LOG CONCAVITY OF THE POWER PARTITION FUNCTION

by

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ABSTRACT

BRENNAN BENFIELD. Log Concavity of the Power Partition Function.(Under the direction of Dr. ARINDAM ROY.)

The main result of this paper is to prove the log concavity of a particular restricted partition $P_k(n)$ that enumerates the partitions of a positive integer into perfect k^{th} powers. Further investigation utilizing MATHEMATICA software yields numerical evidence of certain interesting facts about the function $P_k(n)$.

ACKNOWLEDGEMENTS

The author would like to thank both his advisor, Dr. Arindam Roy, and Madhumita Paul for their invaluable help with the research and proof of this theorem. With their help and support, a new mathematical fact can now be added to the annals of human knowledge.

Table of Contents

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1: INTRODUCTION

1.1 Log Concavity A sequence of non-negative integers $\{a_n\}$ is *log concave* if

$$a_{n-1}a_{n+1} \leq a_n^2$$

for all $n \in \mathbb{N}$. Equivalently, the sequence $\{a_n\}$ is log concave if

$$\log(a_{n-1}) - 2\log(a_n) + \log(a_{n+1}) \leq 0$$

for all $n \in \mathbb{N}$. It is from here that the property gets its name. There are many different applications for log concave sequences and a number of techniques are used to determine if a particular sequence is log concave. Discovering which sequences are log concave has become increasingly popular. Sequences that derive from combinatorial processes are particularly good candidates to test for log concavity.

1.2 Binomial Coefficients The classic example of a log concave sequence is generated by the binomial coefficients. Given non-negative integers n and k, $\binom{n}{k}$ is the coefficient of the x^k term in the polynomial expansion of $(1+x)^n$. The binomial coefficients are given by the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

As an example, suppose $n = 5$. Then the coefficients of the polynomial can be obtained directly by

$$\begin{aligned} (1+x)^5 &= \binom{5}{0}x^0 + \binom{5}{1}x^1 + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5 \\ &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \end{aligned}$$

This example is generalized by the binomial formula:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

The coefficients are found in the famous Pascal Triangle.

$n=0$						1					
$n=1$					1		1				
$n=2$				1		2		1			
$n=3$			1		3		3		1		
$n=4$		1		4		6		4		1	
$n=5$	1		5		10		10		5		1

It is a known result that for a given n, the sequence $\{\binom{n}{k}\}_{k=0}^{n}$ is log concave in k and that the sequence $\{\binom{n}{k}\}_{n=k}^{+\infty}$ is log concave in k. This could be viewed as any row of the Pascal triangle is log concave. In 1978, it was shown by Tanny and Zucker [13] that, for a given n_0, the sequences $\{\binom{n_0-i}{i}\}_i$ and $\{\binom{n_0-id}{i}\}_i$ are log concave in i for some $d \in \mathbb{N}$. In 2007, Belbachir, Bencherif, and Szalay [12] proved the log concavity of the sequence $\{\binom{n_0+i}{id}\}_i$ and made the further conjecture that, for a fixed element of the Pascal triangle $\binom{n_0}{k_0}$ crossed by a ray, the sequence of binomial coefficients is log concave. The sequence is defined for $i = 0, 1, 2, ...$ by

$$C_i = \binom{n_0 + id}{k_0 + i\delta}$$

This conjecture was proven the next year by Su and Wang [11] for $0 < \delta \leq d$.

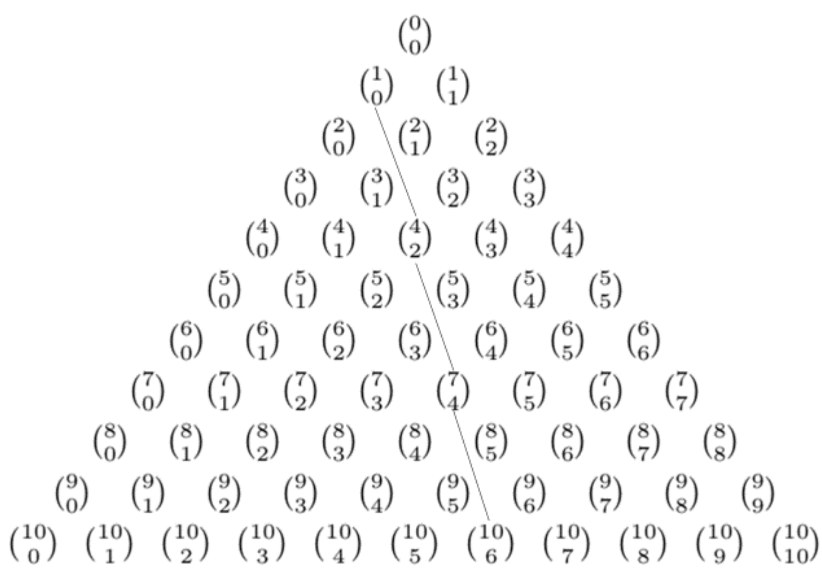

Figure 1: A ray with $d = 3$ and $\delta = 2$

1.3 Stirling Numbers Another classic example of log concave sequences is the Stirling numbers, named after their discovery by James Stirling in the 18^{th} century. Stirling numbers of the first kind count the number of permutations of n elements with k disjoint cycles. Stirling numbers of the first kind are denoted $\left[{n \atop m}\right]$ for nonnegative integers n and m and are defined by the polynomial identity

$$t^{[n]} = t(t+1)(t+2)...(t+n-1) = \sum_m \begin{bmatrix} n \\ m \end{bmatrix} t^m$$

where $0 < m \leq n$ and defined to be zero elsewhere, except $\left[{0 \atop 0}\right] = 1$ by convention. Stirling numbers of the second kind count the number of ways to partition a set of n elements into k nonempty subsets. Denoted $\left\{{n \atop m}\right\}$ the Stirling numbers of the second kind are defined by

$$t^n = \sum_m \left\{ {n \atop m} \right\} t^{(m)}$$

where $n, m \geq 0$ and $t^{(m)} = t(t-1)(t-2)...(t-m+1)$. It is well known that, for a fixed n, Stirling numbers of the first and second kind are log concave sequences in m. In 1985, E. Neuman [14] proved that the sequence $\left(\left\{{n \atop m}\right\}\right)_{n=m}^{\infty}$ is also log concave.

