CALIBRATING SHORT RATE MODELS IN NEGATIVE INTEREST RATE ENVIRONMENTS: AN APPLICATION TO BERMUDAN SWAPTIONS

by

Jasper Peter de Bles

A thesis submitted to the faculty of
The University of North Carolina at Charlotte
in partial fulfillment of the requirements
for the degree of Master of Science in
Economics

Charlotte

2020

Approved by:

____

Dr. Craig A. Depken

____

Prof. Stephen D. Young

____

Prof. Krista J. Saral

ABSTRACT

JASPER PETER DE BLES. Calibrating Short Rate Models in Negative Interest Rate Environments: An Application to Bermudan Swaptions.
(Under the direction of PROF. STEPHEN A. YOUNG)

The evolution of the interest rate has been modelled for a long time by a lognormal distribution. Similar to stock prices, it was generally agreed upon that the lognormal distribution provides the benefit of interest rates not falling below zero. However, governmental interventions after the 2008 finanancial crisis forced interest rates of several currencies to fall below zero. This event opened up a new area of research in mathematical finance. The contemporary negative interest rates can no longer be modelled by the classical lognormal distribution. Rather, distributions that allow for negative values, such as the normal distribution or the shifted lognormal distribution, are being tested.

In order to test for model differences in the years between 2016 and 2020, JPY swaption market implied volatilities based on a lognormal and normal distribution are collected. The moment when the JPY interest rate adopts a negative value, implied volatilities for the lognormal distribution go missing. For the normal distribution, all implied volatilities are returned. This observation confirms the failure of the Black formula in a negative interest rate environment.

Two well-known one-factor short rate models, the Hull-White Extended Vasicek model and the Black-Karasinski model, are calibrated to market data of JPY swaptions. The OIS rate is chosen because it has proven to be superior to the LIBOR in the pricing of financial derivatives. The Hull-White Extended Vasicek model follows a normal distribution and can therefore accomodate negative values. The Black-Karasinski model follows a lognormal distribution and can therefore only accomodate non-negative values.

The interest rate plays a critical role in the pricing of any derivative. The calibrated short-rate models are used to price 1Yx10Y Bermudan style JPY swaptions. The results indicate that calibration with Bachelier implied volatilities is more accurate compared to using Black implied volatilities.

DEDICATION

I dedicate this thesis to my friends from my hometown Alkmaar in the Netherlands, specifically the groups called ”Stadje” and ”Mooikazen”, who have been part of my journey through both my highschool- and undergraduate program and are friends for life. I also dedicate this thesis to all the friendships made in the other universities where I studied, i.e. Delft University of Technology, Groningen University, Renmin University of China, National Taiwan Normal University, and Copenhagen Business School, and to all the other friendships made during my travels. Furthermore, I am grateful for all the people at UNC Charlotte that made this final year a year to remember, especiallly the dual degree students from Cattolica University in Milan, the architects from ”Storrs”, the Computer Science and Engineering students from India, and other friends from the Economics and Mathematical Finance cohort. Moreover, I want to thank my parents for supporting me throughout my academic career. Finally, I dedicate this thesis most of all to my friend Peter Thaleikis, who has showed more than friendship by visiting me across the world and always supported me whenever I needed support.

DECLARATION

I, Jasper Peter de Bles, declare that the Master by Research thesis entitled "Calibrating Short Rate Models in Negative Interest Rate Environments: An Application to Bermudan Swaptions" contains no material that has been submitted previously, in whole or in part, for the award of any other academic degree or diploma. Except where otherwise indicated, this thesis is my own work. Unless noted otherwise, each graph is personally hard-coded in RStudio. The code scripts can be found on page 44. This thesis is part of the dual degree program between the University of North Carolina at Charlotte (UNCC) and Copenhagen Business School (CBS). Upon completion, I will officially receive a Master in Economics from UNCC, in which I followed a concentration in Mathematical Finance, and a Master in Business Administration with a concentration in Applied Economics and Finance from CBS.

ACKNOWLEDGEMENTS

I want to thank all the professors that have contributed to my academic development in the past five years. In every university where I studied, i.e. Delft University of Technology, Groningen University, Renmin University of China, National Taiwan Normal University, Copenhagen Business School, and UNC Charlotte, I have met many professors that were not just excellent teachers of their subjects, but also put effort into getting to know me as a person and sharing their life experiences with the class. One of the teachers that I treasure the most is my traditional Mandarin teacher from the NTNU Mandarin Training Center in Taipei. Finally, I would like to thank professor Stephen Young for not only teaching my friends and me the course 'Asset & Portfolio Management' this year, but also being my supervisor for my last academic assignment ever (at least for now).

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

ATM	At the money
BS	Black & Scholes
CHF	Swiss Franc
ECB	European central bank
ESTR	Euro short term rate
EUR	Euro
IRS	Interest rate swap
ITM	In the money
IV	Implied volatility
JPY	Japanese Yen
LIBOR	London interbank offered rate
OIS	Overnight index swap
OTM	Out of the money
PFS	Payer forward swap
RFS	Receiver forward swap
RW	Random walk
SARON	Swiss average rate overnight
SDE	Stochastic differential equation
SOFR	Secured overnight finance rate
TONAR	Tokyo overnight average rate
USD	United States Dollar
VIX	Volatility index of the S&P500

LIST OF SYMBOLS

C	Price of a call option
D	Discount factor
δ	Shift
K	Strike price or fixed rate
L	Floating rate
$log\mathcal{N}$	Lognormally distributed
$\mathcal{N}$	Normally distributed
μ	Mean
N	Notional amount
P	Price of a put option
r	Interest rate or risk-free rate
σ	Standard deviation or volatility
S	Stock or asset price
τ	Difference between time t_i and t_{i+1}
θ	Shift
W	Brownian motion or Wiener process

INTRODUCTION

Bermudan swaptions are among the most actively traded within the class of fixed income derivatives. The valuation of Bermudan swaptions depends on, among other things, the type of short rate model used to model the interest rate. A first distinction is made between one- and two factor short rate models. In two factor models, the second factor incorporates the joint distribution, and therefore depends on the correlation between two rates with different tenors, as explained in Shreve [19]. Andersen and Andreasen [1] compare the effect of one- and two factor models on the prices of Bermudan swaptions. They conclude that not including a second factor has no significant effect on the price. A second crucial distinction made between the existing short rate models used to model the interest rate is the distribution process that the model follows. The Black-Karasinski model follows a lognormal distribution and can therefore only produce strictly non-negative values. On the contrary, the Hull-White extended Vasicek model follows a normal distribution and can therefore produce both positive and negative rates. Since negatives were never a desired case, practicioners could implement a constraint in the model to disallow negative values. The downside of this constraint is that it influences the results of the calibration.

After the financial crisis at the beginning of the 21^{st} century, governments started to implement policies that forced the interest rate to adopt negative values. However, short rate models that follow a lognormal distribution are not able to model negative values. Russo and Fabozzi [18] were amongst the first to address the problem of calibrating short interest rate models in negative rate environments. Comparing the market implied volatilities of European swaptions from 2014 to 2016 under the Black/lognormal, shifted lognormal, and Bachelier/normal quotes, Russo and Fabozzi show that the classical lognormal Black quotes become unstable when interest rates go negative, and in some cases are not able to produce at all. In contrast, implied volatilities calculated under shifted lognormal or normal distributed formulas are proven to be satisfactory. Furthermore, Russo and Fabozzi callibrated the Hull-White model, the shift-extended CIR model, and the shift-extended Gaussian model and use te Bayesian information criterion (BIC) for evaluation.

The pricing of options started with the doctoral dissertation "Theorie de la speculation" or "Theory of speculating" of French Louis Bachelier in 1900. Although the dissertation is written in the French language, Sullivan and Weither [20] provide an excellent translation of the insights. In this dissertation, Bachelier conducted research toward the mathematical behavior of stocks, futures, and options on *La Bourse*, the Paris stock exchange. Bachelier not only invented the now daily-used payoff diagrams, but he was also able to describe the stock price movements by a normal distribution. Unfortunately, a normal distribution would allow for negative stock prices which is not possible due to the limited liability of stocks. Later, researchers therefore started to describe the stock price movement by means of a lognormal distribution [20]. This formula is called the Black and Scholes (BS) pricing formula and was created by Fischer Black, Robert C. Merton, and Myron Scholes.

One of the assumptions in the BS pricing formula is a constant interest rate. However, in reality the movement of interest rates is, just like the movement of stock prices, random in itself. The same idea, i.e. the assumption that the stock price had to modelled by a lognormal distribution, was applied to the movement of interest rates. As a result, practioners started to describe the development of the interest rate by models with a lognormal distribution, heavily criticizing the normal distribution. After 2008, the market has proven now that interest rates can consistently be negative. Although the normal distribution failed in accurately describing the stock price movement, it is needed back in fixed income.

This thesis is structured as follows. Chapter 1 adresses the relative small history of the negative interest rates. Time series of several currencies over the past 10 years clearly indicate when the interest rates fall below zero. Furthermore, a distinction is made between the London interbank offered rates (LIBOR) and overnight index swap (OIS) rates. Interest rates can be modelled by several mathematical interest rate models. Chapter 2 reviews the mathematics and literature of the most common one-factor short rate models. The basis behind these stochastic differential equations are tools from stochastic calculus. Chapter 3 therefore mentions the most important aspect that distinguishes stochastic calculus from regular calculus, that is the Brownian motion and the Itô Integral. Furthermore, due to the discussion of non-European style derivatives, Chapter 3 briefly mentions the importance of binomial and trinomial trees and stopping times. It is now clear that most, if not all, financial derivatives are to a certain extent, dependent on the interest rate. Therefore, Chapter 4 explains how one can work from the more basic interest rate derivatives, such as interest rate swaps, towards the more complex Bermudan swaption. Several examples are provided, as well as timelines for further clarification. Finally in Chapter 5, a numerical application is provided. The Hull-White extended Vasicek and the Black-Karasinski short rate models are calibrated to market JPY swaption implied volatilities and used to price JPY Bermudan swaptions.

CHAPTER 1: NEGATIVE INTEREST RATES

The pricing of derivatives has been dependent, for years, on models that follow a lognormal distribution. Ever since the interest rates went below zero, practicioners had to adapt their thoughts and reconsider the classical models. Suppose a customer decides to borrow a 1000 USD for one year from the bank and the interest rate is one per cent per year. He or she must repay 1010 USD after one year. However, when the interest rate is negative one per cent, the borrower only has to repay 990 USD. A strange thought that has become reality.

1.1 Historical Facts

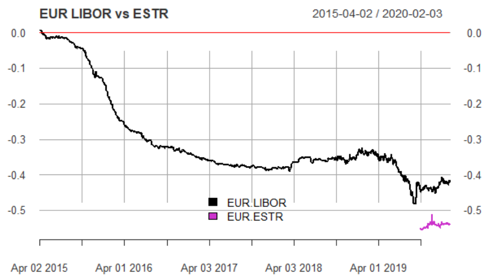
1.1.1 Pre-2008

Purchase and Constantine [16] mention the *Gesell* tax as one of the first ideas of imposing a type of negative interest rate. This so-called ”stamp tax” was named after German economist Silvio Gesell (1862 - 1930). Inspired by the crisis of the 1890s, Gesell argued that the growth of capital is hold back by the interest rate, and that one could reduce the interest rate by imposing carrying-costs on money.

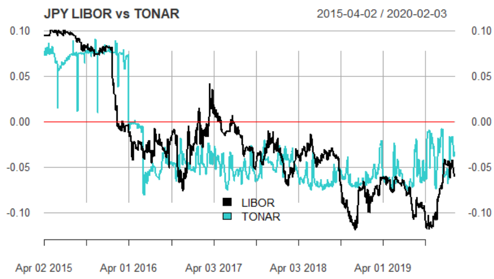
Half a century later, Gesell's unorthodox ideas became reality. In the 1970s, Switzerland imposed a negative interest rate for the first time. Because the Swiss Franc (CHF) acted as a safe-haven currency, investors started to buy CHF which led to an appreciation of the CHF and consequently hurt Swiss exports. Therefore, to discourage investors from buying CHF, the interest rate was set by the Swiss government to negative 2 per cent in 1971 and even negative 12 per cent in 1974 [14].

1.1.2 Post-2008

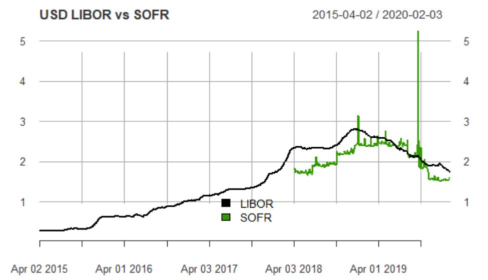
The situation of Switzerland in the 1970s was still an exception on the norm. Nowadays, negative interest rates are a common phenomenon. In 2014, the European Central Bank (ECB) started to force negative rates as an experiment [17]. Due to the 2008 financial crisis, there was a shortage of credit. The rationale is that negative interest rate would punish banks holding cash. The goal is to encourage lending, thereby pushing for inflation. Denmark, Japan, Sweden, and Switzerland followed the ECB's example soon.



