CONSISTENCY RESULTS IN THE THEORY OF CONTINUOUS FUNCTIONS AND SELECTIVE SEPARABILITY

by

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ABSTRACT

DOYEL BARMAN. Consistency results in the theory of continuous functions and selective separability. (Under the direction of DR. ALAN DOW)

We study of the notion of selective separability(SS), which was introduced by Marion Scheepers, and its connection with the game-theoretic strengthening, strategically selective separable spaces (SS$^+$). It is known that every set of countable π-weight is selectively separable, and if X is selectively separable, then all dense subsets of X are selectively separable. We know that some dense countable subsets of $2^{\mathfrak{c}}$ are selectively separable and some are not. It is also known that $C_p(X)$ is selectively separable if and only if it is separable and has countable fan tightness. Here we prove that separable Fréchet spaces are selectively separable. It is also shown that consistently the product of two separable Fréchet spaces might not be selectively separable. Also we show that adding a Sacks real can destroy the property of being selectively separable.

We introduce a notion stronger than selective separability and called it strategically selective separable or SS$^+$ and consider its properties in countable dense subsets of uncountable powers. We show that there is an SS space which fails to be SS$^+$. The motivation for studying SS$^+$ is that it is a property possessed by all separable subsets of $C_p(X)$ for each σ-compact space X. We prove that the winning strategy for countable SS$^+$ spaces can be chosen to be Markov.

We introduce the notion of being compactlike of a collection of open sets in a topological space and with the help of this notion we prove that there are two countable SS$^+$ spaces such that the union fails to be SS$^+$, which contrasts the known result about the union of SS spaces. We also prove that the product of two countable SS$^+$ spaces is again countable SS$^+$.

We prove a very interesting result which consistently contrasts our previous result, that the proper forcing axiom, PFA, implies that the product of two countable Fréchet

spaces is SS. Also we show that consistently with the negation of CH that all separable Fréchet spaces have π-weight at most ω_1.

We also worked on an open question posed by Ohta and Yamasaki in the book "Open Problems in Topology" which is, whether every C^*-embedded subset of a first countable space is C-embedded. It is known that a counterexample can be derived from the assumption $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ and that if the Product Measure Extension Theorem (PMEA) holds then the answer is affirmative in some cases . We show that in the model obtained by adding κ many random reals, where κ is a supercompact cardinal, every C^*-embedded subset of a first countable space (even with character smaller than κ) is C-embedded. The result was derived from the interesting fact that, if two ground model sets are completely separated after adding a random real, then they were completely separated originally.

The dissertation is divided as follows. The first chapter contains the topological properties of selectively separable spaces. The second chapter contains all the results we obtained about SS$^+$ spaces. The third chapter is devoted to the theorems involving CH and forcing extensions. The final chapter contains the results we obtained in the random real model about the C-embedding and C^*-embedding properties.

Any topological term not defined explicitly should be understood as in [1]. The corresponding remark applies to set theoretic notions and [2].

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CHAPTER 1: ON SELECTIVE SEPARABILITY

1.1 Introduction

In this dissertation we focus on the study of the topological properties of the selectively separable spaces which was introduced by Marion Scheepers. Particularly interesting is the naturalness of the SS notion in the function spaces with the pointwise convergent topology, namely $Cp(X)$ for metric spaces X. For a space X, $Cp(X)$ is the subspace of $\mathbb{R}^X$ consisting of the real valued continuous functions on X (i.e. C(X) with the topology of pointwise convergence).

We will let $C_p(X, 2)$ be the subspace of $C_p(X)$ consisting of the 2-valued functions. Since such spaces are dense in the product space 2^X, it is also natural to consider other countable dense subsets of such powers.

Many interesting results and questions were presented in the paper [9] and we consider some of them here. We show that every separable Fréchet space is SS. We prove that there is a dense subspace of 2^{ω_1} which is SS. We exploit the connections found in [3,9] between the Menger Property of a space X and selective separability of $C_p(X)$.

We recall the definitions of some well-known concepts from set theory and topology.

We denote ω as the set of natural numbers. A space X has *countable fan tightness* if, for any $x \in X$ and any sequence $\{A_n : n \in \omega\}$ of subsets of X such that $x \in \bigcap_{n \in \omega} \overline{A_n}$, we can choose a finite set $B_n \subset A_n$ for each $n \in \omega$ in such a way that $x \in \overline{\bigcup\{B_n : n \in \omega\}}$. A space X has *countable tightness* (which is denoted by $t(X) \leqslant \omega$) if for any $x \in X$ and $A \subset X$ if $x \in \overline{A}$, then there is a countable set $B \subset A$ such that $x \in \overline{B}$. A space is *scattered* if every non-empty subspace of X has an isolated point.

Definition 1.1. A *partial order* is a pair $\langle \mathbb{P}, \leqslant \rangle$ such that $\mathbb{P} \neq \emptyset$ and $\leqslant$ is a relation on $\mathbb{P}$ which is transitive and reflexive $(\forall p \in \mathbb{P}(p \leqslant p))$. $p \leqslant q$ usually referred as "p extends q".

Definition 1.2. Let $\langle \mathbb{P}, \leqslant \rangle$ be a partial order. A *chain* in $\mathbb{P}$ is a set $C \subset \mathbb{P}$ such that $\forall p, q \in C(p \leqslant q \vee q \leqslant p)$. p and q are *compatible* if and only if $\exists r \in \mathbb{P}(r \leqslant p \wedge r \leqslant q)$; they are *incompatible* $(p \perp q)$ if and only if $\not\exists r \in \mathbb{P}(r \leqslant p \wedge r \leqslant q)$. An *antichain* in $\mathbb{P}$ is a subset $A \subset \mathbb{P}$ such that $\forall p, q \in A(p \neq q \to p \perp q)$.

A partial order $\langle \mathbb{P}, \leqslant \rangle$ has the *countable chain condition (c.c.c.)* if and only if every antichain in $\mathbb{P}$ is countable.

Definition 1.3. If $\langle \mathbb{P}, \leqslant \rangle$ is a partial order, $D \subset \mathbb{P}$ is *dense* in $\mathbb{P}$ if and only if for all $p \in \mathbb{P}$, there exists a $q \leqslant p$ such that $q \in D$. Now a set $G \subset \mathbb{P}$ is a *filter* in $\mathbb{P}$ if and only if the following conditions are satisfied:

1. $\forall p, q \in G, \exists r \in G \; (r \leqslant p \wedge r \leqslant q)$, and

2. $\forall p \in G, \forall q \in \mathbb{P} \; (q \leqslant p \to q \in G)$.

The Martin's Axiom or $\mathsf{MA}(\kappa)$ is the statement: whenever $\langle \mathbb{P}, \leqslant \rangle$ is a non-empty partial order satisfying the countable chain conditions and $\mathbb{D}$ is a family of $\leqslant \kappa$ dense subsets of $\mathbb{P}$, then there is a filter G in $\mathbb{P}$ such that $\forall D \in \mathbb{D} \; (G \cap D \neq \emptyset)$.

The assumption $\mathsf{MA}_{\mathrm{ctble}}$ is the statement that the well-known statement of Martin's Axiom holds for countable posets (rather than necessarily all ccc posets). This is equivalent to the statement that the real line can not be covered by a family of fewer than $\mathfrak{c}$ many nowhere dense sets and is known to imply that the dominating number $\mathfrak{d}$ is $\mathfrak{c}$. The bounding number $\mathfrak{b}$ is the minimum cardinality of a subset of ω^ω which has no mod finite upper bound. The pseudointersection number, $\mathfrak{p}$, is the minimum cardinality of a free filter base on ω for which there is no infinite set which is mod finite contained in each member of the filter.

For several of our results we require extra set-theoretic hypotheses. Using $\mathsf{MA}_{\mathrm{ctble}}$ we establish that the product of two countable SS spaces may not be SS and that there

is a maximal regular SS space. We seem to require CH to prove that the product of two countable Fréchet spaces may not be SS.

1.2   Selective Separability

Let us start with the definition of selective separability of a topological space X.

Definition 1.4. [3] A space X is called *selectively separable (or SS)* if for each sequence $\{D_n\}_n$ of dense sets, there is a selection $\{E_n \in [D_n]^{<\omega}\}_{n \in \omega}$ with dense union.

Now we have some results concerning π-weight of a space and its relation with selective separability. Before citing those results let us recall the definition.