1.2 The Partition Function For $n \in \mathbb{N}$, the *partition function* $P(n)$ enumerates the number of partitions of n where the partitions are positive integer sequences $\lambda = (\lambda_1, \lambda_2, ...)$ with $\lambda_1 \geq \lambda_2 \geq ... > 0$ and $\sum_{j \geq 1} \lambda_j = n$

For example, $P(4) = 5$ since

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

The first few values for $P(n)$ for $n = 1, 2, ...$ are $1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, ...$ (OEIS A000041).

The origins of the partition function have deep roots in the history of number theory and it has grown to have wide reaching applications in numerous branches of mathematics.

Studied in the 17$^{\text{th}}$ century, Euler gave a generating function for $P(n)$ using q-series. A q-series is commonly denoted $(a;q)_n$ and involves coefficients of the form

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

Certain properties are obeyed by q-series, making them wonderful tools in the theory of partitions, mathematical physics, and especially enumerating possible configurations on a lattice. The generating function for $P(n)$ that Euler invented is closely related to his famous function $\phi(q)$. This is now called simply the Euler function and is given by

$$\phi(q) = \prod_{k=1}^{\infty}(1-q^k) = \sum_{-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}} = 1 - q - q^2 - q^5 - q^7 - q^{12} - q^{15} + q^{22} + q^{26} + ...$$

Then $P(n)$ is given by the generating function

$$\frac{1}{\phi(q)} = \sum_{n=0}^{\infty} P(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + ...$$

Where the coefficients of this series are the partition numbers. It is interesting to note that the exponents of the q-series are the generalized pentagonal numbers. The pentagonal numbers count the number of objects that can be arranged in a regular pentagon. The nth pentagonal number p_n is the number of distinct dots that form a pattern of the outline of regular pentagons with side length up to n dots such that the pentagons all share a single vertex. The first few pentagonal numbers p_n for $n = 1, 2, ...$ are $1, 5, 12, 22, 35, 51, 70, 92, 117, ...$ (OEIS A0000326).

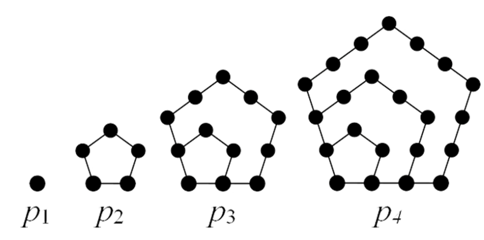

Figure 2: Pentagonal Numbers $p_1 ... p_4$

The pentagonal numbers are given by the formula $p_n = \frac{3n^2-n}{2}$ where $n \in \mathbb{N}$. The exponents found in Euler's q-series are the generalized pentagonal numbers, found by the same formula where $n \in \mathbb{Z}$ (OEIS A001318).

After Euler invented a generating function, a recurrence equation for $P(n)$ was discovered

$$P(n) = \sum_{k=1}^{n} (-1)^{k+1} \left(P\left(n - \frac{1}{2}k(3k-1)\right) + P\left(n - \frac{1}{2}k(3k+1)\right)\right)$$

Further recurrence equations have been found. In 1921, MacMahon [16] found a remarkable recurrence relation where the sum is over the generalized pentagonal numbers $\leq n$. The relation is given by

$$P(n) - P(n-1) - P(n-2) + P(n-5) + P(n-7) - P(n-12) - P(n-15) + ... = 0$$

Another remarkable recurrence equation of $P(n)$ given by Skiena [17] in 1990 involves the divisor function $\sigma(n)$. The recurrence relation is given by

$$P(n) = \frac{1}{n} \sum_{k=0}^{n-1} \sigma(n-k) P(k)$$

where the divisor function counts the number of divisors of an integer and is given by

$$\sigma(n) = \sum_{d|n} d$$

Most interestingly, Euler found earlier another recurrence that involves summing over the generalized pentagonal numbers, namely that

$$\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \sigma(n-12) - \sigma(n-15) + ... = 0$$

In the 20$^{\text{th}}$ century, Srinivasa Ramanujan discovered intriguing patterns using modular arithmetic on the values of the partition function, now known as Ramanujan's congruences. Ramanujan showed that

$$P(5m+4) \equiv 0 \mod 5$$

$$P(7m+5) \equiv 0 \mod 7$$

$$P(11m+6) \equiv 0 \mod 11$$

These congruences have been further studied and numerous other congruences have been found including some general forms of Ramanujan's original congruences. In 2000, K. Ono

[15] proved that for every $n \in \mathbb{N}$ coprime to 6 there exist Ramanujan congruences modulo n.

Ramanujan and G. H. Hardy gave the most famous asymptotic formula for $P(n)$ in 1918 using their newly minted circle method [7].

$$P(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2}{3}n}}$$

1.3 Log Concavity of the Partition Function Since the publication of the asymptotic formula, much work has been done on the partition function. At some point, no exact source is agreed upon, the log concavity of the partition function was conjectured for sufficiently large n, that is,

$$P(n-1)P(n+1) \leq P(n)^2$$

Or alternatively,

$$\log(P(n-1)) - 2\log(P(n)) + \log(P(n+1)) \leq 0$$

In 2010, William Chen [8] made the conjecture that for the partiton function $P(n)$,

$$\frac{P(n-1)}{P(n)}\left(1+\frac{1}{n}\right) > \frac{P(n)}{P(n+1)}$$

Which can be rewritten as

$$P(n-1)P(n+1) < P(n)^2 < \left(1+\frac{1}{n}\right) P(n-1)P(n+1)$$

for sufficiently large n. This was later proven in 2015 by DeSalvo and Pak [1] in a paper that included a proof of the log concavity of the partition function for all $n > 25$. In their paper, DeSalvo and Pak [1] also refined Chen's conjecture to a more precise error bound:

$$P(n-1)P(n+1) < P(n)^2 < \left(1+\frac{240}{(24n)^{\frac{3}{2}}}\right) P(n-1)P(n+1)$$

for all $n > 6$. Which was then further refined by Chen to:

$$P(n-1)P(n+1) < P(n)^2 < \left(1+\frac{\pi}{(24n)^{\frac{3}{2}}}\right) P(n-1)P(n+1)$$

for all $n > 44$.