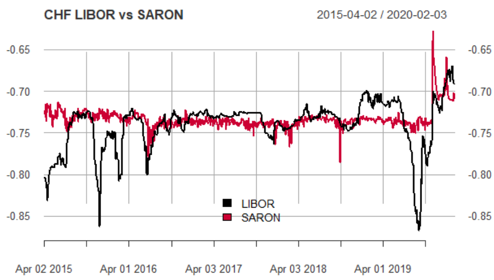
(a) EUR



(b) JPY



(c) USD



(d) CHF

Figure 1.1: LIBOR and OIS rates

Figure 1.1 graphs the interest rate situation of the past 5 years for some of the major currencies. The CHF is clearly negative whereas the EUR and JPY fell below zero, indicated by the red line, in 2015 and 2016 respectively. The USD is still positive but president Donald Trump already called for negative rates for the US at the 2020 World Economic Forum in Davos. The alternative rates for the LIBOR are also introduced in the graphs. Europe has the Euro short term rate (ESTR), Japan the Tokyo overnight average rate (TONAR), the US the secured overnight finance rate (SOFR), and Switzerland the Swiss average rate overnight (SARON).

1.2 LIBOR vs OIS

Whether to use the LIBOR or an OIS rate to price dervative contracts has become an important question after the LIBOR scandal. LIBOR is the average interest rate that banks charge each other while OIS is the interest rate of a country's central bank. Usually, the LIBOR and OIS almost have the same value. During the 2008 financial crisis, however, the gap started to widen significantly. The difference between LIBOR and OIS is used as a measure for credit risk ever since. Traditionally, the LIBOR has been used as a benchmark for the interest rate to price dervatives. In 2012, an investigation started towards the collusion of banks to manipulate the rates during the crisis. This lead to fines to banks totalling up to nine billion USD and even long-term convictions. Since the LIBOR scandal, governments demanded a replacement and this will probably be the OIS, or more specifically the ESTR, TONAR, SOFR, and SARON mentioned above. Also for the pricing of derivatives, the question now is whether practioners should use LIBOR or OIS. According to Hull and White [7], even though banks nowadays use the OIS for collateralized portfolios and the LIBOR for uncollateralized portfolios, the OIS should be used in both cases.

In Figure 1.1a and Figue 1.1c it can be seen that the new replacement rates for the LIBOR in Europe and the US just got introduced in 2019 and 2018 respetively. Thus, not much data for the ESTR and SOFR is available up until 2020. The SOFR has a significant spike in September 2019. One of the three causes pointed out was the fact that Saudi Arabia drew 80 billion USD from the repo market. The great demand for cash and oversupply of Treasuries caused the SOFR to jump. This extremely volatile event made investors start doubting SOFR as the right replacement for LIBOR [9]. In Figure 1.1b and Figure 1.1d the TONAR and SARON are already available for a longer time. These OIS rates seem more stable than their LIBOR counterparts. Furthermore, one can see for the years 2016 and 2017 that there exists a more significant difference between the OIS and LIBOR rates. It is possible that this can be linked to Brexit.

1.3 Probability Distributions

Interest rates that become negative can no longer be modelled by a lognormal distribution. Practioners have to turn back to the normal distribution, defined as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{1.1}$$

Notation:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

A graphical representation of the normal distribution is given in Figure 1.2a. Thus, the normal distribution can accommodate both positive and negative values, which is perfectly suitable for the current situation. The problem of the lognormal formula is that the natural log is strictly non-negative. This is because $ln(x) = y$ is the same as

$e^{ln(x)} = x = e^y$ and the limits of an exponential funciton are infinity and zero. The lognormal distribution is defined as:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{lnx-\mu}{\sigma}\right)^2} \tag{1.2}$$

Notation:

$$ln(X) \sim \mathcal{N}(\mu, \sigma^2)$$

or

$$X \sim log\mathcal{N}(\mu, \sigma^2)$$

A graphical representation of the lognormal distribution is given in Figure 1.2b. Nevertheless, it is possible to take the entire lognormal distribution and shift it to the left by a value θ, thereby pushing it into the negative environment. The shifted lognormal distribution is defined as:

$$f(x) = \frac{1}{(x-\delta)\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{ln(x-\delta)-\mu}{\sigma}\right)^2} \tag{1.3}$$

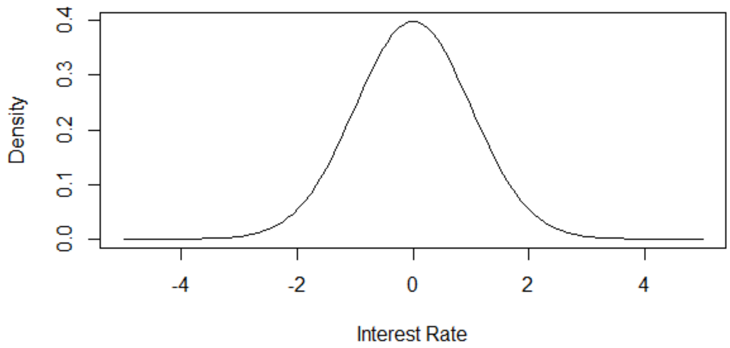
Notation:

$$ln(X + \theta) \sim \mathcal{N}(\mu, \sigma^2)$$

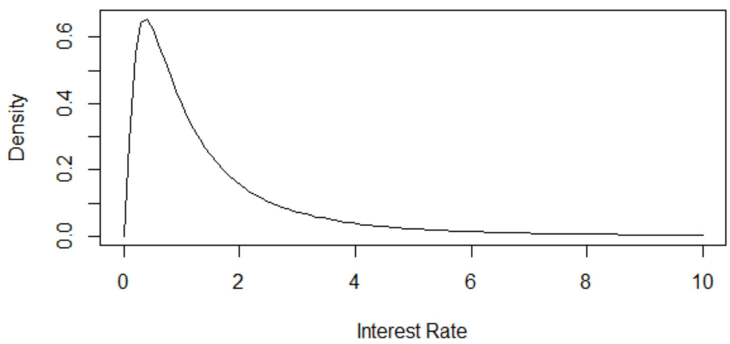
or

$$X + \theta \sim log\mathcal{N}(\mu, \sigma^2)$$

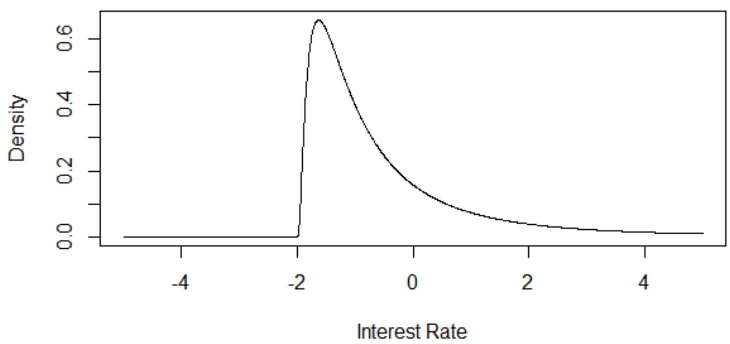
The δ and θ represent the shift in the above formulas. Figure 1.2c shows a shifted lognormal distribution. The bound for the lognormal distribution is zero. Similarly, the bound for the shifted lognormal distribution is the shift itself. Therefore, one has to choose carefully and make the shift more negative than the minimum value of the interest rate.



(a) Normal distribution: r can be negative



(b) Lognormal distribution: r is strictly non-negative



(c) Shifted lognormal distribution: r can be negative but is bounded by the shift θ

Figure 1.2: Probability distributions

CHAPTER 2: STOCHASTIC CALCULUS

Stochastic calculus has contributed significantly to the progression made so far in the field of mathematical finance. For the purpose of this thesis, some crucial tools from stochastic calculus are discussed. The Brownian motion and it's relation to Itô's integral helps us to creat short rate models with a random component. Furthermore, the construction of bionomial or trinomial trees and it's relation to stopping times is important for non-European style derivatives. And, lastly, it is shown how the pricing of derivatives under the risk-neutral measure works.

2.1 Brownian Motion

The interest rate level or stock price of tomorrow, next week, or next year, is uncertain. If investors would know what the level or price is, they would all buy the right financial product and make a certain profit, which is impossible. The uncertainty aspect is characterized by the so-called Brownian motion (BM), named after Scottish botanist Robert Brown (1773-1858). It is a concept that the finance industry borrowed from physics. In physics, the BM is used to describe the motion of particles caused partially by molecules that hit each other infinitely many times and therefore follow a random path. In finance the BM is also knows as a Wiener process, named after American mathematician Norbert Wiener (1894-1964). The BM is constructed from a scaled symmetric random walk (RW) where n goes to infinity. To show this, one first has to understand the rationale behind the simpler symmetric RW.

The symmetric RW is often described as a repeated fair coin toss where there is equal probability on tossing heads (H) or tails (T). The fair coin toss means that the probability p of getting H and the probability $q = 1 - p$ of getting T are $\frac{1}{2}$. In Shreve [19] the symmetric RW is defined as follows: the successive outcomes of the coin tosses are $\omega = \omega_1, \omega_2, ..., \omega_n$. Let

$$X_j = \begin{cases} 1, & \text{if } \omega_j = H. \\ -1, & \text{if } \omega_j = T. \end{cases} \tag{2.1}$$

$$M_k = \sum_{j=1}^{k} X_j, k = 1, 2, ... \tag{2.2}$$

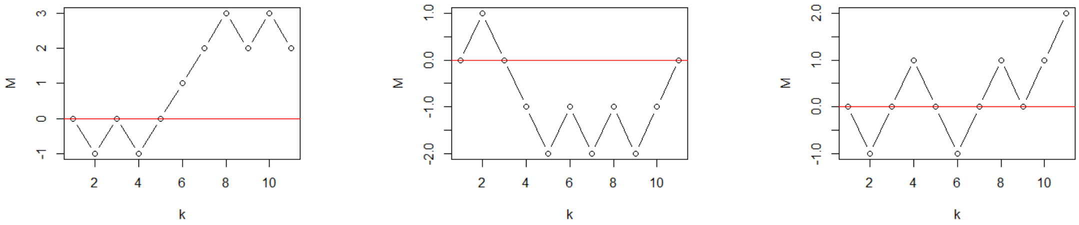


Figure 2.1: Symmetric random walk simulations with $n = 10$

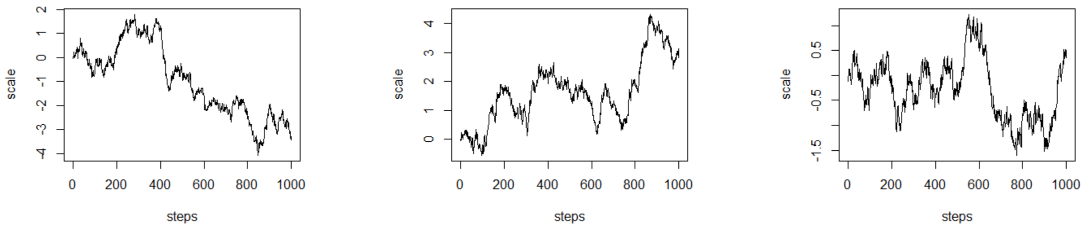


Figure 2.2: Brownian motion simulations with $n = 1000$ and $\sigma = 0.1$

and $M_0 = 0$. Then the process $M_k, k = 0, 1, 2, ...$ is a symmetric RW. Figure 2.1 displays three simulations of the symmetric RW with 10 steps.

A scaled symmetric RW is similar to a symmetric RW. One difference is that the time step is reduced from one to some smaller number. Another difference is that the size of the up- or down movement is reduced from one to some number. In other words, we speed up time and scale down the step size of a symmetric RW. The formula in Shreve [19] is:

$$W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} \tag{2.3}$$

Finally, the Brownian motion can be obtained by taking the limit of the scaled RW, i.e. $n \to \infty$. Thus, with a BM one is able to model by how mucht e.g. the interest rate has increased or decreased in a day, or even in a few hours or minutes. Figure 2.2 displays three simulations of a Brownian motion with 10 steps.

2.2 Itô's Lemma

Ordinary calculus allows us to integrate over a certain function $f(x)$. That is $\int_a^b f(x)dx$ gives the area under, or in other circumstances above, the function $f(x)$ in the range

$[a, b]$. In mathematical finance, the function for the movement of the interest rate or stock price includes a Brownian motion due to its randomness. Therefore, one does not know how the function $f(x)$ exactly will look like and cannot use ordinary calculus to integrate over a function with a BM. To solve this problem, Itô calculus, Itô's integral, or Itô's Lemma was invented. These concepts are named after Japanese mathematician Kiyosi Itô (1915-2008). Itô's Lemma or the Itô-Doeblin formula, in recognition of French-German mathematician Wolfgang Doeblin(1915-1940), is defined as:

$$df(X_t, t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dX_t^2 \tag{2.4}$$

Consider the function $f(t) = t$. Standard integration gives $\int_0^T f(t)dt = \int_0^T tdt = \frac{1}{2}t^2|_0^T = \frac{1}{2}T^2 - 0^2 = \frac{1}{2}T^2$. Now suppose $f(t)$ is a BM instead. Thus, we need to calculate $\int_0^T W_t dt$. Applying Itô gives $d(tW_t) = tdW_t + W_t dt$. Finally, integrating and rearranging results in $\int_0^T W_t dt = tW_t - \int_0^T tdW_t$. The results of integrating over a normal function and integrating over a BM are quite different. Itô calculus helps us to go back and forth between a stochastic differential equation (SDE) and its explicit formula. Chapter 3 shows how to apply Itô on short rate model SDEs.

