$(\pm\ell^{-a}m^{a-2})(\pm\ell^{1-(k-1)-(w-a-1)}m^{(c-a-1)-2(k-1)+3}) = \pm\ell^{3-k-w}m^{c-2k+2}$$

and the LℓHm term of $Q_1(a)P(D_k[\ldots\sigma_{k-1}\ldots\sigma_{k-1}\ldots])$ is

$$(\pm\ell^{1-a}m^{a-1})(\pm\ell^{1-k-(w-a+1)}m^{(c-a+1)-2k+2}) = \pm\ell^{1-k-w}m^{c-2k+2}$$

Therefore the HℓHm term of $P(D_k)$ is $\pm\ell^{k-1-w}m^{c-2k+2}$ and the LℓHm term of $P(D_k)$ is $\pm\ell^{1-k-w}m^{c-2k+2}$.

Case 3, subcase 2. This will be the same as subcase 1 except that here k is even, so $k-1$ is odd and σ_1 is positive.

The HℓHm term of $Q_0(a)P(D_{k-1})$ is

$$(\pm\ell^{-a}m^{a-2})(\pm\ell^{(k-1)-1-(w-a-1)}m^{(c-a-1)-2(k-1)+2}) = \pm\ell^{k-1-w}m^{c-2k+1}$$

and the HℓHm term of $Q_1(a)P(D_k[\ldots\sigma_{k-1}\ldots\sigma_{k-1}\ldots])$ is

$$(\pm\ell^{1-a}m^{a-1})(\pm\ell^{k-1-(w-a+1)}m^{(c-a+1)-2k+3}) = \pm\ell^{k-1-w}m^{c-2k+3}$$

Similarly, the LℓHm term of $Q_0(a)P(D_{k-1})$ is

$$(\pm\ell^{-a}m^{a-2})(\pm\ell^{1-(k-1)-(w-a-1)}m^{(c-a-1)-2(k-1)+2}) = \pm\ell^{3-k-w}m^{c-2k+1}$$

and the LℓHm term of $Q_1(a)P(D_k[\ldots\sigma_{k-1}\ldots\sigma_{k-1}\ldots])$ is

$$(\pm\ell^{1-a}m^{a-1})(\pm\ell^{1-k-(w-a+1)}m^{(c-a+1)-2k+1}) = \pm\ell^{1-k-w}m^{c-2k+1}$$

Therefore the HℓHm term of $P(D_k)$ is $\pm\ell^{k-1-w}m^{c-2k+3}$ and the LℓHm term of $P(D_k)$ is $\pm\ell^{1-k-w}m^{c-2k+1}$.

Case 3, subcase 3. Here k is odd and and D_k is formed from D_{k-1} by adding one string to the braid, one sequence σ_{k-1}^{-a} with $a > 1$ and one crossing σ_{k-1}^{-1} that is separate from the sequence σ_{k-1}^{-a}. Applying the lemma to the sequence σ_{k-1}^{-a}, we have

$$P(D_k[\ldots\sigma_{k-1}^{-a}\ldots\sigma_{k-1}^{-1}\ldots]) = Q_0(-a)P(D_{k-1}) + Q_1(-a)P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-1}\ldots])$$

Notice that the polynomial $P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-1}\ldots])$ is exactly the situation considered in Case 2. Note also that with k odd, $k-1$ is even and σ_1 is positive, and the writhes and crossing numbers of the relevant diagrams are as follows:

$$wr(D_{k-1}) = w + a + 1$$

$$Cr(D_{k-1}) = c - a - 1$$

$$wr(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-1}\ldots]) = w + a - 1$$

$$Cr(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-1}\ldots]) = c - a + 1$$

The HℓHm term of $Q_0(-a)P(D_{k-1})$ is

$$(\pm\ell^{a}m^{a-2})(\pm\ell^{(k-1)-1-(w+a+1)}m^{(c-a-1)-2(k-1)+3}) = \pm\ell^{k-3-w}m^{c-2k+2}$$

and the HℓHm term of $Q_1(-a)P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-1}\ldots])$ is

$$(\pm\ell^{a-1}m^{a-1})(\pm\ell^{k-1-(w+a-1)}m^{(c-a+1)-2k+2}) = \pm\ell^{k-1-w}m^{c-2k+2}$$

