IMPROVED BOUNDS ON A COMBINATORIAL PROBLEM

by

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ABSTRACT

JACOB D PAGE. Improved Bounds on a Combinatorial Problem. (Under the direction of DR. WILL BRIAN)

In this paper, we will define a function $H : \mathbb{N} \to \mathbb{N}$, whose output is the size of an optimized hypergraph based upon the restraints given by its input values. This function is known to be well-defined, however its values are unknown for larger $n \in \mathbb{N}$. Only an upper and lower bound for this function are definitively known. Here, we will use properties of pre-ordered sets to define an improved lower bound for H.

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CHAPTER 1: INTRODUCTION

1.1 H(n): A function regarding Hypergraphs

Recall that a *hypergraph* is a set of *vertices*, V, paired with a collection, $\mathcal{H}$, of subsets of V. These subsets are known as *hyperedges*.

The *size* of a given hypergraph is determined by its total number of vertices, $|V|$.

For a given hypergraph, a vertex $v \in V$ is said to be *isolated* if v is not contained in any $h \in \mathcal{H}$. v is called *restricted* if v is contained in exactly one $h \in \mathcal{H}$.

A *partition*, $(D, \mathcal{G})$, of $(V, \mathcal{H})$ is a subgraph $D \subseteq V, \mathcal{G} \subseteq \mathcal{H}$, such that every $v \in D$ is restricted.

Here, we have an example of a hypergraph with eight vertices and four hyperedges.

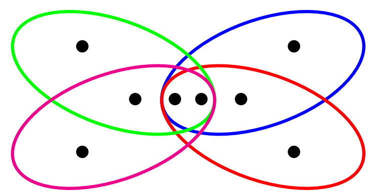


Figure 1.1: A hypergraph, $(V, \mathcal{H})$

Below it is a partition of said hypergraph, which has four vertices and three hyperedges. Particular interest is given to partitions that contain as many vertices as possible. The partition here, for instance, has size 4, and it can be verified that there exists no partition of this hypergraph with size greater than 4.

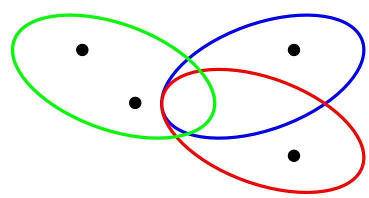


Figure 1.2: A partition of $(V, \mathcal{H})$

Definition 1.1.1. *Define $H(n)$ [1] to be the largest natural number, k, such that there exists a hypergraph, $(V, \mathcal{H})$, with size k such that:*

- *$(V, \mathcal{H})$ has no isolated points.*
- *Every partition of $(V, \mathcal{H})$ has size at most n*

While $H(n)$ is well-defined, only the first six values of the function are known.

- $H(1) = 1$
- $H(2) = 3$
- $H(3) = 5$
- $H(4) = 8$
- $H(5) = 10$
- $H(6) = 14$

1.2 Bounds on H(n)

While the values for larger n are presently unknown, there have been proven upper and lower bounds for the function. In the case of the upper bound[1], we know that for each $n \in \mathbb{N}$,

$$n \ln n + \gamma n + \frac{1}{2}$$

is a known upper bound for $H(n)$, where $\gamma \approx 0.5772156649$ is the *Euler-Mascheroni constant* . This upper bound was derived from the discovery that [1]

$$H(n) \leq \sum_{k=1}^{n} \frac{n}{k}$$

for all $n \in \mathbb{N}$.

