OPTIMAL STRATEGIES IN "LOCKS, BOMBS AND TESTING" (LBT) PROBLEM FOR THE CASE OF INDEPENDENT PROTECTION

by

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ABSTRACT

LI LIU. Optimal Strategies in "Locks, Bombs and Testing" (LBT) Problem for the Case of Independent Protection. (Under the direction of DR. ISAAC SONIN)

This thesis constructs a Defense/Attack resource allocation model. Defender uses "locks" to protect their boxes from Attacker, and Attacker uses "bombs" to destroy as many boxes as possible. The first models of such type were given by E. Borel (1921). Later such models were extensively analyzed at the initial stage of Game Theory development under the general title (Colonel) Blotto game. Previous LBT model focuses on violence patterns produced by attackers with different levels of capacity to see whether rebel capacity influences how rebels fight (the attack timing). We sought to extend this problem into a situation with an extra setting where rebels can test vulnerability of boxes before placing bombs. In previous problem the goal was to find violence patterns produced by rebels. Here, we are interested in the optimal strategy of placing bombs. Further, our problem discusses the optimal strategy for defenders to allocate locks even when attackers have already applied their best strategy for placing bombs.

After posing the basic problem we then examine several specific cases with dependent and independent, identical and non-identical, locks distribution in valued boxes by using Bayes' Posterior distribution and Monte Carlo simulations.

Key words: Defense/Attack model, Blotto game, Search, Testing, game theory.

Classification

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CHAPTER 1: INTRODUCTION

1.1 Motivation and Goal

The LBT model is motivated by the paper "Rebel Capacity, Intelligence Gathering, and the Timing of Combat Operations", K. Sonin, J. Wilson, A. Wright.(SWW)(1). Classic counterinsurgency claims rebel forces execute attacks in an unpredictable manner to limit the government's ability to anticipate and defend against them. SWW focuses on the question whether rebel capacity influences how rebels fight (the attack timing). With the help of data on opium production and farmgate prices from Afghanistan, SWW find high capacity rebels produce patterns of violence that are less random and exhibit temporal clustering.

The LBT model inherits this background setting and adds a new feature where rebels being able to test the vulnerability of the government and take action after receiving signals from test. Let's place above background into the following situation. Suppose two parties are in confrontation.

Defenders: Defenders use locks to protect n sites (battle fields, boxes, cities, cells, targets, time slots...). Due to limited sources, they can only protect some of these sites with locks. The probability that a lock can stop explosion of bombs is 1.

Attackers: Attackers have m bombs, and the probability of explosion for a single bomb is p ($p \leq 1$). They test n sites and receive a signal from each site that help determine the existence of locks. The signal can be positive or negative. If the signal is positive, it indicates that a lock probably exists; otherwise it does not exist. Attackers need to decide where to place m bombs, particularly, how many of them should be in the same site.

Remark 1: Attackers can and will test *every* site trying to find sites without locks. But testing of each site is not perfect: A test can give plus for a site without a lock and minus for a site with a lock. Thus we introduce probability of true positive (sensitivity a) and true negative (specificity b)

Remark 2: The defenders can decide how to distribute the locks. For the case of k locks allocated to n sites, there is a dependency model $A(n, k)$. The case of locks placed into n sites independently with a certain probability is model $B(n, \lambda)$. This paper is mainly focusing on the $B(n, \lambda)$ model .

Attackers have the following main goal:

Functional F1: to maximize the expected number of destroyed sites.

Functional F2: to maximize the expected value of damage.

We discuss two models in this paper.

The first is the Symmetric LBT (**S-LBT**) model, where allocation of locks and testing has a strictly symmetrical structure.

The second is the General LBT (**G-LBT**) model. Where some of the various statements about this model remain true when testing is symmetrical but the prior distribution of locks for Defenders can be different from a uniform distribution and there are different kinds of sites with possibly different values of benefits and costs for

Defender and/or Attacker. This is a natural assumption when the importance, the value of different sites for Defender/Attacker can vary. This immediately transforms the symmetric model into a full-fledged game with equilibrium points defined by randomized strategies, etc. The simplest example of such a game in $A(n,k)$ is a problem where the values of three sites are $(2,1,1)$ and then, having one lock, Defender will distribute it at random with probabilities $(1-2\alpha,\alpha,\alpha)$. In response, Attacker, having for example, one bomb, will use probabilities $(1-2\beta,\beta,\beta)$ to plant a bomb. The unique Nash equilibrium point in this and the more general model can be found in an explicit form.

Remark 3 Game LBT Model is difficult and not solved completely. There is a completely solved case - Symmetric LBT (S-LBT), which consists of two parts: $A(n,k)$ ([SonSon](12)) and $B(n,\lambda)$ in this paper. For General LBT (G-LBT), when model parameters are increasing, the model becomes rather difficult, here we just discuss it under some special settings.

1.2 Symbols and outline

We consider random variables T_i, S_i, C_i, $i = 1, 2, ..., n$ taking two values 0 and 1;

$$T_i = \begin{cases} 1 & \text{when the } i^{th} \text{ site contains a lock} \\ 0 & \text{when the } i^{th} \text{ site contains no lock} \end{cases}$$

$$S_i = \begin{cases} 1(\text{or } +) & \text{when the } i^{th} \text{ site is tested as positive, indicating lock is in present} \\ 0(\text{or } -) & \text{when the } i^{th} \text{ site is tested as negative, indicating lock is not in present} \end{cases}$$

$$C_i = \begin{cases} 1 & \text{when the } i^{th} \text{ site is destroyed} \\ \\ 0 & \text{when the } i^{th} \text{ site is not destroyed} \end{cases}$$

n: Number of sites

m: Number of bombs

x: Number of sites with a minus signal

t: Number of sites containing a lock with minus signal

p: Probability of explosion

$a = P(S = 1|T = 1)$: Sensitivity

$b = P(S = 0|T = 0)$: Specificity

Sometimes, the complement of an event D is denoted as D'.

Dissertation Outline

In this dissertation, Chapter 2-3 consider optimal strategy of Attackers under different settings of LBT model.

In Chapter 4, in the general setting, the Nash Equilibrium point is discussed.

CHAPTER 2: INDEPENDENT IDENTICAL LOCKS ALLOCATION UNDER SYMMERTIC LBT MODEL

2.1 Parameter notation and model building

1 $B(n, \lambda)$ model

Under Symmetric LBT (**S-LBT**) model setting, where allocation of locks and testing has a strictly symmetrical structure, we will discuss posterior distribution of locks given signal, and optimal strategy of attackers.

Define a signal vector $s = (s_1, s_2, ...s_n)$, with s_i either $-$ or $+$. And a r.v. N is the number of sites with minus signals among all n sites.

The symmetry in S-LBT model implies two useful formulas:

$$P(s_1, s_2, ..., s_n) = P(N = x)/\binom{n}{x} \tag{1}$$

$$P(T_i = 0|s_1, s_2, ..., s_n) = P(T_i = 0|s_i, N = x) \tag{2}$$

The $B(n, \lambda)$ model assumes that the chance that a randomly selected site containing a lock is the same (λ). Thus locks are identically and independently distributed in the sites. Notice that when the number of locks (k) is fixed, we would have the $A(n, k)$ model. We will compare results under these two models.

2 Probability Space

We have a probability space $\{(\gamma,\ s)\}$, where γ is a vector of distribution of locks. In the $B(n, \lambda)$ model, the number of locks K is a random variable. Suppose $K = k$ and there are n sites in total, then the locks' position vector is $\gamma = (i_1, i_2, ...i_k)$ with $1 \leq i_1 < i_2 < ... < i_k \leq n$, where i_k stands the k^{th} lock's position among n sites. $s = (s_1, s_2, ..., s_n)$ is a vector of signals. The probability of each outcome $p(\gamma, s) = b_0(\gamma)P(s|\gamma)$, where $b_0(\gamma)$ is the prior distribution of locks, and $P(s|\gamma) = P(S_1 = s_1, ...S_n = s_n|\gamma)$

2.2 Lock's distribution given signal in model $B(n, \lambda)$

2.2.1 Conditional probability of signal given lock's position

Let us introduce r.v.s N_1, the number of minuses in locked sites. N_2, the number of minuses in unlocked sites. $N = N_1 + N_2$ is the total number of minuses after testing. The number of Locks K is a random variable. Suppose we have $K = k$ locks in total, so the probability of having k locks is $p(k) = \binom{n}{k}\lambda^k(1-\lambda)^{(n-k)}$. Then r.v. N_1 (number of false minuses) is a binomial distribution with k trials, and probability of success $1-a$, $N_1 \sim Bin(k, 1-a)$.

r.v. N_2 (number of true minuses) is a binomial distribution with $(n-k)$ trials, and probability of success b. $N_2 \sim Bin(n-k, b)$.

N_1 and N_2 are independent. Thus distribution of N is $P(N = x) = g_B(x) = \sum_k p(k) g_{n,k}(x)$, where $g_{n,k}(x)$ is calculated for a fixed k.

When $K = k$, r.v. N has distribution $g_{n,k}(x) \equiv g(x)$, obtained by the convolution formula. And then $g_B(x) \equiv P(N = x)$ can be calculated by the second formula below

$$\begin{aligned} g(x) \equiv g_{n,k}(x) &= \sum_j p_1(j) p_2(x-j) \\ &= \sum_t p_1(x-t) p_2(t) \\ &= \sum_{i=0}^{min(k,x)} \binom{k}{i} (1-a)^i a^{k-i} \binom{n-k}{x-i} b^{x-i} (1-b)^{n-k-x+i}. \end{aligned} \tag{3}$$

Thus $g_B(x) = \sum_k p(k) g_{n,k}(x)$.

We use notation $t = N_1(\gamma, s), x = N(s)$.

Proposition 1. For the $B(n,\lambda)$ model:

(a). When r.v. $K=k$, and locks' distribution vector is $\gamma(k)$, *for all signal vectors* s *with* $N_1(\gamma,s)=t$, *and* $N(s)=x$, the probability of signal vector s is

$$\begin{aligned} P(s|\gamma(k)) &= P(s|N_1=t,N=x,K=k)=p(t,x|k) \\ &= (1-a)^t a^{(k-t)} b^{(x-t)} (1-b)^{(n-k-(x-t))}. \end{aligned} \tag{4}$$

(b). When r.v. $K=k$, locks' joint distribution

$$\begin{aligned} s(t,x|k) &= P(N_1=t,N_1+N_2=x|k)=P(N_1=t,N_2=x-t|k) \\ &= p_1(t)p_2(x-t) \\ &= \binom{k}{t}(1-a)^t a^{k-t}\binom{n-k}{x-t} b^{x-t}(1-b)^{n-k-x+t}. \end{aligned} \tag{5}$$

(c). Unconditional locks' joint distribution for $B(n,\lambda)$ is

$$\begin{aligned} s_B(t,x) &= \sum_k s(t,x|k)p(k) \\ &= \sum_k \binom{k}{t}(1-a)^t a^{k-t}\binom{n-k}{x-t} b^{x-t}(1-b)^{n-k-x+t}p(k). \end{aligned} \tag{6}$$

(d). Conditional locks' distribution for $B(n,\lambda)$ is

$$\begin{aligned} s_B(t|x) &= \frac{s_B(t,x)}{g_B(x)} \\ &= \frac{\sum_k p(k)s(t,x|k)}{\sum_k p(k)g_{n,k}(x)} \\ &= \frac{\sum_k \binom{k}{t}(1-a)^t a^{k-t}\binom{n-k}{x-t} b^{x-t}(1-b)^{n-k-x+t}p(k)}{\sum_k p(k)\sum_{i=0}^{min(k,x)}\binom{k}{i}(1-a)^i a^{k-i}\binom{n-k}{x-i} b^{x-i}(1-b)^{n-k-x+i}}. \end{aligned} \tag{7}$$

Example 1, $B(n, \lambda)$**.** For $\lambda = .5$, $n = 7$, $a = 7/12$, $b = 9/12$, find conditional distribution $s_B(t|x)$

```
> s_c
  0     1     2      3      4       5       6        7
0 1 0.643 0.413 0.2657 0.1708 0.10979 0.07058 0.045373
1 0 0.357 0.459 0.4428 0.3795 0.30498 0.23527 0.176452
2 0 0.000 0.128 0.2460 0.3163 0.33887 0.32676 0.294087
3 0 0.000 0.000 0.0456 0.1171 0.18826 0.24205 0.272303
4 0 0.000 0.000 0.0000 0.0163 0.05229 0.10085 0.151279
5 0 0.000 0.000 0.0000 0.0000 0.00581 0.02241 0.050426
6 0 0.000 0.000 0.0000 0.0000 0.00000 0.00208 0.009338
7 0 0.000 0.000 0.0000 0.0000 0.00000 0.00000 0.000741
> apply(s_c, 2, sum)
0 1 2 3 4 5 6 7
1 1 1 1 1 1 1 1
```

Figure 1: $s_B(t|x)$ for $\lambda = 0.5$, row is x, column is t

Example 2, $B(n, \lambda)$**.** For $\lambda = .7$, $n = 7$, $a = 7/12$, $b = 9/12$

```
> s_c
  0     1     2      3     4      5       6       7
0 1 0.435 0.190 0.0826 0.036 0.0157 0.00682 0.00297
1 0 0.565 0.492 0.3212 0.186 0.1015 0.05305 0.02695
2 0 0.000 0.319 0.4163 0.363 0.2632 0.17192 0.10482
3 0 0.000 0.000 0.1799 0.313 0.3412 0.29715 0.22646
4 0 0.000 0.000 0.0000 0.102 0.2211 0.28890 0.29356
5 0 0.000 0.000 0.0000 0.000 0.0573 0.14980 0.22832
6 0 0.000 0.000 0.0000 0.000 0.0000 0.03236 0.09866
7 0 0.000 0.000 0.0000 0.000 0.0000 0.00000 0.01827
> apply(s_c, 2, sum)
0 1 2 3 4 5 6 7
1 1 1 1 1 1 1 1
```

Figure 2: $s_B(t|x)$ for $\lambda = 0.7$, row is x, column is t

2.2.2 Posterior distribution for $B(n, \lambda)$

Let s be a signal vector, and suppose that the number of locks $K = \kappa$ is fixed. If the prior distribution $b_0(\gamma(\kappa))$ is uniform, then $b_0(\gamma(\kappa)) = p(\kappa)/\binom{n}{\kappa}$. What would be the distribution of κ locks' position $\gamma = \gamma(\kappa)$?