Similarly, the LℓHm term of $Q_0(-a)P(D_{k-1})$ is

$$(\pm\ell^{a}m^{a-2})(\pm\ell^{1-(k-1)-(w+a+1)}m^{(c-a-1)-2(k-1)+1}) = \pm\ell^{1-k-w}m^{c-2k}$$

and the LℓHm term of $Q_1(-a)P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-1}\ldots])$ is

$$(\pm\ell^{a-1}m^{a-1})(\pm\ell^{1-k-(w+a-1)}m^{(c-a+1)-2k+2}) = \pm\ell^{1-k-w}m^{c-2k+2}$$

Therefore the HℓHm term of $P(D_k)$ is $\pm\ell^{k-1-w}m^{c-2k+2}$ and the LℓHm term of $P(D_k)$ is $\pm\ell^{1-k-w}m^{c-2k+2}$.

Case 3, subcase 4. This will be the same as subcase 3 except that here k is even, so $k-1$ is odd and σ_1 is negative.

The HℓHm term of $Q_0(-a)P(D_{k-1})$ is

$$(\pm\ell^{a}m^{a-2})(\pm\ell^{(k-1)-1-(w+a+1)}m^{(c-a-1)-2(k-1)+2}) = \pm\ell^{k-3-w}m^{c-2k+1}$$

and the HℓHm term of $Q_1(-a)P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-1}\ldots])$ is

$$(\pm\ell^{a-1}m^{a-1})(\pm\ell^{k-1-(w+a-1)}m^{(c-a+1)-2k+1}) = \pm\ell^{k-1-w}m^{c-2k+1}$$

Similarly, the LℓHm term of $Q_0(-a)P(D_{k-1})$ is

$$(\pm\ell^{a}m^{a-2})(\pm\ell^{1-(k-1)-(w+a+1)}m^{(c-a-1)-2(k-1)+2}) = \pm\ell^{1-k-w}m^{c-2k+1}$$

and the LℓHm term of $Q_1(-a)P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-1}\ldots])$ is

$$(\pm\ell^{a-1}m^{a-1})(\pm\ell^{1-k-(w+a-1)}m^{(c-a+1)-2k+3}) = \pm\ell^{1-k-w}m^{c-2k+3}$$

Therefore the HℓHm term of $P(D_k)$ is $\pm\ell^{k-1-w}m^{c-2k+1}$ and the LℓHm term of $P(D_k)$ is $\pm\ell^{1-k-w}m^{c-2k+3}$.

Case 4. The situations considered here are those in which a new string is added to a closed braid D_{k-1} and two sequences each with more than one crossing are added.

Subcase 1. k is odd and $D_k = D_k[\dots\sigma_{k-1}^a \dots \sigma_{k-1}^b \dots]$, with $a > 1$ and $b > 1$. Applying the lemma to the sequence σ_{k-1}^a, we have

$$P(D_k[\dots\sigma_{k-1}^a \dots \sigma_{k-1}^b \dots]) = Q_0(a)P(D_k[\dots\sigma_{k-1}^b \dots]) + Q_1(a)P(D_k[\dots\sigma_{k-1} \dots \sigma_{k-1}^b \dots])$$

Notice that the polynomial $P(D_k[\dots\sigma_{k-1} \dots \sigma_{k-1}^b \dots])$ is exactly the situation considered in Case 3, and the polynomial $P(D_k[\dots\sigma_{k-1}^b \dots]) = P(D_{k-1})P(D_2[\sigma_1^b])$ is the situation considered in Case 1. With k odd, σ_1 is negative, and the writhes and crossing numbers of the relevant diagrams are as follows:

$$wr(D_k[\dots\sigma_{k-1}^b \dots]) = w - a$$

$$Cr(D_k[\dots\sigma_{k-1}^b \dots]) = c - a$$

$$wr(D_k[\dots\sigma_{k-1} \dots \sigma_{k-1}^b \dots]) = w - a + 1$$

$$Cr(D_k[\dots\sigma_{k-1} \dots \sigma_{k-1}^b \dots]) = c - a + 1$$

We are then able to calculate the necessary terms of $P(D_k)$. The HℓHm term of $Q_0(a)P(D_k[\dots\sigma_{k-1}^b \dots])$ is