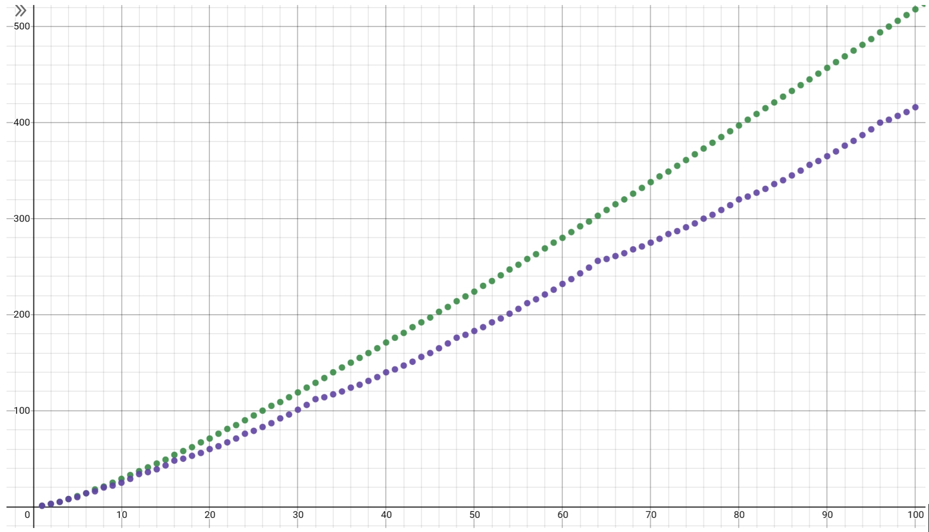
As for the lower bound[1], we know that for each $n \in \mathbb{N}$

$$\frac{1}{2} n \log_2 n - \frac{1}{2} n + \frac{1}{2}$$

is a known lower bound for $H(n)$. This lower bound was attained by proving that for the recursive sequence $\{k_n\}_{n=1}^{\infty}$, where

$$k_1 = 1 \text{ and } k_n = \lfloor \frac{n}{2} \rfloor + k_{\lfloor \frac{n}{2} \rfloor} + k_{\lfloor \frac{n+1}{2} \rfloor} \text{ for } n > 1$$

we have that $H(n) \geq k_n$.



[2]

Figure 1.3: Known Bounds of $H(n)$

1.3 Comparing Bounds

Here we have a display of the bounds of $H(n)$ for $n \leq 100$, where the upper bound is given in green and the lower bound is given in purple. These bounds only agree for the first four values of $H(n)$, and become more sparse as n grows large. Our focus we be towards improving the lower bound of this function. Motivation that a stronger lower bound exists stems from the observation that the known lower bound appears stronger at powers of 2, than at other n.

CHAPTER 2: Hypergraphs and Preordered Sets

In order to construct a stronger lower bound for $H(n)$, we will define a preorder on Hypergraphs with particular qualities, and use that preorder to create corresponding preordered sets, or *prosets*. We will then construct a sequence of partially ordered sets, whose corresponding hypergraphs have size that improves upon the known lower bound for $H(n)$.

2.1 Economical Hypergraphs and Maximal Partitions

In order to define an ordering that is sufficient to what we need, we must first look at hypergraphs that have particular properties.

Definition 2.1.1. *A hypergraph $(V, \mathcal{H})$ is called economical if it contains no isolated points and for all $h \in \mathcal{H}$, there exists a $v \in (V, \mathcal{H}\backslash\{h\})$ that is isolated.*

Definition 2.1.2. *For a given hypergraph, $(V, \mathcal{H})$, a partition $(D, \mathcal{G}) \subseteq (V, \mathcal{H})$ is called maximal if*

$$|D| = |\{v \in V : v \text{ is restricted in } (V, \mathcal{G})\}|$$

Observation 2.1.3. *Let $(D, \mathcal{G})$ be a maximal partition of some hypergraph $(V, \mathcal{H})$. Then for all $(D^*, \mathcal{G})$, such that $(D^*, \mathcal{G})$ is a partition of $(V, \mathcal{H})$, we have that $|D^*| \leq |D|$*

2.2 Hypergraphs as Prosets

Recall that a *preorder* is an ordering on a set of elements that is both reflexive and transitive.

Definition 2.2.1. *For a given hypergraph, $(V, \mathcal{H})$, and $v, u \in V$, we say that $u \leq v$ if for all $h \in \mathcal{H}$, if $v \in h$, then $u \in h$.*

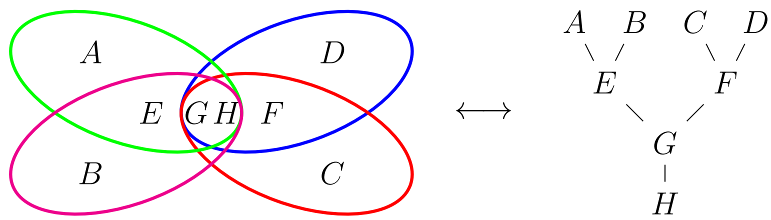


Figure 2.1: A Hypergraph Represented as a Preordered Set

From this definition, we can represent hypergraphs as preordered sets. While this method can be tailored to work for any given hypergraph, we will only be considering hypergraphs that are economical. To begin the representation, place each vertex contained in a single hyperedge in a tier at the top of the proset (This tier will be referred to as T_0). Then, using the preorder, place the remaining vertices below T_0 such that if $u, v \in V$ and $u \leq v$, then u is placed below v in the proset.