In order to solve this problem, we need to introduce ADL (aposterior distribution of locks) first. For both the S- and G-LBT models we described above, our notation implies the following basic equalities:

$$P(S_i = 1|T_i = 1) = a, \quad P(S_i = 0|T_i = 0) = b,$$

$$P(C_i = 1|T_i = 1) = 0, \quad P(C_i = 1|T_i = 0, u_i) = p(u_i), \tag{8}$$

where u_i is the number of bombs in site i, and $p(u)$ is the probability of distribution of an unlocked site with u bombs. The independence of explosions implies that $p(u) = 1 - q^u$, where $q = 1 - p$. Note that the function $p(u)$ is increasing and concave upward, and $\Delta p(u) \equiv p(u + 1) - p(u)$ is decreasing. This property of diminishing utility of each extra bomb plays an important role in the structure of the optimal strategy.

An interesting aspect of all models are the posterior probabilities $P(T_i = 0|s), s = (s_1, ..., s_n)$ and a more general *aposterior distribution of locks* (ADL) with

$$b(\gamma|s) = P(T_i = 1, i \in \gamma, T_i = 0, i \notin \gamma|S_i = s_i, i = 1, ..., n). \tag{9}$$

The following theorem (theorem 1) describes ADL (posterior distribution of locks) $b(\gamma(\kappa)|s)$ for an arbitrary and uniform $b_0(\gamma(\kappa))$. With uniform prior distribution all signals with the same values $N_1 = t, N = x$ have the same probability and as a result

$b(\gamma(\kappa)|s) = b(\gamma(\kappa)|t, x)$. For all possible allocations of κ locks, the ADL $b(\gamma(\kappa)|s)$ is given by elements of an upper triangular $\binom{n}{\kappa} \times 2^n$-dimensional array $B(\gamma(\kappa)|s)$, where $\gamma(\kappa)$ takes all $\binom{n}{\kappa}$ possible values.

In this background setting, the number of locks is a r.v. K with Binomial distribution with n trials and probability of success λ. Thus, rv K has the distribution $p(k) = p(k|n, \lambda), k = 0, 1, ..., n$. When $K = k$, rv N has conditional distribution $g_{n,k}(x)$, and then $g_B(x) \equiv P(N = x)$ can be calculated by the second formula below

$$g_A(x) \equiv g_{n,k}(x) = \sum_j p_1(j)p_2(x-j) \equiv \sum_t p_1(x-t)p_2(t), g_B(x) = \sum_{k=0}^{n} p(k)g_{n,k}(x). \quad (10)$$

Summation over j in the convolution formula above is taken over values j such that $0 \leq j \leq k,\ 0 \leq x - j \leq n - k$. Similarly this holds for summation over t, where $0 \leq x - t \leq k,\ 0 \leq t \leq n - k$. Further, in all convolution formulas we may not specify the exact range of summation since we are assuming that all probabilities involved in sums are well defined.

Theorem 1. ADL in case $B(n,\lambda)$.

a) *For a prior* $b_0(\gamma(\kappa))$*, and any position* γ *and signal* s*, according to the definition of the ADL (formula 9), the ADL* $b(\gamma|s)$ *is given by Bayes' formula*

$$b(\gamma(\kappa)|s) = \frac{b_0(\gamma(\kappa))P(s|\gamma(\kappa))}{\sum_k \sum_\sigma b_0(\sigma(k))P(s|\sigma(k))} \tag{11}$$

where $P(s|\gamma(\kappa)) = P(s|N_1 = t, N = x, K = k) \equiv p(t,x|k)$ *is given by formula (4) with* $t = t(\gamma, s), x = N(s)$

b) *For the* uniform *distribution* $b_0(\gamma(k)) = p(k)/\binom{n}{k}$*, formula (2) holds for any signal* s *and any position* γ*, with* $t(\gamma, s) = t$*,*

$$b(\gamma(\kappa)|s) = b(\gamma(\kappa)|t,x) \equiv \frac{p(\kappa)s_B(t,x|\kappa)}{g_B(x)\binom{x}{t}\binom{n-x}{\kappa-t}}, \tag{12}$$

Proof. **of Theorem 1.** The first equality in point (a) represents Bayes' formula. The equality $b(\gamma|s) = p(t,x)$ and formula (4) were proved in Introduction.

To prove (b), we use

$s_B(t,x) = \sum_k \binom{k}{t}(1-a)^t a^{k-t}\binom{n-k}{x-t} b^{x-t}(1-b)^{n-k-(x-t)}p(k)$, the equality,

$$\binom{n}{k}\binom{k}{t}\binom{n-k}{x-t} = \binom{n}{x}\binom{x}{t}\binom{n-x}{k-t}, \tag{13}$$

and the uniform prior $b_0(\gamma(\kappa)) = p(\kappa)/\binom{n}{\kappa}$. Hence $b(\gamma(\kappa)|s)$ takes the form $b(\gamma(\kappa)|s) = \frac{p(\kappa)P(s|\gamma(\kappa))/\binom{n}{\kappa}}{\sum_k\sum_\sigma p(k)P(s|\sigma(k))/\binom{n}{k}}$. To estimate the sum in the denominator, we prove the following equalities:

$$\begin{aligned}
\sum_k \sum_\sigma b_0(\sigma) P(s|\sigma) &= \sum_k \frac{p(k)}{\binom{n}{k}} \sum_t \sum_{\sigma : t(\sigma,s)=t} p(t,x|k) \\
&= \sum_k \frac{p(k)}{\binom{n}{k}} \sum_t p(t,x|k) |\sigma : t(\sigma,s) = t| \\
&= \sum_k \frac{p(k)}{\binom{n}{k}} \sum_t p(t,x|k) \binom{x}{t} \binom{n-x}{k-t}
\end{aligned}$$

By equality 13

$$\begin{aligned}
&= \sum_k \frac{p(k)}{\binom{n}{k}} \sum_t \frac{s(t,x|k)\binom{n}{k}}{\binom{n}{x}} \\
&= \sum_k \frac{p(k)}{\binom{n}{x}} \sum_t s(t,x|k) \\
&= \sum_k \frac{p(k)}{\binom{n}{x}} g_{n,k}(x) \\
&= \frac{g_B(x)}{\binom{n}{x}},
\end{aligned}$$

where $g_B(x) = \sum_k p(k) g_{n,k}(x)$.

Thus

$$\begin{aligned}
b(\gamma(\kappa)|s) &= \frac{p(\kappa) P(s|\gamma(\kappa))/\binom{n}{\kappa}}{\sum_k \sum_\sigma p(k) P(s|\sigma(k))/\binom{n}{k}} \\
&= \frac{\frac{p(\kappa)}{\binom{n}{\kappa}} \frac{s(t,x|\kappa)}{\binom{\kappa}{t}\binom{n-\kappa}{x-t}}}{\frac{g_B(x)}{\binom{n}{x}}}
\end{aligned}$$

By equality 13

$$\begin{aligned}
&= \frac{\frac{p(\kappa)}{\binom{n}{x}} \frac{s(t,x|\kappa)}{\binom{x}{t}\binom{n-x}{\kappa-t}}}{\frac{g_B(x)}{\binom{n}{x}}} \\
&= \frac{p(\kappa) s(t,x|\kappa)}{g_B(x) \binom{x}{t}\binom{n-x}{\kappa-t}}.
\end{aligned}$$

$\square$

Example 3. $B(n,\lambda)$, for $a = 7/12$, $b = 9/12$, $n = 7$, $\lambda = 0.7$, number of locks $\kappa = 2$, $p = 0.6$. The columns are x (number of minuses), and rows are t (number of minuses in locks)

```
> bb
            0            1            2            3            4            5           6           7
0 6.42109e-05 0.0001802048 5.057360e-04 1.419324e-03 0.0039832630 0.0111788350 0.000000000 0.000000000
1 0.00000e+00 0.0000429059 1.204133e-04 3.379342e-04 0.0009483960 0.0026616274 0.007469728 0.000000000
2 0.00000e+00 0.0000000000 2.866984e-05 8.046052e-05 0.0002258086 0.0006337208 0.001778507 0.004991293
3 0.00000e+00 0.0000000000 0.000000e+00 0.000000e+00 0.0000000000 0.0000000000 0.000000000 0.000000000
4 0.00000e+00 0.0000000000 0.000000e+00 0.000000e+00 0.0000000000 0.0000000000 0.000000000 0.000000000
5 0.00000e+00 0.0000000000 0.000000e+00 0.000000e+00 0.0000000000 0.0000000000 0.000000000 0.000000000
6 0.00000e+00 0.0000000000 0.000000e+00 0.000000e+00 0.0000000000 0.0000000000 0.000000000 0.000000000
7 0.00000e+00 0.0000000000 0.000000e+00 0.000000e+00 0.0000000000 0.0000000000 0.000000000 0.000000000
```

Figure 3: $b(\gamma(\kappa)|s)$

Note, when $N = x = 5$, $N_1 = t = 1$, $b(\gamma(2)|x = 5, t = 1) = \frac{p(2)s(t=1,x=5|2)}{g_B(5)\binom{5}{1}\binom{2}{1}} =$ 0.00266.

Given a fixed x, κ ($x = 5$,$\kappa = 2$), suppose these 5 minus sites are arranged in the first 5 places. Since $B(\gamma(2)|x = 5)$ is the probability of locks' position among n sites.

i	1	2	3	4	5	6	7
$s, x = 5$	$-$	$-$	$-$	$-$	$-$	$+$	$+$
$\gamma, t = 0$						$\otimes$	$\otimes$

$$\begin{aligned} B(\gamma(2)|x=5) &= P(L_1 = i_1, L2 = i_2|x = 5), i_1 < i_2 \\ &= b(\gamma(2)|x = 5, t = i) \\ &= \begin{cases} b(\gamma(2)|x=5,t=0) & i_1 = 6, i_2 = 7 \\ b(\gamma(2)|x=5,t=1) & i_2 \text{ is selected from site } 6,7 \\ b(\gamma(2)|x=5,t=2) & \text{None of } i_1, i_2 \text{ are selected from site } 6,7 \end{cases} \end{aligned}$$

We get table and histogram for $B(\gamma(2)|x=5)$ (probability that a lock is in position i_1, i_2)

\> B

	1	2	3	4	5	6	7
1	0	0.0006337208	0.0006337208	0.0006337208	0.0006337208	0.002661627	0.002661627
2	0	0.0000000000	0.0006337208	0.0006337208	0.0006337208	0.002661627	0.002661627
3	0	0.0000000000	0.0000000000	0.0006337208	0.0006337208	0.002661627	0.002661627
4	0	0.0000000000	0.0000000000	0.0000000000	0.0006337208	0.002661627	0.002661627
5	0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.002661627	0.002661627
6	0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.000000000	0.011178835
7	0	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.000000000	0.000000000

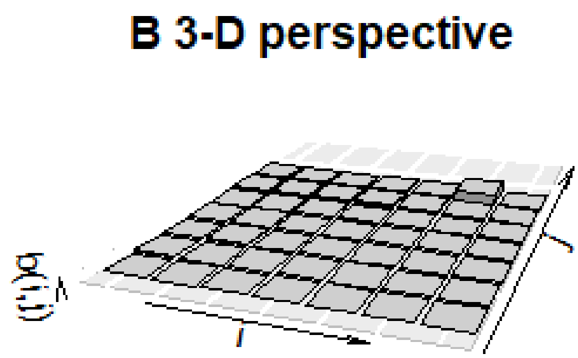


Figure 4: B table and histogram

While in case $A(n,k)$, when all settings are the same as Example 3, ($a = 7/12$, $b = 9/12$, $n = 7$, $k = 2$, $p = 0.6$) and $N = x = 5$, $N_1 = t = 1$, we have

```
> bb #b(i,j|x)
           0          1           2           3           4          5          6          7
0 0.04761905 0.06086957 0.080401094 0.110776187 0.161361142 0.25330270 0.00000000 0.00000000
1 0.00000000 0.01449275 0.019143118 0.026375283 0.038419319 0.06031017 0.10447761 0.00000000
2 0.00000000 0.00000000 0.004557885 0.006279829 0.009147457 0.01435956 0.02487562 0.04761905
3 0.00000000 0.00000000 0.000000000 0.000000000 0.000000000 0.00000000 0.00000000 0.00000000
4 0.00000000 0.00000000 0.000000000 0.000000000 0.000000000 0.00000000 0.00000000 0.00000000
5 0.00000000 0.00000000 0.000000000 0.000000000 0.000000000 0.00000000 0.00000000 0.00000000
6 0.00000000 0.00000000 0.000000000 0.000000000 0.000000000 0.00000000 0.00000000 0.00000000
7 0.00000000 0.00000000 0.000000000 0.000000000 0.000000000 0.00000000 0.00000000 0.00000000
```

Figure 5: $b(\gamma|s)$

Note, when $N = x = 5$, $N_1 = t = 1$, $b(\gamma|x = 5, t = 1) = P(L_1 = i_1, L_2 = i_2|x = 5, t = 1) = \frac{s(t=1|x=5)}{\binom{5}{1}\binom{2}{1}} = 0.06031017$.