Because of the interest in the partition function, $P(n)$, questions began to arise about other types of partitions, partitions restricted by some parameter. These functions can be expressed by $P_A(n)$ where A is some restriction on λ. Of course, if $A = \mathbb{N}$ then $P_A(n) = P(n)$, but something interesting happens when A properly restricts λ. For instance, notice what happens if λ is restricted to powers of 2. This is called the *binary partition function*, $b(n)$. In this case, $b(n)$ is log concave at every even index, $n = 2k$ but fails (and is *log convex*, that is, $b_{k-1}b_{k+1} \geq b_k^2$) at every odd index, $n = 2k+1$. Indeed, the set of restricted partitions $P_A(n)$ where A restricts λ to powers of $m \in \mathbb{N}$ is log concave for all indicies $n \equiv 0 \mod m$, is log convex for all indicies $n \equiv m-1 \mod m$, and is both(that is, $P_A(k-1)P_A(k+1) = P_A(k)^2$) at all indicies inbetween. A natural question arises, what types of restrictions A of λ preserve log concavity and would it be possible to classify all such A?

An interesting restricted partition function called the Andrews smallest parts partition function and denoted $spt(n)$, counts the number of smallest parts among $P(n)$. For example, when $n = 4$, the partition function with the smallest part underlined is

$$\begin{aligned}
4 &= \underline{4} \\
&= 3 + \underline{1} \\
&= \underline{2} + \underline{2} \\
&= 2 + \underline{1} + \underline{1} \\
&= \underline{1} + \underline{1} + \underline{1} + \underline{1}
\end{aligned}$$

And so, $spt(4) = 10$. This function is particularly interesting because its has many analogus properties to $P(n)$. For instance, its generating function is given by q-series. In 2008, Andrews [18] proved that there are spt analogues to Ramanujan congruences, namely that

$$\begin{aligned}
spt(5n+4) &\equiv 0 \mod 5 \\
spt(7n+5) &\equiv 0 \mod 7 \\
spt(13n+6) &\equiv 0 \mod 13
\end{aligned}$$

Further, its asymptotic formula was obtained by Bringmann[19] in 2008 which closely resembles the asymptotic of $P(n)$, namely that

$$spt(n) \sim \frac{1}{\pi\sqrt{8n}} e^{\pi\sqrt{\frac{2n}{3}}}$$

Notwithstanding the close relationship between the properties of $spt(n)$ and $P(n)$, it is not obvious that $spt(n)$ is log concave. However, in 2017, in a paper by Dawsey and Masri [20] it was proven that the smallest parts partition function is indeed log concave.

Another example of a log concave partition function arises from a conjecture by Z. W. Sun [21] in 2013. He claimed that for $q(n) = \frac{P(n)}{n}$ the sequence $\{q(n)\}_n \geq 31$ is log concave, that is,

$$\left(\frac{P(n)}{n}\right)^2 \geq \left(\frac{P(n-1)}{(n-1)}\right)\left(\frac{P(n+1)}{(n+1)}\right)$$

This was eventually proven in 2015 by DeSalvo and Pak [1] in the same paper that included the proof of the log concavity of $P(n)$.

1.4 The Power Partition Function One particular A that restricts λ to perfect k^{th} powers is denoted $P_k(n)$, and is known as the *power partition function.* Note that $P(n) = P_k(n)$ when $k = 1$, but for $k = 2$, $P_2(n)$ restricts λ to perfect squares, that is, $P_2(n)$ enumerates the number of partitions of n where the partitions are positive integer sequences $\lambda = (\lambda_1, \lambda_2, ...)$ with $\lambda_1 \geq \lambda_2 \geq ... > 0$ and $\sum_{j\geq 1} \lambda_j = n$ such that each λ is a perfect square.

For example, $P_2(4) = 2$ since

$$\begin{aligned} 4 &= 2^2 \\ &= 1^2 + 1^2 \end{aligned}$$

Similarly, $P_k(n)$ restricts λ to perfect k^{th} powers, that is, $P_k(n)$ enumerates the number of partitions of n where the partitions are positive integer sequences $\lambda = (\lambda_1, \lambda_2, ...)$ with $\lambda_1 \geq \lambda_2 \geq ... > 0$ and $\sum_{j\geq 1} \lambda_j = n$ such that each λ is a perfect k^{th} power. The first asymptotic formula for the power partition function was given in 1918 by Hardy and Ramanujan [7] using the circle method. They stated, without proof, the following asymptotic equivalence:

$$\log P_k(n) \sim (k+1)\left(\frac{1}{k}\Gamma\left(1+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right)^{\frac{k}{(k+1)}} n^{\frac{1}{(k+1)}}$$

The power partition function was further studied by Wright [9] in 1934 who produced a more precise asymptotic formula utilizing more complicated terms. Then, in 2015, R. C. Vaughan [6] gave an asymptotic formula for the case where $k = 2$. The next year, A. Gafni [2] generalized this asymptotic formula for the power partition function $P_k(n)$. It is this asymptotic formula that is utilized in this thesis and given in the second lemma. It is the purpose of this thesis to prove that the power partition function is in the class of restricted partitions having the property of log concavity.

CHAPTER 2: THEOREMS AND CONJECTURES

Theorem. *For each $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that $P_k(n)$ is log concave for all $n \geq N_k$*

Note that N_k depends on k, but for sufficiently large n, $P_k(n)$ is log concave. In their paper, DeSalvo and Pak [1] showed that $P(n)$ is log concave for all $n > 25$. We have computed the smallest N_k for which $P_k(n)$ is log concave for all $n > N_k$ for $k = 2, 3$. The smallest N_k has an interesting property that leads to the following conjecture.