By labeling, we can see in 2.1 how the vertices in a given hypergraph map to its relative proset. Note that since G and H are both in the center of the hypergraph, their position in the proset is interchangeable. We will consider vertices contained in all $h \in \mathcal{H}$ to be in the same 'tier', which we will call T_c.

Now that we have an interpretation of hypergraphs as prosets, we can reinterpret our previous definitions for hypergraphs in terms of prosets.

Observation 2.2.2. *For a given hypergraph, $(V, \mathcal{H})$, a vertex $v \in V$ is restricted if under its corresponding proset, either: $v \in T_0$ or there exists exactly one $u \in T_0$, such that $v \leq u$.*

Observation 2.2.3. *A partition $(D, \mathcal{P})$ of a hypergraph is maximal if and only if under its corresponding proset, it is the case that for all vertices $v \in T_0 \subseteq D$, we have that for all $u \leq v$, if*

$$D \cap \{v^* \in T_0 : u \leq v^*\} = \{v\}$$

then $u \in D$

2.3 P_n: A Sequence of Posets

Definition 2.3.1. *Define* $\{P_n\}_{n=1}^{\infty}$ *to be the sequence of partially ordered sets, constructed as follows:*

For $n \geq 4$, *note that there exist unique* $k, m \in \mathbb{N}$ *such that*
$2 \leq m \leq 2^n + 1$ *and* $n = 2^k + m$.

1. *Begin with a row of* n *nodes. This will be row* T_0.

2. *Add a row,* T_1, *below* T_0, *containing* n_1 *nodes, where* $n_1 = \lfloor \frac{n}{2} \rfloor$. *For* $i \in \{1, 2, ..., n_1\}$, *declare the following order:*

   - $v_i^1 \leq v_{2i-1}^0$
   - $v_i^1 \leq v_{2i}^0$
   - $v_{n_1}^1 \leq v_n^0$

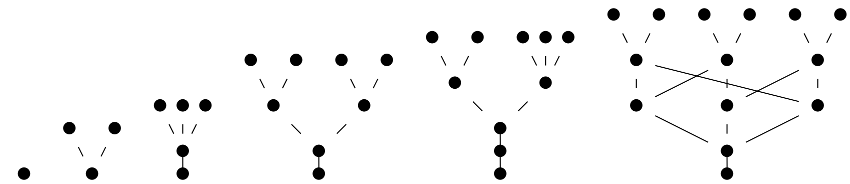
3. *for* $k \geq 2$, *continue adding rows down to* T_k, *each with* n_1 *nodes. For* $i \in \{1, 2, ..., n_1\}$ *and* $j \in \{2, ..., k\}$, *declare the following order:*

   - $v_i^j \leq v_i^{j-1}$
   - $v_i^j \leq v_{i+2^{j-2}}^{j-1}$, *when* $i + 2^{j-2} \leq n_1$
   - $v_i^j \leq v_{i+2^{j-2}-n_1}^{j-1}$, *when* $i + 2^{j-2} > n_1$

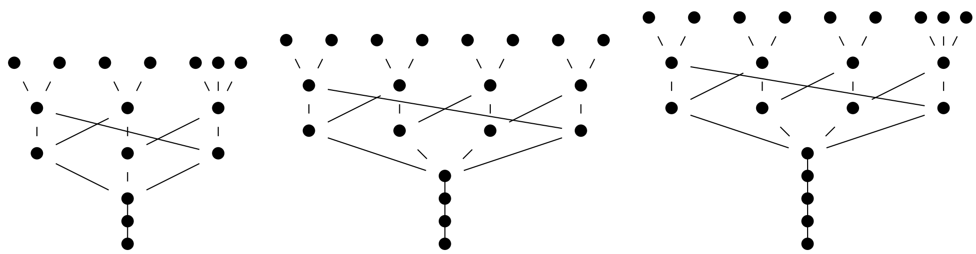
4. *Add* m *nodes below the lowest row. This will be* T_c, *and declare the following order:*