Fix x ($x = 5$), and suppose these 5 minus sites are arranged in the first 5 places. Since $B(\gamma|x = 5)$ is the probability of locks' position among n sites.

i	1	2	3	4	5	6	7
$s, x = 5$	$-$	$-$	$-$	$-$	$-$	$+$	$+$
$\gamma, t = 0$						$\otimes$	$\otimes$

$$
\begin{aligned}
B(\gamma|x = 5) &= P(L_1 = i_1, L2 = i_2|x = 5), i_1 < i_2 \\
&= b(\gamma(2)|x = 5, t = i) \\
&= \begin{cases} b(\gamma|x = 5, t = 0) & i_1 = 6, i_2 = 7 \\ b(\gamma|x = 5, t = 1) & i_2 \text{ is selected from site } 6, 7 \\ b(\gamma|x = 5, t = 2) & \text{None of } i_1, i_2 \text{ are selected from site } 6, 7 \end{cases}
\end{aligned}
$$

We get the following table and histogram for $B(\gamma|x=5)$ (probability that lock is in position i_1, i_2). After comparing the histogram for model $A(n,k)$ and $B(n,\lambda)$, we find that the $A(n,k)$ model has a more obvious difference for those two locks' positions between pairs $(1,2),(1,3),(1,4),(1,5)$ and $(6,7)$.

```
> B
  1          2          3          4          5          6          7
1 0 0.01435956 0.01435956 0.01435956 0.01435956 0.06031017 0.06031017
2 0 0.00000000 0.01435956 0.01435956 0.01435956 0.06031017 0.06031017
3 0 0.00000000 0.00000000 0.01435956 0.01435956 0.06031017 0.06031017
4 0 0.00000000 0.00000000 0.00000000 0.01435956 0.06031017 0.06031017
5 0 0.00000000 0.00000000 0.00000000 0.00000000 0.06031017 0.06031017
6 0 0.00000000 0.00000000 0.00000000 0.00000000 0.00000000 0.25330270
7 0 0.00000000 0.00000000 0.00000000 0.00000000 0.00000000 0.00000000
```

Figure 6: when $k=2$, likelihood of lock's position in 7 sites

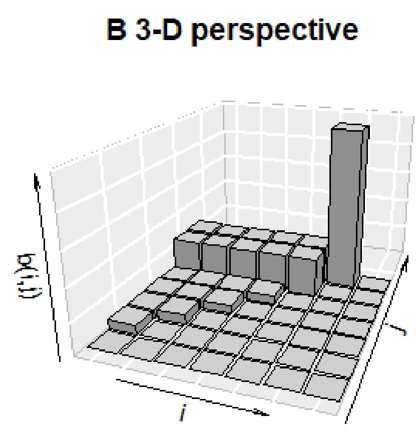


Figure 7: when $k=2$, histogram of lock's position

2.3 Optimal strategy for attackers

Now we know the posterior distribution of locks given information in the signal. The next thing to consider is Goal F1: How to maximize the expected number of destroyed sites?

2.3.1 Ratio for signal in $B(n, \lambda)$

Since the signal test is not perfect (sensitivity and specificity are less than 1), we introduce the ratio $r(\lambda)$ here, to obtain more efficiently the information of signals by comparing the probability of no lock in this position given a negative signal with a positive signal.

Proposition 2. a) The ratio $r \equiv r_B(\lambda)$ is given by formula

$$r_B(\lambda) = \frac{P(T=0|S=0)}{P(T=0|S=1)} \equiv \frac{p^-(\lambda)}{p^+(\lambda)} = \frac{b}{(1-b)} \frac{\lambda a + (1-\lambda)(1-b)}{\lambda(1-a) + (1-\lambda)b}, \tag{14}$$

b) The probabilities used in (14) are given by formulas

$$p^-(\lambda) = \frac{(1-\lambda)b}{\lambda(1-a) + (1-\lambda)b}, \qquad p^+(\lambda) = \frac{(1-\lambda)(1-b)}{\lambda a + (1-\lambda)(1-b)}, \tag{15}$$

c) If $a + b > 1$, then function $r_B(\lambda)$ is increasing from 1 to $\frac{a}{1-a}\frac{b}{1-b} = c_1 c_2 = c > 1$, when λ increases from 0 to 1, otherwise, it is decreasing from 1 to $c < 1$.

Here $c_1 = \frac{a}{1-a}$ and $c_2 = \frac{b}{1-b}$ represent the quality of sensitivity and specificity, and $c = c_1 c_2$ represents the combined quality of testing.

Remark 1. Note that parameters a and b in function $r_B(\lambda)$ are not symmetrical, i.e., though $r(.5|a, b)r(.5|b, a) = c$ and $r_B(\lambda|a, b) \approx r_B(\lambda|b, a) \approx c$ for λ close to 1, generally $r_B(\lambda|a, b) \neq r_B(\lambda|b, a)$ for all $\lambda < 1$. This asymmetry property is in contrast to the symmetry of a and b for $r_A(x)$ in the model $A(n,k)$, see of Proposition 2(b). The visual plot of $\{a, b : r_A(\lambda|a,b) = constant\}$ is given in (Figure 9).

Example 1. $B(n, \lambda)$**,** for $\lambda = .5$ with $a = \frac{7}{12}, b = \frac{9}{12}$. By formula (1) we obtain $r = \frac{15}{7} \approx 2.143$, and with $a = \frac{9}{12}, b = \frac{7}{12}$, we obtain $r = \frac{49}{25} = 1.96$, and hence $r(.5|a,b)r(.5|b,a) = \frac{21}{5} = 4.2 = c$.

For $\lambda = .7$ with $a = \frac{7}{12}, b = \frac{9}{12}$, we obtain $r = \frac{87}{31} \approx 2.806$, and with $a = \frac{9}{12}, b = \frac{7}{12}$, we obtain $r = \frac{13}{5} = 2.6$, and $r(.7|a,b)r(.7|b,a) \approx 7.297$.

Example 2. $B(n, \lambda)$**,** let $a = \frac{7}{12}, b = \frac{9}{12}$, probability of explosion $p = 0.6$ by formula (1), we obtain graph of ratio r w.r.t. λ as follows:

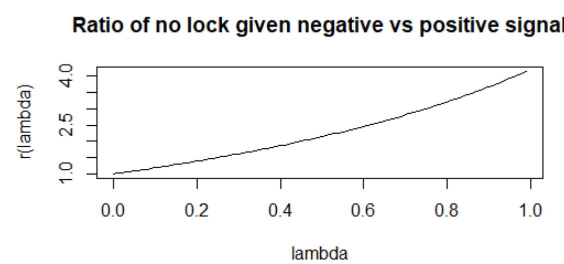


Figure 8: ratio for a=7/12,b=9/12, p=0.6

Example 3. $B(n, \lambda)$**,** find a and b for a fixed value r, such as $\{(a,b) : r(a,b) = c\}$

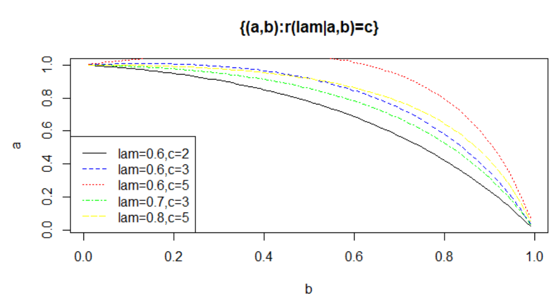


Figure 9: $r = c$

For $A(n,k)$, we have the following conclusion that the symmetry of Defense strategy implies that k locks are allocated at random between n sites. Let us show that the probability that a particular site has a lock is $\lambda = k/n$. The number of combinations of k locks having one lock on a fixed position and the other $k-1$ locks having any of remaining $n-1$ positions is $\binom{n-1}{k-1}$. Then $\lambda = \binom{n-1}{k-1}/\binom{n}{k}$. The first of the two trivial equalities for binomial coefficients below with $m = n$ yields $\lambda = k/n$. The second equality in (16) will be used later.

$$\binom{m-1}{k-1}/\binom{m}{k} = \frac{k}{m}; \qquad \binom{m}{k-1}/\binom{m}{k} = \frac{k}{m+1-k}. \tag{16}$$

For the case $A(n,k)$ we obtain two different representations for $r_{n,k}(x)$ using total probability formula for *different* partitions.

Theorem 2. a) The crucial ratio $r_{n,k}(x|a,b) \equiv r_{n,k}(x),\ 0 < x < n$, is given by the formula

$$r_{n,k}(x) \equiv \frac{P(T=0|S=0,x)}{P(T=0|S=1,x)} \equiv \frac{p^-(x)}{p^+(x)} = \frac{b}{(1-b)} \frac{(n-x)}{x} \frac{g_{n-1,k}(x-1)}{g_{n-1,k}(x)}, \tag{17}$$

b) The probabilities used in (17) $p^-(x) \equiv P(T=0|S=0,x)$ and $p^+(x) \equiv P(T=0|S=1,x))$ for $0 < x < n$ are given by formulas

$$p^-(x) = \frac{n-k}{x} * b * \frac{g_{n-1,k}(x-1)}{g_{n,k}(x)}, \quad p^+(x) = \frac{n-k}{n-x} * (1-b) * \frac{g_{n-1,k}(x)}{g_{n,k}(x)}. \tag{18}$$

c) The functions $r_{n,k}(x) \equiv r_{n,k}(x|a,b)$ as functions of parameters a,b for all $n,k,0 < x < n$ depend only on parameter $c = \frac{a}{1-a}\frac{b}{1-b}$, (see Remark 1), and hence satisfy the equality $r_{n,k}(x|a,b) = r_{n,k}(x|b,a) = r_{n,k}(x|\theta,\theta)$, where $\theta = \frac{\sqrt{c}}{1+\sqrt{c}}$.

d) The functions $r_{n,n-1}(x) = c$ for all x and the functions $r_{n,k}(x)$ for $k < n-1$ are monotonically increasing in x for $0 < x < n$, and $r(x) > 1$ when $a + b > 1$, and < 1 when $a + b \leq 1$.

e) The functions $r_{n,k}(x)$ are monotonically decreasing for all fixed $k, 0 < x < n$ when n is increasing.

Proof. **of Theorem 2(c). Proof that $r(x)$ depends on a, b through c,** First, we can represent $g_{n,k}(x)$ as

$$\begin{aligned} g_{n,k}(x) &= \sum_i \binom{k}{i} a^{k-i} \binom{n-k}{x-i} b^{x-i} (1-b)^{n-k-x+i} \\ &= a^k b^x (1-b)^{n-k-x} \sum_i \binom{k}{i} \binom{n-k}{x-i} \frac{1}{c^i}, \end{aligned} \tag{19}$$

where $d_1(x) = max(0, x - n + k) \leq i \leq min(k, x) = d_2(x)$. Then we can represent $r_{n,k}(x)$ as

$$r_{n,k}(x) = \frac{n-x}{x} \frac{\sum_{d_1(x-1) \leq i \leq d_2(x-1)} \binom{k}{i} \binom{n-k-1}{x-i-1} c^{-i}}{\sum_{d_1(x) \leq i \leq d_2(x)} \binom{k}{i} \binom{n-k-1}{x-i} c^{-i}}. \tag{20}$$

Therefore $r(x)$ depends only on c. We also have $\binom{n-k-1}{x-i-1} = \binom{n-k-1}{x-i} \frac{n-(x-i)}{x}$. Using these equalities and formula (20), we can show that $r_{n,k}(x)$ grows in c as a function of c.

□

To compare with Example 1 on page 19, we can look at the $A(n = 10, k = 5)$ model, with the same values $a = 7/12$, $b = 9/12$, and the number of minus sites $x = 0, 1, 2, ...10$. $r(x)$ is given by formula 17

	1	2	3	4	5	6	7	8	9	10
x	1.000000	2.000000	3.0000	4.000000	5.000000	6.000000	7.000000	8.000	9.000000	10
r(x)	1.733945	1.832666	1.9515	2.090505	2.243105	2.394689	2.536165	2.664	2.777778	0

Figure 10: $r(x)$ for $A(10, 5)$

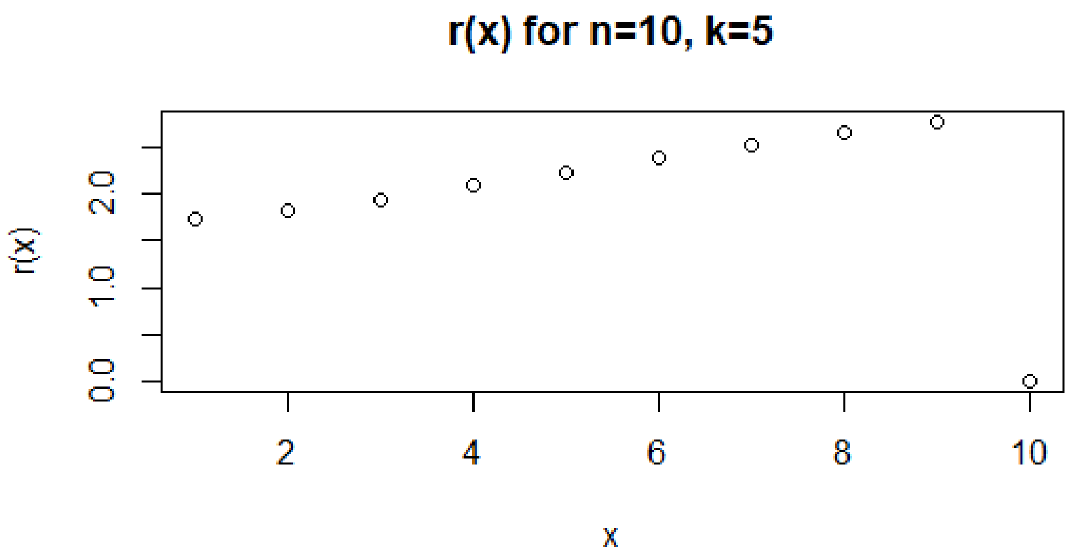


Figure 11: The graph of $r(x)$ for $A(10, 5)$

2.3.2 Threshold for the number of bombs

Again, for S–LBT models the problem description above and our notation imply the following basic equalities:

$$P(S_i = 1|T_i = 1) = a, \quad P(S_i = 0|T_i = 0) = b,$$

$$P(C_i = 1|T_i = 1) = 0, \quad P(C_i = 1|T_i = 0, u_i) = p(u_i), \tag{21}$$

The independence of explosions implies that $p(u) = 1 - q^u, q = 1 - p$. Note that the function $p(u)$ is increasing and concave upward, and $\Delta p(u) \equiv p(u+1) - p(u)$ is decreasing. This property of diminishing utility of each extra bomb plays an important role in the structure of the optimal strategy.

We consider strategy $\pi = (u_1, ..., u_n|s)$ as an allocation of m bombs between sites, given the signal $s = (s_1, ..., s_n)$, $\sum_{j=1}^n u_j = m$, and we introduced $U^-(\pi|s) = \{u_j, j \in B^-(s)\}$ and $U^+(\pi|s) = \{u_j \in B^+(s)\}$ as two possible sets of the values of u_j in minus $B^-(s)$ and plus $B^+(s)$ sites. By symmetry of the prior distribution of locks and testing, all strategies with the same pair of sets $(U^-(\pi|s), U^+(\pi|s))$ can be obtained using permutations of these sets among corresponding sites, and they all have the same value, denoted as $w^\pi(x, m)$ for a problem $N(s) = x$. We denoted also $v(x, m) = \sup_\pi v^\pi(x, m)$, the value function over all strategies, given m and x, and $v(m)$, the overall value function.