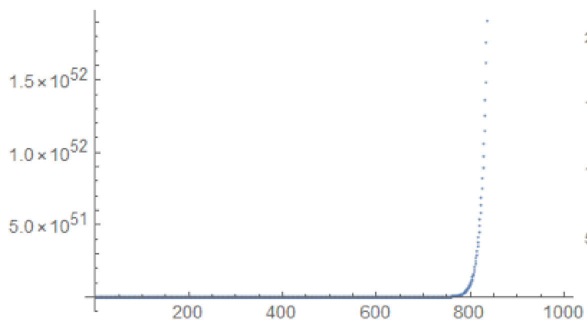

Figure 3: $N_1 = 25$

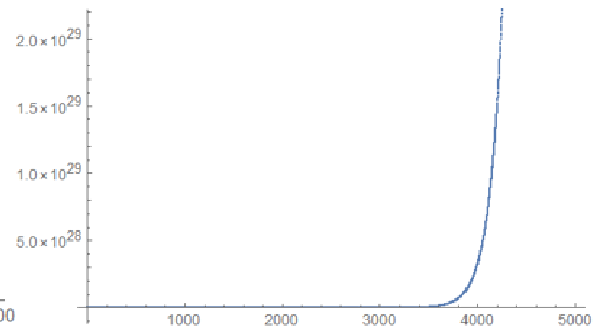

Figure 4: $N_2 = 1042$

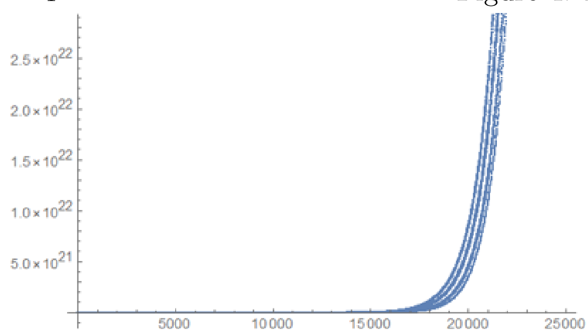

Figure 5: $N_3 = 15656$

Conjecture. *The smallest N_k for which $P_k(n)$ is log concave for all $n > N_k$ is computable for every k, for instance, $N_1 = 25$, $N_2 = 1042$, $N_3 = 15656$*

which generates the sequence $\{N_k\} = 25, 1042, 15656, ...$

Which leads to the following open question:

Question. *Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $N_k = f(k)$?*

Before beginning the proof of the main theorem of this paper, it is necessary to first consider two lemmas.

CHAPTER 3: LEMMATA

Lemma. *Suppose $f(x)$ is a positive, increasing function with two continuous derivatives for all $x > 0$, that $f'(x) > 0$ and decreasing for all $x > 0$, and that $f''(x) < 0$ and increasing for all $x > 0$. Then $f''(x-1) < f(x-1) - 2f(x) + f(x+1) < f''(x+1)$ for all $x > 1$*

This is the same lemma found in the paper by DeSalvo and Pak [1].

Lemma. *Let n be a sufficiently large natural number, and choose positive numbers X and Y satisfying*

$$n = \frac{\alpha_k}{k+1} X^{\frac{1}{k}+1} - \frac{X}{2} - \frac{1}{2}\zeta(-k) \quad \textit{and} \quad Y = \frac{\alpha_k}{2k} X^{\frac{1}{k}} - \frac{1}{4}, \tag{1}$$

where $\alpha_k := \frac{k+1}{k^2}\zeta\left(\frac{k+1}{k}\right)\Gamma\left(\frac{1}{k}\right)$. Then, for each $J \in \mathbb{N}$ there are real numbers $c_1, c_2, ..., c_J$ (independent of n), so that

$$P_k(n) = \frac{\exp\left(\alpha_k X^{\frac{1}{k}} - \frac{1}{2}\right)}{(2\pi)^{\frac{k+2}{2}} X^{\frac{3}{2}} Y^{\frac{1}{2}}} \left(\sqrt{\pi} + \sum_{i=1}^{J} \frac{c_i}{Y^i} + O\left(\frac{1}{Y^{J+1}}\right)\right). \tag{2}$$

This is the asymptotic formula given by Gafni [2] that we will use in this paper. Note that α_k is treated as a constant that only depends on k, and that the c_i terms are computable, if one is patient enough. In his paper, Gafni devises a way that one could compute each c_i up to $i = J$, however, it is an arduous task that requires computing an enormous amount of polynomials. For $J = 1$, Gafni computes 29 polynomials and obtains the first coefficient $c_1 = -\frac{\sqrt{\pi}}{24k^2}(k^2 + \frac{5}{2}k + 1)$. For our purpose, it is sufficient to know that they are indeed independent of n.

CHAPTER 4: PROOF OF THE MAIN THEOREM

From (1) we find

$$X^{\frac{1}{k}} = \frac{2k}{\alpha_k}\left(Y + \frac{1}{4}\right). \tag{3}$$

Hence from (2) and (3) one has

$$P_k(n) = \frac{A_k \exp(2kY)}{Y^{\frac{1}{2}}(Y+\frac{1}{4})^{\frac{3k}{2}}}\left(1 + \sum_{i=1}^{J}\frac{d_i}{Y^i} + O\left(\frac{1}{Y^{J+1}}\right)\right) \tag{4}$$

where, $A_k := \frac{\sqrt{\pi}(\alpha_k)^{\frac{3k}{2}}\exp\left(\frac{k-1}{2}\right)}{(2\pi)^{\frac{k+1}{2}}(2k)^{\frac{3k}{2}}}$ and $d_i := \frac{c_i}{\sqrt{\pi}}$.

Rewriting $P_k(n)$ as $P_k(n) = T_k(n)\left(1 + \frac{R_k(n)}{T_k(n)}\right)$, where

$$T_k(n) = A_k\frac{\exp(2kY)}{Y^{\frac{1}{2}}\left(Y+\frac{1}{4}\right)^{\frac{3k}{2}}}\left(1 + \sum_{j=1}^{J}\frac{d_i}{Y^i}\right) \quad \text{and} \quad R_k(n) = O_k\left(\frac{\exp(2kY)}{Y^{J+\frac{3}{2}}\left(Y+\frac{1}{4}\right)^{\frac{3k}{2}}}\right). \tag{5}$$

Now define an operator $\mathcal{T}$ by

$$\mathcal{T}(g(n)) = 2\log g(n) - \log g(n+1) - \log g(n-1). \tag{6}$$

We will prove that $\mathcal{T}(P_k(n)) \geq 0$ for all $k \in \mathbb{N}$ and for sufficiently large n, which will prove the statement of the theorem. Define $\mathcal{T}(P_k(n)) = \mathcal{T}(f(n)) + \mathcal{T}(h(n))$ where

$$f(n) := \log T_k(n) \quad \text{and} \quad h(n) := \log\left(1 + \frac{R_k(n)}{T_k(n)}\right). \tag{7}$$

Since both f and h are also function of Y, from (5), we find

$$f(n) = \log A_k + 2kY - \frac{1}{2}\log Y - \frac{3k}{2}\log\left(Y + \frac{1}{4}\right) + \log\left(1 + \sum_{i=1}^{J}\frac{d_i}{Y^i}\right). \tag{8}$$

From (1) one finds that Y increases with n. Hence $f(n) > 0$ for large n.