2.3 Binomial and Trinomial Trees

The price of a European style option can directly be approximated by using formulas such as Black & Scholes. This is possible due to the fact that there is only one date to be evaluated, that is the payoff at maturity time T. On this date, the holder decides to exericise the option if it is in the money (ITM) or not exercise if the option is out of the money (OTM), thereby losing the premium. Pricing gets more complicated if options are of an American or Bermudan style. American options can be exercised at any moment between the start of the contract and maturity. This means that the payoff of the option has to be evaluated at any point in time and this makes a direct pricing formula such as the BS obtain unreliable results. A common solution is the construction of a binomial or trinomial tree.

A binomial tree starts with the price level of the underlying S at a certain time t, the time at which we want to know the price of the option. At time $t+1$, the level of S goes up or down by a certain amount, depending on the volatility σ of S, with probability p

and $q = 1 - p$ respectively. These up- and down movements at each node continue in the same fashion until time T. At these final nodes, the pricing of the option starts by calculating the payoff at each possible node. After discounting, the payoff at each node at time $T-1$ is calculated. This process continues until we are back at time t to get the price of the option and is called backward induction. For a European option, it can be shown that the binomial tree, when the number of steps goes to infinity, approximates the BS model. For American options, one has to compare the value at each node with the value of immediately exercising the option. If the value of exercising right now is superior, the value at the node has to be replaced. In this way, one can distinguish the price of an American option from a European option.

The trinomial tree is similar to a binomial tree with the only difference that there is also a possibility of a middle movement p_m. Thus, it is possible that the the interest rate or stock price does not go up or down in the next period, but remains at the same level. Two simple examples are given in Figure 2.3a and Figue 2.3b. The probabilities of going up, middle, or down, are given by the Hull-White tree formulas as:

$$X_j = \begin{cases} p_{up} = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} + \frac{\eta_{j,k}}{2\sqrt{3}V_i} \\ p_{middle} = \frac{2}{3} - \frac{\eta_{j,k}^2}{3V_i^2} \\ p_{down} = \frac{1}{6} + \frac{\eta_{j,k}^2}{6V_i^2} - \frac{\eta_{j,k}}{2\sqrt{3}V_i} \end{cases} \tag{2.5}$$

For Bermudan swaptions, it is pointed out that the discretizations are finer in those sections of the tree where early exercise should be evaluated, and more coarse in other sections such as the section following the last possible date to exercise [3].

2.4 Stopping Times

Related to the construction of trees and non-European style financial derivatives are stopping times. The optimal stopping time is a time t between the start of the contract t_0 and the end of the contract T at which it is considered to be optimal to exercise early. In stochastic calculus, the definition is as follows: a stopping time τ is a random variable that takes a value in the space $[0, \infty]$ and must satisfy the condition:

$$\tau \leq t \in F(t) \text{ for all } t \geq 0 \tag{2.6}$$

(a) Binomial tree (b) Trinomial tree

Figure 2.3: Construction of lattices

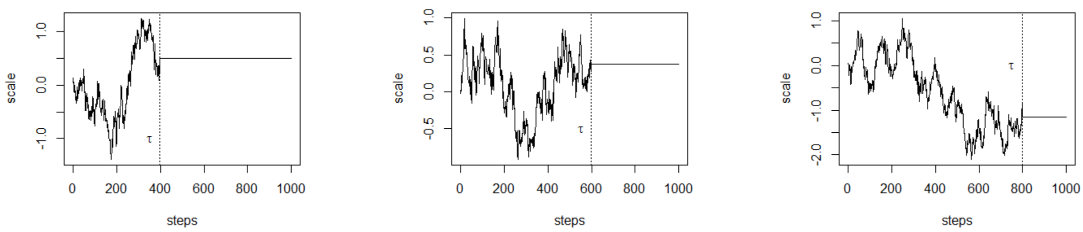


Figure 2.4: Stopped process simulations with $n = 1000$ and $\sigma = 0.1$

Three simulations of a stopped process, that is simulations of a BM as in Figure 2.2 stopped at different stopping times, are graphed in Figure 2.4. Kolodko and Schoenmakers [12] provide a mathematical derivation to find the optimal stopping time for Bermudan swaptions.

2.5 Risk-Neutral Pricing

A key aspect in the pricing of financial derivatives is the concept of pricing in the risk-neutral world. The probabilities for the up- and down states are the real world probabilities. However, risk-neutral probabilities cause the expected discounted value of the payoff of a derivative to equal the market price. Thus, the risk-neutral measure incorporates risk. The connection between the real world probability measure and the risk-neutral probability measure is the so-called *Radon-Nikodým* derivative [19], defined as:

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)} \tag{2.7}$$

One can change the SDE of a stock price from the real-world measure to the risk-neutral measure as follows [15]:

$$dS_t = \mu S_t dt + \sigma S_t dW_t = \mu S_t dt + \sigma S_t (d\tilde{W}_t + \frac{r-\mu}{\sigma} dt) = r S_t dt + \sigma S_t d\tilde{W}_t$$

where the connection between the two probability measures is:

$$\tilde{W}_t = W_t - \frac{r-\mu}{\sigma} t$$

Thus, it is assumed that the SDEs of the short rate models in Chapter 3 are defined under the risk-neutral probability measure $\tilde{\mathbb{P}}$ as well [5]. This is indicated by $\tilde{W}$ or W^*.

CHAPTER 3: ONE FACTOR SHORT RATE MODELS

Various mathematical models for interest rates have been developed over the years. In general, either one- or two-factor models are applied. Principal component analysis has been used to determine what factors drive the term structure of interest rates. The first factor is a shift of the entire term structure and accounts for around 80-90 per cent of the variance. The second factor, the twist, is a situation in which the short-term rate and long-term rate move in opposite directions and accounts for an additional 5-10 per cent of the variance. Finally, the third factor called the butterfly is a situation in which the mid-term structure moves in the opposite direction of the short- and long-term rates [5]. Andersen and Andreasen [1] analyzed in their 2001 paper the factor dependence of Bermudan swaptions and conclude that the Wall Street practice of using calibrated single factor models does not lead to significant mispricing of Bermudan swaptions. Therefore, the focus of this thesis is limited to one factor models.

For the purpose of this research, models that allow for negative interest rates are of interest. This goal is in sharp contrast with most literature, where the positive probability of getting a negative value for the interest rate is being regarded as a drawback, as seen in the quotes below:

> "...the theoretical possibility of r going below zero is a clear drawback of the model..." [3]

> "Unfortunately, by definition, Gaussian interest rate models do not prevent the interest rate from becoming negative, which is economically unrealistic." [5]

In general, short rate models have the form:

$$dr_t = \mu_t dt + \sigma_t dW_t \tag{3.1}$$

Where μ is the drift, σ is the volatility, and W is a Brownian motion. Therefore, the evolution of the interest rate level depends on time and a BM. This chapter gives an overview of the most famous one factor short rate models, their SDEs, and probability distributions.

3.1 Merton

In his 1974 paper, Merton notes that "While a number of theories and empirical studies has been published on the term structure of interest rates, there has been no systematic development of a theory for pricing bonds when there is a significant probability of default" [13]. He assumes the general SDE like in equation 3.1:

$$dr_t = r_t dt + \sigma dW_t^* \tag{3.2}$$

The explicit solution is:

$$r_t = r_0 + r_t t + \sigma W_t^* \tag{3.3}$$

The Merton model is normally distributed and can therefore accommodate negative values.

3.2 Vasicek

Vasicek [21] assumed in his 1977 model that the spot rate evolves as an Ornstein-Uhlenbeck process, which is a stochastic proces that tends to drift, or revert, towards its long-term mean. The SDE is given by:

$$dr_t = k[\theta - r_t]dt + \sigma dW_t^* \tag{3.4}$$

Where θ is considered to be the long-term mean. One can see that the drift is negative when $r_t > \theta$ and positive when $\theta > r_t$. The process r_t is normally distributed and can therefore accommodate negative values. Despite the fact that past literature sees this as a drawback, the possibility for negative values is of interest to this research.

Applying Itô to $f(t, r(t)) = e^{kt}(r_t - \theta)$:

$$df(t, r(t)) = ke^{kt}(r_t - \theta)dt + e^{kt}dr_t$$

Substitution for $dr(t)$ gives:

$$df(t, r(t)) = e^{kt}(r_t - \theta)dt + e^{kt}(k[\theta - r(t)]dt + \sigma dW(t)) = \sigma e^{kt}dW_t$$

Integration gives:

$$e^{kt}(r_t - \theta) - e^{k0}(r_0 - \theta) = \sigma \int_0^t e^{ku} dW_u^*$$

Taking exponentials on both sides and rearrangement results in :

$$r(t) = e^{-kt} r_0 + \theta(1 - e^{-kt}) + \sigma e^{-kt} \int_0^t e^{ku} dW_u^* \tag{3.5}$$

3.3 Dothan

One year later, Dothan [4] introduced a model that followed a geometric Brownian motion. Initially, the model was presented without drift but the drift was later added under the following SDE:

$$dr_t = a r_t dt + \sigma r_t dW_t^* \tag{3.6}$$

Where a is a constant. In the case of the Dothan model, $r(t)$ is lognormally distributed and results therefore in strictly non-negative values. Consequently, pricing interest rate derivatives with the Dothan model in countries where interest rates are negative leads to flawed results. The process is only mean-reverting if $a < 0$.

Applying Itô to $f(t, r(t)) = ln(r)$:

$$df(t, r(t)) = \frac{1}{r} dr_t - \frac{1}{2} \frac{1}{r^2} dr_t^2$$

Substitution for $dr(t)$ gives:

$$df(t, r(t)) = \frac{1}{r}[ardt + \sigma r dW_t^*] - \frac{1}{2} \frac{1}{r^2}[\sigma^2 r^2 dt] = (a - \frac{1}{2}\sigma^2)dt + \sigma dW_t^*$$

Integration gives:

$$ln(r(t)) - ln(r(0)) = \int_0^t (a - \frac{1}{2}\sigma^2) du + \int_0^t \sigma dW_u^*$$

Taking exponentials on both sides and rearrangement results in the explicit solution:

$$r(t) = r(0) e^{\int_0^t (a - \frac{1}{2}\sigma^2) du + \int_0^t \sigma dW_u^*} \tag{3.7}$$

3.4 CIR

In 1985, Cox, Ingersoll, and Ross [10] presented an adapted version of the Vasicek model. This so-called CIR model has an additional square-root term in the diffusion coefficient and its SDE is given as:

$$dr_t = k[\theta - r_t]dt + \sigma\sqrt{r_t}dW_t^* \tag{3.8}$$

The process $r(t)$ follows a noncentral Chi-squared distribution. Specifically, the condition $2k\theta > \sigma^2$ is imposed to ensure positive values [3]. Consequently, if one does not include this condition, values can become negative and the model can be used in a negative interest rate environment.

Applying Itô to $f(t, r(t)) = e^{kt}r(t)$:

$$df(t, r(t)) = kr_te^{kt}dt + e^{kt}dr_t$$

Substitution for $dr(t)$ gives:

$$df(t, r(t)) = kr_te^{kt}dt + e^{kt}[k[\theta - r(t)]dt + \sigma\sqrt{r(t)}dW_t^*] = k\theta e^{kt}dt + e^{kt}\sigma\sqrt{r_t}dW_t^*$$

Integration gives:

$$e^{kt}r(t) - e^{k0}r(0) = \theta k\int_0^t e^{ku}du + \sigma\int_0^t e^{ku}\sqrt{r(u)}dW_u^*$$

Taking exponentials on both sides and rearrangement results in the explicit solution:

$$r(t) = e^{-kt}r(0) + \theta[1 - e^{-kt}] + \sigma e^{-kt}\int_0^t e^{ku}\sqrt{r_u}dW_u^* \tag{3.9}$$

3.5 Hull-White Extensions

In 1990, Hull and White [8] showed that the Vasicek and CIR models can be extended so that they are consistent with the term structure of interest rates and the term structure of either spot rate or forward rate volatilities. The HW extended Vasicek SDE is:

$$dr_t = [\vartheta - a_t r_t]dt + \sigma_t dW_t^* \tag{3.10}$$

This model is, just like the Vasicek model, is normally distributed and can therefore accommodate negative values. It is also a mean-reverting Ornstein Uhlenbeck process. ϑ is the long-term mean and a represents the speed of mean-reversion. The drift is negative when $a_t r_t > \vartheta$ and positive when $\vartheta > a_t r_t$.