Let J be a subset of sites and $C(J)$ the event that *all* sites in J are destroyed and let C_j be the event that site j is destroyed. Then, given strategy π, we have $w^\pi(m|x) = \sum_{i=1}^n P(C_i|u_i, s_i, x)$. The conditional independece of testing and explosions, formula

(21), and total probability formula imply the following formula for the conditional probability of the destruction of a particular site with u bombs, and for any event F generated by testing (signals), $P(C|u,F) = P(C|u,T=0)P(T=0|F) = p(u)P(T=0|F)$, $u \geq 1$. Using this formula and the definitions of $r(\lambda)$, $p^-(\lambda)$ and $p^+(\lambda)$, we have:

$$\begin{aligned} P(C|u,S=1,\lambda) &= P(T=0|S=1,\lambda)P(C|u,T=0) = p^+(\lambda)p(u), \quad p(u) = 1-q^u, \\ P(C|u,S=0,\lambda) &= P(T=0|S=0,\lambda)P(C|u,T=0) = p^-(\lambda)p(u) = r(\lambda)p^+(\lambda)p(u). \end{aligned} \tag{22}$$

The next Proposition justifies our claim that the optimal strategy in all problems are (separately) UAP in minus and plus sites.

Lemma 1 (This lemma will be proved in theorem 5). *If $0 < \lambda < 1$, then the optimal strategy is to distribute all bombs between minus and plus sites $d(\lambda)$-UAP, where $d(\lambda)$ is defined by the formula*

$$d(\lambda) = min(i \geq 1 : r(\lambda)q^i < 1). \tag{23}$$

Theorem 3. *Let $\pi(\lambda) = (u_l, l = 1,2,...,n)$ be an optimal strategy. Then $|u_s - u_t| \leq 1$ when the signals in sites s,t have the same sign.*

Theorem 4. *Let $\pi(\lambda) = (u_l, l = 1,2,...,n)$ be a strategy, $0 < \lambda < 1$, $u^- = i$ be the number of bombs in some minus site, $u^+ = j$ be the number of bombs in some plus site, and $d = d(\lambda)$ is defined by formula (23). Then, if $i - j > d$ or, if $j \geq 1$ and $i - j < d - 1$, then strategy π is not optimal, or, equivalently, if π is optimal, and $j = 0$, then $1 \leq i \leq d$, and if $j \geq 1$, then $i - j = d$ or $d - 1$.*

Proof. **of Theorem 3.** Suppose that theorem 3 is not true and let us say $u_s = i$, $u_t = j$, $i < j - 1$ and $S_s = S_t = 1$. The concavity of the function $p(\cdot)$ implies that $p(i+1) + p(j-1) > p(i) + p(j)$. Then, using the formulas in (22), we have

$$\begin{aligned} & P(C=1|i+1, S=1, \lambda) + P(C=1|j-1, S=1, \lambda) \\ = \;& p^{+}(\lambda)[p(i+1) + p(j-1)] \\ > \;& p^{+}(\lambda)[p(i) + p(j)] \\ = \;& P(C=1|i, S=1, \lambda) + P(C=1|j, S=1, \lambda). \end{aligned}$$

Thus our initial strategy is not optimal. The proof for $S_s = S_t = 0$ is similar with $p^{+}(\lambda)$ replaced by $p^{-}(\lambda) = r(\lambda)p^{+}(\lambda)$.

□

Proof. **of Theorem 4.** Let $d(\lambda) = d$. As always, we assume that $a + b > 1$ and then $r(\lambda) > 1$ for $0 < \lambda < 1$, and hence $u^{-} \geq u^{+}$. Let us denote the incremental utilities for minus and plus sites as $\Delta C^{-}(i|\lambda) = P(C=1|i+1, S=0, \lambda) - P(C=1|i, S=0, \lambda)$, $\Delta C^{+}(j|\lambda) = P(C=1|j+1, S=1, \lambda) - P(C=1|j, S=1, \lambda)$. Using formulas in (22), it is easy to check that $\Delta C^{+}(j|\lambda) = pp^{+}(\lambda)q^{j}$ and $\Delta C^{-}(i|\lambda) = pr(\lambda)p^{+}(\lambda)q^{i}$, and then their difference for $0 \leq j \leq i$ is

$$\Delta(i-1, j) = \Delta C^{-}(i-1|\lambda) - \Delta C^{+}(j|\lambda) = pq^{j}p^{+}(\lambda)[r(\lambda)q^{i-j-1} - 1]. \tag{24}$$

By definition of $d(\lambda)$, we have $r(\lambda)q^{d(\lambda)-1} \geq 1$ and $r(\lambda)q^{d(\lambda)} < 1$. Then if $i - j > d$, then formula (24) implies that $\Delta(i-1, j) < 0$, i.e., a transfer of one bomb from a minus site from this pair to a plus site will increase the value of a strategy. Similarly, if $j \geq 1$ and $i - j < d - 1$ for such pairs, then using the formula for $\Delta(i, j-1)$ similar to formula (24), we can show that the inverse transfer will increase the value.

Note also that if $r(\lambda)q^{d(\lambda)-1} = 1$, then $d(\lambda)$-UAP strategy remains optimal but is no longer unique since then in formula (24) gives zero for $i - j = d(\lambda)$. Note also, that if $p = 1$, i.e. $q = 0$, then $d(\lambda) = 1$ for all $0 < \lambda < 1$, and if p is decreasing to zero, then $d(\lambda)$ tends to infinity. $\square$

Example 1. $B(n, \lambda)$, with $a = 7/12$, $b = 9/12$, $n = 8$, number of bombs $m = 50$, and $p = 0.6$

From the Figure 12, we can find that when $\lambda = 0.5$, $r(0.5) = 2.143$ and $d = 1$.

```
> data
            0     1     2     3     4     5     6     7      8      9
lambda      0 0.100 0.200 0.300 0.400 0.500 0.600 0.700 0.8000 0.9000
r(lambda)   1 1.186 1.390 1.615 1.865 2.143 2.455 2.806 3.2069 3.6667
p_minus     1 0.942 0.878 0.808 0.730 0.643 0.545 0.435 0.3103 0.1667
p_plus      1 0.794 0.632 0.500 0.391 0.300 0.222 0.155 0.0968 0.0455
d           1 1.000 1.000 1.000 1.000 1.000 2.000 2.000 2.0000 1.0000
```

Figure 12: $d(\lambda)$ with 50 bombs and probability of explosion is 0.6

Thus when $\lambda = 0.5$, if there is a total of 50 bombs, the attacker will place them into 8 sites according the following strategy to maximize damage.

i	1	2	3	4	5	6	7	8
$s, x = 5$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
bombs	7	7	6	6	6	6	6	6

$A(n,k)$, with $a = 7/12$, $b = 9/12$, $n = 8$, $k = 4$ number of bombs $m = 50$, and $p = 0.6$

From the Figure 13, we can find that when $x = 5$, $r(5) = 2.5066767$ and $d = 2$.

```
> data
                1         2         3         4         5         6         7   8
x       1.0000000 2.0000000 3.0000000 4.0000000 5.0000000 6.0000000 7.0000000 8.0
r(x)    1.7710843 1.9132821 2.0944363 2.3066439 2.5066767 2.6808081 2.8285714 0.0
p_minus 0.8076923 0.7788204 0.7424901 0.6975786 0.6454941 0.5929401 0.5439560 0.5
p_plus  0.4560440 0.4070599 0.3545059 0.3024214 0.2575099 0.2211796 0.1923077 NaN
d       1.0000000 1.0000000 1.0000000 1.0000000 2.0000000 2.0000000 2.0000000 1.0
```

Figure 13: threshold for $A(8,4)$

Thus when $x = 5$, if there is a total of 50 bombs, the attacker will place them into 8 sites according the following strategy to maximize damage.

i	1	2	3	4	5	6	7	8
$s, x = 5$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$
bombs	7	7	7	7	7	5	5	5

2.3.3 Value function

Theorem 5. Let, given signal s, the total number of minuses $N = x, 0 \leq x \leq n$. Then

a) *if $x = 0$ or n, then the optimal strategy is to distribute all bombs between sites UAP and the value function $v(m|0) = v(m|n)$ for $m = n*i+e,\ i = 0, 1, ...,\ 0 \leq e < n$, is given by formula*

$$v(m|0) \quad = \quad v(m|n) = \sum_{k=0}^{n} \frac{n-k}{n}[ep(i+1) + (n-e)p(i)]P(k). \tag{25}$$

Where $P(k) = \binom{n}{k}\lambda^k(1-\lambda)^{(n-k)}$

b) *If $0 < x < n$, then the optimal strategy is to distribute all bombs between minus and plus sites $d(\lambda)$-UAP, where $d(\lambda)$ is defined by formula*

$$d(\lambda) = min(i \geq 1 : r(\lambda)q^i < 1), \tag{26}$$

$q = 1 - p$ *and* $r(\lambda) = r_A(\lambda)$ *is defined by formula* (14).

The value function $v(x, m)$ for $m = m^- + m^+ = i * x + e + j * (n - x) + e'$, where the tuple (i, e, j, e') is (uniquely) defined by the value x and $d(\lambda)$-UAP strategy, is given by the formula

$$v(x, m) = p^+(\lambda)[r(\lambda)(ep(i+1) + (x-e)p(i)) + (e'p(j+1) + (n-x-e')p(j))]. \tag{27}$$

c) *The value function $v(m), m = 1, 2, ...$ is given by the formula*

$$\begin{aligned} v(m) \quad &= \quad \sum_{x=0}^{n} P(N = x)v(x, m) \\ &= \quad \sum_{x=0}^{n} v(x, m)g_A(x). \end{aligned} \tag{28}$$

Proof. **of Theorem 5.** (a) If $x = 0$ or n then all sites have the same sign and Proposition 4 implies that it is optimal to distribute all bombs between all sites UAP. When $m = n * i + e$, where $0 \leq e < n$, then UAP means that e sites have $i + 1$ bombs each, and $n - e$ sites have i bombs each. When $K = k$, the probability that a particular site has no lock is $\frac{n-k}{n}$. Then, using the last equality in formula (21), we obtain that the expected damage in all n sites is

$\sum_{k=0}^{n} \frac{n-k}{n}[eP(C|i+1, T = 0) + (n-e)P(C|i, T = 0)]P(k) = \sum_{k=0}^{n} \frac{n-k}{n}[ep(i+1) + (n-e)p(i)]P(k)$, i.e., $v(m|0) = v(m|n)$ is given by formula (25).

(b) Let $0 < x < n$, and $u = (u_1, ..., u_n)$ be the allocation of bombs, defined by an optimal strategy π and signal s with $N(s) = x$, and m^-, m^+ be the total number of bombs in minus and plus sites. We assume that $a + b > 1$ and hence $r(\lambda) > 1$ for $0 < \lambda < 1$. By theorem 3, if π is optimal, then the allocation of bombs in minus and plus sites is UAP, and hence m^-, m^+ satisfy the equalities $m^- = l^- * x + e^-$, $m^+ = l^+ * (n - x) + e^+$. Using shorthand notation $l^- = i, e^- = e,\ \ l^+ = j, e^+ = e'$, we have $m^- = i * x + e,\ m^+ = j * (n - x) + e', 0 \leq e < x, 0 \leq e' < n - x$. If $j = e' = 0$, then theorem 4 implies that the maximum number of bombs in a minus site, $i \leq d(\lambda)$, and this allocation is in agreement with $d(\lambda)$-UAP strategy with $m = m^- \leq x * d(\lambda)$ bombs. Note, that if $e > 0$, then there are two minus sites with different number of bombs, $i + 1$, and i. Similarly, if $e' > 0$, then there are two plus sites with different number of bombs, $j + 1$, and j. By theorem 4, the difference between the number of bombs in any pair of (minus, plus) sites can have only two values, $d(\lambda)$ or $d(\lambda) - 1$. Then only one of $e,\ e'$ can be positive. If $e' > 0$, then $e = 0$ and by theorem 4, $\ i - j = d(\lambda)$. Similarly, if $e > 0, j > 0$, then $e' = 0$ and by theorem 4

$i - j = d(\lambda) - 1$. In both cases this allocation is in agreement with $d(\lambda)$-UAP strategy with $m^- = x * i + e,\ m^+ = (n - x) * j + e'$ bombs. The optimality of $d(\lambda)$-strategy is proven. We mentioned earlier that if $r(\lambda)q^{d(\lambda)-1} = 1$, then there are other optimal strategies. Of course, they have the same value function $v(x, m)$.

Consider the first time to transfer the extra one bomb from minus sites to plus sites, thus we have $j = 0$, from Equation 24 , we have

$$\Delta(i, 0) = \Delta C^-(i|\lambda) - \Delta C^+(0|\lambda) = pp^+(\lambda)[r(\lambda)q^i - 1]. \tag{29}$$

Thus we have the first threshold $d(\lambda) = min_i\{i : r(\lambda)q^i < 1\}$

Now we can analyze $v(x, m)$ using an optimal $d(\lambda)$-UAP strategy. Let $m = m^- + m^+ = i * x + e + j * (n - x) + e'$, where the tuple i, e, j, e' is (uniquely) defined by value x and $d(\lambda)$-UAP strategy. We also proved in theorem 4 that $e * e' = 0$. Then, the each of e minus sites have $i + 1$ bombs each, and $x - e$ minus sites have i bombs each, and in plus sites e' sites have $j + 1$ bombs each, and $n - x - e'$ sites have j bombs each. Then the expected damage in all n sites is

$eP(C|i + 1, S = 0, x) + (x - e)P(C|i, S = 0, x) + e'P(C|j + 1, S = 1, x) + (n - x - e')P(C|j, S = 1, x) = p^+(x)[r(\lambda)(ep(i+1) + (x-e)p(i)) + e'p(j+1) + (n-x-e')p(j\}),$

i.e., $v(x, m)$ is given by formula (27). We proved point b) of Theorem 1.