Definition 1.5. Let (X, τ) be a topological space. A family $\zeta \subset \tau$ is a π-*base* of X if for each $U \in \tau$, there is a $B \in \zeta$ such that $B \subset U$.

The cardinal $\pi w(X)$ (called the π-weight of X) is the minimal cardinality of a π-base of the space X. The following results are already known.

Proposition 1.1. [9] Assume that X is selectively separable. Then

1. every dense subspace of X is selectively separable and hence separable;

2. every open subspace of X is selectively separable;

3. every open continuous image of X is selectively separable;

4. every closed irreducible continuous image of X is selectively separable.

Proof.   Properties 1 and 2 are straightforward from the definition. Now, if a continuous onto map $f : X \to Y$ is either open or closed irreducible, then for any sequence $\{E_n : n \in \omega\}$ of dense subsets of Y, the set $D_n = f^{-1}(E_n)$ is dense in X, so we can choose finite $F_n \subset D_n$ such that $\bigcup_n F_n$ is dense in X. Then $G_n = f(F_n)$ is finite subset of E_n for every $n \in \omega$ and $\bigcup_{n \in \omega} G_n$ is dense in Y, which proves 3 and 4.   □

Proposition 1.2. [9] Each space with countable π-*weight* is selectively separable.

Proof.   Suppose that $\pi \omega(X) = \omega$ and fix a π-base $\{B_n : n \in \omega\}$ in X. If $\{D_n : n \in \omega\}$ is a sequence of dense subsets of X, then we can choose a point $x_n \in D_n \cap B_n$ for each $n \in \omega$; now $\{x_n : n \in \omega\}$ is dense in X, so X is selectively separable.   □

Proposition 1.3. [3] Each countable space with π-weight $< \mathfrak{d}$ is selectively separable.

Proof. Let us fix a sequence of indexed dense sets $\{D_n = \{d(n, l) : l \in \omega\} : n \in \omega\}$.

Fix a π-base $\mathcal{U}$ of cardinality less than $\mathfrak{d}$. For each $U \in \mathcal{U}$ there is a function $f_U \in \omega^\omega$

satisfying, for each $n \in \omega$, $U \bigcap \{d(n, \ell) : \ell < f_U(n)\} \neq \emptyset$. Since $\mid \mathcal{U} \mid < \mathfrak{d}$, there is

a function $g \in \omega^\omega$ such that $f_U <^* g$ for all U. Now let $E_n = \{d(n, l) : l < g(n)\}$.

Then it is easy to check that, for each $U \in \mathcal{U}$, $U \cap E_n \neq \emptyset$ for all such n such that

$f_U(n) < g(n)$, and so $U \cap \bigcup_n E_n$ is not empty. □

A space is said to be *crowded* if it has no isolated points. For convenience we will

often assume that the spaces under discussion are crowded. Spaces which are not

crowded are easily handled by the following observation.

Lemma 1.4. A space X is SS if and only if the set I of isolated points is countable

and $X \setminus \overline{I}$ is SS.

We recall, and generalize, the notion of countable fan-tightness.

Definition 1.6. A space X has *countable (dense) fan-tightness* at $x \in X$, if for each

sequence (of dense sets) $\{Y_n\}_n$ with $x \in \bigcap_n \overline{Y}_n$, there is a selection $\{W_n \in [Y_n]^{<\omega} :$

$n \in \omega\}$ such that $x \in \overline{\bigcup_n W_n}$. A space X has countable (dense) fan-tightness if it has

countable (dense) fan-tightness at each point $x \in X$.

It is immediate that each SS space has countable dense fan-tightness, but it is useful

to make note of the partial converse.

Lemma 1.5. For a space X, the following conditions are equivalent:

1. X is SS,

2. X is separable and has countable dense fan-tightness,

3. X has countable dense fan-tightness at each point of some countable dense
   subset.

Proof. It suffices to prove that condition 3 implies condition 1. We may assume

that the space X is crowded. Let $\{A_n : n \in \omega\}$ be a partition of ω into infinite

sets. Let $D = \{d_n : n \in \omega\}$ be a dense subset of X such that X has countable

dense fan-tightness at each $d \in D$. Let $\{D_n : n \in \omega\}$ be a sequence of dense subsets

of X. Now for each $n \in \omega$ and for each $k \in A_n$ we use countable fan-tightness to select $\{F_k^n : n \in \omega, k \in A_n\}$ so that $d_n \in \overline{\bigcup_{k \in A_n} F_k^n}$. Since $X = \overline{\{d_n : n \in \omega\}}$ and $\{d_n : n \in \omega\} \subseteq \overline{\bigcup_n \bigcup_{k \in A_n} F_k^n}$, we have $X = \overline{\bigcup_n \bigcup_{k \in A_n} F_k^n}$. $\qquad\square$

One of our main results shows the surprising connection between the Fréchet property and selective separability. Let us recall the definition of a Fréchet Space:

Definition 1.7. A space is called *Fréchet* if it is the case that a point is in the closure of a subset of X iff there is a sequence from the set converging to that point.

Theorem 1.6. Each separable Fréchet space is selectively separable.

Proof. We may assume that the space X is crowded. Let D be the postulated countable dense subset of X and let $d \in D$. By Lemma 1.5, it suffices to show that X has countable dense fan-tightness at (each point) d (of D). Since $d \in \overline{X \setminus \{d\}}$ there exists a sequence $\{d_n : n \in \omega\} \subseteq X \setminus \{d\}$ which converges to d. Let $\{D_n : n \in \omega\}$ be a family of dense subsets of X. If we replace each D_n by $\bigcup_{k \geqslant n} D_k$, we may assume that the sequence $\{D_n : n \in \omega\}$ is descending. For all n, $d_n \in \overline{D_n}$, which implies that there exists a sequence $S_n \subseteq D_n$ which converges to d_n. Now we observe that $d \in \overline{\bigcup_n S_n}$ and, therefore, we can select a sequence $S_d \subseteq \bigcup_n S_n$ such that $S_d \to d$. Now for all $n \in \omega$, $S_d \cap S_n$ is finite since S_d and S_n converge to distinct points. Let $F_n = S_d \cap S_n$, which is a finite subset of D_n. Now $S_d = \bigcup_n F_n$ and $d \in \overline{S_d}$, which implies $d \in \overline{\bigcup_n F_n}$. Therefore, by Lemma 1.5, X is selectively separable. $\qquad\square$

We present the following example because it seems to us to be a very natural example of a countable space with minimal π-weight (namely $\mathfrak{d}$) which fails to be selectively separable. An example using $C_p(X)$ theory was given in [9].

Example 1.1. Consider the box topology on the countable power $(\omega+1)^\omega$ where $\omega+1$ is the usual compact ordinal topology. Let $S = \{f \in \square(\omega+1)^\omega : (\exists n)\ f(k) = \omega$ iff $k \geqslant n\}$.

Let $D_n = \{f \in S : f(k) \neq \omega \forall k \leqslant n\}$. It is easily seen that D_n is a dense subset of S which is moreover open. We will show that the sequence $\{D_n : n \in \omega\}$ is a witness to

the fact that S is not SS. Assume that $F_n \in [D_n]^{<\omega}$ for each $n \in \omega$. Define a function $h \in \omega^\omega$ so that $f(n) < h(n)$ for each $f \in F_n$. Now the basic open set $\Pi_{k \in \omega}[h(k), \omega]$ in $\square(\omega + 1)^\omega$ does meet S but it is clearly disjoint from $\bigcup_n F_n$. Therefore S is not selectively separable.

To show $\pi w(S) = \mathfrak{d}$, let $\mathcal{D} \subset \omega^\omega$ be a dominating family of functions of cardinality $\mathfrak{d}$. Then basic open sets are of the form: $W(t, f) = \Pi_{i < dom(t)}\{t(i)\} \times \Pi_{i \geqslant dom(t)}[f(i), \omega]$, $t \in \omega^{<\omega}$ and $f \in \omega^\omega$. For any open $U(s, g)$, where $s \in \omega^{<\omega}$ and $g \in \omega^\omega$, we can take $W(s, f) \subset U(s, g)$ where f dominates g , $f \in \mathcal{D}$. Let $\kappa < \mathfrak{d}$, then for $\{f_\alpha : \alpha < \kappa\}$, $\exists g$ such that $|\{n : f_\alpha(n) < g(n)\}| = \omega$. Then $U_\alpha = U(\emptyset, f_\alpha) \not\subset W(\emptyset, g)$, which shows that $\{U_\alpha : \alpha < \kappa\}, \kappa < \mathfrak{d}$ is not a π-base. Therefore $\pi\omega(S) = \mathfrak{d}$.