Applying Itô to $f(t, r(t)) = e^{\int_0^t \alpha_t dt} r_t$:

$$df(t, r(t)) = e^{\int_0^t \alpha_t dt}(r_t \alpha_t dt + dr_t)$$

Substitution for dr_t gives:

$$df(t, r(t)) = e^{\int_0^t \alpha_t dt}(\vartheta_t dt + \sigma_t dW_t^*)$$

Integration gives:

$$e^{\int_0^t \alpha_t dt} r_t = e^{\int_0^t \alpha_t dt} r_0 + \int_0^t e^{\int_0^t \alpha_t dt} \vartheta_t du + \int_0^t e^{\int_0^t \alpha t dt} \sigma_t dW_u^*$$

Finally, the explicit solution is:

$$r_t = e^{-\alpha t} r_0 + \frac{\vartheta}{\alpha}(1 - e^{-\alpha t}) + \sigma e^{-\alpha t} \int_0^t e^{au} dW_u^* \tag{3.11}$$

3.6 Black-Karasinski

In 1991, Black and Karasinski [2] came up with a lognormal short rate model that did not produce, what were deemed, undesirable negative rates. It is also called the extended exponential Vasicek model. The SDE is given by:

$$dln(r_t) = [\theta_t - a_t ln(r_t)]dt + \sigma_t dW_t^* \tag{3.12}$$

The explicit solution is:

$$r_t = e^{lnr_0 e^{-at} + \int_0^t e^{-au}\theta_u du + \sigma \int_0^t e^{-au} dW_u^*} \tag{3.13}$$

Originally, Black Karasinski calibration was executed with a binomial tree but it has

Table 3.1: Summary of one-factor short rate models

Author	SDE	Distribution	Can r <0?
Merton (1974)	$dr_t = r_t dt + \sigma dW_t^*$	Normal	Yes
Vasicek (1977)	$dr_t = k[\theta - r_t]dt + \sigma dW_t^*$	Normal	Yes
Dothan (1978)	$dr_t = ar_t dt + \sigma r_t dW_t^*$	Lognormal	No
CIR (1985)	$dr_t = k[\theta - r_t]dt + \sigma\sqrt{r_t} dW_t^*$	Chi-squared	Yes
HW ext. Vasicek (1990)	$dr_t = [\vartheta - a_t r_t]dt + \sigma_t dW_t^*$	Normal	Yes
BK ext. exp. Vasicek (1991)	$dln(r_t) = [\theta_t - a_t ln(r_t)]dt + \sigma_t dW_t^*$	Lognormal	No

become common practice to use a Hull White trinomial tree.

The short rade models discussed are summarized in Table 3.1. Merton, Vasicek, and CIR are affine or time-invariant models, in which the term structure is a linear function of the short rate. In the real world, term structures are never linear. The extended Vasicek and Black Karasinski are time-variant models and are able to exactly fit the term structure due to the time-varying parameters. As a consequence, these type of models will provide better calibration results.

CHAPTER 4: TOWARDS BERMUDAN SWAPTIONS

A Bermudan swaption is an option on an underlying interest rate swap that can be exercised on multiple dates, i.e. Bermudan style.

4.1 Related Instruments

This section discusses how the rather complex Bermudan swaption is related to- and constructed from the more basic financial derivatives that are traded on the market.

4.1.1 Swaps

Interest rate swaps (IRS) are derivative constracts where a fixed interest rate is swapped for a floating interest rate or vice versa. The IRS as underlying is the focus of this research but other swaps exist such as credit default swaps (CDS), commodity swaps, currency swaps, and even variance or volatility swaps. The volatility index (VIX) traces the volatility of the S&P500 and e.g. a variance swap exchanges a fixed variance for the floating variance or vice versa.

For an IRS, let the fixed interest rate be:

$$N\tau_i K$$

where N is the nominal value, K is the fixed interest rate set before the contract, and τ_i is the time fraction. The floating interest rate is:

$$N\tau_i L(T_{i-1}, T_i)$$

where the part $L(T_{i-1}, T_i)$ is the interest rate resetting after every payment. The fixed interest rate is acting as the strike price and is determined by looking at the expected interest rate from the start date to the end date of the contract in order to have a value of zero at inception. If one pays a fixed interest rate and receives the floating interest rate, it is called a Payer IRS (PFS). Similarly, if one pays the floating interest rate and receives a fixed interest rate, it is called a Receiver IRS (RFS). The discounted payoff of a PFS is [3]:

$$\sum_{i=\alpha+1}^{\beta} D(t,T_i)N\tau_i(L(T_{i-1},T_i)-K) \tag{4.1}$$

The discounted payoff of an RFS is [3]:

$$\sum_{i=\alpha+1}^{\beta} D(t,T_i)N\tau_i(K-L(T_{i-1},T_i)) \tag{4.2}$$

One needs to take into account that fixed and floating payments might not occur at the same date and with the same frequency.

Figure 4.1a displays a 2-Year IRS with the JPY LIBOR as floating rate and a fixed rate of 1.2 per cent. The first year, the holder of a PFS would make a profit because the LIBOR exceeds the fixed rate of the swap contract, as can be seen in Figure 4.1b. However, at the second quarterly payment of 2017, the PFS starts to lose money and the RFS starts to gain.

4.1.2 Options

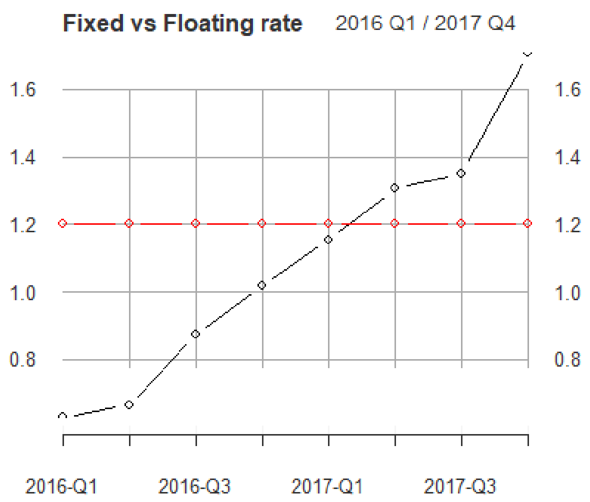
Options are derivative contracts where the owner has the right, but not the obligation, to exercise the option and buy or sell the underlying for a strike price K. A call option gives the owner the right to buy the underlying. Thus, if at maturity the stock price is $S > K$, the owner can buy the stock for K and sell immediately for S, thereby making a profit of $S - K$. If $K > S$ the option will not be exercised and the payoff will be 0. Thus, the payoff for a call option is of the form:

$$(S-K,0)^+$$

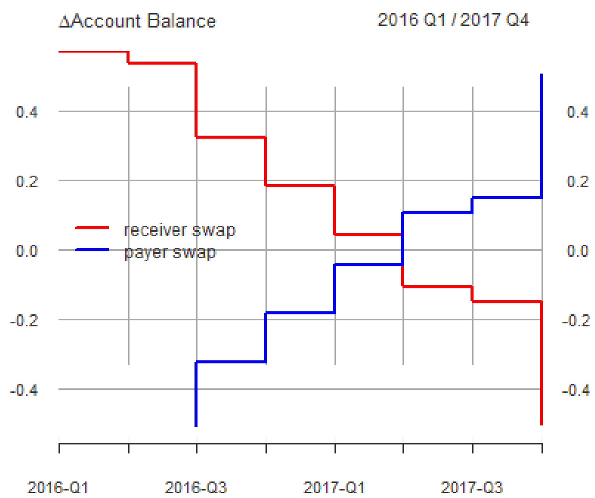
Similarly, a put option gives the owner the right to sell the underlying. Thus, if at maturity the stock price is $S > K$, the owner will not exercise the option. However, if $K > S$ the owner can buy the stock for S and sell for K, thereby making a profit of $K - S$. Thus, the payoff for a put option is of the form:

$$(K-S,0)^+$$

Because options give the upside but not the downside, as is the case with forwards and futures, one needs to pay a premium when buying the option. The price of options



(a) LIBOR IRS with K = 1.2%



(b) Change in Account Balance

Figure 4.1: 2-year JPY IRS

depends on the time to maturity, the strike price, the stock price, the interest rate, and the implied volatility and is calculated by the Black & Scholes pricing formula or an extension thereof. The price of a European call option is:

$$C = SN(d_1) - e^{-r(T-t)}KN(d_2) \tag{4.3}$$

The price of a European put option is:

$$P = e^{-rt(T-t)}KN(-d_2) - SN(-d_1) \tag{4.4}$$

where

$$d_1 = \frac{ln\frac{S}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

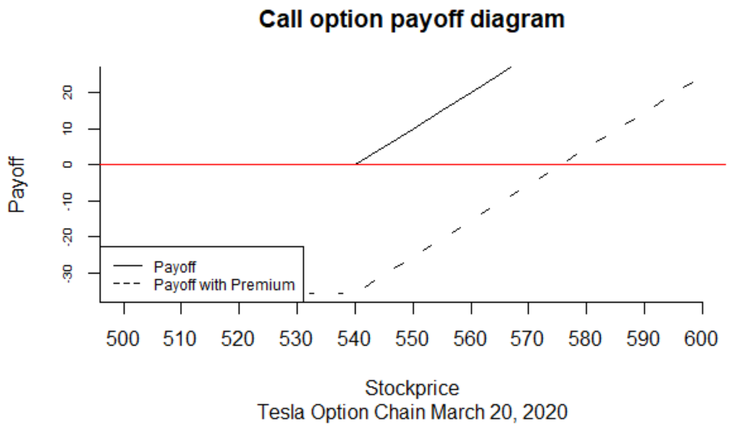
and

$$d_2 = \frac{ln\frac{S}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

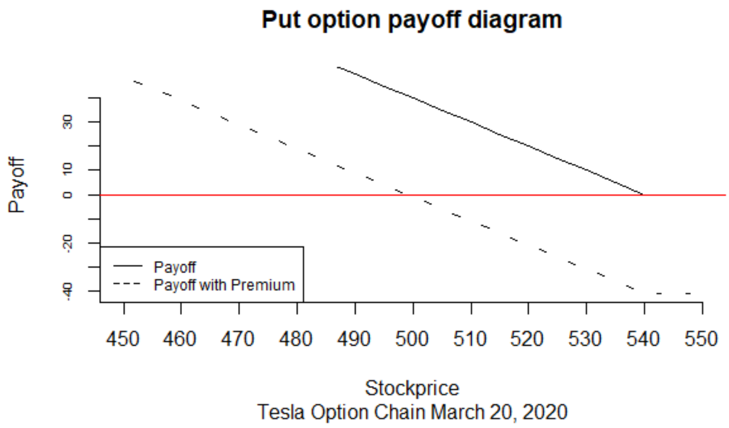
If the strike price K is far out of the money (OTM), the probability of making money is lower and the price of the option will therefore be low. In contrast, if the strike price is already in the money (ITM) one has to pay a significant premium to buy the option. Furthermore if the volatility is high, the probability that asset price S will either go up or drop increases, and hence the proability that $S - K$ or $K - S$ will have a higher value increases. Because there is no downside, a higher volatility is associated with a higher premium for both calls and puts.

Finally, a distinction is made between European style and American style options. European options can only be exercised at maturity. American options can be exercised at any time between the start of the contract and maturity. Due to the possibility of early exercise, the Black & Scholes model is not appropriate for American style options. As an alternative, the construction of binomial or trinomial trees is used. This will play an important role in the pricing of Bermudan swaptions.

Figure 4.2 shows the payoffdiagrams for Tesla (ticker: TSLA) put and call options with the same strike price, $K = 540$, expiring March 20, 2020. Because the stock price was 546.62 on March 13, it means that the call option is ITM and the put option is



(a) Tesla Call with K=540 and C=35.5



(b) Tesla Put with K = 540 and P=41

Figure 4.2: TSLA option payoff diagram

OTM. Nevertheless, the premium of the put is higher than that of the call.

4.1.3 Caplets and Floorlets

Caplets and floorlets, or caps and floors, are a special type of interest rate swap in which the holder only receives the positive cashflow. Thus, in the case of a cap, the holder will receive the difference between the fixed and floating leg only if the the fixed leg is smaller than the floating rate. The payoff for a cap is given as [3]:

$$\sum_{i=\alpha+1}^{\beta} D(t,T_i) N \tau_i (L(T_{i-1},T_i) - K)^+$$

Similarly, a floor will only pay out when the fixed leg is bigger than the floating rate. The payoff for a floor is [3]:

$$\sum_{i=\alpha+1}^{\beta} D(t,T_i)N\tau_i(K - L(T_{i-1},T_i))^+$$

Because caps and floors have the upside but not the downside, it is required to pay a premium for such a contract.

4.1.4 Swaptions

A swaption is an option on a swap. In the case of a European swaption, the owner has the right but not the obligation to exercise at maturity and enter the underlying swap, the IRS. A receiver swaption is a put- and a payer swaption a call option on a bond. In forward rate agreements (FRA) a swap is usually quoted in "Maturity x Tenor". The maturity refers to the time to maturity of the option and the tenor to the length of the underlying swap. For example, a 5Yx10Y swaption gives the owner an option that matures in five years the right to enter a 10 year swap. In Brigo and Mercurio [3] the swaption notation of the option's maturity is T_α. When one assumes that that date coincides with the start of the underlying IRS, the reset date, the IRS length is $T_\beta - T_\alpha$. Just like with any option, the holder will exercise only if the value at maturity is positive. Therefore, the payer swaption payoff is given by [3]:

$$ND(t,T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\tau_i(F(T_\alpha;T_{i-1},T_i) - K)\right)^+ \tag{4.5}$$

where N is the notional amount, $D(t,T_\alpha)$ is the discount factor between now and the reset date T_α, K is the fixed interest rate of the underlying IRS, P the pure discount bond price between time α and i, $F(T_\alpha;T_{i-1},T_i)$ is the forward rate at the reset date α for the expiry-maturity pair $i-1$ and i. For a receiver swaption, the floating rate and strike change sign to get to [3]:

$$ND(t,T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} P(T_\alpha,T_i)\tau_i(K - F(T_\alpha;T_{i-1},T_i))\right)^+ \tag{4.6}$$

Figure 4.3a shows the timeline of a European swaption. T_i is the start date of the swaption and at time T_l the holder has to decide whether to exercise and enter the underlying interest rate swap until T_n.