Point c) is straightforward.

$\square$

Example 2: $B(n, \lambda)$, $a = 7/12$, $b = 9/12$, $n = 8$, $\lambda = 0.5$, number of bombs $m = 50$, and $p = 0.6$

This results in the following table where the first row is the value of x (number of locks in minus sites), the value function is $\sum_{x=0}^{8} vg_x = 3.27$, where $vg_x = v(x, m)g_A(x)$.

> data

	v0	1	2	3	4	5	6	7	8
x	0.00e+00	1.0000	2.0000	3.000	4.000	5.000	6.000	7.0000	8.0000
i	0.00e+00	7.0000	7.0000	6.000	6.000	6.000	6.000	6.0000	6.0000
e_minus	0.00e+00	0.0000	0.0000	2.000	2.000	2.000	2.000	2.0000	2.0000
m_minus	0.00e+00	7.0000	14.0000	20.000	26.000	32.000	38.000	44.0000	50.0000
j	6.00e+00	6.0000	6.0000	6.000	6.000	6.000	6.000	6.0000	0.0000
e_plus	2.00e+00	1.0000	0.0000	0.000	0.000	0.000	0.000	0.0000	0.0000
m_plus	5.00e+01	43.0000	36.0000	30.000	24.000	18.000	12.000	6.0000	0.0000
v_m	3.99e+00	2.0943	2.4170	2.739	3.061	3.383	3.705	4.0265	3.9861
g_B	9.08e-04	0.0102	0.0499	0.140	0.244	0.274	0.192	0.0766	0.0134
vg	3.62e-03	0.0213	0.1205	0.382	0.748	0.926	0.710	0.3085	0.0534

Figure 14: $B(8, 0.5)$ with 50 bombs and probability of explosion $p = 0.6$

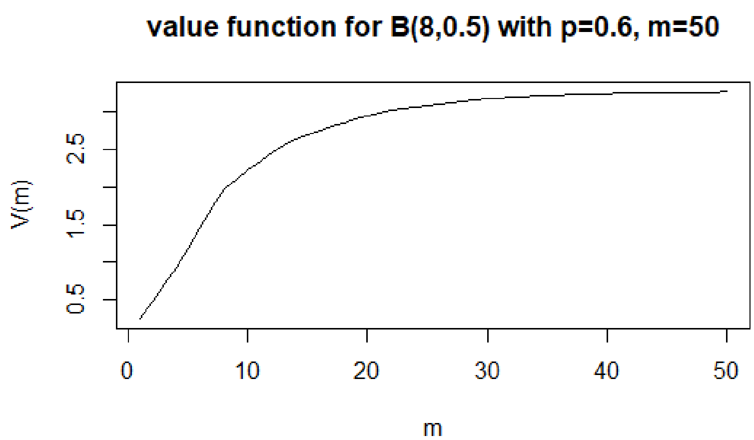


Figure 15: $B(8, 0.5)$ value function w.r.t number of bombs when probability of explosion $p = 0.6$

For $A(n,k)$ with $n = 8$, $k = 4$, we get the following table where the first row is the value of x (number of locks in minus sites), and the value function is $\sum vg = 3.99$

```
> data
```

	v0	1	2	3	4	5	6	7	8
x	0.00e+00	1.00000	2.0000	3.000	4.000	5.000	6.000	7.0000	8.00000
r(x)	NaN	1.77108	1.9133	2.094	2.307	2.507	2.681	2.8286	0.00000
p_minus	NaN	0.80769	0.7788	0.742	0.698	0.645	0.593	0.5440	0.50000
p_plus	5.00e-01	0.45604	0.4071	0.355	0.302	0.258	0.221	0.1923	NaN
d	1.00e+00	1.00000	1.0000	1.000	1.000	2.000	2.000	2.0000	1.00000
i	0.00e+00	7.00000	7.0000	6.000	6.000	7.000	6.000	6.0000	6.00000
e_minus	0.00e+00	0.00000	0.0000	2.000	2.000	0.000	4.000	3.0000	2.00000
m_minus	0.00e+00	7.00000	14.0000	20.000	26.000	35.000	40.000	45.0000	50.00000
j	6.00e+00	6.00000	6.0000	6.000	6.000	5.000	5.000	5.0000	0.00000
e_plus	2.00e+00	1.00000	0.0000	0.000	0.000	0.000	0.000	0.0000	0.00000
m_plus	5.00e+01	43.00000	36.0000	30.000	24.000	15.000	10.000	5.0000	0.00000
v_m	3.99e+00	3.98672	3.9874	3.987	3.987	3.987	3.987	3.9864	3.98607
g_B	4.52e-04	0.00672	0.0413	0.136	0.259	0.291	0.190	0.0661	0.00954
vg	1.80e-03	0.02679	0.1647	0.542	1.033	1.161	0.756	0.2636	0.03801

Figure 16: $A(8,4)$ with 50 bombs and prob of explosion $p = 0.6$

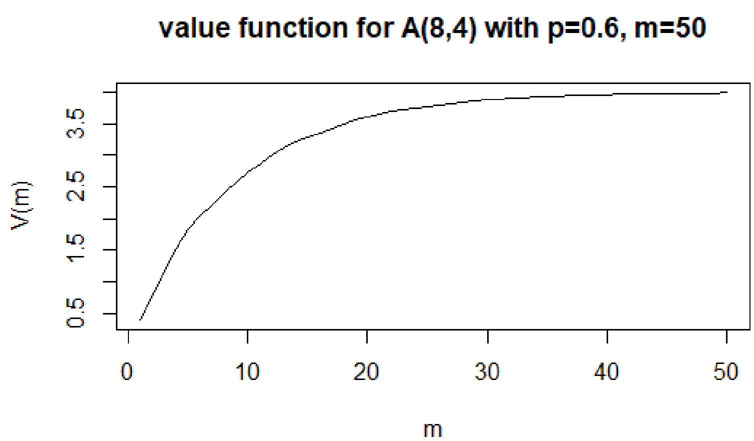


Figure 17: $A(8,4)$ value function w.r.t number of bombs when prob of explosion $p = 0.6$

CHAPTER 3: INDEPENDENT LOCKS ALLOCATION UNDER GENERAL LBT MODEL

3.1 Notations And Conditions

Under the setting of the general **G-LBT** model, when there are different kinds of sites, locks and bombs, with possibly different values of benefits and costs for Defender and/or Attacker, and testing is not uniform with respect to different sites, e.g., when defender can test only a subset of all sites, or parameters of testing a and b depend on the site number, we can construct a posterior distribution of locks and obtain an optimal strategy for attackers.

G-LBT model $G_B(n, \Lambda, m)$.

Defender: There are a total of n sites and each site has a value c_i, such that $c = (c_1, c_2, c_3, ...c_n)$. Defenders allocate locks independently with non-identical probability for different sites, i.e. $\Lambda = (\lambda_1, \lambda_2, ...\lambda_n)$, where λ_i indicates probability of containing a lock in the i^{th} site, with the restriction $\sum_i \lambda_i = k$ and $0 \leq \lambda_i \leq 1$ for $i = 1, 2, ...n$. So there are a total of $\kappa = 2^n$ allocations of locks. Let's redefine locks' allocation vector to be $\gamma = (\gamma_1, \gamma_2, ...\gamma_\kappa)$.

Attacker: There are m bombs, among which u bombs are placed into sites with $u = (u_1, u_2, ...u_n)$, and $\sum u_i = m$, and each site is tested to obtain a signal vector $s = (s_1, s_2, ...s_n)$, with s_i either be $-$ or $+$.

Assume the sites themselves have different values ($c = c_1, c_2, c_3, ...c_n$) with $c_1 \geq c_2 \geq c_3 > ... \geq c_n$. Attacker's goal is to maximize the expected value of damage(loss).

Remark: Here we have two cases.

Case 1: Attackers know each λ_i, $i = 1, 2, ...n$.

Case 2: Attackers only know k.

Due to the complexity of this model, here we just discuss a little about Case 1 and leave Case 2 for future work.

3.2 Parameter r in model $G_B(n, \Lambda, m)$

Instead of computing $r(\lambda)$, we calculate for $r_i(s)$,

$$r_i(s) = c_i P(T_i = 0|s)p, \tag{1}$$

where $P(T_i = 0|s)$ is given by a Posterior Distribution, such that $P(T_i = 1|s) = \sum_{\gamma_i : i \in \gamma} b(\gamma_i|s)$, thus $P(T_i = 0|s) = 1 - P(T_i = 1|s)$.

For attackers, when the number of bombs $m = 1$, for any observed signal, they can compare the parameter r among different sites, and choose the site with highest r to maximize expected value of the total damage.

Here we define the function for the real damage as $d(x) = \sum_s p(s|x)d(s|x))$, where $d(s|x)$ is the real damage for the optimal placement of a unique bomb given by the maximum potential damage $d_i(s|x)$ for each signal, where $d_i(s|x) = c_i p P(T_i = 0|s)$ for $i = 1, 2, ...n$. $p(s|x)$ is the total probability of the signal given by $p_0(s) = \sum_i p(s|\gamma_i) b_0(\gamma_i)$, x is the parameter in Λ. When $n = 2$, $k = 1$ $\Lambda = (x, 1 - x)$.

Example 1. $G_B(2, \Lambda, 1)$, where the number of sites is $n = 2$, the probability of having a lock in each site is $\Lambda = (\lambda, 1-\lambda)$, and with site values $c = (2, 1)$, $a = 7/12$, $b = 9/12$. The probability of an explosion is $p = 1$ and the number of bombs is $m = 1$. (This example will be compared to the dependent case model $G_A(n, k, m)$ where $k = 1$).

Here is the summary of the prior probability for lock's location (Table 1) and the distribution of the signal vector (S) (Table 2).

Table 1: A summary of Locks' location and prior probability for $G_B(n, \Lambda, m)$.

Lock' Location (γ)	Probability of γ ($b_0(\gamma)$)
$\gamma_1 = (0, 0)$	$(1-\lambda)\lambda$
$\gamma_2 = (0, 1)$	$(1-\lambda)^2$
$\gamma_3 = (1, 0)$	λ^2
$\gamma_4 = (1, 1)$	$\lambda(1-\lambda)$

Table 2: A summary of signal vector and distribution for $G_B(n, \Lambda, m)$.

Signal (S)	$p(S\|\gamma_1)$	$p(S\|\gamma_2)$	$p(S\|\gamma_3)$	$p(S\|\gamma_4)$
$s_1 = (-, -)$	b^2	$b(1-a)$	$(1-a)b$	$(1-a)^2$
$s_2 = (-, +)$	$b(1-b)$	ab	$(1-a)(1-b)$	$(1-a)a$
$s_3 = (+, -)$	$b(1-b)$	$(1-a)(1-b)$	ab	$a(1-a)$
$s_4 = (+, +)$	$(1-b)^2$	$a(1-b)$	$a(1-b)$	a^2

Signal (S)	$p_0(S) = \sum_i p(S\|\gamma_i)b_0(\gamma_i)$
$s_1 = (-, -)$	$-(4\lambda-9)(4\lambda+5)/144$
$s_2 = (-, +)$	$(4\lambda-7)(4\lambda-9)/144$
$s_3 = (+, -)$	$(4\lambda+3)(4\lambda+5)/144$
$s_4 = (+, +)$	$-(4\lambda+3)(4\lambda-7)/144$

Example 1a. The $G_B(2, \Lambda, 1)$ model. Specially, let's take a look at the posterior distribution and destruction with $\lambda = 0.7$. The posterior distribution for locks is shown in Table 3. Parameter $r_i(S)$ and real damage $d(s|x)$, $i = 1, 2$ are in Table 4.

When $\lambda = 0.7$, destruction $d = d_1 + d_2 = 0.788$.

If attacker sees signal vector s_1 or s_2, she should choose site 1 to place a bomb.

Table 3: A summary of signal and Posterior Distribution when $\lambda = 0.7$.

Lock (γ)	Signal (S)	$b(\gamma_i\|S) = p(S\|\gamma_i)b_0(\gamma_i)/p_0(S)$
$\gamma_1 = (0, 0)$	$s_1 = (-, -)$	$-81\lambda(1-\lambda)/((4\lambda-9)(4\lambda+5)) = 0.352$
	$s_2 = (-, +)$	$27\lambda(1-\lambda)/((4\lambda-7)(4\lambda-9)) = 0.218$
	$s_3 = (+, -)$	$27\lambda(1-\lambda)/((4\lambda+3)(4\lambda+5)) = 0.125$
	$s_4 = (+, +)$	$-9\lambda(1-\lambda)/((4\lambda+3)(4\lambda-7)) = 0.078$
$\gamma_2 = (0, 1)$	$s_1 = (-, -)$	$-45(1-\lambda)^2/((4\lambda-9)(4\lambda+5)) = 0.084$
	$s_2 = (-, +)$	$63(1-\lambda)^2/((4\lambda-7)(4\lambda-9)) = 0.218$
	$s_3 = (+, -)$	$15(1-\lambda)^2/((4\lambda+3)(4\lambda+5)) = 0.03$
	$s_4 = (+, +)$	$-21(1-\lambda)^2/((4\lambda+3)(4\lambda-7)) = 0.078$
$\gamma_3 = (1, 0)$	$s_1 = (-, -)$	$-45\lambda^2/((4\lambda-9)(4\lambda+5)) = 0.456$
	$s_2 = (-, +)$	$15\lambda^2/((4\lambda-7)(4\lambda-9)) = 0.282$
	$s_3 = (+, -)$	$63\lambda^2/((4\lambda+3)(4\lambda+5)) = 0.682$
	$s_4 = (+, +)$	$-21\lambda^2/((4\lambda+3)(4\lambda-7)) = 0.422$
$\gamma_4 = (1, 1)$	$s_1 = (-, -)$	$-25\lambda(1-\lambda)/((4\lambda-9)(4\lambda+5)) = 0.109$
	$s_2 = (-, +)$	$35\lambda(1-\lambda)/((4\lambda-7)(4\lambda-9)) = 0.282$
	$s_3 = (+, -)$	$35\lambda(1-\lambda)/((4\lambda+3)(4\lambda+5)) = 0.162$
	$s_4 = (+, +)$	$-49\lambda(1-\lambda)/((4\lambda+3)(4\lambda-7)) = 0.422$

Table 4: A summary of destruction.
Notice: Site 1 has larger r for signal s_1 and s_2 and site 2 has larger r for signal s_3 and s_4.

site	Signal (S)	$r_i(S) = c_i P(T_i = 0\|S)p$	$d(s\|\lambda = 0.7)$
1	$s_1 = (-, -)$	$2(b(\gamma_1\|s_1) + b(\gamma_2\|s_1)) = 0.871$	$2b_0(\gamma_1)(p(s_1\|\gamma_1) + p(s_2\|\gamma_1))$
	$s_2 = (-, +)$	0.871	$+2b_0(\gamma_2)(p(s_1\|\gamma_2) + p(s_2\|\gamma_2)))$
	$s_3 = (+, -)$	0.31	$= 0.45$
	$s_4 = (+, +)$	0.31	
2	$s_1 = (-, -)$	$(b(\gamma_1\|s_1) + b(\gamma_3\|s_1)) = 0.808$	$b_0(\gamma_1)(p(s_3\|\gamma_1) + p(s_4\|\gamma_1))$
	$s_2 = (-, +)$	0.5	$+b_0(\gamma_3)(p(s_3\|\gamma_3) + p(s_4\|\gamma_3)))$
	$s_3 = (+, -)$	0.808	$= 0.338$
	$s_4 = (+, +)$	0.5	

Example 1b. We contrast this to the dependent case $G_A(n = 2, k = 1, m = 1)$, with valued site $c = (2, 1)$, $a = 7/12$, $b = 9/12$ and the probability of allocating the lock in site 1 is λ, the probability of allocating the lock in site 2 is $1 - \lambda$.