The elegant and natural connections between properties of a space X and the selective separability of its function space $C_p(X)$ was discovered in [3] and explored further in [9]. The connection is the Menger Property.

Definition 1.8. A space X has the *Menger Property (or is Menger)* if for each sequence $\{U_n\}_n$ of open covers, there is a selection $\{W_n \in [U_n]^{<\omega}\}_n$ such that $\bigcup_n (\bigcup W_n)$ is a cover.

For example, any σ-compact space, such as $\mathbb{R}$ or 2^ω, has the Menger Property but it is known that $\omega^\omega \approx \mathbb{R} \setminus \mathbb{Q}$ does not.

Theorem 1.7. [3,9] For a space X, $C_p(X)$ is selectively separable if and only if $C_p(X)$ is separable and X^n is Menger for each $n \in \omega$.

The following theorem is due to Arhangelskii.

Theorem 1.8. [5] X^n is Menger for each n if and only if $C_p(X)$ has countable fan tightness.

We shall need one direction of the above result, so we include a proof for the reader's convenience.

Proposition 1.9. If a space X has the property that X^n is Menger for each n, then $C_p(X)$ has countable fan tightness.

Proof. Since $C_p(X)$ is homogeneous, it suffices to show that $C_p(X)$ has countable fan-tightness at the constant zero function $\underline{0}$. Let $\{D_n\}_n$ be the sequence of sets each with the constant $\underline{0}$ function as a $C_p(X)$-limit. For each n, let U_n be the collection of open sets $\{(d^{-1}(-\frac{1}{n}, \frac{1}{n}))^k : d \in D_n, k \leqslant n\}$. We show that U_n contains an open cover of X^k for each $k \leqslant n$. Fix any $k \leqslant n$ and $\langle x_i \rangle_{i<k} \in X^k$. Since $\underline{0}$ is a limit of D_n, there exists a $d \in D_n$ such that $d(x_i) \in (-\frac{1}{n}, \frac{1}{n})$ for each $i < k$. This, in turn, means that $\langle x_i \rangle_{i<k} \in (d^{-1}(-\frac{1}{n}, \frac{1}{n}))^k$ which is a member of U_n. Thus it follows that U_n contains an open cover of X^k. Applying the Menger Property (for X^k for each k and open covers $\{U_n : k \leqslant n \in \omega\}$) we can select $E_n \in [D_n]^{<\omega}$ for each n so that the finite subcollection W_n, of U_n we get from the elements $d \in E_n$ yields a cover of each X^k. In fact, we can, and do, ensure that for each $k < n$, the collection $\bigcup_{n \leqslant m} W_m$ contains a cover of X^k. To show that $\underline{0}$ is a limit of $\bigcup_n E_n$, let us fix any k, $\{x_i : i < k\} \subset X$ and $n \geqslant k$. Now we need an $e \in \bigcup_{n \leqslant m} E_m$ such that $e(x_i) \in (-\frac{1}{n}, \frac{1}{n})$ for $i < k$. Since $\langle x_i \rangle_{i<k}$ is covered by the collection $\bigcup_{n \leqslant m} W_m$, we get one such e in $\bigcup_{n \leqslant m} E_n$. $\square$

These next results, also from [9], reveal some of the interesting behavior of SS in products and subspaces.

Corollary 1.10. $2^{\mathfrak{c}}$ has a dense selectively separable subspace, namely $C_p(2^\omega, 2)$.

Proof. Countable fan-tightness is easily seen to be hereditary and $C_p(2^\omega, 2)$ is separable. Therefore it is SS. It is well-known that $C_p(2^\omega, 2)$ is dense in 2^{2^ω}. $\square$

Similarly we have the existence of a countable dense non-SS subspace.

Corollary 1.11. $2^{\mathfrak{c}}$ has a countable dense non-selectively separable subspace, namely $C_p(\omega^\omega, 2)$.

Let us mention here that G. Gruenhage [14] has established the non-trivial fact that a finite union of SS spaces is again SS. On the other hand, it is interesting to note that the union of the two countable dense subsets of the product space $2^{\mathfrak{c}}$, namely $C_p(2^\omega, 2)$ and $C_p(\omega^\omega, 2)$, results in a countable space which is not SS and yet which has a dense SS subset. Certainly a countable discrete space is SS, hence the continuous image of

an SS space need not be SS. A more revealing example of this is to consider a dense copy, X, of the irrationals in 2^ω, and to observe that $\{f \restriction X : f \in C_p(2^\omega, 2)\}$ is a continuous image (by the projection map from 2^{2^ω} onto 2^X) of the SS space $C_p(2^\omega, 2)$ which is itself not SS . Similarly, as noted in [9], the non-SS space $C_p(\omega^\omega, 2)$ has a countable dense SS subspace consisting of those functions which are continous with respect to a coarser (compact Hausdorff) topology on ω^ω.

The following result was shown to hold for countable π-weight in [9].

Theorem 1.12. If X and Y are both countable, selectively separable and $\pi w(Y) < \mathfrak{b}$, then $X \times Y$ is selectively separable.

Proof. Let $\{\mathcal{B}_\alpha : \alpha < \kappa\}$ where $\kappa < \mathfrak{b}$ be a π-base for Y. Let $\{D_k = \{d_{k,m} : m \in \omega\} : k \in \omega\}$ be a countable sequence of dense subsets of $X \times Y$. Let π_x and π_y be the natural projection onto the spaces X and Y respectively. For each $\alpha < \kappa$, the set $G_k^\alpha = \pi_x[D_k \cap (X \times B_\alpha)]$ is dense in X. Since X is selectively separable, there is a selection $F_k^\alpha \subseteq D_k$ ($k \in \omega$) so that $\pi_x[F_k^\alpha] \subseteq G_k^\alpha$ and $\overline{\bigcup \pi_x[F_k^\alpha]} = X$. Since F_k^α is finite, $\exists f_\alpha(k) \in \omega$ so that $F_k^\alpha \subseteq \{d_{k,m} : m < f_\alpha(k)\}$. Therefore we have a sequence $\{f_\alpha : \alpha < \kappa\}$ where $f_\alpha : \omega \to \omega$. Since $\kappa < \mathfrak{b}$, there exists a function $f \in \omega^\omega$ such that $\forall \alpha < \kappa$, $f_\alpha <^* f$. Let us define $F_k = \{d_{k,m} : m < f(k)\} \subset D_k$. We claim that $\overline{\bigcup_{k \in \omega} F_k} = X \times Y$. Let us choose a basic open set $U \times B_\alpha$ of $X \times Y$, then $\exists l \in \omega$ such that $\forall i > l, f(i) > f_\alpha(i)$. Since $U \cap \bigcup \pi_x[F_k^\alpha] \neq \emptyset$, there exists a $z \in F_k$ such that $\pi_x(z) \in U \cap \bigcup \pi_x[F_k^\alpha]$, which implies that $z \in F_k \cap (U \times B_\alpha)$. Therefore $\bigcup F_k$ is dense in $X \times Y$. $\square$

It is established in [9] that it is independent of the usual axioms that 2^{ω_1} has a dense non-selectively separable subspace. On the other hand, the following result answers a natural question posed in [9]. One should recall, as noted above, that one cannot conclude that the projection of an SS subspace of $2^{\mathfrak{c}}$ will remain SS.

Theorem 1.13. 2^{ω_1} does have a dense SS subspace.

Proof. If $\mathfrak{b} > \omega_1$ then every countable subset of 2^{ω_1} is selectively separable. Otherwise,

let $Y = \{f_\alpha : \alpha \in \mathfrak{b}\} \subset \omega^\omega$ be $<^*$-unbounded family of increasing functions. Let $Q = \{q \in (\omega + 1)^\omega : q$ is monotone and is eventually equal to $\omega\}$.

Now we make use the following result from [10].

Proposition 1.14. $X = Q \cup Y$ has all the finite powers Menger.

Again for the reader's convenience, we include the proof. Let us define basic open sets in the product space $(\omega + 1)^\omega$:

$$[s : n] = \{f \in X : s \subset f \text{ and } f(|s|) > n\} \subset (\omega + 1)^\omega$$

where $s \in \omega^{<\omega}$ and $n \in \omega$.