Suppose a JPY OIS swaption trades in the market with expiration at $02-03-2020$,

Table 4.1: Discount factors for a 1-year swap

Expiration (days)	Term structure	PV discount factors
90	-0.0425	1.010739103
180	-0.06	1.030927835
270	-0.071	1.056245049
360	-0.081	1.088139282

the option to enter a 1-year swap with exercise rate -0.05 . To calculate the price of the swaption at expiration, one needs the market price of entering into a 1-year swap at expiration. The term structure values in Table 4.1 are taken from the JPY OIS rate for 02-03-2020 in Table C1 in the appendix. Because swaps usually have quarterly payments and it is a 1-year swap, four rates are necessary. The discount factors are positive because the OIS rates are all negative. The market swap is calculated as $\frac{1-PVF_n}{\sum_i^n PVF_i}$ and adjusted for the payment frequency, which results in this case in -0.084. Usually for a payer swaption, $S > K$ is required in order to make a profit. However, because the interest rates of the JPY are negative it turns out that $K > S$ for a payer swaption. Since $-0.084 > -0.05$ the payer swaption at maturity is worth zero. However, the receiver swaption is worth 0.0358. With a notional amount of $100.000.000,00$ JPY, the swaption is worth 3581364 JPY.

The Black & Scholes model for swaptions changes to the Black '76 model where d_1 is specified as:

$$d_1 = \frac{ln\frac{F}{K} + (\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

In a negative interest rate environment, it is not possible to take the natural log in the formula for d_1. Therefore, one has to use the normal model or the shifted lognormal model. The Black model with a shift is specified as:

$$C = (S-\theta)N(d_1) - e^{-r(T-t)}(K-\theta)N(d_2) \tag{4.7}$$

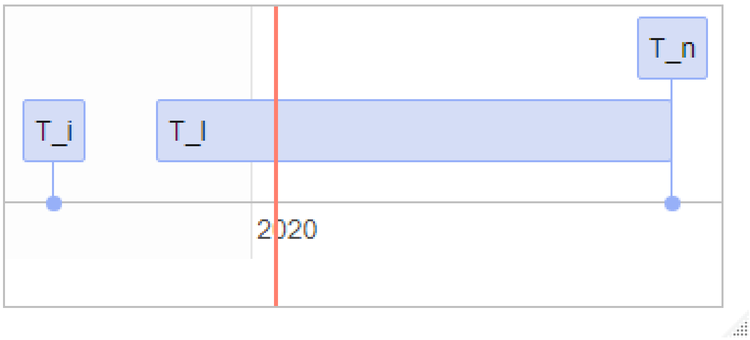
where

$$d_1 = \frac{ln\frac{F-\theta}{K-\theta} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

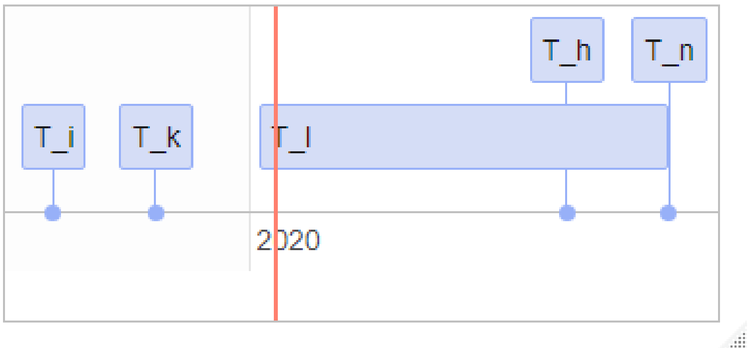
4.2 Bermudan Swaptions

Bermudan swaptions are unique in the sense that the swaptions cannot only be exercised at maturity of the option and the first reset date of the underlying swap, but also on subsequent reset dates, mainly coupon paying dates. This makes the Bermudan style swaption an intermediate case between its European and American counterparts. A Bermudan swaption is more valuable than a European swaption. Suppose if on the first reset date of the underlying swap, the payer swaption is OTM, i.e. $(F(T_\alpha; T_{i-1}, T_i) < K$, a European swaption holder would not exercise the option on the underlying swap because it is out of the money. However, in the same case a Bermudan swaption holder has the possibility to wait until the next reset dates and will not exercise until $(F(T_\alpha; T_{i-1}, T_i) > K$. This is where stopping times from Section 2.4 come in. When the Bermudan Swaption goes ITM, there will be an exercise at stopping time τ. Obviously, the possibility of exercising prior to maturity makes the Bermudan swaption more expensive.

Figure 4.3b shows the timeline of a Bermudan swaption. T_i is the start date of the swaption and between T_k and T_h the holder has to decide whether to exercise and enter the underlying interest rate swap until T_n. Thus, the longest potential swap is when $T_l = T_k$ and the shortest when $T_l = T_n$.



(a) European swaption



(b) Bermudan swaption

Figure 4.3: Swaption timelines

This chapter adresses a numerical application of the calibration of two short rate models, the Hull-White extended Vasicek and the Black-Karasinsky models, to market swaption implied volatilities. Consequently, the calibration results are used to price Bermudan swaptions. Research has generally focused on numerical applications with EUR swaptions. This is the reason that this application uses another currency that finds itself in a negative interest rate environment, the JPY. Khwaja [11] displays that the most traded tenors for swaptions are 10Y, the most traded maturities 1M, 3M, 6M, 1Y, and 3Y, and the most traded contracts 1Mx10Y and 3Mx10Y. 3Yx10Y, 3Yx30Y, 5Yx10Y, and 10Yx10Y contracts traded more frequently as well. In this application is chosen for a 1Yx10Y Bermudan swaption contract.

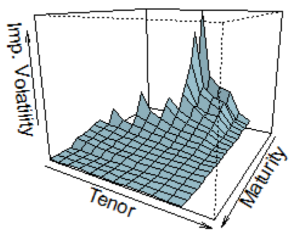
5.1 Data

Three types of datasets were collected from Bloomberg. First, the LIBOR and OIS rates for the EUR, JPY, USD, and CHF were collected for the plots in Chapter 1. The plot for the JPY is useful to keep as a reference to see when the JPY interest rate goes negative. Second, JPY swaption implied volatilities for the 1^{st} of February 2016, the 1^{st} of February 2018, and the 3^{rd} of February 2020 were collected. For the year 2020, the 1^{st} of February was not a business day and is therefore shifted to the 3^{rd}. The maturities range from one month to 30 years and the tenors from 1 year to 30 years. Because of the aforementioned research from Hull and White [7], the swaption volatilities are gathered under the OIS rate. The volatility swaptions are collected for both the Black/lognormal and the Bachelier/normal. The Black implied volatilities can be viewed in Table A1, Table A2, and Table A3 of the appendix. The Bachelier implied volatilities can be viewed in Table B1, Table B2, and Table B3 of the appendix. The swaption implied volatilities are used for creating volatility surfaces and the calibration of short rate models to the market. Finally, OIS rates on the 1^{st} of February 2016, the 1^{st} of February 2018, and the 3^{rd} of February 2020 were collected for different tenors. These are used to construct the zero curves for the JPY OIS on the aforementioned dates. The data can be found in Table C1 of the appendix.

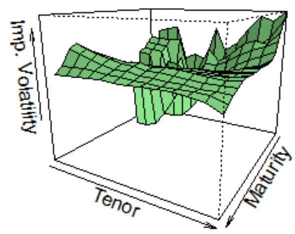
5.2 Results

5.2.1 Volatility Surfaces

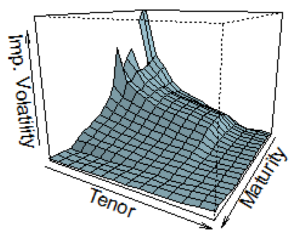
Figure 5.1 displays six graphs: the Black and Bachelier volatatility surfaces, each plotted for the years 2016, 2018, and 2020. The normal implied volatilities are all present for each maturity and tenor combination. Furthermore, they are also relatively stable as can be seen from the flatness of the surface. Long-term implied volatility is more influenced by the long-term average. Short-term implied volatility is higher during high- and lower during low realized volatility periods. The JPY interest rate seems to deal with the latter case with the Bachelier approach. The more interesting plots are those of the lognormal implied volatilities. For all years, data is missing, as can be seen in the appendix as well. In 2016, a significant amount of data is missing; in 2018, only one entry is missing; and in 2020, even more data compared to 2016 is missing. How can this be explained? By looking back at Figure 1.1b, one can see that the JPY LIBOR and TONAR suddenly drop below zero at the beginning of 2016. In 2017, the LIBOR goes up again to a positive value but after 2018 the rates decrease even more below zero than before. The Black formula returns NA for the implied volatilities because due to its lognormal character it is not able to accomodate negative the negative interest rates. Thus, it is confirmed that the lognormal approach fails when interest rates go negative. Why is it that some values are available, especially for the year 2018? This can be explained by the zero curves in subsection 5.2.2. Finally, it should be noted that the swaption volatility matrix may be biased because some contracts are more liquid than others [3].



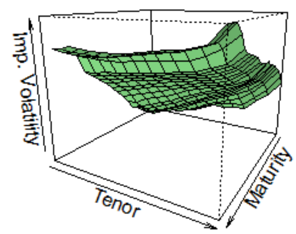
(a) Black 2016



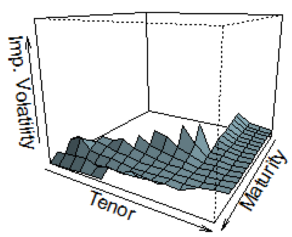
(b) Bachelier 2016



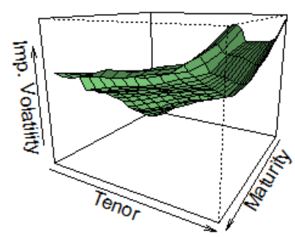
(c) Black 2018



(d) Bachelier 2018



(e) Black 2020



(f) Bachelier 2020

Figure 5.1: JPY OIS swaption IV surfaces

5.2.2 Term Structures

The zero curve, the yield curve, or the term structure of interest rates is a curve that is constructed from similar contracts, and plots the interest rate but with different contract lengths at a specific point in time. For the pricing of swaptions, usually a quarterly payment frequency is set. Thus, one needs the discount factors calculated from the yield curve on these dates. Because some contracts do not exist for specific maturities, interpolation is used. The straightforward interpolation formula is given as:

$$y_i = y_s + \frac{(y_t) - y_s}{x_t - x_s} \cdot (x_i - x_s) \tag{5.1}$$

Where y is the zero rate, x the number of days until expiration, and $s < i < t$. More advanced interpolation methods are discussed by Hagan and West [6].

Figure 5.2 plots the term structure of the JPY OIS on the three dates that were selected for this research. Lengths range from a day, to a week, to a month, up to 40 years. The data can be reviewed in Table C1 of the appendix. The term structures in 2016, 2018, and 2020, all have a similar shape. The JPY OIS starts negative but goes the longer the lenght of the contract, the higher the rate. The longer tenors have a concave shape, meaning that the yield increases the longer the tenor, but at a decreasing rate. The rate becomes positive between a tenor of 10 and 15 years for 2020 and a tenor of 3 years for 2018. For 2016, the rate is positive for the tenors from 1 day to 1 month and from a tenor of 7 years onwards. These results are directly connected with the development of the JPY LIBOR and TONAR in Figure 1.1b and the volatility surfaces in Figure 5.1. In February 2016, the interest rate is still positive but goes negative around April 2016. The term structures help explain why the Black formula is still able to return implied volatilities in negative environments: because for longer terms, the investor is rewarded and gets a positive rate nonetheless. It also explains why more data is missing for 2020 than for 2016. The term structure of 2016 initially started with a positive value and goes back to positive before 2020 does. Finally, the Black implied volatilities for 2018 are all available with the exception of one value for a 1Y tenor. This does not completely agree with the term structure, where the value goes positive after a 3 year tenor.

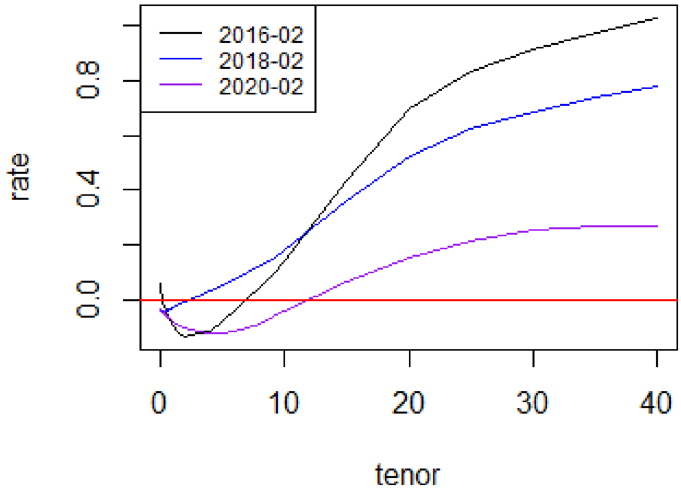


Figure 5.2: JPY OIS term structures

5.2.3 Calibration with Bermudan Swaptions

The calibration of a model means choosing the values of the parameters of the model in order to minimize the square root of the sum of the squares of the relative differences between market and model prices [18]. In the models, the parameters θ or ϑ are calculated from the yield curve whereas σ is calculated from the swaption volatility matrix. The calibration formula is:

$$argmin\sqrt{\sum_{i=1}^{n}\left(\frac{Swpt_i^M - Swpt_i}{Swpt_i}\right)^2} \tag{5.2}$$

The results of the calibration of the Hull-White extended Vasicek and the Black-Karasinski to the volatility surface of JPY swaptions on 02-01-2018 in *RQuantLib* are displayed in Table 5.1, Table 5.2, Table 5.3, and Table 5.4. The comparison between the use of the Black or Bachelier implied volatilities clearly shows that the calibration of both the models is more accurate when using the normal implied volatilities. The differences between model prediction and market value, as displayed in the third column, are all a hundredth value in Table 5.3, and Table 5.4, and a tenth in Table 5.1 and Table 5.2. Thus, the Bachelier/normal swaption implied volatilities should be used in favor of the Black/lognormal implied volatilities.