   - *For all* $i \in \{1, 2, ..., n_1\}$, $v_1^c \leq v_i^k$
   - *For all* $j \in \{2, ..., m\}$, $v_j^c \leq v_{j-1}^c$

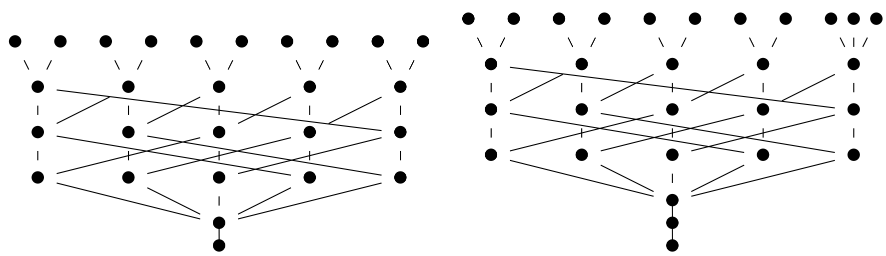
For $n < 4$, $n = 2^k + m$, *where* $k = 0$ *and* $m = 0, 1, 2$. *In this case, Construct* P_n *by only performing steps 1 and 4, as stated above.*



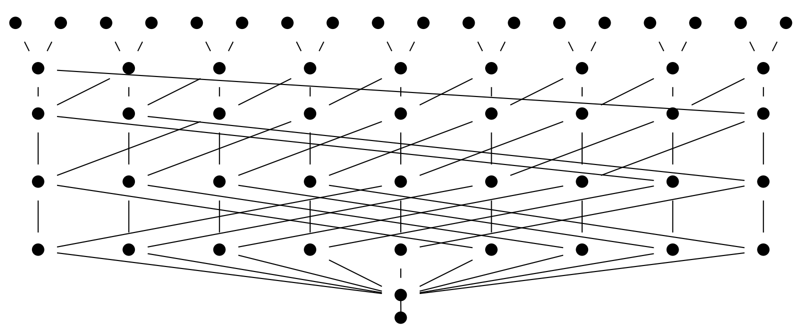
P_1 P_2 P_3 P_4 P_5 P_6



P_7 P_8 P_9



P_{10} P_{11}



P_{18}

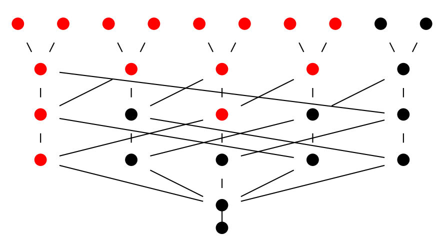
Figure 2.2: Early Examples of P_n

CHAPTER 3: A New Lower Bound

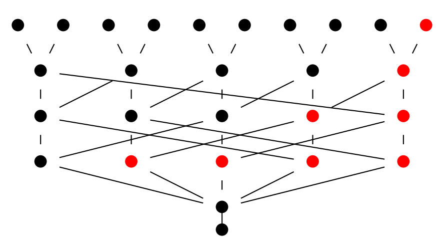
3.1 Properties of P_n

Now that we have constructed a sequence of partially ordered sets, we will show that this sequence yields a new lower bound for $H(n)$. To do this, we will make some observations about the construction of these sets, and then use the observations to prove that for any natural number n, P_n does not have a partition of size greater than n.