Differentiating Y with respect to n and from (1) and (3) one has

$$Y' = \frac{(2\alpha_k)^k}{4k^{k+1}}\frac{1}{(4Y+1)^{k-1}}. \tag{9}$$

Now differentiate both sides of (8) and from (9), we have

$$f'(n) = 2kY' - \frac{1}{2}\frac{Y'}{Y} - \frac{3k}{2}\frac{Y'}{Y+\frac{1}{4}} - \frac{\sum_{i=1}^{J} id_i \frac{Y'}{Y^{i+1}}}{1+\sum_{i=1}^{J}\frac{d_i}{Y^i}} \tag{10}$$

$$= \frac{(2\alpha_k)^k}{2k^k}\frac{1}{(4Y+1)^{k-1}} - \frac{(2\alpha_k)^k}{8k^{k+1}}\frac{1}{Y(4Y+1)^{k-1}} \tag{11}$$

$$- \frac{3.2^{k-1}\alpha_k^k}{k^k}\frac{1}{(4Y+1)^k} - \frac{(2\alpha_k)^k}{4k^{k+1}Y(4Y+1)^{k-1}}\frac{\sum_{i=1}^{J}\frac{id_i}{Y^i}}{1+\sum_{i=1}^{J}\frac{d_i}{Y^i}}. \tag{12}$$

Differentiating again one has

$$f''(n) = -(k-1)\left(\frac{2^{k-1}(\alpha_k)^k(k+1)^k}{k^k}\right)\left(\frac{4Y'}{(4Y+1)^k}\right) \tag{13}$$

$$- \frac{(k+1)^k(\alpha_k)^k 2^{k-3}}{k^{k+1}}\left(-\frac{Y'}{Y^2(4Y+1)^{k-1}} - (k-1)\frac{4Y'}{Y(4Y+1)^k}\right) \tag{14}$$

$$+ \frac{3.2^{k-1}(k+1)^k(\alpha_k)^k}{k^k}(k)\frac{4Y'}{(4Y+1)^{k+1}} - \frac{\sum_{i=1}^{J} d_i \frac{(k+1)^k(\alpha_k)^k 2^{k-2}}{k^{1+k}Y^{i+1}(4Y+1)^{k-1}}}{1+\sum_{i=1}^{J}\frac{d_i}{Y^i}}. \tag{15}$$

For sufficiently large n, one finds

$$|1+\sum_{i=1}^{J}\frac{d_i}{Y^i}| \geq \frac{1}{2}. \tag{16}$$

Use (9) in (15) and consider the large order terms of $f''(Y)$, we have

$$f''(n) = -\left(\frac{k-1}{k^{2k+1}}2^{-2k+1}(\alpha_k)^{2k}(k+1)^{2k}\right)\frac{1}{Y^{2k-1}} + O_k\left(\frac{1}{Y^{2k}}\right) \tag{17}$$

for large n. Similarly, differentiating (15) and from (9), we have

$$f'''(n) = \left(\frac{(2k-1)(k-1)}{k^{3k+2}}2^{-3k+1}(\alpha_k)^{2k}(k+1)^{2k}\right)\frac{1}{Y^{3k-1}} + O_k\left(\frac{1}{Y^{3k}}\right) \tag{18}$$

for $n \to \infty$. Hence from (8), (12), (17) and (18) we find $f(n) \geq 0$, $f'(n) \geq 0$, $f''(n) \leq 0$ and $f'''(n) \geq 0$ for large value of n. Therefore, by Lemma 1

$$-f''(n+1) \leq \mathcal{T}(f(n)) \leq -f''(n-1). \tag{19}$$

From (1), one finds that

$$X = \left(\frac{n(k+1)}{\alpha_k}\right)^{\frac{k}{k+1}}(1+o(1)). \tag{20}$$

Hence from (1)

$$Y = \frac{1}{2k}\alpha_k^{\frac{k}{k+1}}(n(k+1))^{\frac{1}{k+1}}(1+o(1)). \tag{21}$$

as $n \to \infty$. Combining (21) with (17), we have

$$f''(n) = -c_k\left(\frac{1}{n}\right)^{\frac{2k-1}{k+1}}(1+o(1)) \tag{22}$$

as $n \to \infty$, where

$$c_k = \frac{(k-1)\alpha_k^{3k}(k+1)^{2k^2+1}}{k^2}. \tag{23}$$

Hence

$$f''(n-1) = -c_k\left(\frac{1}{n}\right)^{\frac{2k-1}{k+1}}(1+o(1)) \quad \text{and} \quad f''(n+1) = -c_k\left(\frac{1}{n}\right)^{\frac{2k-1}{k+1}}(1+o(1)). \tag{24}$$

Using above in (19), we have

$$\frac{c_k}{n^{\frac{2k-1}{k+1}}}(1+o(1)) \le \mathcal{T}(f(n)) \le \frac{c_k}{n^{\frac{2k-1}{k+1}}}(1+o(1) \tag{25}$$

Next, let $z_n := \frac{R_k(n)}{T_k(n)}$. Then from (5) and (21), one has $|z_n| \ll \frac{1}{Y^{2k}} \ll \frac{1}{n^{\frac{2k}{k+1}}}$. Therefore $z_n \to 0$ for sufficiently large n. Note that $\log(1+x) \sim x$ as $x \to 0$. Hence $\log(1+z_n) \sim z_n$ as $n \to \infty$. From (7), we have

$$\mathcal{T}(h(n)) \sim 2z_n - z_{n+1} - z_{n-1}. \tag{26}$$

This gives us

$$|\mathcal{T}(h(n))| \ll \frac{1}{n^{\frac{2k}{k+1}}}. \tag{27}$$

Combining (25) and (27), one deduces

$$\frac{c_k}{n^{\frac{2k-1}{k+1}}}(1+o(1)) \le \mathcal{T}(P_k(n)) \le \frac{c_k}{n^{\frac{2k-1}{k+1}}}(1+o(1)) \tag{28}$$

for sufficiently large values of n. Thus, for given $\epsilon > 0$, we have

$$1 \le \frac{(P_k(n))^2}{(P_k(n+1)(P_k(n-1))} \le 1 + \frac{(1+\epsilon)c_k}{n^{\frac{2k-1}{k+1}}}. \tag{29}$$

This completes the proof of the theorem.