We prove by induction on m that X^m is Menger. Given a sequence $\langle U_n \rangle_n$ of open covers of X^m by basic open sets, we may assume that each basic open subset of a member of U_n is also in U_n. Now let us define $g(\ell)$ by recursion on ℓ. Given $l = mk^3$ and i such that $l + i < m(k + 1)^3$, so that $g(l + i)(i < mk)$ has been defined, set $n = g(l + i + 1)$ large enough so that for each sequence $\{s_j : j < m\} \subset (g(l + i)^{<k})$ the set

$$[s_0 : n] \times [s_1 : n] \times \ldots \times [s_{m-1} : n] \in U_k$$

(and is added to W_k). Such a value for n exists, since we are just asking for a member of U_k which contains the point $\langle x_j : j < m \rangle$ where for $j < m$, x_j is the unique member of Q extending s_j such that $x_j(|s_j|) = \omega$. Let $\bar{g}(k) = g(m(k + 1)^3)$ for each k. Let us assume that $f_{\alpha_0} \not<^* \bar{g}$ and let $\alpha_0 \leqslant \alpha_1 \leqslant \ldots \leqslant \alpha_{m-1} < \mathfrak{b}$.

Now fix any k so that

$$g(mk^3) < \bar{g}(k) < f_{\alpha_0}(k) \leqslant \ldots \leqslant f_{\alpha_{m-1}}(k).$$

For each $i < mk$ and $j < m$, there is a minimal $s_j^i \subset f_{\alpha_j} \upharpoonright k$ such that $f_{\alpha_j}(|s_j^i|) \geqslant n_i = (mk^3 + i)$. It follows that

$$\{s_j^i : j < m\} \subset (g(mk^3 + i))^{<k}$$

and that

$$[s_0^i : n_{i+1}] \times [s_1^i : n_{i+1}] \times \ldots \times [s_{m-1}^i : n_{i+1}] \in W_k.$$

Given such an i, if $\langle f_{\alpha_j} \rangle_{j<m}$ is not in $[s_0^i : n_{i+1}] \times [s_1^i : n_{i+1}] \times \ldots \times [s_{m-1}^i : n_{i+1}]$, then

for some $j < m$, the domain of s_j^{i+1} is strictly bigger than the domain of s_j^i. As this can only happen at most mk times, there is an $i < mk$ such that

$$\langle f_{\alpha_j} \rangle_{j<m} \in [s_0^i : n_{i+1}] \times [s_1^i : n_{i+1}] \times \ldots \times [s_{m-1}^i : n_{i+1}] \in W_k$$

The same argument shows that any rearrangement of the order of the elements in $\langle f_{\alpha_j} \rangle \in X^m$ will be covered by a member of this choice for W_k.

If follows therefore that we are able to choose the sequence $\langle W_n \rangle_n$ to be a cover of $(X \setminus \{f_\beta : \beta < \alpha_0\})^m$. It is rather immediate that a Lindelof space which is the union of fewer than $\mathfrak{b}$ many Menger subspaces is again Menger. Therefore if follows by induction on m, that the complement in X^m of $(X \setminus \{f_\beta : \beta < \alpha_0\})^m$ is Menger. This completes the proof that X^m is Menger.

Thus $C_p(X, 2)$ is a selectively separable subspace of $2^\mathfrak{b}$. $\qquad\square$

CHAPTER 2: ON SS$^+$ SPACES

2.1   Introduction

While studying selective separability, we were interested to explore the game theo-retic strengthening, strategically selective separable spaces, namely SS$^+$ spaces. The motivation for studying SS$^+$ is that it is a property possessed by all separable subsets of $C_p(X)$ for each σ-compact space X.

Let us begin by the definition,

Definition 2.1. A space has the property SS^+, if player II has a winning strategy for the natural game: player I picks a dense set D_n; player II picks a finite set $E_n \subset D_n$. Player II wins if $\bigcup_n E_n$ is dense.

Dr. Gary Gruenhage posed the question whether player II would always have a Markov strategy in each SS$^+$ space. A strategy is Markov if it only depends on which move it is and the other player's previous move.

Since SS seems to have arisen in the study of fan tightness in the spaces of the form $C_p(X)$, it is natural to introduce the idea of strategic fan tightness. We observe that if a space X is σ-compact, then $C_p(X)$ has strategic fan tightness, so all the separable subsets will be SS$^+$. Pursuing the duality between the properties of a space X and the base properties of $C_p(X)$, we introduce the idea of a collection of open subsets from a space being compactlike and notice that the property of being SS$^+$ is not finitely additive while it is productive in case of countable spaces. In pursuit of an answer to Gruenhage's question, we are able to show that if an SS$^+$-space is countable, then it has a Markov strategy for being SS$^+$.

2.2   On Strategic Fan Tightness and SS$^+$

Let us start this section by recalling the definition of countable fan tightness of a topological space X.

Definition 2.2. [3] A space S has *countable fan tightness* at x if for each sequence $\langle A_n : n \in \omega \rangle$ of subsets of S each with x in the closure, then there is a sequence of finite sets $\langle a_n : n \in \omega \rangle \in \Pi_n[A_n]^{<\omega}$ such that x is in the closure of $\bigcup_n a_n$.

We let *countable dense fan tightness* refer to the property we get by restricting each A_n to be dense. Using this definition for countable fan tightness we introduce the natural game, namely, strategic fan tightness at a point defined as:

Definition 2.3. A space S has *strategic fan tightness* at a point $x \in S$ if Player II has winning strategy for the following game:

- Player I plays A_n with $x \in \overline{A}_n$.

- Player II selects $a_n \in [A_n]^{<\omega}$.

- Player II wins if $x \in \overline{\bigcup_n a_n}$

This definition leads to the following immediate Lemma.

Lemma 2.1. A space S is SS (SS$^+$) if it is separable and has (strategic) countable dense fan tightness at each point.

Clearly each SS$^+$ space is separable and SS.

Lemma 2.2. Any crowded SS$^+$ space has an uncountable almost disjoint family of dense subsets; thus no maximal space is SS$^+$.

Proof.   Let $\mathcal{D}$ be the collection of all dense subsets of a space which is SS$^+$ and assume, for convenience, that ω is a dense subset. Fix a winning stragegy σ for Player II; i.e. σ is a function with domain $\bigcup_n \mathcal{D}^n$, and for $\vec{D} \in dom(\sigma)$ and $D \in \mathcal{D}$, $\sigma(\vec{D}, D) \in [D]^{<\omega}$. Since we are assuming that the space is crowded, it follows that $D_k = \omega \setminus k$ is dense for any $k \in \omega$. Let $J_0 = 0$ and, recursively define for $n \in \omega$,

$$J_{n+1} = 1 + \max \bigcup \{ \sigma(\langle D_{k_i} \rangle_{i<m}) : k_0 < \cdots < k_m, \text{ for } m, k_m < J_n \} \ .$$

Notice that if I is any infinite subset of ω, then $D_I = \bigcup_{n \in I}[J_n, J_{n+1})$ contains the union of the responses of σ for the run of the game given by the dense sets $\{D_{J_n} : n \in I\}$. Therefore D_I is dense. □

We record the following observation.

Proposition 2.3. Each space with countable π-*weight* is SS$^+$.

The following theorem shows an important result about SS and SS$^+$.

Theorem 2.4. SS does not imply SS$^+$.

Proof. We consider the space $X = \mathbb{Q} \cup \{f_\alpha : \alpha < \mathfrak{b}\} \subset (\omega + 1)^\omega$ from Proposition 1.14. Let $S = \{f \mid_X : f \in C_p((\omega + 1)^\omega, 2)\}$ which is SS. We claim that S is not SS$^+$. To prove this, let $\mathcal{D}$ be the collection of all dense subsets of X. Assume that σ is a strategy for Player II, i.e. for each sequence $\{D_0, D_1, \ldots, D_n\} \subset \mathcal{D}$, $\sigma(D_0, D_1, \ldots, D_n) \in [D_n]^{<\omega}$. Let us fix a sequence $\vec{D} = \{D_0, D_1, \ldots, D_{n-1}\} \subset \mathcal{D}$ and set $K_{\vec{D}} = \bigcap_{D \in \mathcal{D}} \left(\overline{\bigcup_{d \in \sigma(\vec{D}, D)} d^{-1}(1)} \right)$ where the closure is taken in $(\omega + 1)^\omega$.