$$(\pm\ell^{-a}m^{a-2})(\pm\ell^{k-1-(w-a)}m^{(c-a)-2k+2}) = \pm\ell^{k-1-w}m^{c-2k}$$

and the HℓHm term of $Q_1(a)P(D_k[\dots\sigma_{k-1} \dots \sigma_{k-1}^b \dots])$ is

$$(\pm\ell^{1-a}m^{a-1})(\pm\ell^{k-1-(w-a+1)}m^{(c-a+1)-2k+2}) = \pm\ell^{k-1-w}m^{c-2k+2}$$

Similarly, the LℓHm term of $Q_0(a)P(D_k[\dots\sigma_{k-1}^b \dots])$ is

$$(\pm\ell^{-a}m^{a-2})(\pm\ell^{1-k-(w-a)}m^{(c-a)-2k+2}) = \pm\ell^{1-k-w}m^{c-2k}$$

and the LℓHm term of $Q_1(a)P(D_k[\ldots\sigma_{k-1}\ldots\sigma_{k-1}^b\ldots])$ is

$$(\pm\ell^{1-a}m^{a-1})(\pm\ell^{1-k-(w-a+1)}m^{(c-a+1)-2k+2}) = \pm\ell^{1-k-w}m^{c-2k+2}$$

Therefore the HℓHm term of $P(D_k)$ is $\pm\ell^{k-1-w}m^{c-2k+2}$ and the LℓHm term of $P(D_k)$ is $\pm\ell^{1-k-w}m^{c-2k+2}$.

Subcase 2. k is even and $D_k = D_k[\ldots\sigma_{k-1}^a\ldots\sigma_{k-1}^b\ldots]$, with $a > 1$ and $b > 1$. This is the same as subcase 1 except with k even, σ_1 is positive.

The HℓHm term of $Q_0(a)P(D_k[\ldots\sigma_{k-1}^b\ldots])$ is

$$(\pm\ell^{-a}m^{a-2})(\pm\ell^{k-1-(w-a)}m^{(c-a)-2k+3}) = \pm\ell^{k-1-w}m^{c-2k+1}$$

and the HℓHm term of $Q_1(a)P(D_k[\ldots\sigma_{k-1}\ldots\sigma_{k-1}^b\ldots])$ is

$$(\pm\ell^{1-a}m^{a-1})(\pm\ell^{k-1-(w-a+1)}m^{(c-a+1)-2k+3}) = \pm\ell^{k-1-w}m^{c-2k+3}$$

Similarly, the LℓHm term of $Q_0(a)P(D_k[\ldots\sigma_{k-1}^b\ldots])$ is

$$(\pm\ell^{-a}m^{a-2})(\pm\ell^{1-k-(w-a)}m^{(c-a)-2k+1}) = \pm\ell^{1-k-w}m^{c-2k-1}$$

and the LℓHm term of $Q_1(a)P(D_k[\ldots\sigma_{k-1}\ldots\sigma_{k-1}^b\ldots])$ is

$$(\pm\ell^{1-a}m^{a-1})(\pm\ell^{1-k-(w-a+1)}m^{(c-a+1)-2k+1}) = \pm\ell^{1-k-w}m^{c-2k+1}$$

Therefore the HℓHm term of $P(D_k)$ is $\pm\ell^{k-1-w}m^{c-2k+3}$ and the LℓHm term of $P(D_k)$ is $\pm\ell^{1-k-w}m^{c-2k+1}$.