Table 5.1: Calibration of Black-Karasinsky to 2018 swaption Black IV

Calibration with Black/Lognormal IV		
Black Karasinsky	Market	Difference
0.309164	0.536955	-0.227791
0.322614	0.620961	-0.298347
0.337289	0.695717	-0.358428
0.352864	0.730815	-0.377951
0.982673	0.775065	0.207608
1.030179	0.808926	0.221253
0.970220	0.889253	0.080967
1.011325	0.798431	0.212894

Table 5.2: Calibration of HW ext. Vasicek to 2018 swaption Black IV

Calibration with Black/Lognormal IV K=-0.05 ts=0.01		
HW ext. Vasicek	Market	Difference
0.461607	0.536955	-0.075348
0.461666	0.620961	-0.159295
0.461771	0.695717	-0.233946
0.461863	0.730815	-0.268952
0.955154	0.775065	0.180089
0.955391	0.808926	0.146465
0.999477	0.889253	0.110224
1.051966	0.798431	0.253535

Table 5.3: Calibration of Black-Karasinsky to 2018 swaption normal IV

Calibration with Bachelier/Normal IV		
Black Karasinsky	Market	Difference
0.066699	0.149667	-0.082968
0.066769	0.154499	-0.087730
0.066832	0.159028	-0.092196
0.066886	0.152206	-0.085320
0.177153	0.144104	0.033049
0.177327	0.148242	0.029085
0.183458	0.152117	0.031341
0.184873	0.157676	0.027197

Table 5.4: Calibration of HW ext. Vasicek to 2018 swaption normal IV

Calibration with Bachelier/Normal IV K=-0.05 ts=0.01		
HW ext. Vasicek	Market	Difference
0.090381	0.149667	-0.059286
0.090395	0.154499	-0.064104
0.090406	0.159028	-0.068622
0.090422	0.152206	-0.061784
0.180350	0.144104	0.036246
0.180377	0.148242	0.032135
0.180998	0.152117	0.028881
0.181376	0.157676	0.023700

Table 5.5: Model parameters and price Bermudan swaption

Summary with Black/Lognormal IV			
Model	Price Bermudan Swaption	a	sigma
HW ext. Vasicek	890.5963	0.0004001	0.009224
Black Karasinsky	724.7189	1.056e-06	1.112

Summary with Bachelier/Normal IV			
Model	Price Bermudan Swaption	a	sigma
Black Karasinsky	723.6029	4.606e-06	0.1838
HW ext. Vasicek	723.6031	0.0003026	0.001806

CONCLUSION

This thesis adressed a relatively new area in finance: the appearance of negative interest rates for multiple currencies. Negative interest rates caused the return of the normal distribution, the downfall of the lognormal distribution, and the rise of the shifted lognormal distribution.

A brief history of negative interest rates was given. Before the 2008 financial crisis, there existed negative interest rates but these appearances were very much short-term. The general sense was that negative interest rates could not, and should not exist. However, the 2008 financial crisis was the cause of changes in government regulations that made several interest rates go negative. These appearances are already lasting for a longer time than ever before.

Stochastic calculus plays a crucial role in the mathematics behind finance. The randomness of a stock price or in this case, the interest rate, is directly associated with the Brownian Motion and Itô's Integral. The Brownian Motion is used in all the short-rate models that are used to model the interest rate. An overview of the most important short rate models and their characteristics was provided.

The interest rate can be found in basically any formula of a financial product. Therefore, it's negativity has a direct impact on the pricing of these financial products. The most important interest rate products are interest rate swaps, swaptions and, lastly, Bermudan swaptions. Bermudan swaptions are options on interest rate swaps where there is a possibility of exercise on multiple predetermined dates before expiration. As a numerical application, two short rate models, the Black Karasinski model and the Hull White model, are calibrated to market data to price Bermudan swaptions. For this, JPY OIS market swaption implied volatilitities for different tenors and maturities were collected at three dates, February 2016, 2018, and 2020, to construct implied volatility surfaces. Furthermore, the OIS rates were collected to construct the term structure of JPY interest rates on these specific dates. The JPY is of interest because the interest rate finds itself in a negative environment. The implied volatility surfaces and the term structures are used for calibration to determine the parameters of the two short rate models. With the defined model parameters, 1Yx10Y Bermudan swaptions are priced. The results indicate that the calibration with Bachelier implied volatilities is superiour to calibration with Black implied volatilitities.

The possibilities in this new area of research seem endless. First, there are multiple currencies that experience a negative interest rate at the moment. The CHF, DKK, JPY, EUR, can all be used for research and results can be compared between currencies. Second, there are many different short rate models, both one- and two factor. The one factor short rate models did not prove to generate unrealistic prices than their two factor counterparts. Third, there are different distributions that can all be compared: normal, lognormal, and shifted lognormal distributions. Fourth, different timeframes can be applied in the numerical applications. And finally, as stated before, all the financial products are partially dependent on the interest rate. Numerical applications can be executed for any of these financial derivatives such as swaps, caps, floors, or European swaptions. The models can then be used in the hedging of portfolios.

REFERENCES

Journal Articles

[1] Leif Andersen and Jesper Andreasen. "Factor dependence of Bermudan swaptions: fact or fiction?" In: *Journal of Financial Economics* 62 (2001), pp. 3–37.

[2] Fischer Black and Piotr Karasinski. "Bond and option pricing when short rates are lognormal". In: *Financial Analysts Journal* 47 (1991), pp. 52–59.

[4] L. Uri Dothan. "On the term structure of interst rates". In: *Journal of Financial Economics* 6 (1978), pp. 59–69.

[5] Lhabitant Gibson and Talay. "Modeling the term structure of interest rates: a review of the literature". In: (2001).

[6] Patrick S. Hagan and Graeme West. "Methods for constructing a yield curve". In: *WILMOTT Magazine* (2008), pp. 70–81.

[7] John Hull and Alan White. "LIBOR vs OIS: The Derivatives Discounting Dilemma". In: *Journal of Investment Management* 11 (2013), pp. 14–27.

[8] John Hull and Alan White. "Pricing interest-rate-derivative securities". In: *The Review of Financial Studies* 3 (1990), pp. 573–592.

[10] Jonathan E. Ingersoll John C. Cox and Stephen A. Ross. "A theory of the term structure of interst rates". In: *Econometrica* 53 (1985), pp. 385–407.

[12] Anastasia Kolodko and John Schoenmakers. "Iterative construction of the optimal Bermudan stopping time". In: *Finance and Stochastics* 10 (2006), pp. 27–49.

[13] Robert C. Merton. "On the pricing of corporate debt: the risk structure of interest rates". In: *The Journal of Finance* 29 (1974), pp. 449–470.

[18] Vincenzo Russo and Frank J. Fabozzi. "Calibrating short interest rate models in negative environments". In: *Journal of Derivatives* 24 (2017).

[20] Edward J. Sullivan and Timothy M. Weithers. "Louis Bachelier: the father of modern option pricing theory". In: *The Journal of Economic Education* 22 (2014), pp. 165–171.

[21] Oldrich Vasicek. "An equilibrium characterication of the term structure". In: *Journal of Financial Economics* 5 (1977), pp. 177–188.

Books

[3] Damiano Brigo and Fabio Mercurio. *Interest Rate Models - Theory and Practice.* 2007.

[15] Adriana Ocejo. *Lecture Notes on Stochastic Calculus for Finance II.* 2019.

[19] Steven E. Shreve. *Stochastic Calculus for Finance II - Continuous-Time Models.* 2008.

Online

[9] S&P Global Market Intelligence. *After repo rates spike, leveraged loan investors raise concern about sofr volatility.* 2019. URL: `https://www.spglobal.com/marketintelligence/en/news-insights/latest-news-headlines/leveraged-loan-news/after-repo-rates-spike-leveraged-loan-investors-raise-concern-about-sofr-volatility`.

[11] Amir Khwaja. *Swaptions clearing, a more detailed look.* 2013. URL: `https://www.clarusft.com/swaptions-clearing-a-more-detailed-look/`.

[14] Stephen Mihm. *Switzerland tried negative rate in the 1970s - it got very ugly.* 2019. URL: `https://www.swissinfo.ch/eng/switzerland-tried-negative-rates-in-the-1970s--it-got-very-ugly/45179144`.

[16] James Purchase and Nick Constantine. *Life below zero - the impact of negative rates on derivatives activity.* 2016. URL: `https://bankunderground.co.uk/2016/05/25/life-below-zero-the-impact-of-negative-rates-on-derivatives-activity/`.

[17] Jana Randow and Yuko Takeo. *Negative interest rates.* 2019. URL: `https://www.bloomberg.com/quicktake/negative-interest-rates`.

APPENDIX A: BLACK SWAPTION IMPLIED VOLATILITIES

Table A1: JPY swaption Black IV - (02/01/2016)

	1M	2M	3M	6M	9M	1Y	1.5Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
30Y	63.08	59.36	53.54	49.46	44.92	41.05	40.08	38.93	37.47	36.19	35.38	33.66	32.97	32.21	31.86	30.91	28.83	28.89	28.19	27.05
25Y						33.79														
20Y	73.89	63.50	59.67	58.06	51.55	46.29	43.85	42.39	40.64	39.03	37.05	34.64	33.57	32.04	31.52	30.55	28.80	29.28	30.23	29.71
15Y	111.01	83.32	87.77	77.88	67.10	60.06	52.75	50.34	45.68	42.35	39.60	36.76	34.97	33.19	32.11	31.03	28.16	29.37	30.58	30.58
10Y	217.29	164.73	183.47	155.37	128.95	108.48	90.88	85.33	67.37	58.52	51.61	44.55	40.66	36.93	34.42	32.90	27.82	29.26	31.38	31.70
9Y	242.68	166.94	208.32	179.38	136.72	135.61	104.81	96.09	79.27	64.10	55.48	47.57	43.27	38.27	35.35	34.13	28.27	29.31	31.37	31.73
8Y	296.15	166.14	252.72	227.78	162.64	174.34	125.25	113.54	87.93	73.98	60.75	50.14	46.59	40.28	36.73	35.16	28.48	29.52	31.45	31.84
7Y	660.19	169.19	546.18	386.53	292.29	236.62	156.42	138.84	101.98	81.21	69.95	54.38	49.37	43.29	38.67	35.98	28.35	29.28	31.49	32.04
6Y		317.21			249.36		216.64	184.43	126.95	95.95	76.88	63.26	54.34	46.17	41.92	38.59	29.17	30.15	31.68	32.65
5Y							209.73		156.89	114.40	88.30	69.08	62.37	50.79	44.52	41.02	29.02	29.83	31.78	33.59
4Y							201.05		252.06	136.98	99.87	76.46	65.74	58.62	48.53	43.59	29.98	30.74	32.20	33.30
3Y											113.44	84.77	74.32	62.12	57.29	48.54	31.16	31.93	32.53	32.97
2Y											160.21	88.73	84.14	70.35	59.86	60.42	32.29	33.34	32.60	32.72
1Y												101.87	91.02	80.48	68.22	62.89	33.95	34.92	32.34	32.58

Table A2: JPY swaption Black IV - (02/01/2018)