For each $y \in (\omega + 1)^\omega \setminus X$ the set $D_y = \{d \in S : y \notin \overline{d^{-1}(1)}\}$ is dense in S. To see this let us fix any finite partial function s from X into 2 which defines a basic clopen set $[s] = \{d \in S : s \subset d\}$. Since $C_p((\omega + 1)^\omega, 2)$ is dense in $2^{(\omega+1)^\omega}$, there is an $f \in C_p((\omega + 1)^\omega, 2)$ such that $s \subset f$ and $f(y) = 0$. It follows that $d = f \mid_X$ is in D_y, hence $[s] \cap D_y$ is non-empty. The set $K_{\vec{D}}$ is a compact subset of X, hence it is countable.

Now let us fix a countable $M \prec H_\theta$ for $\theta = 2^{\mathfrak{c}^+}$ such that σ, X, and $\mathcal{D}$ are in M. Let $x \in X \setminus M$. If $\vec{D} \in M \cap [\mathcal{D}]^{<\omega}$ then, since it is countable, $K_D \subset X \cap M$. Therefore there is some $D' \in \mathcal{D}$ such that $x \notin \bigcup_{d \in \sigma(\vec{D}, D')} d^{-1}(1)$. Since $\sigma(\vec{D}, D')$ is simply a finite subset of D, there is some $D^* \in \mathcal{D} \cap M$ so that $\sigma(\vec{D}, D^*) = \sigma(\vec{D}, D')$. From this it follows that we may inductively choose a play of the game $\{D_0, D_1, \ldots\} \subset \mathcal{D} \cap M$ such that for all $\vec{D}_n = \{D_0, D_1, \ldots, D_n\}, x \notin \bigcup_{d \in \sigma(\vec{D}_n)} d^{-1}(1)$. Now we have ensured that if $d \in E_n = \sigma(\vec{D}_n)$ for any n, then $d(x) = 0$. Then $\bigcup_n E_n$ is not dense in S, since it misses the clopen set $\{d \in S : d(x) = 1\}$. Therefore σ is not a winning strategy,

which shows that S is not SS^+. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

It is natural to ask which dense subsets of 2^{ω_1} are SS^+ and which are not. The following result provides an interesting insight from $\mathsf{MA}_{\text{ctble}}$.

Theorem 2.5. ($\mathsf{MA}_{\text{ctble}}$) If $\omega < \kappa < \mathfrak{c}$ and D is a countable dense subset of 2^κ, then D is not SS^+.

Proof. Let $\mathcal{B}$ be the Boolean algebra generated by $\{d^{-1}(0) : d \in D\} \subseteq P(\kappa)$. Now since D is dense, $\mathcal{B}$ separates points. Let us identify κ as (the fixed ultrafilters) $X \subset \mathcal{S}(\mathcal{B})$ where $\mathcal{S}(\mathcal{B})$ is the Stone space generated by $\mathcal{B}$. With this identification, we can view D as a countable dense subset of $C_p(X, 2)$; but we may also view members of D as continuous functions on all of $\mathcal{S}(\mathcal{B})$ because they do have uniquely defined continuous extensions. For all $y \in \mathcal{S}(\mathcal{B})$ we ask if $D_y = \{d : y \in \overline{d^{-1}(0)}\}$ is dense in D or not. If not then there is finite partial function $\tau_y : X \to 2$ so that the corresponding basic open set $[\tau_y] \cap C_p(X, 2)$ is disjoint from D_y, i.e. $d \supset \tau_y$ implies $d(y) = 1$. Now given a finite partial function $\tau : X \to 2$, let $Y_\tau = \{y : \tau = \tau_y\}$.

Let us assume that for some compact crowded set $K \subset \mathcal{S}(\mathcal{B})$, we have that $X \cap \overline{Y_\tau \cap K}$. is infinite. If so then $\exists x \in \overline{Y_\tau \cap K}$, $x \notin dom(\tau)$, so we can find $d \supset \tau$ such that $d \in D$ and $d^{-1}(0)$ is a neighbourhood of x, i.e., $d(x) = 0$, which implies that $d^{-1}(0) \cap Y_\tau$ is non-empty. Let $y \in d^{-1}(0) \cap Y_\tau$. Then $d \supset \tau = \tau_y$ implies $d(y) = 0$ and $d(y) = 1$ - a contradiction. It follows then that for each τ, $\overline{Y_\tau} \cap X$ is countable.

Since there are only κ many such functions τ, we can conclude from $\mathsf{MA}_{\text{ctble}}$ that there exists uncountably many $y \in \mathcal{S}(\mathcal{B}) \setminus X$ such that $y \notin Y_\tau$ for all τ. For each such y we note that the corresponding set D_y is dense.

Now let $\mathcal{D}$ be the collection of all dense subsets of $C_p(X, 2)$ and assume that σ is a strategy for Player II, i.e. for each sequence $\langle D_0, D_1, \ldots, D_n \rangle \subset \mathcal{D}$, $\sigma(\langle D_0, D_1, \ldots, D_n \rangle) \in [D_n]^{<\omega}$. Consider a sequence $\vec{D} = \langle D_0, D_1, \ldots, D_{n-1} \rangle \subset \mathcal{D}$ and set,

$$K_{\vec{D}} = \bigcap_{D \in \mathcal{D}} \left(\bigcup_{d \in \sigma(\vec{D}, D)} \overline{d^{-1}(0)} \right)$$

We showed above that $K_{\vec{D}} \cap X$ is countable. Now let us again fix a countable $M \prec H_\theta$ where $\theta = 2^{\mathfrak{c}^+}$ where σ, X, κ, and D are in M. Let $x \in X \setminus M$. Since it is countable, if $\vec{D} \in M$, then $(X \cap K_D) \subset X \cap M$. Arguing as in the proof of Theorem 2.4, there is a play of the game $\{D_0, D_1, ...\} \subset \mathcal{D} \cap M$ such that for all $\vec{D}_n = \{D_0, D_1, ..., D_n\}$, $x \notin \bigcup_{d \in \sigma(\vec{D}_n)} d^{-1}(0)$. If $d \in E_n = \sigma(\vec{D}_n)$ for any n, then $d(x) = 1$. Then $\bigcup_n E_n$ is not dense in S, since it misses the clopen set $\{d \in D : d(x) = 0\}$. Therefore σ is not a winning strategy, which shows that S is not SS$^+$. $\qquad\square$

We noted in the previous section that if X has countable dense fan tightness, then $C_p(X)$ is SS. Here is a similar result for SS$^+$.

We recall the following definition,

Definition 2.4. A space X is Menger if given a sequence $\langle U_n : n \in \omega \rangle$ of open covers of X, there is a sequence $\langle W_n \rangle_n \in \Pi_n[U_n]^{<\omega}$ such that $\bigcup_n W_n$ is again a cover.

The next result is due to Arhangelski [5],

Theorem 2.6. For an arbitrary space X the following are equivalent:

1. X^n is Menger for all $n \in \omega$,

2. $C_p(X)$ has countable fan tightness.

Proof. $(2) \Rightarrow (1)$: Let $n \in \omega$ and $\{U_k : k \in \omega\}$ a countable open cover of X^n. A system μ of covers of X is called U_k-small if for any $V_1, ..., V_n \in \mu$ there is a $G \in U_k$ such that $V_1 \times ... \times V_n \subset G$. Denote by ε_k the family of all finite U_k-small systems of open sets in X. For $\mu \in \varepsilon_k$, we put $F_\mu = \{f \in C_p(X) : f(X \setminus \mu) = \{0\}\}$.

We show that the set $A_k = \cup\{F_\mu : \mu \in \varepsilon_k\}$ is everywhere dense in $C_p(X)$.

Let $f \in C_p(X)$ and $K \subset X$, K is finite. There is a finite family Θ of open sets in X such that for any $(y_1, ..., y_n) \in K^n$ there are $V_1, ..., V_n \in \Theta$ and a $G \in U_k$ satisfying the conditions: $y_i \in V_i$ and $V_1 \times ... \times V_n \subset G$. Clearly $K \subset \cup\mu_k$.