Subcase 3. k is odd and $D_k = D_k[\ldots\sigma_{k-1}^{-a}\ldots\sigma_{k-1}^{-b}\ldots]$, with $a > 1$ and $b > 1$. Applying the lemma to the sequence σ_{k-1}^{-a}, we have

$$\begin{aligned} P(D_k[\ldots\sigma_{k-1}^{-a}\ldots\sigma_{k-1}^{-b}\ldots]) &= Q_0(-a)P(D_k[\ldots\sigma_{k-1}^{-b}\ldots]) \\ &\quad +Q_1(-a)P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-b}\ldots]) \end{aligned}$$

As before, the polynomial $P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-b}\ldots])$ was considered in Case 3, and the polynomial $P(D_k[\ldots\sigma_{k-1}^{-b}\ldots]) = P(D_{k-1})P(D_2[\sigma_1^{-b}])$ was considered in Case 1. With k odd, σ_1 is positive, and the writhes and crossing numbers of the relevant diagrams are as follows:

$$wr(D_k[\ldots\sigma_{k-1}^{-b}\ldots]) = w + a$$

$$Cr(D_k[\ldots\sigma_{k-1}^{-b}\ldots]) = c - a$$

$$wr(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-b}\ldots]) = w + a - 1$$

$$Cr(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-b}\ldots]) = c - a + 1$$

The HℓHm term of $Q_0(-a)P(D_k[\ldots\sigma_{k-1}^{-b}\ldots])$ is

$$(\pm\ell^{a}m^{a-2})(\pm\ell^{k-1-(w+a)}m^{(c-a)-2k+2}) = \pm\ell^{k-1-w}m^{c-2k}$$

and the HℓHm term of $Q_1(-a)P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-b}\ldots])$ is

$$(\pm\ell^{a-1}m^{a-1})(\pm\ell^{k-1-(w+a-1)}m^{(c-a+1)-2k+2}) = \pm\ell^{k-1-w}m^{c-2k+2}$$

Similarly, the LℓHm term of $Q_0(-a)P(D_k[\ldots\sigma_{k-1}^{-b}\ldots])$ is

$$(\pm\ell^{a}m^{a-2})(\pm\ell^{1-k-(w+a)}m^{(c-a)-2k+2}) = \pm\ell^{1-k-w}m^{c-2k}$$

and the LℓHm term of $Q_1(-a)P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-b}\ldots])$ is

$$(\pm\ell^{a-1}m^{a-1})(\pm\ell^{1-k-(w+a-1)}m^{(c-a+1)-2k+2}) = \pm\ell^{1-k-w}m^{c-2k+2}$$

Therefore the HℓHm term of $P(D_k)$ is $\pm\ell^{k-1-w}m^{c-2k+2}$ and the LℓHm term of $P(D_k)$ is $\pm\ell^{1-k-w}m^{c-2k+2}$.

Subcase 4. k is even and $D_k = D_k[\ldots\sigma_{k-1}^{-a}\ldots\sigma_{k-1}^{-b}\ldots]$, with $a > 1$ and $b > 1$. This is the same as subcase 3 except with k even, σ_1 is negative.

The HℓHm term of $Q_0(-a)P(D_k[\ldots\sigma_{k-1}^{-b}\ldots])$ is

$$(\pm\ell^{a}m^{a-2})(\pm\ell^{k-1-(w+a)}m^{(c-a)-2k+1}) = \pm\ell^{k-1-w}m^{c-2k-1}$$

and the HℓHm term of $Q_1(-a)P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-b}\ldots])$ is

$$(\pm\ell^{a-1}m^{a-1})(\pm\ell^{k-1-(w+a-1)}m^{(c-a+1)-2k+1}) = \pm\ell^{k-1-w}m^{c-2k+1}$$

Similarly, the LℓHm term of $Q_0(-a)P(D_k[\ldots\sigma_{k-1}^{-b}\ldots])$ is

$$(\pm\ell^{a}m^{a-2})(\pm\ell^{1-k-(w+a)}m^{(c-a)-2k+3}) = \pm\ell^{1-k-w}m^{c-2k+1}$$

and the LℓHm term of $Q_1(-a)P(D_k[\ldots\sigma_{k-1}^{-1}\ldots\sigma_{k-1}^{-b}\ldots])$ is

$$(\pm\ell^{a-1}m^{a-1})(\pm\ell^{1-k-(w+a-1)}m^{(c-a+1)-2k+3}) = \pm\ell^{1-k-w}m^{c-2k+3}$$

Therefore the HℓHm term of $P(D_k)$ is $\pm\ell^{k-1-w}m^{c-2k+1}$ and the LℓHm term of $P(D_k)$ is $\pm\ell^{1-k-w}m^{c-2k+3}$. □