	1M	2M	3M	6M	9M	1Y	1.5Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
30Y	25.08	27.34	29.59	30.28	31.30	31.44	31.18	31.98	30.72	30.06	29.59	29.93	29.78	29.53	29.32	29.98	30.96	33.34	34.25	35.09
25Y	26.74	28.92	31.00	31.44	32.00	31.97	31.64	31.88	30.51	30.10	29.50	29.42	29.15	28.90	28.73	29.05	29.95	31.77	32.53	33.20
20Y	26.42	28.96	31.42	32.45	32.96	33.07	32.65	32.23	30.66	30.50	29.66	29.07	28.65	28.35	28.17	28.12	29.01	30.27	30.90	31.40
15Y	30.81	33.83	36.69	37.98	38.86	39.38	37.94	36.64	34.02	32.20	30.87	29.85	29.13	28.47	27.99	27.70	27.72	29.58	30.12	30.48
10Y	42.90	47.66	53.60	56.20	56.12	55.98	53.53	51.59	45.29	40.59	37.69	34.62	32.49	30.74	29.45	28.56	26.55	28.86	29.64	29.81
9Y	47.66	53.70	57.89	60.63	60.47	60.60	56.99	54.46	48.77	42.79	39.10	36.40	33.53	31.33	29.74	28.64	26.33	28.48	29.25	29.46
8Y	53.17	58.55	62.10	64.88	64.44	64.44	61.48	59.14	51.42	46.25	41.00	37.44	35.00	32.16	30.17	28.81	26.12	28.05	28.83	29.09
7Y	62.18	64.41	66.87	69.57	68.64	68.29	65.75	63.34	55.94	48.88	43.86	38.98	35.96	33.55	30.91	29.12	25.93	27.51	28.32	28.64
6Y	67.40	69.12	71.02	74.03	73.08	72.37	68.91	66.63	59.14	52.60	45.99	42.02	37.50	34.50	32.46	29.98	25.81	27.02	27.88	28.34
5Y	74.09	74.80	75.78	80.26	78.84	77.51	72.87	70.66	62.77	55.73	50.15	43.80	40.41	35.81	32.97	31.29	25.71	26.68	27.67	28.36
4Y	89.59	89.24	86.51	87.42	86.77	86.12	80.89	79.61	67.84	60.04	53.53	47.78	42.24	38.71	34.35	31.90	25.66	26.61	27.61	28.30
3Y	115.38	111.71	106.81	101.81	98.94	97.23	89.65	88.93	74.94	64.56	57.18	50.77	46.04	39.98	37.02	32.93	25.65	26.64	27.65	28.34
2Y	136.01	128.19	122.29	117.23	116.29	116.89	101.75	104.41	79.84	71.87	61.99	54.38	49.51	44.31	38.00	35.63	25.67	26.75	27.70	28.44
1Y	215.27	217.84	198.75		152.84	137.18	121.94	160.13	97.76	81.25	68.67	57.68	53.34	47.85	42.94	35.45	25.91	27.20	28.22	28.93

Table A3: JPY swaption Black IV - (02/03/2020)

	1M	2M	3M	6M	9M	1Y	1.5Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y
30Y	103.95	102.30	100.62	98.62	97.42	95.98	95.65	95.07	95.03	94.87	94.62	94.27	94.33	95.22	97.09	100.63			
25Y	113.35	111.02	108.67	106.02	103.82	98.61	99.64	97.43	93.93	90.34	86.85	86.94	87.50	88.59	90.69	94.52			
20Y	126.30	123.31	120.58	118.23	115.68	113.54	111.47	108.89	104.65	100.47	96.45	91.77	87.96	84.62	82.19	80.91	110.17		
15Y	190.93	184.09	177.50	168.35	160.07	155.42	145.23	134.50	117.51	105.39	95.14	89.11	84.52	81.58	79.82	79.58	77.85		
10Y	893.71									263.91	163.67	119.46	98.84	88.58	82.42	79.49	81.13	113.61	
9Y											264.85	147.90	108.32	92.40	83.46	78.87	77.20	126.88	
8Y												193.66	127.36	99.16	85.66	78.89	73.45	133.74	
7Y												216.77	148.12	112.87	89.98	79.62	69.43	107.53	
6Y													188.86	128.14	102.77	84.77	67.38	121.47	72.64
5Y														174.90	108.96	94.75	64.97	98.45	76.55
4Y														144.77	125.79	98.84	63.69	123.86	70.91
3Y														154.58	145.08	107.72	62.58	121.12	66.20
2Y															132.97	168.06	61.43	116.14	62.72
1Y															136.82	148.99	60.75	106.87	62.05

APPENDIX B: NORMAL SWAPTION IMPLIED VOLATILITIES

Table B1: JPY swaption normal IV - (02/01/2016)

	1M	2M	3M	6M	9M	1Y	1.5Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
30Y	65.59	61.89	56.04	52.22	47.99	44.39	44.30	44.00	44.14	44.26	44.70	43.96	44.31	44.38	44.87	44.37	42.81	41.48	39.58	37.65
20Y	60.80	52.63	49.84	49.38	44.87	41.25	40.81	41.15	42.56	43.70	43.98	43.24	43.75	43.34	44.04	43.86	42.91	41.90	41.76	40.51
15Y	62.92	47.98	51.24	47.17	42.52	39.75	37.87	38.92	40.19	41.52	42.49	42.57	43.23	43.36	44.03	44.27	42.99	41.85	41.87	41.30
10Y	55.65	43.34	48.92	43.99	39.73	36.28	34.93	36.83	38.26	40.55	41.60	41.92	43.25	43.40	44.03	44.95	43.53	42.71	41.94	42.11
9Y	48.33	34.41	43.31	39.84	33.77	36.08	33.92	36.09	37.95	39.79	41.44	41.24	43.36	43.17	43.96	45.65	44.25	42.89	42.04	41.99
8Y	43.78	25.71	38.84	37.35	30.67	35.12	32.34	34.71	37.19	39.09	40.88	40.61	43.12	43.00	44.00	45.86	44.65	43.34	42.27	41.90
7Y	55.86	17.02	44.91	37.22	34.58	33.35	30.79	33.24	35.80	38.40	40.18	39.98	42.87	42.83	44.04	45.44	44.62	43.28	42.50	41.81
6Y	37.74	15.08	33.25	31.48	19.24	32.56	28.47	32.14	35.44	37.97	39.88	39.83	42.91	43.03	44.37	46.41	45.95	44.67	42.96	41.96
5Y	39.64	6.97	31.42	28.29	40.57	37.90	18.55	29.50	33.82	37.07	38.86	39.75	42.28	43.22	44.72	46.07	46.02	44.68	43.41	42.11
4Y	36.97	1.53	49.38	27.67	22.61	38.38	9.31	26.04	31.06	35.72	37.06	38.08	41.40	42.73	44.85	46.79	47.39	45.84	43.87	41.84
3Y	55.31	0.02	48.97	45.54	42.97	23.96	1.76	22.98	27.87	34.36	35.51	36.37	40.50	42.33	45.03	48.02	49.03	47.34	44.23	41.53
2Y	53.97	0.02	49.25	46.19	43.54	41.23	0.03	34.09	26.05	30.26	33.94	34.01	39.50	41.33	44.99	49.21	50.60	49.08	44.30	41.29
1Y	52.40	0.02	49.64	46.81	44.07	42.37	0.02	37.92	36.06	28.85	32.62	31.15	38.95	39.86	44.52	50.63	52.84	50.98	44.02	41.16

Table B2: JPY swaption normal IV - (02/01/2018)

	1M	2M	3M	6M	9M	1Y	1.5Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
30Y	22.19	24.29	26.39	27.28	28.52	28.94	29.28	30.63	30.54	30.94	31.46	32.74	33.43	33.94	34.41	35.74	38.25	40.32	40.34	40.44
20Y	18.46	20.39	22.29	23.48	24.35	24.93	25.55	26.15	26.60	28.08	28.82	29.59	30.36	31.13	31.90	32.67	35.70	36.97	37.00	37.15
15Y	16.06	17.85	19.63	21.01	22.28	23.29	23.85	24.37	25.06	25.92	26.85	27.88	28.90	29.77	30.63	31.49	34.23	36.01	36.05	36.23
10Y	13.38	15.17	17.39	19.22	20.22	21.13	22.03	22.93	24.02	24.84	25.90	26.96	28.00	28.83	29.65	30.45	32.95	35.29	35.19	35.45
9Y	13.04	14.97	16.50	18.19	19.09	20.04	21.02	22.08	23.25	24.28	25.47	26.51	27.52	28.38	29.22	30.04	32.62	34.94	34.80	35.06
8Y	12.66	14.25	15.45	17.08	17.92	18.82	19.98	21.08	22.48	23.66	24.86	25.97	26.94	27.85	28.74	29.60	32.29	34.54	34.36	34.62
7Y	12.65	13.47	14.34	15.90	16.66	17.48	18.90	20.05	21.59	23.06	24.07	25.29	26.46	27.30	28.22	29.10	31.95	34.05	33.82	34.07
6Y	11.41	12.09	12.81	14.40	15.22	16.02	17.37	18.63	20.43	22.01	23.42	24.71	25.91	26.98	27.92	28.81	31.65	33.62	33.32	33.58
5Y	10.26	10.75	11.28	12.98	13.74	14.41	15.82	17.21	19.32	21.01	22.70	24.05	25.34	26.41	27.43	28.40	31.30	33.38	33.01	33.27
4Y	9.96	10.36	10.41	11.59	12.49	13.29	14.82	16.34	18.35	20.29	21.95	23.39	24.85	25.93	27.11	28.21	31.23	33.31	32.95	33.20
3Y	9.79	10.07	10.17	10.74	11.43	12.12	13.71	15.21	17.32	19.37	21.20	22.64	24.11	25.37	26.57	27.86	31.22	33.34	32.99	33.23
2Y	8.91	9.14	9.26	9.90	10.62	11.32	12.84	14.24	15.77	18.55	20.59	22.11	23.61	24.88	26.21	27.38	31.23	33.46	33.05	33.36
1Y	7.48	8.33	8.07	8.93	10.00	11.03	12.47	13.20	15.81	17.93	19.72	21.43	23.19	24.49	25.92	27.23	31.48	33.96	33.56	33.85

Table B3: JPY swaption normal IV - (02/03/2020)

	1M	2M	3M	6M	9M	1Y	1.5Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
30Y	33.00	32.45	31.91	31.25	30.89	30.49	30.46	30.42	30.61	30.73	30.84	30.98	31.13	31.30	31.46	31.62	33.12	33.75	33.73	33.58
20Y	28.02	27.45	26.96	26.70	26.48	26.39	26.65	26.91	27.46	27.92	28.37	28.78	29.18	29.45	29.72	29.99	30.57	31.60	31.58	31.37
15Y	24.68	24.04	23.47	22.97	22.73	22.96	23.23	23.50	24.01	24.68	25.27	25.73	26.19	26.67	27.14	27.59	28.98	30.08	30.05	29.82
10Y	21.72	21.33	20.65	20.39	20.35	20.47	20.86	21.09	21.85	22.68	23.34	23.93	24.52	25.31	26.10	26.87	27.75	28.95	28.69	28.43
9Y	21.09	20.75	20.20	19.66	19.51	19.97	20.38	20.61	21.37	22.09	23.01	23.55	24.06	24.79	25.51	26.22	27.29	28.50	28.34	28.04
8Y	20.71	20.12	19.59	19.04	19.24	19.32	19.78	20.05	21.12	21.73	22.62	23.05	23.56	24.24	24.91	25.56	26.83	28.04	27.99	27.64
7Y	19.65	19.57	18.97	18.41	18.44	18.55	19.13	19.73	20.56	21.10	22.06	22.50	23.13	23.68	24.27	24.83	26.28	27.50	27.57	27.15
6Y	18.62	18.37	18.36	17.74	17.50	17.50	18.38	18.97	19.88	20.86	21.80	22.33	22.98	23.49	24.01	24.51	26.13	27.22	27.47	26.94
5Y	17.55	17.34	17.19	17.16	16.90	16.76	17.34	18.23	19.18	20.34	21.16	21.85	22.68	22.99	23.59	24.13	25.97	26.91	27.41	26.71
4Y	15.91	15.84	15.80	15.78	15.88	16.21	16.51	17.22	18.57	19.67	20.77	21.22	22.11	22.69	23.16	23.84	25.67	26.57	26.82	26.67
3Y	14.31	14.21	14.13	14.62	14.75	15.46	15.99	16.74	18.01	19.15	20.19	21.02	21.66	22.13	22.60	23.38	25.41	26.22	26.18	26.62
2Y	12.91	12.91	12.92	13.18	14.02	14.62	15.43	16.05	17.19	18.61	19.69	20.59	21.28	21.75	22.20	22.92	25.13	25.92	25.63	26.57
1Y	11.86	11.95	12.22	12.59	13.27	14.27	15.10	15.95	16.86	18.24	19.26	20.00	20.93	21.36	21.99	22.41	24.95	25.75	25.51	26.35

APPENDIX C: JPY OIS RATES

Table C1: JPY OIS rates (2016, 2018, 2020)

	Time	Year	JPY.OIS.02.03.20.Mid.YTM	JPY.OIS.02.01.18.Mid.YTM	JPY.OIS.02.01.16.Mid.YTM
1	1D	0.00	-0.03	-0.04	0.06
2	1W	0.02	-0.04	-0.05	0.06
3	2W	0.04	-0.04	-0.05	0.06
4	3W	0.06	-0.04	-0.05	0.03
5	1M	0.08	-0.05	-0.05	0.02
6	2M	0.17	-0.05	-0.05	-0.01
7	3M	0.25	-0.04	-0.05	-0.03
8	4M	0.33	-0.05	-0.05	-0.03
9	5M	0.42	-0.06	-0.04	-0.04
10	6M	0.50	-0.06	-0.04	-0.05
11	7M	0.58	-0.06	-0.04	-0.06
12	8M	0.67	-0.07	-0.04	-0.06
13	9M	0.75	-0.07	-0.04	-0.07
14	10M	0.83	-0.07	-0.03	-0.08
15	11M	0.92	-0.08	-0.03	-0.09
16	1Y	1.00	-0.08	-0.03	-0.10
17	18M	1.50	-0.10	-0.02	-0.12
18	2Y	2.00	-0.10	-0.01	-0.14
19	3Y	3.00	-0.12	0.01	-0.13
20	4Y	4.00	-0.12	0.03	-0.12
21	5Y	5.00	-0.12	0.05	-0.08
22	6Y	6.00	-0.12	0.07	-0.04
23	7Y	7.00	-0.10	0.10	0.00
24	8Y	8.00	-0.09	0.12	0.05
25	9Y	9.00	-0.07	0.15	0.09
26	10Y	10.00	-0.04	0.18	0.14
27	15Y	15.00	0.06	0.36	0.44
28	20Y	20.00	0.15	0.52	0.70
29	25Y	25.00	0.21	0.62	0.84
30	30Y	30.00	0.25	0.69	0.91
31	35Y	35.00	0.26	0.74	0.98
32	40Y	40.00	0.27	0.78	1.03