The family μ_k is U_k-small. In fact, take an arbitrary $W_{x_1} \times ... \times W_{x_n}$, where $x_i \in K$. There are $V_1, ..., V_n \in \Theta$ and a $G \in U_k$ such that $y_i \in V_i$ and $V_1 \times ... \times V_n \subset G$. Since $W_{x_i} \subset V_i$ for $i = 1, ..., n$, we have that $W_{x_1} \times ... \times W_{x_n} \subset G$. Take a function $g \in C_p(X)$ such that $g \upharpoonright K = f \upharpoonright K$ and $g(X \setminus \cup\mu_k) = \{0\}$. Clearly, $g \in F_{\mu_K} \subset A_k$

and g lies in all standard neighborhoods of f based on K. Let $f \equiv 1$ on X. By the above, $f \in \overline{A_k}$ for all $k \in \omega$. Since $C_p(X)$ has countable fan tightness, there are finite sets $B_k \subset A_k$ for which $f \in \overline{B_k}$. there is a finite subfamily $\zeta_k \subset \varepsilon_k$ such that each function $g \in B_k$ is μ-small with respect to some $\mu \in \zeta_k$.

Let $\mu \in \zeta_k$. For each $\xi = (V_1, ..., V_n) \in \mu^n$ we choose a set $G_\xi \in U_k$ such that $V_1 \times ... \times V_n \subset G_\xi$.

The family $\lambda_k = \{G_\xi : \xi \in \mu^n, \mu \in \zeta_k\}$ is finite, since ζ_k is finite and every $\mu \in \zeta_k$ is finite. Clearly $\lambda_k \subset U_k$. We show that the family $\cup_k \lambda_k$ covers X.

Take an arbitrary $(x_1, ..., x_n) \in X^n$ and put $U = \{f \in C_p(X) : f(x_i) > 0, i = 1, ..., n\}$. The set U is open in $C_p(X)$, and $f \in U$. Since $f \in \overline{\cup_k B_k}$, there is a $k^* \in \omega$ for which $U \cap B_{k^*} \neq \emptyset$. Then $U \cap F_{\mu^*} \neq \emptyset$ for some $\mu^* \in \zeta_{k^*}$, i.e. there is a μ^*-small function $g \in U$. We have $g(x_i) > 0$ for $i = 1, ..., n$, and $g(x) = 0$ for all $x \in X \setminus \cup \mu^*$. Take $V_i \in \mu^*$ such that $x_i \in V_i$, for $i = 1, ..., n$. Then $(x_1, ..., x_n) \in V_1 \times ... \times V_n \subset G_\xi$ for some $G_\xi \in \lambda_{k^*}$. Hence $(x_1, ..., x_n) \in \cup \cup_k U_k$.

$(1) \Rightarrow (2)$: Let X^n be Menger for all $n \in \omega$. Fix $f \in C_p(X)$ and a family $\{A_k : k \in \omega\}$ of sets in $C_p(X)$ such that $f \in \cap_k \overline{A_k}$. Fix also $n \in \omega$ and $k \in \omega$. For each $\overline{x} = (x_1, ..., x_n) \in X^n$ there are $g_{\overline{x},k}, k \in A_k$, such that $| g_{\overline{x},k}(y_i) - f(y_i) | < 1/n$ for all $i = 1, ..., n$.

Since the functions $g_{\overline{x},k}$ and f are continuous, there is a neighborhood O_i of x_i such that $| g_{\overline{x},k}(y_i) - f(y_i) | < 1/n$ for all $y_i \in O_i$. The set $V_{\overline{x},k} = O_1 \times ... \times O_n$ is a neighborhood of $\overline{x}$ in X^n.

Thus $\gamma_{n,k} = \{V_{\overline{x},k} : \overline{x} \in X^n\}$ covers X^n, and $| g_{\overline{x},k}(y_i) - f(y_i) | < 1/n$ for all $(y_1, ..., y_n) \in V_{\overline{x},k}$. Since X^n is Menger, there are finite sets $P_{n,k} \subset X^n$ such that the family $\cup\{\gamma_{n,k} : k \in \omega, k \geqslant n\}$, where $\gamma_{n,k} = \{V_{\overline{x},k} : \overline{x} \in P_{n,k}\}$, covers X^n. The set $B_{n,k} = \{g_{\overline{x},k} : \overline{x} \in P_{n,k}\}$ is finite, and $B_{n,k} \subset A_k$. But the $B_k = \cup\{B_{n,k} : n \leqslant k\}$ is a finite set, and $B_k \subset A_k$. We show that $f \in \overline{\cup_k B_k}$.

Take arbitrary $y_1, ..., y_n \in X$ and an $\epsilon > 0$. We may assume that $1/n < \epsilon$. There is

a $K^* \geqslant n$ for which $(y_1, ..., y_n) \in \cup \lambda_{n,K^*}$. Then $(y_1, ..., y_n) \in V_{\bar{x},k^*}$ for some $\bar{x} \in P_{n,k^*}$. We have $g_{\bar{x},k^*} \in B_{n,k^*}$ and $| g_{\bar{x},k^*}(y_i) - f(y_i) | < 1/n$ for all $i = 1, ..., n$. But $B_{n,k^*} \subset B_{k^*}$, since $n \leqslant k^*$. Thus $g_{\bar{x},k^*} \in B_{k^*}$ and $f \in \overline{\cup_k B_k}$. The theorem has been proved. $\quad \square$

Our investigation is inspired by the connection between strategic fan tightness in $C_p(X)$ and the σ-compactness of X. We include this next result for motivation and the reader's convenience.

Proposition 2.7. If X is σ-compact then $C_p(X)$ has strategic fan tightness at each point; and so separable subsets of $C_p(X)$ are SS$^+$.

Proof. Since $C_p(X)$ is homogeneous, it suffices to show that $C_p(X)$ has strategic fan tightness at the constant zero function $\underline{0}$. Let $\{X_k : k \in \omega\}$ be an increasing chain of compact sets which cover X. We recall that, $C_p(X)$ is simply a subspace of $\mathbb{R}^X$; where a basic open subset (neighborhood of $f \in C_p(X)$) is, $[f \upharpoonright \{x_i : i < n\}; \epsilon] = \{g \in C_p(X) : | g(x_i) - f(x_i) | < \epsilon$ for $i < n\}$. Now player I chooses $A_n \subset C_p(X)$ with $\underline{0} \in \overline{A_n}$.

Let

$$U_n = \{(a^{-1}(-\tfrac{1}{n}, \tfrac{1}{n}))^k : k \leqslant n \text{ and } a \in A_n\}$$

We claim that U_n contains an open cover of $(X_k)^k$ for each $k \leqslant n$. Indeed, for any $k \leqslant n$ and $H \in (X_k)^k$, $[\underline{0} \upharpoonright H; \tfrac{1}{n}]$ is a neighborhood of $\underline{0}$ and so must intersect A_n. Thus, as required, there is some $a \in A_n$ satisfying that $H \in (a^{-1}(-\tfrac{1}{n}, \tfrac{1}{n}))^k$. Now, since each $(X_k)^k$ is compact, player II may select a finite $e_n \subset A_n$ so that the finite subcollection $W_n = \{(a^{-1}(-\tfrac{1}{n}, \tfrac{1}{n}))^k : k \leqslant n \text{ and } a \in e_n\}$ is a cover of $(X_k)^k$ for each $k \leqslant n$. Now we are left to show that $\underline{0} \in \overline{\cup_n e_n}$. To show that, let us fix any k, $\{x_i : i < k\} \subset X$ and $\epsilon > 0$. We need to show there is an $a \in \cup_n e_n$ such that $a \in [\underline{0} \upharpoonright \{x_i : i < k\}; \epsilon]$. Choose $n \geqslant k$ so large that $\{x_i : i < k\} \subset X_n$ and $\tfrac{1}{n} < \epsilon$. It follows then that there is an $a \in e_n$ such that $\langle x_i : i < k \rangle \in \left(a^{-1}(-\tfrac{1}{n}, \tfrac{1}{n}) \right)^k$; and therefore, $a \in [\underline{0} \upharpoonright \{x_i : i < k\}; \epsilon]$ as required. $\quad \square$

It is routine to generalize Lemma 1.5 to obtain the following.

Corollary 2.8. If X is σ-compact and metrizable, then $C_p(X)$ is SS$^+$.

Our next example shows that it is consistent that Fréchet does not imply SS$^+$.

Example 2.1. Let $X \subset 2^\omega$ be such that $\omega_1 \leqslant |X| < \mathfrak{p}$. Let $D = C_p(2^\omega, 2)$ Then $D' = D \upharpoonright X \subset C_p(X)$ and $wt(D') < |X| < \mathfrak{p} < \mathfrak{d}$ which says it is SS and with the similar argument in Theorem 2.4 shows that it is not SS$^+$, whereas the space is Fréchet since it has countable tightness (indeed it is countable) and character less than $\mathfrak{p}$.