Let's take a look at the cooresponding posterior distribution and parameter r for $\lambda = 0.7$. We have destruction $d = d_1 + d_2 = 0.8896$.

When attacker sees signal vector s_2, she should place the bomb in site 1, otherwise site 2. See table 8

Table 5: A summary of Locks' location and prior probability for $G_A(n, k, m)$.

Lock' Location (γ)	Probability of γ ($b_0(\gamma)$)
$\gamma_1 = (1, 0)$	λ
$\gamma_2 = (0, 1)$	$1 - \lambda$

Table 6: A summary of signal vector and distribution for $G_A(n, k, m)$.

Signal (S)	$p(S\|\gamma_1)$	$p(S\|\gamma_2)$	$p_0(S) = \sum_i p(S\|\gamma_i)b_0(\gamma_i)$
$s_1 = (-, -)$	$b(1 - a)$	$b(1 - a)$	$45/144$
$s_2 = (-, +)$	$(1 - a)(1 - b)$	ab	$(63 - 48\lambda)/144$
$s_3 = (+, -)$	ab	$(1 - a)(1 - b)$	$(15 + 48\lambda)/144$
$s_4 = (+, +)$	$a(1 - b)$	$a(1 - b)$	$21/144$

Table 7: A summary of signal and Posterior Distribution when $\lambda = 0.7$.

Lock (γ)	Signal (S)	$b(\gamma_i \vert S) = p(S \vert \gamma_i) b_0(\gamma_i)/p_0(S)$
$\gamma_1 = (1, 0)$	$s_1 = (-, -)$	$\lambda = 0.7$
	$s_2 = (-, +)$	$15\lambda/(63 - 48\lambda) = 0.357$
	$s_3 = (+, -)$	$63\lambda/(63\lambda + 15(1 - \lambda)) = 0.907$
	$s_4 = (+, +)$	$\lambda = 0.7$
$\gamma_2 = (0, 1)$	$s_1 = (-, -)$	$1 - \lambda = 0.3$
	$s_2 = (-, +)$	$63(1 - \lambda)/(63 - 48\lambda) = 0.643$
	$s_3 = (+, -)$	$(15 + 48\lambda) = 0.093$
	$s_4 = (+, +)$	$1 - \lambda = 0.3$

Table 8: A summary of destruction.
Notice: Site 1 has larger r for signal s_2 and site 2 has larger r for signal s_1,s_3 and s_4.

site	Signal (S)	$r_i(S) = c_i P(T_i = 0 \vert S) p$	$d(s \vert \lambda = 0.7)$
1	$s_1 = (-, -)$	$2b(\gamma_2 \vert s_1) = 0.6$	
	$s_2 = (-, +)$	1.286	$2b_0(\gamma_2)p(s_2 \vert \gamma_2)$
	$s_3 = (+, -)$	0.185	$= 0.2625$
	$s_4 = (+, +)$	0.6	
2	$s_1 = (-, -)$	$b(\gamma_1 \vert s_1) = 0.7$	
	$s_2 = (-, +)$	0.357	$b_0(\gamma_1)(p(s_1 \vert \gamma_1) + p(s_3 \vert \gamma_1) + p(s_4 \vert \gamma_1))$
	$s_3 = (+, -)$	0.907	$= 0.627$
	$s_4 = (+, +)$	0.7	

Example 2. $G_B(n, \Lambda, m)$, assuming $n = 3$, $c = (7, 5, 3)$, $\Lambda = (0.2, 0.1, 0.7)$, $a = 7/12$, $b = 9/12$, probability of explosion $p = 0.8$ and let parameter k here be 1.

When $S = \{-, -, +\}$, $m = 1$, from Table 9, Attacker should place a bomb to site 1.

Table 9: A summary of destruction.
Notice: When $S = s_5 = \{-, -, +\}$, site 1 has larger r (which is $r = 9.926$) than site 2 (which is $r = 7.605$) and site 3 (which is $r = 0.752$).

site	Signal (S)	$r_i(S) = c_i P(T_i = 0\|S)p$	$d(s\|\Lambda) = c_i P(T_i = 0\|s)p$
1	$s_1 = (-, -, -)$	4.917	$d_1 = 3.677$
	$s_2 = (+, -, -)$	1.752	
	$s_3 = (-, +, -)$	4.917	
	$s_4 = (+, +, -)$	1.752	
	$s_5 = (-, -, +)$	9.926	
	$s_6 = (+, -, +)$	3.537	
	$s_7 = (-, +, +)$	9.926	
	$s_8 = (+, +, +)$	3.537	
2	$s_1 = (-, -, -)$	3.767	$d_2 = 0.855$
	$s_2 = (+, -, -)$	1.866	
	$s_3 = (-, +, -)$	3.176	
	$s_4 = (+, +, -)$	1.574	
	$s_5 = (-, -, +)$	7.605	
	$s_6 = (+, -, +)$	3.767	
	$s_7 = (-, +, +)$	6.412	
	$s_8 = (+, +, +)$	3.176	
3	$s_1 = (-, -, -)$	1.045	$d_3 = 0$
	$s_2 = (+, -, -)$	0.518	
	$s_3 = (-, +, -)$	1.045	
	$s_4 = (+, +, -)$	0.518	
	$s_5 = (-, -, +)$	0.752	
	$s_6 = (+, -, +)$	0.372	
	$s_7 = (-, +, +)$	0.752	
	$s_8 = (+, +, +)$	0.372	

CHAPTER 4: NASH EQUILIBRIUM POINTS

In game theory, the Nash equilibrium, named after the mathematician John Forbes Nash Jr., is a proposed solution of a non-cooperative game involving two or more players in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy. It is a concept within game theory where the optimal outcome of a game is where there is no incentive to deviate from their initial strategy. More specifically, the Nash equilibrium is a concept of game theory where the optimal outcome of a game is one where no player has an incentive to deviate from his chosen strategy after considering an opponent's choice. Overall, an individual can receive no incremental benefit from changing actions, assuming other players remain constant in their strategies. A game may have multiple Nash Equilibria or none at all.

In our case, even when Attackers have already selected the optimal strategy of bombs placement for any signal received, Defenders can still minimize their potential loss by choosing an optimal strategy of locks allocation

This is the case of Nash Equilibrium

4.1 Notations And Conditions

Under the setting of General **G-LBT** model when there are different kinds of sites, locks and bombs, with possibly different values of benefits and costs for Defender and/or Attacker, and testing is not uniform with respect to different sites, e.g., when Defender can test only a subset of all sites, or parameters of testing a and b depend on the site number, we can construct posterior distribution of locks and obtain an optimal strategy for attackers.

G-LBT model $G_B(n, \Lambda, m)$:

Defender: There is a total of n sites and each site has a value c_i, with $c = (c_1, c_2, c_3, ...c_n)$. Defenders allocate locks independently with non-identical probability in different sites, i.e. $\Lambda = (\lambda_1, \lambda_2, ...\lambda_n)$, where λ_i indicates the probability of containing a lock in i^{th} site, with restriction $\sum_i \lambda_i = k$ and $0 \leq \lambda_i \leq 1$ for $i = 1, 2, ...n$. So there is a total of $\kappa = 2^n$ allocations of locks. Let's redefine locks' allocation vector to be $\gamma = (\gamma_1, \gamma_2, ...\gamma_\kappa)$.

Attacker: There are m bombs, among which u bombs are placed into sites with $u = (u_1, u_2, ...u_n)$, and $\sum u_i = m$, and each site is tested to obtain a signal vector $s = (s_1, s_2, ...s_n)$, with s_i either be $-$ or $+$.

Assume the sites themselves have different values ($c = (c_1, c_2, c_3, ...c_n)$), i.e. $c_1 \geq c_2 \geq c_3 \geq ... \geq c_n$. Attacker's goal is to maximize expected value of damage (loss).

4.2 Nash Equilibrium Point for the $G_B(n, \Lambda, m)$ model

Parameter k can be fixed or random. Fixed k is used when we have the dependent model $G_A(n, k, m)$, and the strategy of how to allocate locks is defined by a probability vector $X = (x_1, x_2, ..., x_k)$. In our paper, the parameter k; $k < n$, is random, then the strategy of the Defender is a probability distribution $b(\gamma)$ on a set of all possible positions of locks. To be noticed, in general LBT model, calculation would much easier to analyze with lock's allocation vector for all sites rather than a single site. Since there are n sites in total, and each position can either have a lock or not, so there is a total of $\kappa = 2^n$ allocations of locks, thus locks' allocation vector is $\gamma = (\gamma_1, \gamma_2, ...\gamma_\kappa)$.

In our Bayesian setting we assume that this prior distribution $b(\gamma)$ is known to the attacker, though, of course, the real positions of the locks are not. After the locks are allocated, Attacker receives signal s and, having m bombs, distributes them among n sites deterministically or using some randomization trying to maximize the expected sum of values of all destroyed sites. Without loss of generality, we can assume that this distribution is deterministic and the optimal strategy of the attacker is $\pi(m|b(\gamma))$, with respect to a strategy of the defender, $b(\gamma)$, is a collection of her optimal responses $u(s|m, b(\gamma)) = u(s) = (u_1(s), u_2(s), ...u_n(s))$ to each signal s, where $u_i(s)$ is the number of bombs placed into site i; $i = 1, 2, ...n$. Using the prior distribution $b(\gamma)$, the probabilities of signals $p(s)$, given this distribution, the posterior distribution of the positions of locks $b(\gamma|s)$ and the total expected damage (loss), $L(b(\gamma), \pi(m|b(\gamma))$ can be calculated. The goal of Defender is to select a prior distribution of locks $b_*(\gamma)$ to minimize this loss. Then the pair $(b_*(\gamma), \pi_*)$, where π_* is an optimal response of

Attacker to strategy $b_*(\gamma)$ forms a classical Nash equilibrium (NE) point. The corresponding value of the game is $v_* = L(b_*(\gamma), \pi_*)$. As we will see, though the $b_*(\gamma)$'s are not unique, they all have common properties that result in a unique (up to some randomization) Attacker's strategy π_*, and thus a specific value of v_*.

We denote by $G_B = G_B(n, \Lambda, m|a, b, c)$ this general Bayesian game, where n is the number of sites, n dimensional vectors a and b represent the quality of testing, (the sensitivities and the specificities), and vector $c = (c_1, ..., c_n)$ describes the values of each site.

With only one available bomb, $m = 1$, Attacker will place it into the next valuable site, and if $m > 1$ she should solve the problem of discrete optimization by placing the next available bomb into the site with the maximal marginal utility.

The other extreme situation is when testing is not informative, i.e., when $a_i = b_i = 1/2$; $i = 1, 2, ..., n$, and then the posterior distribution $b(\gamma|s)$ coincides with the prior distribution $b(\gamma)$ for all signals s. Given the prior distribution $b(\gamma)$, let us introduce the probability $\alpha_i = P(T_i = 0)$.

For model $G_B(n = 2, \Lambda, m = 1)$, assume $p = 1$ and valued sites has value $c = (c, 1)$, $\Lambda = (\lambda, 1 - \lambda)$. At first glance, it seems that if c is much larger than 1, defender should place a unique lock into the most valuable site and then her loss is 1. But simple calculations show that the optimal distribution of locks is $\Lambda = (\frac{c}{c+1}, \frac{1}{c+1})$ and $v = \frac{c}{c+1} < 1$. Attacker can place her unique bomb into any site or place it at random. Similarly, for the game $G_B(n = 3, \Lambda, m = 1)$ with vector of values $c = (4, 3, 2)$, and $\Lambda = (\lambda_1, \lambda_2, 1 - \lambda_1 - \lambda_2)$. We obtain that the optimal distribution of a unique lock is given by $\Lambda = (7/13, 5/13, 1/13)$ and $\alpha = (6/13, 8/13, 12/13)$, $c_i * \alpha_i = v_* = 24/13$

for $i = 1, 2, 3$. But if the vector of values is $c = (4, 3, 1)$, then $\Lambda = (4/7, 3/7, 0)$ and $\alpha = (3/7, 4/7, 1)$, $c_i * \alpha_i = v_* = 12/7$ for $i = 1, 2, 3$.