Theorem 6.4. *Let L be a link and let D_k be a reduced alternating k-string closed braid representation of L, with at most two sequences of consecutive crossings between each pair of adjacent strings. If $P(L)$ is the HOMFLY polynomial of L, then the minimum exponent of ℓ in $P(L)$ is $1-k-w$, and the maximum exponent of ℓ in $P(L)$ is $k-1-w$, where w is the writhe of D_k.*

Proof. If D_k has at least one sequence of consecutive crossings between each pair of adjacent strings, then the result follows directly from Theorem 6.3. So suppose D_k has at least one pair of consecutive strings between which there are no crossings. Let n be the number of pairs of consecutive strings between which there are no crossings. Then the diagram of D_k could be separated into $n+1$ distinct links, each of which fits the conditions required by Theorem 6.3. Let L_1, L_2, $\ldots$, L_{n+1} be these links, then we have $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_{n+1}$. Let k_i and w_i be the number of strings and the writhe, respectively, in each link L_i, for $1 \leq$

$i \leq n+1$. Let w be the writhe of L, and note that $\sum_{i=1}^{n+1} k_i = k$ and $\sum_{i=1}^{n+1} w_i = w$. As shown in Example 3.1, the HOMFLY polynomial of L is $P(L) = [-m^{-1}(\ell^{-1} + \ell)]^n \prod_{i=1}^{n+1} P(L_i)$. Therefore the highest power of ℓ in $P(L)$ is

$$\begin{aligned} \ell^n \prod_{i=1}^{n+1} \ell^{k_i-1-w_i} &= \ell^n \ell^{\sum_{i=1}^{n+1}(k_i-1-w_i)} \\ &= \ell^n \ell^{k-(n+1)-w} \\ &= \ell^{k-1-w} \end{aligned}$$

and the lowest power of ℓ in $P(L)$ is

$$\begin{aligned} \ell^{-n} \prod_{i=1}^{n+1} \ell^{1-k_i-w_i} &= \ell^{-n} \ell^{\sum_{i=1}^{n+1}(1-k_i-w_i)} \\ &= \ell^{-n} \ell^{(n+1)-k-w} \\ &= \ell^{1-k-w} \end{aligned}$$

□

Corollary 6.5. *Let L be a link having a reduced alternating k-string closed braid representation, with at most two sequences of consecutive crossings between each pair of adjacent strings. Then the braid index of L is k.*

Proof. Since the diagram of L has k strings, the braid index of L cannot be greater than k. From Theorem 6.3, the highest ℓ power in the HOMFLY polynomial of L is ℓ^{k-1-w} and the lowest power of ℓ is ℓ^{1-k-w}, where w is the writhe of the diagram. By Theorem 6.2 we know that the braid index must be at least

$$\begin{aligned} \frac{1}{2}(E_\ell - e_\ell) + 1 &= \frac{1}{2}((k-1-w) - (1-k-w)) + 1 \\ &= \frac{1}{2}(2k-2) + 1 \\ &= k \end{aligned}$$

Therefore the braid index of L is k. □

CHAPTER 7: CONCLUSIONS AND FUTURE RESEARCH

If a k-string braid has at most two sequences of crossings between each pair of adjacent strings, and if its closure is a reduced, alternating link diagram, it has been shown that the link L represented by the closed braid D_k has braid index k. This also means that L has deficiency zero, for $Cr(L) = Cr(D_k)$ by Theorem 1.2, and $s(D_k) = s(L) = b(L) = k$ by Theorem 2.3, so the genus $g(L) = \frac{2-s(D_k)+Cr(D_k)-\mu(L)}{2} = \frac{2-k+Cr(L)-\mu(L)}{2}$, and then

$$\begin{aligned} d(L) &= Cr(L) - b(L) - 2g(L) - \mu(L) + 2 \\ &= Cr(L) - k - (2 - k + Cr(L) - \mu(L)) - \mu(L) + 2 \\ &= 0 \end{aligned}$$

Thus all of the properties listed in Chapter 2 for zero-deficiency links apply to closed braids of this class. If we know a link L_1 has a closed braid representation in this class, then its connected sum with any other zero-deficiency link L_2 will have the property that $Cr(L_1 \# L_2) = Cr(L_1) + Cr(L_2)$. Also, if connected with another link that has deficiency greater than zero, Theorem 2.6 gives a lower bound on the crossing number of the connected sum.