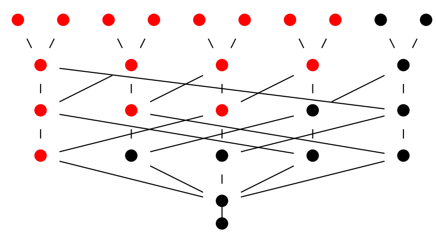
Observation 3.1.1. *For a given P_n, recall from the construction that $n = 2^k + m$ for specific k and m. Let $v \in T_j, j \leq k$. Then we can observe that there exist at least 2^j total $v^* \in T_0$ such that $v \leq v^*$. Furthermore, for each $u \in T_i, i < j$, if $v \leq u$, then there exist at least $2^j - 2^i$ total $v^* \in T_0$ such that $v \leq v^*$, but u and v^* are not comparable.*



Observation 3.1.2. *Let $v^* \in T_0$. Then for all $i = 1, ..., k$, we have that $|\{v \in T_i : v \leq v^*\}| = 2^{i-1}$. Furthermore, for all $u \in T_j$, where $j = 1, ..., i$, we have that $|\{v \in T_i : v \leq u\}| = 2^{i-j}$*



Observation 3.1.3. *For a given P_n, Let A_0 be a collection of vertices $v \in T_0$. For $i \geq 1$, Define A_i to be the set of all $u \in P_n$ such that $\{v \in T_0 : u \leq v\} \subseteq A_0$. Then $|A_i| \leq \frac{|A_0|}{2} - 2^{i-1} + 1$.*



3.2 $|P_n| \leq H(n)$

Lemma 3.2.1. *Let D be a partition for some P_n. Then $|D| \leq n$.*

Proof. First, we will consider $|D \cap T_0|$.

Clearly $|D \cap T_0| \leq |T_0| = n$.

Next, we shall consider $|D \cap (T_0 \cup T_1)|$.

Let $v \in D \cap T_1$. By 3.1.1, there exist 2 $v^* \in T_0$ such that $v \leq v^*$. Since $v \in D$, this implies that one of those vertices cannot be in D. Since this is true for all $v \in T_1 \cap D$, we have that

$$|D \cap (T_0 \cup T_1)| = |D \cap T_0| + |D \cap T_1| \leq |T_0| - |D \cap T_1| + |D \cap T_1| = |T_0| = n$$

We will now look at $|D \cap (T_0 \cup T_1 \cup T_2)|$.

Let $v \in T_2 \cap D$. By 3.1.1 there exist 4 $v^* \in T_0$ such that $v \leq v^*$. Note that 2 of these vertices were not observed in the previous cases.

Since D is maximal, there exists a $u \in T_1$ such that $v \leq u$. Define

$$A = \{v^* \in T_0 : u \leq v^*\}$$

and

$$A_0 = \{v^* \in T_0 : v \leq v^*\} \backslash A$$

By 3.1.3, $|A_1| \leq 1$. Define

$$B = \{u \in D \cap T_2 : u \leq a \in A_1\}$$

Note that $v \in B$. Then, by 3.1.2, $|B| \leq 2$. Since this is true for all $v \in D \cap T_2$, we have that $|D \cap T_0| \leq |T_0| - |D \cap T_1| - |D \cap T_2|$. So

$$|D \cap (T_0 \cup T_1 \cup T_2)| = |D \cap T_0| + |D \cap T_1| + |D \cap T_2|$$

$$\leq |T_0| - |D \cap T_1| - |D \cap T_2| + |D \cap T_1| + |D \cap T_2| = |T_0| = n$$

Induct on these subsets of D up to $T_j, j \leq k$. Consider

$$|D \cap (\bigcup_{i=0}^{j} T_i)|$$

where $|D \cap T_0| \leq |T_0| - \sum_{i=1}^{j-1} |D \cap T_i|$.

Let $v \in D \cap T_j$. By 3.1.1, there exist 2^j total $v^* \in T_0$ such that $v \leq v^*$. Note that 2^{j-1} of these vertices have not yet been observed.

Since D is maximal, there exists a $u \in T_1$ such that $v \leq u$. Define

$$A = \{v^* \in T_0 : u \leq v^*\}$$

and

$$A_0 = \{v^* \in T_0 : v \leq v^*\} \backslash A$$

Then $|A_0| \leq 2^j - 2$ and, by 3.1.3,

$$|A_{j-1}| \leq \frac{A_0}{2} - 2^{j-2} + 1 = \frac{2^j - 2}{2} - 2^{j-2} + 1 = 2^{j-1} - 1 - 2^{j-2} + 1 = 2^{j-2}$$