Theorem 6. **(Non-Informative Case).** For $m = 1$; $c = (c_1, c_2, c_3, ...c_n)$ with $c_1 \geq c_2 \geq c_3 \geq ... \geq c_n$,

a) The class of optimal strategies Λ has the following structure; there exists k_{*1}, k_{*2}, $0 \leq k_{*1} \leq k \leq k_{*2} \leq n$ and constant $v_* = v(c)$ such that: $c_i\alpha_i = v_*$ for $k_{*1} \leq i \leq k_{*2}$, and $v_* > c_i$; $\alpha_i = 0$ for $0 \leq i < k_{*1}$; $\alpha_i = 1$ for $k_{*2} < i \leq n$. This relationship is represented as follows:

$$\begin{cases} \alpha_i = 0 & \text{if } i < k_{*1} \\ c_i\alpha_i = v_* & \text{if } k_{*1} \leq i \leq k_{*2} \\ \alpha_i = 1 & \text{if } i > k_{*2} \end{cases}$$

b) The optimal strategy for attacker is to place a bomb at random among the sites with numbers $k_{*1}, ...k_{*2}$.

c) The value of the game is $v_* = (n-k)/C1_{k_{*1}}$, where $k_{*1} = max\{j : j \leq k, c_j > (n-k)/C1_j\}$ and $C1_j = \sum_{i=j}^{n} 1/c_i$.

d) The value of the game is $v_* = (k_{*2}-k)/C2_{k_{*2}}$, where $k_{*2} = max\{j : j \geq k, c_j > (j-k)/C2_j\}$ and $C2_j = \sum_{i=1}^{j} 1/c_i$.

Proof. **of Theorem 6(c).** When $k_{*1} \leq i \leq k_{*2}$, we have $c_i \alpha_i = v_*$ for $i = k_{*1}, ...k_{*2}$.

When $i < k_{*1}$, we have $\alpha_i = 0$.

Hence, in general, we have

$$\begin{aligned}
E(\sum_{i=k_{*1}}^{n} 1_{(T_i=1)}) = \sum_{i=k_{*1}}^{n} (1-\alpha_i) &= k - k_{*1} + 1 \\
n - k_{*1} + 1 - \sum_{i=k_{*1}}^{n} \frac{v_*}{c_i} &= k - k_{*1} + 1 \\
\text{Let } C_{*1} &= \sum_{i=k_{*1}}^{n} \frac{1}{c_i} \\
\text{Thus } v_* &= \frac{n-k}{C_{*1}}
\end{aligned}$$

Hence, when $i = k_{*1}, ...k_{*2}$, $\alpha_i = \frac{v_*}{c_i} < 1$

Thus, we have $\frac{n-k}{C_{*1} c_i} < 1$, or, $c_i > \frac{n-k}{C_{*1}}$.

So $k_{*1} = max\{j : j \leq k, c_j > (n-k)/C1_j\}$. □

Proof. **of Theorem 6(d).** When $k_{*1} \leq i \leq k_{*2}$, we have $c_i \alpha_i = v_*$ for $i = 1, 2, ...k_{*2}$.

When $k_{*2} < i \leq n$, we have $\alpha_i = 1$.

Hence, in general, we have

$$\begin{aligned}
E(\sum_{i=1}^{k_{*2}} 1_{(T_i=0)}) = \sum_{i=1}^{k_{*2}} \alpha_i &= k_{*2} - k \\
\sum_{i=1}^{k_{*2}} \frac{v_*}{v_i} &= k_{*2} - k \\
\text{Let } C_{*2} &= \sum_{i=1}^{k_{*2}} \frac{1}{c_i} \\
\text{Thus } v_* &= \frac{k_{*2} - k}{C_{*2}}
\end{aligned}$$

Hence, when $i = 1, 2, ...k_{*2}$, $\alpha_i = \frac{v_*}{c_i} < 1$

Thus, we have $\frac{k_{*2}-k}{C_{*2}c_i} < 1$, or, $c_i > \frac{k_{*2}-k}{C_*}$.

So $k_{*2} = max\{j : j \geq k, c_j > (j-k)/C2_j\}$. $\square$

In other words, if $k_* < n$, then the sites with numbers greater than k_* should not be protected at all and the distribution of k locks in the first k_* sites should make all sites *equally desirable for attack*. In layman terms, if the strength of a chain is defined by the strength of the weakest link, and the resources to make links strong are limited, then make all links of equal strength. This is a special case of a more general "Chain-Link Optimization Principle".

Note also that for *any vector of values* c, the number $k < k_*$, the optimal strategy $b_*(\gamma)$ is always randomized and value $v_* > c_{k_*+1}$. As a result, the attacker will allocate her m bombs among the first k_* sites if $m \leq k_*$.

Of course, the main interest in the Bayesian LBT game problem is the case of imperfect but informative testing. For simplicity we will assume that this means that $a_i > 1/2$, $b_i > 1/2$, $i = 1, 2, ...n$, though in the general case this property should be described using vectors a and b. To obtain the description of NE points, we have to solve three problems.

The first problem, is, given a strategy of defender, $b(\gamma)$, describe the optimal strategy (response) of attacker, $\pi(m|b(\gamma))$, i.e., describe the optimal allocation of bombs $u(s|m)$ given signal s and m available bombs. The full answer to this problem is given by a recursive procedure S described. The expected value of damage (loss) for the pair of strategies $(b(\gamma), \pi(m|b(\gamma)))$ can also be obtained.

The second (more difficult) problem is to find the optimal strategy or strategies, $b_*(\gamma)$, of the defender, minimizing this loss. So far, the proof of corresponding Theorem 2 is not 100% complete but its heuristic meaning is similar to the meaning of Theorem 1: These strategies have to make the potential expected losses in the sites, that are worth protecting, equal when attacker applies her optimal response to $b_*(\gamma)$. The difficulty here lies in the fact that in the informative case the optimal response depends on signal s. Given a strategy of the defender, $b(\gamma)$, let us denote by $L_i(m)$ the expected loss at site i when the attacker applies her optimal strategy given signal s, $p(s)$ the probability of signal s and $L_i(m) = \sum_s p(s) L_i(s|m)$ the corresponding expected loss.

Theorem 7. **(Informative Case)** Given $m = 1, 2, ...$, the class of optimal strategies $b_*(\gamma|m)$ can be obtained using the Principle of Indifference, that takes the following specific form: the values of $L_i(m) = v_*$ must be equal for all $i = 1, 2, ..., k_*(m)$, where $k_*(m), k < k_*(m) \leq n$, is the number of sites worth protecting, and v_* id the value of the game.

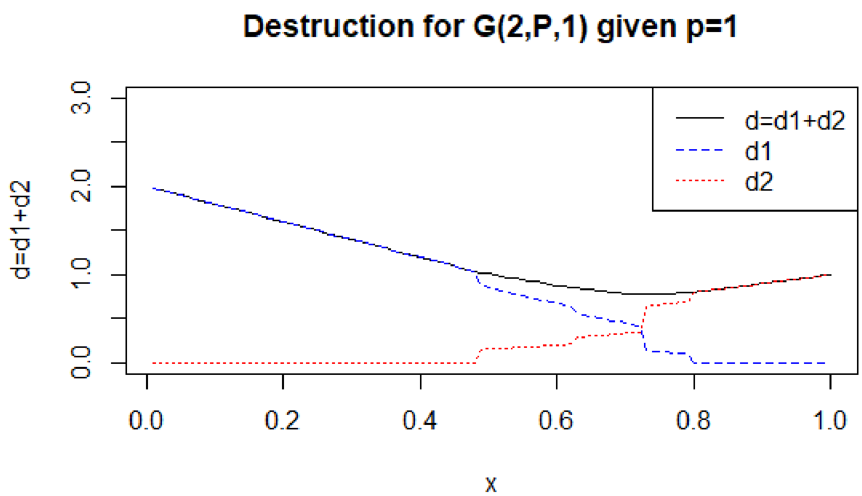
The difficulty in applying Theorem 7 lies in the fact that in informative case the optimal response depends on signal s and calculation of $L_i(m)$ is nontrivial. In the next section we provide an example of an application of Theorem 3 to find the optimal strategy.

The third problem to be solved is to obtain the full description of all NE points, i.e. to describe all $b_*(\gamma)$ delivering the equality of $L_i(m)$ in Theorem 2.

Remark 1. The description of $b_*(\gamma)$ is based on the following interesting property of a general game: to obtain the optimal response of attacker given any $b(\gamma)$ and signal s, i.e. to use procedure R, the attacker needs to know only the marginal probabilities $\alpha_i(s)$ for all s, but the defender, trying to obtain $b_*(\gamma)$, needs to know v(m), and to calculate the total expected loss, she needs to have $p(s)$ based on the whole distribution $b(\gamma)$. There is the simple example that shows that two distinct $b(\gamma)$ can have the same probabilities $\alpha_i(s)$. Thus, one of a side problems is to obtain the description of all $b(\gamma)$ having the same probabilities $\alpha_i(s)$.

Remark 2. The statements and interpretation of Theorems 1 and 2 can be expressed also using the concepts of *information* and *entropy*. Loosely speaking, the optimal defender's strategy is to create the situation for attacker with maximal possible entropy.

Example 1. Informative Case: for $G_B(2, \Lambda, 1)$, where $\Lambda = (\lambda, 1-\lambda)$ with valued site $c = (2, 1)$, $a = 7/12$, $b = 9/12$. (This example will be compared to the dependent case model $G_A(n, k, m)$ where $k = 1$). From (Figure 18), we find at $\lambda = 0.72$, destruction will be minimized with $d = 0.7728$.



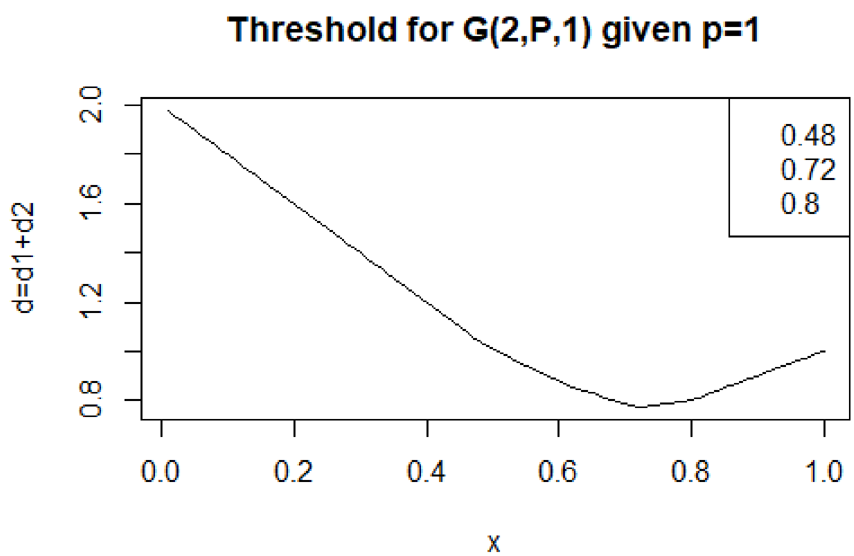


Figure 18: Destruction and threshold for $G_B(n, \Lambda, m)$

For the dependent model $G_A(n, k, m)$ with $n = 2$, $k = 1$, $m = 1$ and with valued site $c = (2, 1)$, $a = 7/12$, $b = 9/12$ and the probability of allocating the lock in site 1 is λ, the probability of allocating the lock in site 2 is $1 - \lambda$. From Figure (19) destruction is minimized at $\lambda = 0.67$ with value $d = 0.89$.

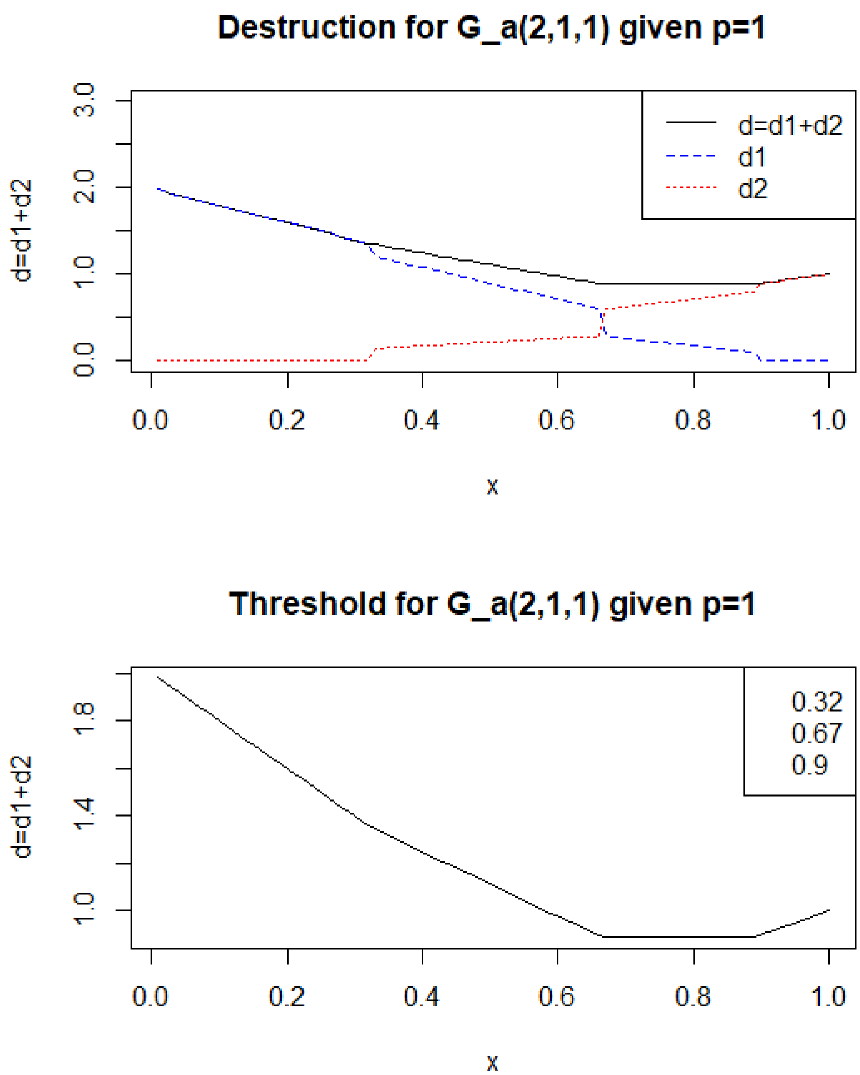
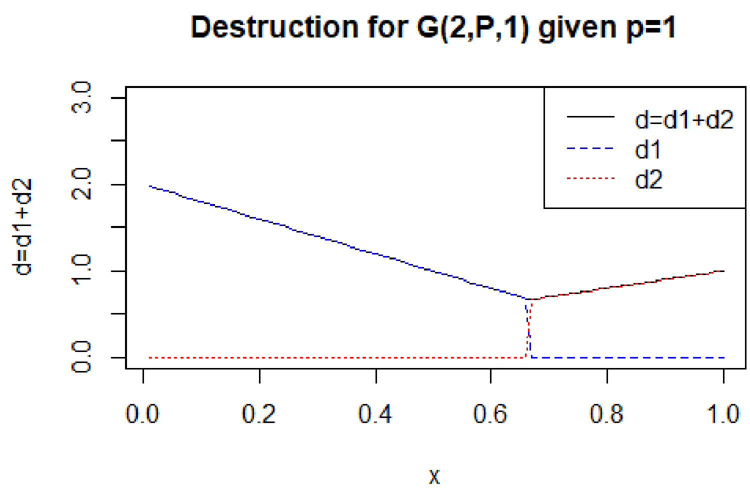


Figure 19: Destruction and threshold for $G_A(n, k, m)$

Example 2. Non-informative Case: $G_B(2, \Lambda, 1)$, where $\Lambda = (\lambda, 1-\lambda)$ with valued site $c = (2, 1)$, $a = 1/2$, $b = 1/2$. (This example will be compared to the dependent case $G_A(n, k, m)$ where $k = 1$). See figure (20), the destruction is minimized at $\lambda = 2/3 = 0.67$.