R SCRIPT

Probability Distributions

```
#Load packages
library(greybox)

# Normal Distribution (Bachelier)
grid1 = seq(-5,5, length = 100)
plot(grid1, dnorm(grid1), type="l", lty=1, xlab="Interest Rate"
    ,
      ylab="Density")

# Lognormal Distribution (Black)
grid2 = seq(0,10, length = 100)
plot(grid2,dlnorm(grid2),type="l", lty=1,xlab="Interest Rate",
      ylab="Density")

# Shifted Lognormal Distribution
x <- dtplnorm(c(-1000:1000)/200, 0, 1, -2)
plot(c(-1000:1000)/200, x, type="l", lty = 1, xlab="Interest
    Rate",
      ylab = "Density")
```

OIS and LIBOR

```r
# Cleaning the workspace
rm(list=ls())
suppressMessages(library(stargazer))
suppressMessages(library(ggplot2))
suppressMessages(library(timeSeries))
suppressMessages(library(xts))
suppressMessages(library(graphics))
suppressMessages(library(zoo))

# import the data
LIBOR = read.csv("LIBOR.csv")
LIBOR$Date = as.Date(LIBOR$Date, format = "%m/%d/%Y")
LIBOR.xts = as.xts(LIBOR, order.by = LIBOR$Date)
LIBOR.xts = LIBOR.xts[,-1]

#create xts for all the currencies
CHF.LIBOR = as.xts(LIBOR$CHF.LIBOR.Ask.Price, order.by = LIBOR$
    Date)
CHF.LIBOR = setNames(CHF.LIBOR, "LIBOR")
CHF.SARON = as.xts(LIBOR$SSARON.Mid.Price, order.by = LIBOR$
    Date)
CHF.SARON = setNames(CHF.SARON, "SARON")
US.LIBOR = as.xts(LIBOR$US.LIBOR.Ask.Price, order.by = LIBOR$
    Date)
US.LIBOR = setNames(US.LIBOR, "LIBOR")
US.SOFR = as.xts(LIBOR$SOFR.Mid.Price, order.by = LIBOR$Date)
US.SOFR = setNames(US.SOFR, "SOFR")
JPY.LIBOR = as.xts(LIBOR$JPY.LIBOR.Ask.Price, order.by = LIBOR$
    Date)
JPY.LIBOR = setNames(JPY.LIBOR, "LIBOR")
```

```
JPY.TONAR = as.xts(LIBOR$TONAR.Mid.Price, order.by = LIBOR$Date
    )
JPY.TONAR = setNames(JPY.TONAR, "TONAR")
LIBOR.xts$zeroline = 0
zeroline = LIBOR.xts$zeroline
EUR.LIBOR = as.xts(LIBOR$EUR.LIBOR.Ask.Price, order.by = LIBOR$
    Date)
EUR.ESTR = as.xts(LIBOR$ESTR.Last.Price, order.by = LIBOR$Date)

#plot the data
plot.xts(merge.xts(CHF.LIBOR, CHF.SARON), legend.loc = "bottom"
    , col = c("black", "#CC0033"), main = "CHF LIBOR vs SARON")
plot.xts(merge.xts(US.LIBOR, US.SOFR), legend.loc = "bottom",
    col = c("black", "#339900"), main = "USD LIBOR vs SOFR")
plot.xts(merge.xts(JPY.LIBOR, JPY.TONAR), legend.loc = "bottom"
    , col = c("black", "#33CCCC"), main = "JPY LIBOR vs TONAR")
lines(zeroline, col = "red")
plot.xts(merge.xts(EUR.LIBOR, EUR.ESTR), legend.loc = "bottom",
    col = c("black", "#CC33CC"), main = "EUR LIBOR vs ESTR")
lines(zeroline, col = "red")
```

RW, BM, and Stopping Times Simulations

```
# Symmetric Random Walk
SymRW = function(k = 10,initial.value = 0) {
  samples = rbinom(k,size = 1, prob = 0.5)
  samples[samples==0] = -1
  initial.value + c(0, cumsum(samples))
}
plot(SymRW(), type = 'b', xlab = 'k', ylab = "M")
abline(h=0, col = 'red')

# Brownian Motion
N = 1000
BM = rnorm(N, 0, 0.1)
BM = cumsum(BM)
plot(BM,type = 'l', xlab = "steps", ylab = "scale")

# A stopped process
N = 1000
BMstop = rnorm(N, 0, 0.1)
BMstop = cumsum(BMstop)
BMstop = data.frame(BMstop)
BMstop[400:1000,] = BMstop[399,]
BMstop = data.matrix(BMstop, rownames.force = NA)
plot(BMstop,type = 'l', xlab = "steps", ylab = "scale")
abline(v = 399, lty = 3)
text(350,-1, expression(tau))
```

Timelines European and Bermudan Swaption

```r
# Timeline Swaption and Bermudan Swaption

# Load packages
suppressMessages(library(htmltools))
suppressMessages(library(timevis))

# Timeline for a European swaption
time.eur <- data.frame(
  id      = 1:3,
  content = c("T_i", "T_l", "T_n"),
  start   = c("2018-02-01", "2019-02-01", "2024-02-01"),
  end     = c(NA,"2024-02-01", NA))

timevis(time.eur, showZoom = FALSE)


# Timeline for a Bermudan swaption
time.berm <- data.frame(
  id      = 1:5,
  content = c("T_i", "T_k", "T_l", "T_h", "T_n"),
  start   = c("2018-02-01", "2019-02-01", "2020-02-01","
     2023-02-01", "2024-02-01"),
  end     = c(NA, NA, "2024-02-01", NA, NA))

timevis(time.berm, showZoom = FALSE)
```

```
# Import the data and adjust to R and plot the volatility
    surface

# Black Volatility surface 2016
black16 = read.csv('jpyswapvolblack02.01.16.csv')

black16 = black16[,-1]                                    # Remove
    tenors column 1
black16 = black16[c('14', '13', '12', '11', '10', '9', '8', '7'
    , '6', '5', '4', '3', '2', '1'),] # Last row starts first
    now, etc.
black16 = t(black16)                                      # Switch
    maturity with tenor
black16 = black16/100                                     # Make
    values percentage

persp(black16, theta = 120, phi = 15, ticktype = "simple", col
   = 'lightblue',
      xlab = 'Maturity', ylab = 'Tenor', zlab = 'Imp.␣
         Volatility',
      shade = 0.2, expand = 0.8)

# Black Volatility surface 2018
black18 = read.csv('jpyswapvolblack02.01.18.csv')

black18 = black18[,-1]
black18 = black18[c('14', '13', '12', '11', '10', '9', '8', '7'
    , '6', '5', '4', '3', '2', '1'),]
black18 = t(black18)
black18 = black18/100
```

```
persp(black18, theta = 120, phi = 15, ticktype = "simple", col
  = 'lightblue',
      xlab = 'Maturity', ylab = 'Tenor', zlab = 'Imp. 
        Volatility',
      shade = 0.4, expand = 0.8)

# Black volatility surface 2020
black20 = read.csv('jpyswapvolblack02.03.20.csv')

black20 = black20[,-1]
black20 = black20[c('14', '13', '12', '11', '10', '9', '8', '7'
  , '6', '5', '4', '3', '2', '1'),]
black20 = t(black20)
black20 = black20/100

persp(black20, theta = 120, phi = 15, ticktype = "simple", col
  = 'lightblue',
      xlab = 'Maturity', ylab = 'Tenor', zlab = 'Imp. 
        Volatility',
      shade = 0.6, expand = 0.8)

# Bachelier volatility surface 2016
bach16 = read.csv('jpyswapvolnorm02.01.16.csv')

bach16 = bach16[,-1]
bach16 = bach16[c('14', '13', '12', '11', '10', '9', '8', '7',
  '6', '5', '4', '3', '2', '1'),]
bach16 = t(bach16)
bach16 = bach16/100

persp(bach16, theta = 120, phi = 15, ticktype = "simple", col =
```

```
    'lightgreen',
      xlab = 'Maturity', ylab = 'Tenor', zlab = 'Imp.␣
         Volatility',
      shade = 0.2, expand = 0.8)

# Bachelier volatility surface 2018
bach18 = read.csv('jpyswapvolnorm02.01.18.csv')

bach18 = bach18[,-1]
bach18 = bach18[c('14', '13', '12', '11', '10', '9', '8', '7',
   '6', '5', '4', '3', '2', '1'),]
bach18 = t(bach18)
bach18 = bach18/100

persp(bach18, theta = 120, phi = 15, ticktype = "simple", col =
    'lightgreen',
      xlab = 'Maturity', ylab = 'Tenor', zlab = 'Imp.␣
         Volatility',
      shade = 0.4, expand = 0.8)

# Bachelier volatility surface 2020
bach20 = read.csv('jpyswapvolnorm02.03.20.csv')

bach20 = bach20[,-1]
bach20 = bach20[c('14', '13', '12', '11', '10', '9', '8', '7',
   '6', '5', '4', '3', '2', '1'),]
bach20 = t(bach20)
bach20 = bach20/100

persp(bach20, theta = 120, phi = 15, ticktype = "simple", col =
    'lightgreen',
      xlab = 'Maturity', ylab = 'Tenor', zlab = 'Imp.␣
```

```
    Volatility',
shade = 0.6, expand = 0.8)
```

Calibration with Bermudan Swaptions

```
# CALIBRATION OF Hull-White EXTENDED VASICEK MODEL
# Hull-White is normal and allows for negative rates

# Load packages
suppressMessages(library(RQuantLib))

# Specify the parameters
parameters = list(tradeDate = as.Date('2018-02-01'),  # Start
    date of Bermudan Swaption
                  settleDate = as.Date('2018-02-04'), # 2
                      business days between tradeDate and
                      settleDate
                  startDate = as.Date('2019-02-04'),  # First
                      time possible to exercise
                  maturity = as.Date('2029-02-04'),   # Last
                      time possible to exercise
                  strike = 0.03,                      # Strike
                      rate
                  dt = 0.25,                          #
                      Quarterly reset dates
                  payFixed = TRUE,                    # Payer
                      or Receiver Swaption, no put-call parity
                      for Bermudan swaption!
                  method = "BKTree",                # Hull-White
                      model, BKTree is also possible
                  interpWhat = "zero",              # Curve
                      construction
                  interpHow = "linear")             #
                      Interpolation method
```

```r
# Set the date to be evaluated
setEvaluationDate(as.Date('2018-02-01'))

 tsQuotes = list(d1w  = 0.0001,
                 s3y  =0.004,
                 s5y  =0.005,
                 s7y = 0.006,
                 s10y =0.007,
                 s20y =0.008)

 times = seq(0, 19.75, 0.25)                          #

 swcurve=DiscountCurve(parameters,tsQuotes,times)     #

#Swaption tenors, maturities, implied volatilities from
    Bloomberg market data
# These can and should be adjusted based on the maturity and
    tenor of your product
swaptionMaturities = c(1/12, 2/12, 3/12, 6/12, 9/12, 1, 1.5,
    2,3,4,5, 6, 7, 8, 9, 10,15, 20, 25, 30) # rows
swaptionMaturities = swaptionMaturities[2:9]
    # adjust rows
swaptionTenors = c(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, 20,30)
    # columns
swaptionTenors = swaptionTenors[2:9]
    # adjust columns

# Adjusting the raw data of the implied volatilities
# volsurf is normal IV
volsurf = read.csv("jpyswapvolnorm02.01.18.csv") # Read data
    from Bloomberg
volsurf = volsurf[,-1]                              # Remove
```

```
    tenors column 1
volsurf = volsurf[c('13', '12', '11', '10', '9', '8', '7', '6',
     '5', '4', '3', '2', '1'),] # Last column starts first now,
    etc.
volsurf = t(volsurf)                                # Last row
    starts first now, etc.
volsurf = volsurf/100                              # Data from
    Bloomberg is in bp but should be in R in percentage
volsurf.select = volsurf[2:9, 2:9]                # Select part
    of the IV surface to be used

# volsurf2 is lognormal IV
volsurf2 = read.csv("jpyswapvolblack02.01.18.csv")
volsurf2 = volsurf2[,-1]
volsurf2 = volsurf2[-2,] # Remove 25Y becaue it is missing for
    Normal
volsurf2 = volsurf2[c('14','13', '12', '11', '10', '9', '8', '7
    ', '6', '5', '4', '3','1'),]
volsurf2 = t(volsurf2)
volsurf2 = volsurf2/100
volsurf2.select = volsurf2[2:9, 2:9]



# Calibration and Pricing of the Bermudan Swaption
price = BermudanSwaption(parameters, swcurve,
    swaptionMaturities, swaptionTenors, volsurf.select)
summary(price)

price2 = BermudanSwaption(parameters, swcurve,
    swaptionMaturities, swaptionTenors, volsurf2.select)
summary(price2)
```