Define

$$B = \{u \in D \cap T_j : u \le a \in A_{j-1}\}$$

Note that $v \in B$. Then, by 3.1.2, $|B| \le 2|A_{j-1}| = 2^{j-1}$. Since this is true for all $v \in D \cap T_j$, we have that $|D \cap T_0| \le |T_0| - |\bigcup_{i=1}^{j} D \cap T_i|$. So

$$\left|D \cap \big(\bigcup_{i=0}^{j} T_i\big)\right| = \sum_{i=0}^{j} |D \cap T_i| = |D \cap T_0| + \sum_{i=1}^{j} |D \cap T_i|$$

$$\le |T_0| - \sum_{i=1}^{j} |D \cap T_i| + \sum_{i=1}^{j} |D \cap T_i| = |T_0| = n$$

Lastly, Consider $|D|$. If there are no $v \in D \cap T_c$, then it follows from above that $|D| \le n$. Suppose that there exists a $v \in D \cap T_c$. Then we know that

- $T_c \subset D$
- $|D \cap T_0| = 1$

Thus,

$$|D| = 1 + \sum_{i=1}^{k} |D \cap T_i| + |T_c|$$

$$= 1 + \sum_{i=1}^{k} 2^{i-1} + m$$

$$= 1 + 2^k - 1 + m$$

$$= 2^k + m = n$$

$\square$

Theorem 3.2.2. *For any natural number, n, $|P_n| \le H(n)$, and as a result*

$$H(n) > 2n + \frac{n-1}{2}(\log_2(n-2) - 1) - 2^{\log_2(n-2)-1} \text{ for } n \ge 4$$

Proof. Let $n \in \mathbb{N}$. Let $(V_n, \mathcal{H})$ be the hypergraph whose proset conversion is P_n. Then by 3.2.1, $(V_n, \mathcal{H})$ has no partition of size greater than n and $|V_n| = |P_n|$. Thus, $|P_n| \leq H(n)$.

Now assume that $n \geq 4$. Recall that from the construction of P_n, there exist $k \geq 1$ and $m \in [2, 2^k + 1]$ such that $n = 2^k + m$. Then,

$$|P_n| = |T_0| + \sum_{j=1}^{k} |T_j| + |T_c| = n + \sum_{j=1}^{k} \lfloor \frac{n}{2} \rfloor + m$$

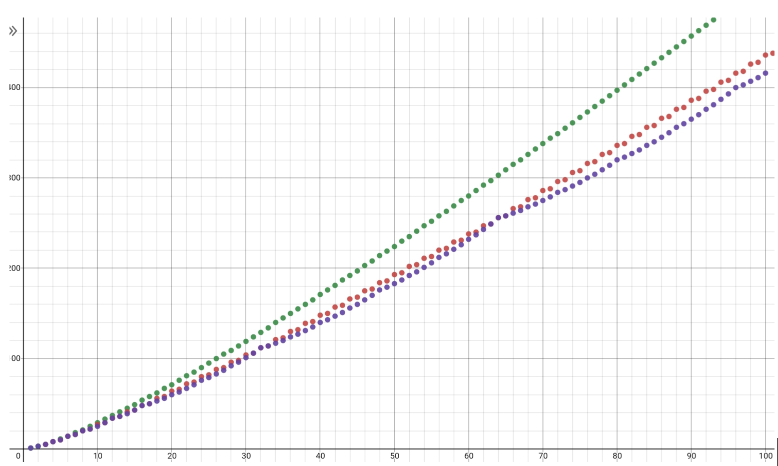
$$= n + k\lfloor \frac{n}{2} \rfloor + m = n + k\lfloor \frac{n}{2} \rfloor + n - 2^k \geq 2n + \frac{n-1}{2}k - 2^k$$

$$> 2n + \frac{n-1}{2}(\log_2(n-2) - 1) - 2^{\log_2(n-2)-1}$$

$\square$

3.3 Comparing Lower Bounds

Here, we have all three bounds, with the new lower bound shown in red. As we can see by comparing this lower bound with the original, the new one is at least as large as the original at all times. In fact, the only time when they agree is when n is near a power of two.



[2]

Figure 3.1: $|P_n|$ alongside the bounds of $H(n)$

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[1] W. Brian and P. B. Larson, "Choosing between incompatible ideals," 2019.

[2] "Desmos graphing calculator," accessed: 2021.