One of the most obvious questions is whether the method used here can be extended to prove the same result for more general alternating, reduced, closed braids. Preliminary investigation indicates that it most likely can be extended to at least the case of such closed braids having up to three sequences of crossings between each pair of adjacent strings, and work is being done to verify this and complete the proof. The polynomials Q_0 and Q_1 introduced in Lemma 5.2 simplify the calculations a great deal in the quest for the highest and lowest degrees of ℓ in the HOMFLY polynomial. In the case of a closed braid with three sequences of crossings between each pair of adjacent strings, the enhanced resolving tree seen in Figure 6.1 would simply have one more split of each diagram on the right, and many of the eight resulting diagrams can be evaluated inductively as in the proof of Theorem 6.3. However, as more sequences of crossings are assumed, some of those resulting diagrams (after applications of Lemma 5.2) become more challenging. It is not certain whether the same methods will suffice, or if additional techniques will be needed. Of course, if the result can be extended, then all links to which the result would apply will have also been shown to have deficiency zero.

This result, particularly if it can be extended to include more closed braids, could also provide more options for the possible use of closed braids as a means of generating random knots and links. For a k-string closed braid, the braid word could consist of symbols $\sigma_i^{\pm 1}$ where i is chosen randomly such that $1 \leq i < k$. It would be easy to stipulate that the closed braid be alternating, and if it is also reduced then both the crossing number and the braid index would be known directly from the diagram.

It might also be interesting to investigate whether the lower bound for crossing numbers found by M. Lackenby can be improved for certain types of links. Recall from Chapter 2 that his result reveals that $\frac{1}{152}(Cr(K_1) + Cr(K_2)) \leq Cr(K_1 \# K_2) \leq Cr(K_1) + Cr(K_2)$ for any two knots K_1 and K_2. Can a stronger lower bound be found for $Cr(K_1 \# K_2)$ provided that one of the knots is a zero-deficiency knot, for instance?

Finally, it seems worth pointing out that a general k-string closed braid diagram whose braid index is exactly k does not necessarily represent a link with deficiency zero. There exist non-alternating reduced closed braids with minimal number of strings but without the minimal number of crossings for their link type. For example, the closed braid $D_3[\sigma_1^{-2}\sigma_2^{-1}\sigma_1\sigma_2^{-1}]$ (which we will call D_3) has HOMFLY polynomial $-m^{-1}(\ell^3 + \ell^5) + m(-\ell + \ell^3)$, hence its braid index is 3. Let L be the link represented by D_3. The genus of D_3 is thus $g = \frac{2-s(D_3)+Cr(D_3)-\mu(L)}{2} = \frac{2-3+5-2}{2} = 1$. However, L has another diagram D that is alternating and has only four crossings. D is not a closed braid representation (Somewhat surprisingly, L is an alternating link that is not expressible as an alternating closed braid). This diagram has four Seifert circles, so its genus is $g = \frac{2-s(D)+Cr(D)-\mu(L)}{2} = \frac{2-4+4-2}{2} = 0$, and thus $g(L) = 0$. Hence the deficiency of L is $d(L) = Cr(L) - b(L) - 2g(L) - \mu(L) + 2 = 4 - 3 - 0 - 2 + 2 = 1$.

However, some links represented by non-alternating closed braids do have deficiency zero. For example, most torus knots are not alternating. It would be interesting to investigate whether specific criteria can be found under which a non-alternating closed braid represents a zero-deficiency link.

REFERENCES

[1] C. Adams, *The Knot Book* (W. H. Freeman and Company, New York, 1994).

[2] J. W. Alexander, A lemma on a system of knotted curves, *Proc. National Acad. Sciences USA* **9** (1923), 93–95.

[3] J. W. Alexander, Topological invariants of knots and links, *Trans. Amer. Math. Soc.* **30** (1928), 275–306.