CHAPTER 5: FURTHER RESULTS

5.1 Chen's Conjecture A consequence of the result in this paper is that, not only does Chen's conjecture (mentioned in the introduction) also hold true for the power partition function for sufficiently large n, but it is also possible to determine a precise error bound for the log concavity of the power partition function. Recall that Chen originally conjectured

$$\frac{P_k(n-1)}{P_k(n)}\left(1+\frac{1}{n}\right) > \frac{P_k(n)}{P_k(n+1)}$$

where $k = 1$. This is not the best error bound on the log concavity of the partitition function and, as k increases, the error bound is still not optimal for the power partition function. As k increases, the smallest number that does not satisfy this inequality also increases. Let C_k indicate the smallest n such that the previous inequality holds true for all $C_k < n$. For instance, $C_1 = 0$, $C_2 = 107$, $C_3 = 929$, $C_4 = 3046$ which generates the increasing sequence $\{C_k\} = 0, 107, 929, 3046, ...$ Below are the graphs generated by MATHEMATICA of

$$\frac{P_k(n-1)}{P_k(n)}\left(1+\frac{1}{n}\right) - \frac{P_k(n)}{P_k(n+1)}$$

for $k = 1, 2, 3, 4$ and for $n \leq 5000$. Notice how large C_k grows as k gets large. This allows for much improvement of the error bounds of the log concavity of $P_k(n)$. Because C_k increases so rapidly as k increases, it is useful to have a more precise error bound. As a consequence of the proof of the log concavity of $P_k(n)$, the following corollary can be made.

Corollary. *For a given $\epsilon > 0$ and for sufficiently large n,*

$$\frac{P_k(n-1)}{P_k(n)}\left(1+\frac{(1+\epsilon)c_k}{n^{\frac{2k-1}{k+1}}}\right) > \frac{P_k(n)}{P_k(n+1)}$$

where

$$c_k = \frac{(k-1)\alpha_k^{3k}(k+1)^{2k^2+1}}{k^2}$$

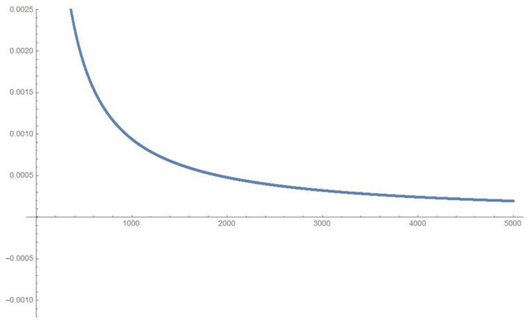

Figure 6: $C_1 = 0$

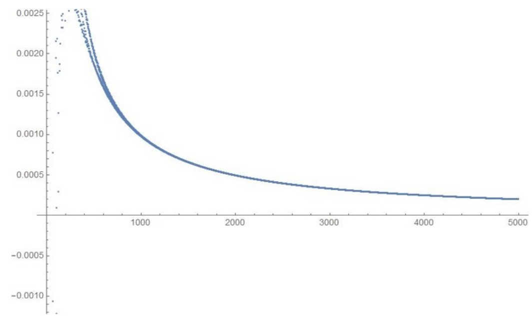

Figure 7: $C_2 = 107$

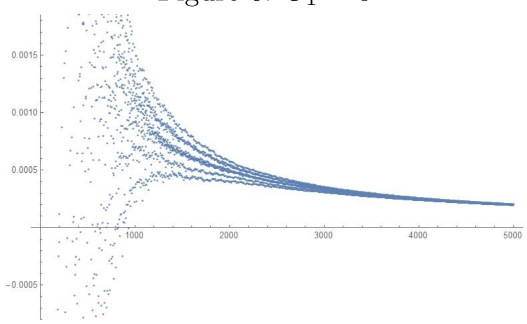

Figure 8: $C_3 = 929$

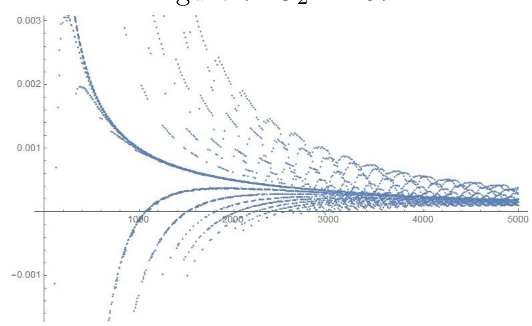

Figure 9: $C_4 = 3046$

Because C_k increases so rapidly as k increases, it is useful to have a more precise error bound. As a consequence of the proof of the log concavity of $P_k(n)$, the following corollary can be made.

Corollary. *For a given $\epsilon > 0$ and for sufficiently large n,*

$$\frac{P_k(n-1)}{P_k(n)}\left(1 + \frac{(1+\epsilon)c_k}{n^{\frac{2k-1}{k+1}}}\right) > \frac{P_k(n)}{P_k(n+1)}$$

where

$$c_k = \frac{(k-1)\alpha_k^{3k}(k+1)^{2k^2+1}}{k^2}$$

5.2 Monotonicity Another interesting property of the power partition function is that for each $n \in \mathbb{N}$ the number of ways that n can be written as the sum of perfect k^{th} powers decreases as k increases.

Definition. *A family of functions $\{f_n\}$ is monotone decreasing if for all x, $f_{n-1}(x) \leq f_n(x)$ for all x and for all n.*

Further, one can see that, for a fixed $n \in \mathbb{N}$, as n increases the number of ways that n can be written as the sum of perfect k^{th} powers decreases until there exists some K such that, for all $k > K$, the only way to represent n as the sum of perfect k^{th} powers will be $n = 1^k + 1^k + ... + 1^k$, and thus, for all $k > K$, $P_k(n) = 1$. For example,

TABLE 1. Representations of $P_k(4)$ for $k = 1, 2, ...$

$P_k(4)$	total number representations	value
$P(4)$	$(4) = 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$	$P(4) = 5$
$P_2(4)$	$(4) = 2^2 = 1^2+1^2+1^2+1^2$	$P_2(4) = 2$
$P_3(4)$	$(4) = 1^3+1^3+1^3+1^3$	$P_3(4) = 1$
$P_4(4)$	$(4) = 1^4+1^4+1^4+1^4$	$P_4(4) = 1$
$P_{k>2}(4)$	$(4) = 1^k+1^k+1^k+1^k$	$P_{k>2} = 1$

Now, considering the family of power partition functions, $\{P_k\}$, it can be seen that for all $k \in \mathbb{N}$, $P_{k-1}(n) \leq P_k(n)$ for all $n \in \mathbb{N}$. The following graphs $P_k(n)$ for $k = 1, 2, ..., 6$.