We end this discussion by observing that, in contrast to Theorem 2.5, the space $2^\mathfrak{c}$ contains countable dense SS$^+$ subspace, namely $C_p(2^\omega, 2)$.

As mentioned previously, Gruenhage asked whether there is always a Markov strategy in SS$^+$ spaces. In such a case let us say that the space is Markov SS. We show that there is always a connection if the space is countable.

When studying SS or SS$^+$ for the spaces like $S = C_p(X, 2) \subset 2^X$, the role of X can be thought of as enumerating the base for S, and a compact subset of X plays a crucial role in SS$^+$. Keeping that in mind we define the notion of a subcollection of open sets being compactlike in a space, which we define as follows:

Definition 2.5. Suppose S is a space and $\mathcal{C}$ is a collection of (open) subsets of S. We say that $\mathcal{C}$ is compactlike, if for all dense $D \subset S$, there is a finite $e \subset D$ such that $e \cap C \neq \emptyset$ for all $C \in \mathcal{C}$.

It is immediate from the definition, that if $\mathcal{E}$ is a family of finite subsets of a space S satisfying that each dense set contains a member of $\mathcal{E}$, then any family $\mathcal{C}$ of open sets which meets every member of $\mathcal{E}$ will be a compactlike family.

The notion of σ-compactlike is defined as follows:

Definition 2.6. A space S is σ-compactlike, if the topology τ related with S is σ-compactlike, that is, if τ can be written as countable union of compactlike open subcollections of τ.

Lemma 2.9. If a space S is σ-compactlike, then S has a Markov strategy for being SS$^+$, i.e., S will be Markov SS.

Proof.   Since S is σ-compactlike, it has a σ-compactlike base, say $\mathcal{B}$. Let $\mathcal{B} = \cup_n \mathcal{B}_n$, where $\langle \mathcal{B}_n \rangle_n$ is an increasing family and each of them is compactlike. So for each dense $D \subset S$ and each n, there exists a finite $e_n \in D$ such $e_n \cap B \neq \emptyset$ for each $B \in \mathcal{B}_n$. We show that this selection, $e_n \subset D$ at stage n is the desired Markov strategy for Player II. Indeed, let, at stage n, player I plays A_n, where A_n is dense in S. Player II will choose a finite set $e_n \subset A_n$ as above, i.e. so that $e_n \cap B \neq \emptyset$ for all $B \in \mathcal{B}_n$. It is immediate that $\overline{\cup_n e_n}$ is dense since it meets every member of the base $\mathcal{B}$. $\square$

Also we have the next result,

Theorem 2.10. If S is Markov SS, then S is σ-compactlike.

Proof.   Let $\mathcal{D}$ be the collection of all dense subsets of S. Since S is Markov SS there is a winning strategy σ with domain $\mathcal{D} \times \omega$, where, for each $(D, n) \in \mathcal{D} \times \omega$, $\sigma(D, n)$ is a finite subset of D. Now let us consider the collection $\mathcal{C}_n = \{C \in \mathcal{B} :$ for $D \in \mathcal{D}, C \cap \sigma(D, n) \neq \emptyset\}$. From the definition of $\mathcal{C}_n$, it is clear that each of them is compactlike, so $\cup_n \mathcal{C}_n$ is σ-compactlike. So we just need to prove that the collection $\cup_n \mathcal{C}_n$ is a base. To show this, let $x \in S$ and U be any open set such that $x \in U$. If no member of $\mathcal{C}_n$ is contained in U, then for some $D_n \in \mathcal{D}, \sigma(D_n, n)$ misses U. If we can find D_n for each n, then the fact that $\cup_n \sigma(D_n, n)$ misses an open set, contradicts that it is to be a dense union. Therefore $\cup_n \mathcal{C}_n$ is a σ-compactlike base. $\square$

Theorem 2.11. If a space S is countable and SS$^+$, then it is Markov SS.

Proof.   The space is SS$^+$, so there is a SS$^+$ strategy σ on S. Let $\mathcal{D}$ denote the family of dense subsets of S. Our assumption on σ is that it is a function with domain consisting of finite sequences $\langle D_i : i \leqslant n \rangle$ from $\mathcal{D}$, satisfying that $\sigma(\langle D_i : i \leqslant n \rangle)$ is a finite subset of D_n and, for all infinite sequences $\langle D_i : i \in \omega \rangle$ from $\mathcal{D}$, the sequence $\{\sigma(\langle D_i : i \leqslant n \rangle) : n \in \omega\}$ of finite subsets of S will have dense union.

We now show that S is σ-compactlike. We will recursively define a tree T consisting of finite sequences of finite subsets of S which result from partial plays of the game

following the strategy σ. Thus, if $t \in T$ there is an integer $\ell = dom(t)$, and for each $i < \ell$, $t(i)$ is a finite subset of S. Furthermore, $t \in T$ if and only if there is a fixed sequence $\langle D_i^t : i < \ell = dom(t) \rangle \in \mathcal{D}^\ell \rangle$ such that for each $i < \ell$, $t(i) = \sigma(\langle D_j^t : j \leqslant i \rangle)$. An important additional assumption is that if $t \subset s$ are both in T, then for $i \in dom(t)$, $D_i^t = D_i^s$.

We begin with the empty sequence as an element of T. It follows easily that for each $t \in T$,

$$\mathcal{E}_t = \{\sigma(\langle D_0^t, \ldots, D_{dom(t)-1}^t, D \rangle) : D \in \mathcal{D}\}$$

is a family of finite subsets of S satisfying that every dense set includes one. Let $dom(t) = \ell$, and for each $e \in \mathcal{E}_t$, we have that $s_e \in T$ where $dom(s_e) = \ell + 1$, $s_e \supset t$ and $s_e(\ell) = e$. In addition, for $s = s_e$, $D_i^s = D_i^t$ for $i \in dom(t)$, and D_ℓ^s is chosen to be any $D \in \mathcal{D}$ such that $\sigma(\langle D_i^s : i \leqslant \ell \rangle) = e$. Therefore, the collection $\mathcal{C}_t$ (or $\mathcal{D}_{\mathcal{E}_t}$) is compactlike, where an open subset U of S is in $\mathcal{C}_t$ if and only if it meets every member of $\mathcal{E}_t$.

We show that every non-empty open set is in $\bigcup_{t \in T} \mathcal{C}_t$; thus showing that the topology on S is σ-compactlike. Assume otherwise, and assume that $U \notin \mathcal{C}_t$ for all $t \in T$. By a simple recursion, choose an increasing chain $\{t_n : n \in \omega\}$ in T so that $U \cap t_{n+1}(n)$ is empty for each n. It follows easily that $\langle D_n^{t_{n+1}} : n \in \omega \rangle$ is a play of the game that the strategy σ fails to defeat by virtue of the fact that the union of Player II's play will miss U. $\square$

The above connections between countable SS$^+$-spaces and the property of being σ-compactlike is instrumental in our approach to discovering that the union of two SS$^+$ spaces need not be SS$^+$. This is quite surprising since it was shown in [14, 15] that the property SS is finitely additive. We include the proof here,

Theorem 2.12. If X is a finite union of selectively separable spaces, then X is selectively separable.

Proof. It suffices to show that the union of two selectively separable spaces is selec-

tively separable. So suppose $X = A \cup B$, where A and B are selectively separable, and that D_n, $n \in \omega$, is a sequence of dense subsets of X. We need to show that there are finite $F_n \subset D_n$ such that $\bigcup_{n \in \omega} F_n$ is dense.

For each n, let

$$U_n = X \setminus \overline{(\cup_{i \geqslant n} D_i) \cap A}$$

and let $U = \bigcup_{n \in \omega} U_n$.

Claim 1: For each $i \geqslant n$, $D_i \cap B \cap U_n$ is dense in U_n .

Proof of Claim 1: Suppose $x \in U_n$. Then there exists an open neighborhood N_x of x contained in U_n such that

$$N_x \cap (\cup_{i \geqslant n} D_i) \cap A = \emptyset,$$

and so

$$N_x \cap (\cup_{i \geqslant n} D_i) \subset B.$$

Thus for each $i \geqslant n$, $D_i \cap N_x = D_i \cap B \cap N_x$. It follows that $x \in \overline{D_i \cap B \cap N_x} \subset \overline{D_i \cap B \cap U_n}$, and the Claim 1 is proved.