[4] J. Arsuaga, T. Blackstone, Y. Diao, K. Hinson, E. Karadayi, and M. Saito, Sampling Large Random Knots in a Confined Space, *Journal of Physics A: Mathematical and General* **40** (2007), 11697–11711 .

[5] J. H. Conway, An enumeration of knots and links, and some of their algebraic properties, *Computational Problems in Abstract Algebra* (Proc. Conference Oxford 1970), ed. J. Leech, Pergamon Press (1967), 329–358.

[6] P. Cromwell, *Knots and Links* (Cambridge University Press, Cambridge UK, 2004).

[7] R. H. Crowell, Genus of alternating link types, *Ann. of Math.* **69** (1959), 258–275.

[8] J. Birman, Braids, links and mapping class groups, *Ann. Math. Studies No. 82* (Princeton Univ. Press, Princeton, NJ, 1976).

[9] J. Birman and T. Brendle, Braids: A Survey. In W. Menasco and M. Thistlethwaite, editors, *Handbook of Knot Theory* (Elsevier, Amsterdam, 2005).

[10] Y. Diao, The Additivity of Crossing Numbers, *Journal of Knot Theory and its Ramifications* **13** (2004), 857–866.

[11] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu, A new polynomial invariant of knots and links, *Bull. Amer. Math. Soc. (N.S.)* **12**(2) (1985), 239–246.

[12] J. Franks and R. F. Williams, Braids and the Jones-Conway polynomial, *Trans. Amer. Math. Soc.* **303** (1987), 97–108.

[13] V. F. R. Jones, A polynomial invariant for knots and links via Von Neumann algebras, *Bull. Amer. Math. Soc.* **12** (1985), 103–111.

[14] L. Kauffman, New Invariants in the Theory of Knots, *Amer. Math. Mon.* **95**(3) (1988), 195–242.

[15] M. Lackenby, The Crossing number of composite knots, *Journal of Topology 2* (2009), 747–768.

[16] W. B. R. Lickorish, *An Introduction to Knot Theory*, Graduate Texts in Mathematics, Vol. 175 (Springer-Verlag, New York, 1997).

[17] W. B. R. Lickorish and M. Thistlethwaite, Some links with non-trivial polynomials and their crossing-numbers, *Comment. Math. Helv.*, **63**(4) (1988) 527–539.

[18] A. A. Markov, Über die freie Äquivalenz der geschlossenen Zöpfe, *Recueil Math. Moscou* **1** (1936), 73–78.

[19] J. W. Milnor, Most knot are wild, *Fundamenta Math.* **54** (1964), 335–338.

[20] H. Morton, Seifert circles and knot polynomials, *Proc. Cabridge Phil. Soc.* **99** (1986), 107–110.

[21] K. Murasugi, On the genus of the alternating knot, B-5, B-6, *J. Math. Soc. Japan* **10** (1958), 94–105, 235–248.

[22] K. Murasugi, Jones polynomials and classical conjectures in knot theory, *Topology* **26**(2) (1987), 187–194.

[23] K. Murasugi, On the braid index of alternating links, *Trans. Amer. Math. Soc.* **326** (1991), 237–260.

[24] K. Reidemeister, *Knotentheorie*, Springer-Verlag, 1932. English translation: *Knot Theory*, BCS Associates, 1983.

[25] D. Rolfsen, *Knots and Links* (Publish or Perish Press, Berkeley, CA, 1976).

[26] H. Schubert, Knotten und Vollringe, *Acta Math.* **90** (1953), 131–286.

[27] H. Seifert, Uber das Geschlecht von Knoten, *Math. Ann.* **110** (1934), 571–592.

[28] M. Thistlethwaite, A spanning tree expansion of the Jones polynomial, *Topology* **26**(2) (1987), 297–309.

[29] R. Varela, K. Hinson, J. Arsuaga and Y. Diao, A Fast Ergodic Algorithm for Generating Ensembles of Equilateral Random Polygons, *Journal of Physics A: Mathematical and General* **42** (2009), 095204.

[30] S. Yamada, The minimal number of Seifert circles equals the braid index of a link, *Invent. Math.* **89**(2) (1987) 347–356.