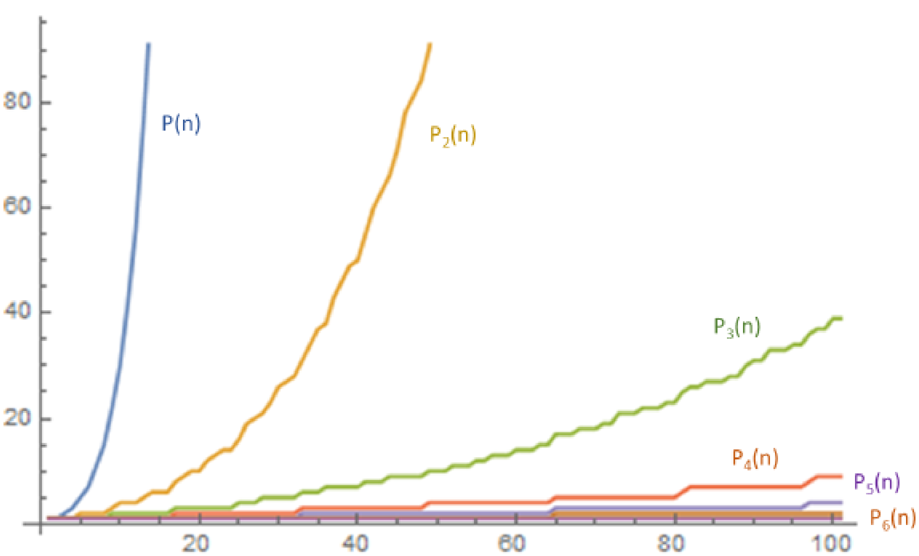

Figure 10: $P(n) \geq P_2(n) \geq P_3(n) \geq P_4(n) \geq P_5(n) \geq P_6(n)$

5.3 An Analytic Inequality In 2014, it was shown by Bessenrodt and Ono [10] that the partition function $P(n)$ satisfies the inequality $P(n)P(m) \geq P(n+m)$ for all $n, m > 1$ where $n+m > 8$ and where equality holds only for the values $\{(2,7),(2,6),(3,4)\}$. Here it is possible to utilize MATHEMATICA to test if the same property is true for $P_k(n)$, that is, if $P_k(n)P_k(m) \geq P_k(n+m)$ for all $k \in \mathbb{N}$. The following are graphs of $P_k(n)P_k(m) - P_k(n+m)$ for $k = 1, 2, ..., 5$ and for $n \leq 1000$. Notice that for most values, the graph is positive. This indicates that the inequality holds true, but the particular n and m for which it fails is still unknown.

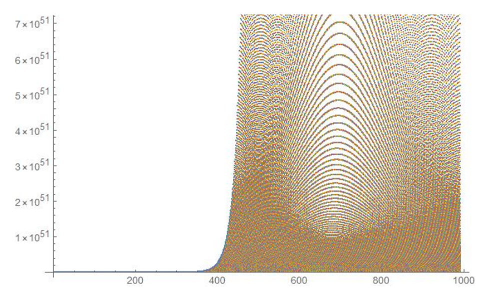

Figure 11: k=1

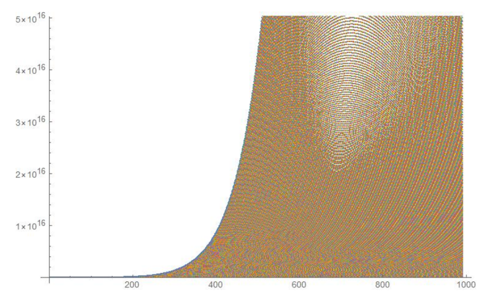

Figure 12: k=2

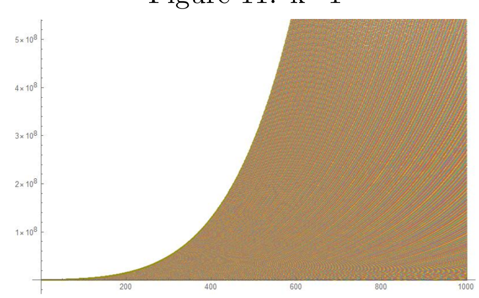

Figure 13: k=3

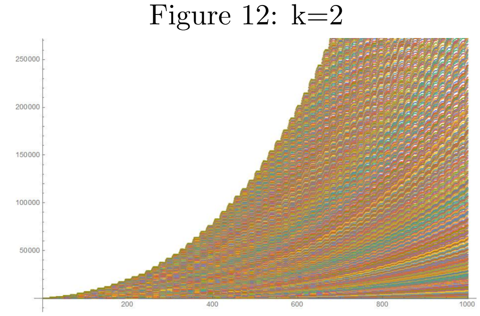

Figure 14: k=4

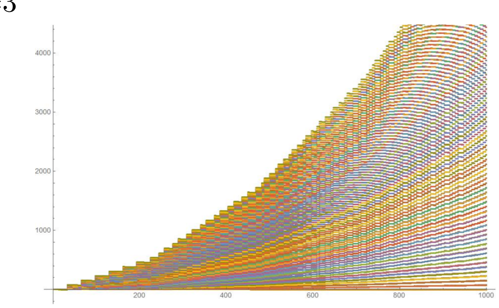

Figure 15: k=5

5.4 Sun's Conjecture As mentioned earlier, Z. W. Sun conjectured in 2013 that for $q(n) = \frac{P(n)}{n}$, the sequence $\{q(n)\}_{n\geq 31}$ is log concave, that is

$$\left(\frac{P(n)}{n}\right)^2 \geq \left(\frac{P(n-1)}{(n-1)}\right)\left(\frac{P(n+1)}{(n+1)}\right)$$