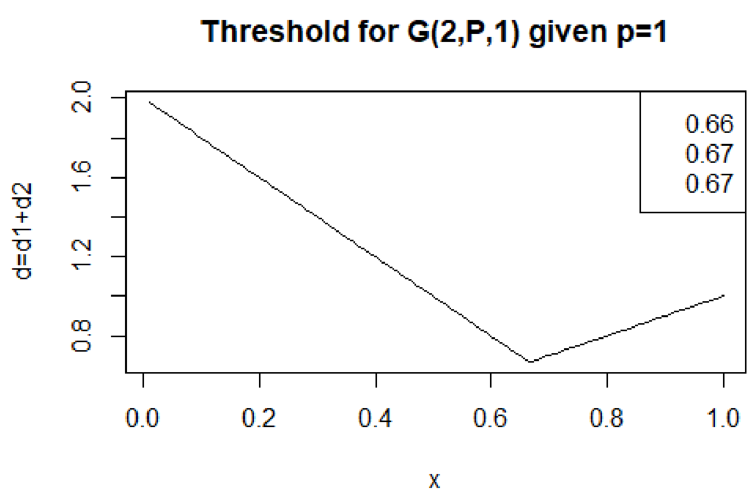


Figure 20: Destruction and Threshold for non-inform-general case in model $G_B(2, \Lambda, 1)$

For the dependent case $G_A(2,1,1)$, we have the same conclusion as above with the destruction minimized at $\lambda = 2/3 = 0.67$.

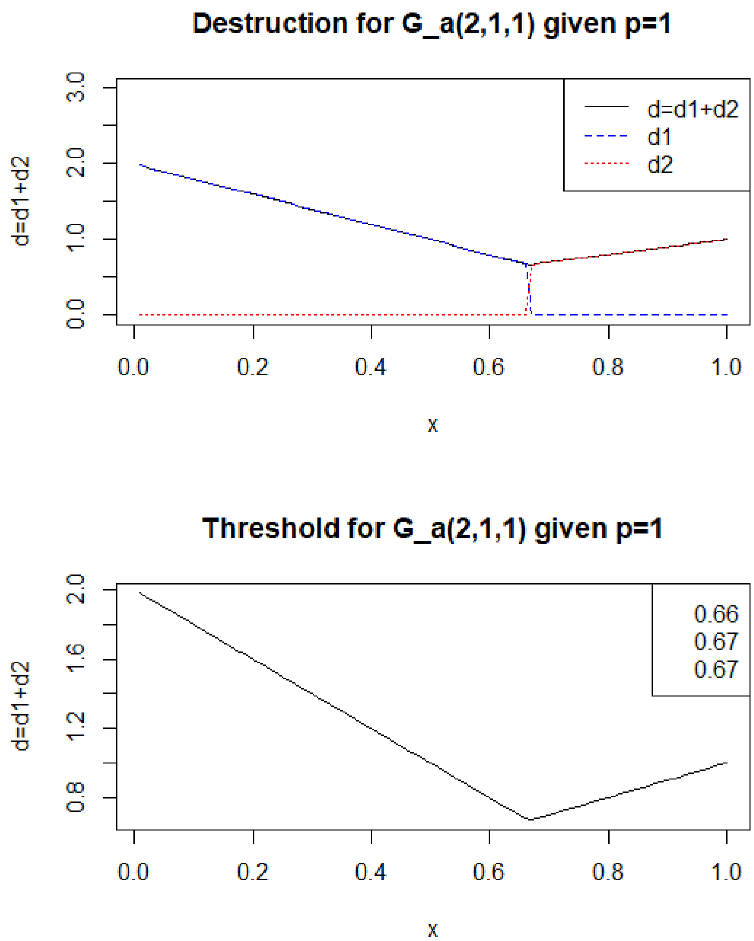


Figure 21: Destruction and Threshold for non-inform-general case in model $G_A(2,1,1)$

Example 3. Non-informative Case: $G_B(3, \Lambda, 1)$, where $\Lambda = (\lambda_1, \lambda_2, 1-\lambda_1-\lambda_2)$ with valued site $c = (4, 3, 1)$, $a = 1/2$, $b = 1/2$. Here parameter $k = 1$.

From theorem, we get $k_{*1} = 1$ and $k_{*2} = 2$. $\alpha = (3/7, 4/7, 1)$.

See figure (22) and (23), and destruction is minimized at $\Lambda = (4/7, 3/7, 0) = (0.57, 0.43, 0)$ with value $v_* = 12/7 = 1.72$.

```
> Defender_strategy(nn,P1,V)
$result
   box signal1 signal2 signal3    r joint_p    d.ps
1    1       0       0       0 1.72 0.05375 0.05375
2    1       1       0       0 1.72 0.05375 0.05375
3    1       0       1       0 1.72 0.05375 0.05375
4    1       1       1       0 1.72 0.05375 0.05375
5    1       0       0       1 1.72 0.05375 0.05375
6    1       1       0       1 1.72 0.05375 0.05375
7    1       0       1       1 1.72 0.05375 0.05375
8    1       1       1       1 1.72 0.05375 0.05375
9    2       0       0       0 1.71 0.07125 0.00000
10   2       1       0       0 1.71 0.07125 0.00000
11   2       0       1       0 1.71 0.07125 0.00000
12   2       1       1       0 1.71 0.07125 0.00000
13   2       0       0       1 1.71 0.07125 0.00000
14   2       1       0       1 1.71 0.07125 0.00000
15   2       0       1       1 1.71 0.07125 0.00000
16   2       1       1       1 1.71 0.07125 0.00000
17   3       0       0       0 1.00 0.12500 0.00000
18   3       1       0       0 1.00 0.12500 0.00000
19   3       0       1       0 1.00 0.12500 0.00000
20   3       1       1       0 1.00 0.12500 0.00000
21   3       0       0       1 1.00 0.12500 0.00000
22   3       1       0       1 1.00 0.12500 0.00000
23   3       0       1       1 1.00 0.12500 0.00000
24   3       1       1       1 1.00 0.12500 0.00000
```

Figure 22: r and destruction for signals in each site

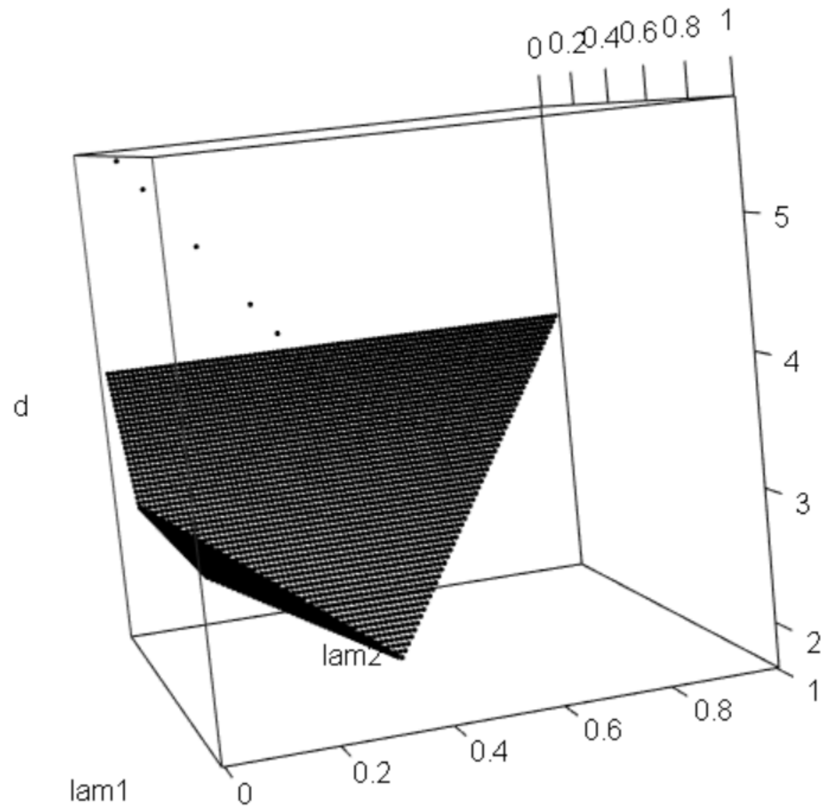


Figure 23: graph for destruction on different lambdas

Non-informative Case: for $G_A(n = 3, k = 1, m = 1)$, with valued site $c = (4, 3, 1)$, $a = 1/2$, $b = 1/2$. Here parameter $k = 1$.

From theorem, we get $k_* = 2$ and $\alpha = (3/7, 4/7, 1)$.

It is easy to obtain that the optimal distribution of a unique lock is given by $b(\gamma) = (\frac{4}{7}, \frac{3}{7}, 0)$ with value $v_* = 12/7$.

Thus, under optimal strategy of defender site 3 is not protected at all.

Locks' allocation vector(γ)	$b(\gamma)$
$(1, 0, 0)$	$4/7$
$(0, 1, 0)$	$3/7$
$(0, 0, 1)$	0

Example 4. **Non-informative Case**: $G_B(n = 3, \Lambda, m = 1)$, where $\Lambda = (\lambda_1, \lambda_2, 2 - \lambda_1 - \lambda_2)$ with valued site $c = (4, 3, 1)$, $a = 1/2$, $b = 1/2$. Here parameter $k = 2$.

From theorem, we get $k_{*1} = 2$ and $k_{*2} = 3$. $\alpha = (0, 1/4, 3/4)$.

Thus the destruction is minimized at $\Lambda = (1, 3/4, 1/4)$ with value $v_* = 3/4$.

Non-informative Case: for $G_A(n = 3, k = 2, m = 1)$, with valued site $c = (4, 3, 1)$, $a = 1/2$, $b = 1/2$. Here parameter $k = 2$.

From theorem, we get $k_* = 3$ and $\alpha = (3/19, 4/19, 12/19)$

It is easy to obtain that the optimal distribution of two locks is given by $b(\gamma) = (3/19, 4/19, 12/19)$ with value $v_* = 12/19$.

Locks' allocation vector(γ)	$b(\gamma)$
$(0, 1, 1)$	$3/19$
$(1, 0, 1)$	$4/19$
$(1, 1, 0)$	$12/19$

CHAPTER 5: CONCLUSION AND FUTURE WORK

In Chapter 3, we developed General LBT model (**G-LBT**) under special condition of valued site and nonidentical probability of locks allocation when number of bombs is 1. In the future, we can also extend this model to a more general condition where number of bombs could be a nonnegative continuous variable.Similarly, we can convert integer locks to a continuous protection resource.

Moreover, **G-LBT** model could tolerate different kinds of bombs and locks, i.e. different kinds of bombs and locks have different kinds of power. Testing is not uniform with respect to different sites, in this case, Defender can test only a subset of all sites, or parameters of testing a and b depend on the site number.

REFERENCES

[1] K. Sonin, J. Wilson, A. Wright (2018). Rebel Capacity, Intelligence Gathering, and the Timing of Combat Operations, manuscript.

[2] Gross, O.,Wagner,(1950).A continuous Colonel Blotto game. RAND Corporation RM-408

[3] E. Presman, (2018). Some Property of an important parameter in Locks and Bombs model, manuscript.

[4] Kemeny J. G., Snell J. L. *Finite Markov Chains*, 1976. Springer, Princeton, NJ.

[5] Shubik M., Weber R.J. (1981). Systems Defense Games: Colonel Blotto, Command and Control. Nav Res Log Quart. 28, p. 281-287.

[6] Kvasov, D. (2007). "Contests with Limited Resources". Journal of Economic Theory. 136: 738748.

[7] Sequential Control with Incomplete Information: The Bayesian Approach to Multi-Armed Bandit Problems, London: Academic Press, (1990).

[8] Robert Powell(2007). Defending against Terrorist Attacks with Limited Resources, American Political Science Review. Vol. 101, No. 3, August 2007

[9] Sergiu Hart(2015). Allocation games with caps: from Captain Lotto to all-pay auctions, Springer: Int J Game Theory, DOI 10.1007/s00182-015-0485-7

[10] Sergiu Hart(2007). Discrete Colonel Blotto and General Lotto games, Springer: Int J Game Theory(2008), 36:441460 DOI 10.1007/s00182-007-0099-9

[11] Brian Roberson(2006). The Colonel Blotto game, Economic Theory (2006) 29: 124 DOI 10.1007/s00199-005-0071-5

[12] Sonin I., Sonin K., (2019). Game of Locks, Bombs and Testing, manuscript.

[13] Presman E., (2018). Some Property of an Important Parameter in Locks and Bombs Model, manuscript.

[14] Shubik M., Weber R.J. (1981). Systems Defense Games: Colonel Blotto, Command and Control. Nav Res Log Quart. 28, p. 281-287.

[15] Presman E., Sonin I., Sequential Control with Incomplete Information: The Bayesian Approach to Multi-Armed Bandit Problems, London: Academic Press, (1990).