A natural question is to wonder if for $q_k(n) = \frac{P_k(n)}{n}$, the sequence $\{q_k(n)\}_{n>N}$ is log concave all k and for some $N \in \mathbb{N}$, that is

$$\left(\frac{P_k(n)}{n}\right)^2 \geq \left(\frac{P_k(n-1)}{(n-1)}\right)\left(\frac{P_k(n+1)}{(n+1)}\right)$$

It turns out that this result is very similar to the log concavity of $P(n)$. DeSalvo and Pak [1] showed that the smallest N for which $P(n)$ is log concave for all $N > n$ is 25 and that the smallest N for which $q(n)$ is log concave for all $n > N$ is 31. Something analogous happens with $q_k(n)$. Let N_k^m denote the smallest N such that for all $n > N$, $q_k^m = \frac{P_k(n)}{n^m}$ is log concave, that is,

$$\left(\frac{P_k(n)}{n^m}\right)^2 \geq \left(\frac{P_k(n-1)}{(n-1)^m}\right)\left(\frac{P_k(n+1)}{(n+1)^m}\right)$$

Then the following table can be constructed:

TABLE 2. Smallest N_k^m for $k = 1, 2, 3$ and $m = 1, ..., 6$

$P_k(n)$	m=0	m=1	m=2	m=3	m=4	m=5	m=6
$P(n)$	$N_1^0 = 25$	$N_1^1 = 31$	$N_1^2 = 42$	$N_1^3 = 50$	$N_1^4 = 66$	$N_1^5 = 86$	$N_1^5 = 116$
$P_2(n)$	$N_2^0 = 1042$	$N_2^1 = 1086$	$N_2^2 = 1150$	$N_2^3 = 1218$	$N_2^4 = 1294$	$N_2^5 = 1386$	$N_1^5 = 1631$
$P_3(n)$	$N_3^0 = 15656$	$N_3^1 = 16368$	$N_3^2 = 17160$	$N_3^3 = 18032$	$N_3^4 = 19176$		

The rows of this table are the smallest N_k^m such that, for a fixed k and for all $n > N_k^m$, q_k^m is log concave. The columns of this table are the smallest N_k^m such that, for a fixed m and for all $n > N_k^m$, q_k^m is log concave.

This generalizes Sun's conjecture and the conjecture stated previously in the introduction. When $m = 0$, Sun's conjecture is simply log concavity. When $m = 1$, this is properly Sun's conjecture. But the pattern seems to hold for higher powers of m, which leads to the following conjecture.

Conjecture. *For every power $m \in \mathbb{N}$, and for all k, there exists an N_k^m such that for all $n > N_k^m$, $q_k^m = \frac{P_k(n)}{n^m}$ is log concave, that is,*

$$\left(\frac{P_k(n)}{n^m}\right)^2 \geq \left(\frac{P_k(n-1)}{(n-1)^m}\right)\left(\frac{P_k(n+1)}{(n+1)^m}\right)$$

holds for all k and for all m.

It is interesting to note the growth of N_k^m for a fixed k or for a fixed m. Taking the value of m out to $m = 20$ for $k = 1$ yields the sequence

$$\{25, 31, 42, 50, 66, 86, 116, 152, 193, 239, 290, 346, 407, 472, 543, 618, 698, 784, 874, 968, 1068\}$$

and taking the value of m out to $m = 10$ yields the sequence

$$\{1042, 1086, 1150, 1218, 1294, 1386, 1631, 1951, 2275, 2783, 3556\}$$

Below are scatterplots of N_k^m for a fixed $k = 1, 2$ and for $m = 10$ and $m = 20$ respectively.

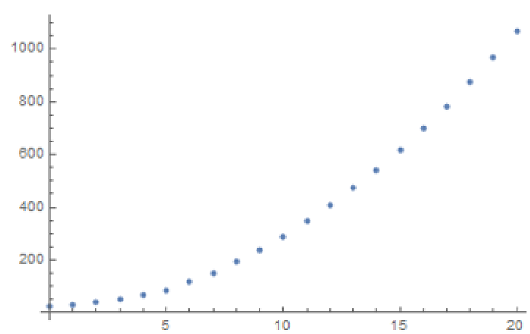

Figure 16: $N_1^0 \ldots N_1^{20}$

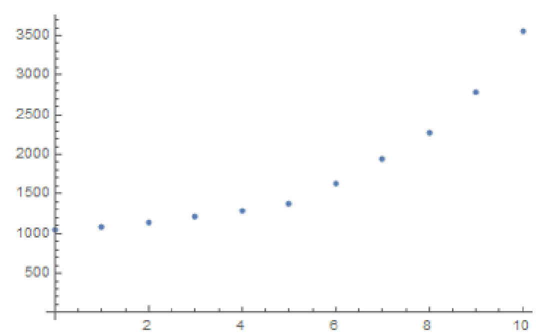

Figure 17: $N_2^0 \ldots N_2^{10}$

A natural question arises, what type of function models the growth of N_k^m. Regression analysis suggests that N_1^m and N_2^m are modeled well by polynomials. Running a regression analysis in MATHEMATICA, one can find quartic equations that fit the data with a coefficient of determination very close to 1.

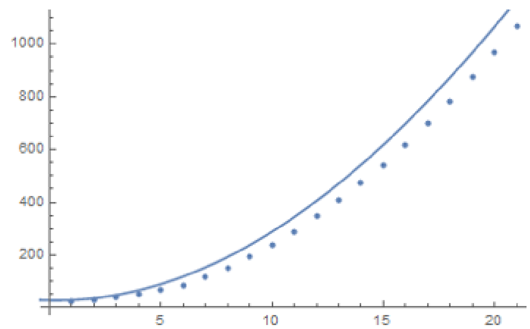

Figure 18: $R^2 \sim 0.9999635$

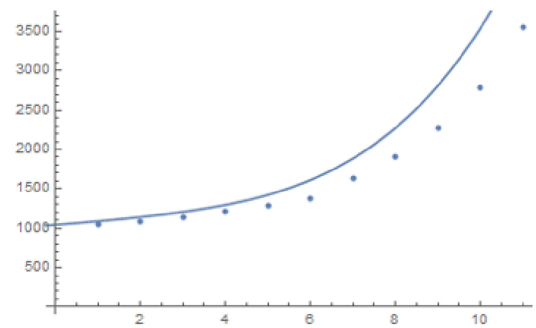

Figure 19: $R^2 \sim 0.9989421$

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