Claim 2 : There are finite $G_n \subset D_n$ such that $\bigcup_{n \in \omega} G_n$ is dense in U

Proof of Claim 2 : Since open subsets of selectively separable spaces are selectively separable, $B \cap U_n$ is selectively separable for each n. Then it follows fron Claim 1 that there are finite subsets G_i^n of $D_i \cap B \cap U_n$, $i \geqslant n$, such that $\bigcup_{i \geqslant n} G_i^n$ is dense in $B \cap U_n$ and hence also dense in U_n. Now let $G_i = \bigcup_{i \geqslant n} G_i^n$. Then G_i is a finite subset of D_i and $\bigcup_{i \in \omega} G_i$ is dense in U_n for all n, and hence in U. This proves Claim 2.

Now let $V = X \setminus \overline{U}$. Clearly each $x \in V$ is in $\overline{(\cup_{i \geqslant n} D_i) \cap A}$ for each n, and so $(\cup_{i \geqslant n} D_i) \cap A$ is dense in V, and also $A \cap V$, for each n. $A \cap V$ is selectively separable, so there are finite $H_n \subset (\cup_{i \geqslant n} D_i) \cap A$ such that $\bigcup_{n \in \omega} H_n$ is dense in $V \cap A$, hence in V.

For each $x \in H_n$, let $i_n(x) \in \omega \setminus n$ such that $x \in D_{i_n(x)}$.

Let $K_i = \{x : \exists n \in \omega(x \in H_n \wedge i_n(x) = i)\}..$ Then K_i is a finite subset of D_i and $\bigcup_{i \in \omega} K_i = \bigcup_{n \in \omega} H_n$ and hence is dense in V.

Finally, if G_n is as in Claim 2, then $\bigcup_{n \in \omega} G_n \cup K_n$ is dense in X. Thus X is selectively separable. $\square$

In [8], we produced an example of a space being SS but not SS$^+$. By the next result we now have another example of an SS$^+$ space which is not SS, namely the union of the two SS$^+$ spaces.

Theorem 2.13. There are countable SS$^+$ spaces A, B such that $A \cup B$ is not SS$^+$.

Proof. For $x \in 2^\omega$, let us define $x^\dagger$ by flipping the first value, i.e.,

$$x^\dagger = \langle 1 - x(0), x(1), x(2), ... \rangle$$

Let $Z \subset 2^{2^\omega}$ be defined by,

$$Z = \{z \in 2^{2^\omega} : z(x) \cdot z(x^\dagger) = 0 \ \forall x \in 2^\omega\} = 2^{2^\omega} \setminus \bigcup_{x \in 2^\omega} ([x;1] \cap [x^\dagger;1]),$$

where $[x;i]$ is the basic open neighborhood of a function which takes x to i for $i \in \{0,1\}$. Let $A = C_p(2^\omega, 2) \cap Z$. Since $C_p(2^\omega, 2)$ is SS$^+$, A is SS$^+$. To identify the set B, we first define a new topology $\tau^\dagger$ on 2^ω. Let Q denote the countable dense set of rationals in 2^ω (the sequences that are eventually 0). The basic open sets in $\tau^\dagger$ are of the form $[s]_{\tau^\dagger} = [s] \setminus Q \cup ([s^\dagger] \cap Q)$ for any $s \subset 2^{<\omega} = \bigcup_n 2^n$. It is immediate that this space is just another copy of the Cantor set obtained by a simple permutation on the elements of Q. Now we define $B = C_p((2^\omega, \tau^\dagger), 2) \cap Z$. Again it follows immediately that B is SS$^+$.

We claim that A and B are mutually dense in Z. We show that A is dense in Z and omit the simple modification necessary to show that B is also dense in Z. Let us consider any $l \in \omega$ and let $\cap_{i<l}([x_i;0] \cap [y_i;1])$ be a basic open set meeting Z. Note that since this basic open set does meet Z, we have that $\forall i \neq j$, $y_i^\dagger$ is not equal y_j. To show that this basic open set hits A, we pick m so large that, first, if any of the members of $\{x_i, y_i : i < l\}$ are rationals, they are constant above m, and secondly, any two distinct elements of $\{x_i, y_i, x_i^\dagger, y_i^\dagger : i < l\}$ will differ somewhere below m. Let $a \in C_p(2^\omega, 2)$ be defined so that whenever $t \in 2^m$, $a[t] = 1$ if and only if there is an $i < l$ such that $t = y_i \upharpoonright m$. It is clear that $a \in \cap_{i<l}([x_i;0] \cap [y_i;1])$. To show $a \in Z$,

let $x \in 2^{<\omega}$ and $a(x) = 1$. We need to prove that $a(x^\dagger) = 0$. Let $t = x \restriction m$, therefore $a[t] = 1$ and $t \subset y_i$ for some $i < l$. Now if $a(x^\dagger) = 1$, then there must be some $i \neq j < l$ such that $x^\dagger \restriction m \subset y_j$. Of course it now follows that $y_i^\dagger \restriction m = y_j \restriction m$ which contradicts the assumptions that $y_i^\dagger \neq y_j$ for all $i \neq j$, and that m is large enough to distinguish these elements. Therefore A is dense in Z.

As mentioned above, each of A and B are SS^+. We claim that $A \cup B$ does not have σ-compactlike topology. Assume that $\mathcal{B} = \{[x; 1] \cap Z : x \in 2^\omega\}$ can be written as countable union of compactlike sets. By the Baire category theorem then, there is an $I \subset 2^\omega \setminus Q$ which is dense in some Cantor basic clopen set $[s]$ such that $I' = \{[x; 1] : x \in I\}$ is compactlike. Let us pick any rational $q \in Q \cap [s]$ and let us define the set

$$D = (A \cap [q; 0]) \cup (B \cap [q^\dagger; 0]).$$

Fix any m_q so that q is constantly 0 above m_q. Since the union of the two open sets $Z \cap [q; 0]$ and $Z \cap [q^\dagger; 0]$ is dense in Z, it follows immediately that D is dense in Z. Now if $d \in D$, then either $d \in A \cap [q; 0]$ or $d \in B \cap [q^\dagger; 0]$. Since $q \in [s] = \overline{I}$, there is a sequence $\langle x_n \rangle_n \subset I$ converging to q. We show that d is in only finitely many of the sets from the collection $\{[x_n; 1] : n \in \omega\}$ and so no finite subset of D can meet every member of the collection I'. Notice that this is equivalent to proving that $d(x_n) = 0$ for all but finitely many n.

First suppose that $d \in A \cap [q; 0]$; hence d is continuous with respect to the usual topology on 2^ω. It follows then that there is an $m > m_q$ such that d sends the entire basic open set $[q \restriction m]$ to 0. Since all but finitely many of the x_n's are in $[q \restriction m]$, this completes the proof of the case $d \in D \cap A$. Now suppose that $d \in B \cap [q^\dagger; 0]$. Now d is continuous with respect to $\tau^\dagger$. In this new topology, it is easy to see that the sequence $\{x_n : n \in \omega\}$ converges to the point $q^\dagger$. Thus, since $d(q^\dagger) = 0$, it follows again that $d(x_n) = 0$ for all but finitely many n.

Therefore $A \cup B$ is not σ-compactlike, and so, by Theorems 2.11 and 2.10, this space is not SS^+. $\qquad\square$

Now we will prove that Markov SS is finitely productive. For that we need the following lemma,

Lemma 2.14. Let S be any space and $\mathcal{C}$ be any collection of open sets. Then $\mathcal{C}$ is compactlike if and only if for each ultrafilter $\mathcal{U}$ on $\mathcal{C}$, the collection $S(\mathcal{C}, \mathcal{U}) = \{s \in S : \mathcal{C}_s = \{C \in \mathcal{C} : s \in C\} \in \mathcal{U}\}$ has interior.

Proof. If $S(\mathcal{C}, \mathcal{U})$ does not have interior, then $D = S \setminus S(\mathcal{C}, \mathcal{U})$ is dense and therefore for any finite $F \subset D$, $a \in F$ implies $\mathcal{C}_a = \{C \in \mathcal{C} : a \in C\} \notin \mathcal{U}$, so F does not even meet $\mathcal{U}$-many elements of $\mathcal{C}$. Conversely, assume that for each ultrafilter $\mathcal{U}$ on $\mathcal{C}$, $S(\mathcal{C}, \mathcal{U})$ has interior. Let D be any dense subset of S. Then, for each $d \in D \cap \text{int}(S(\mathcal{C}, \mathcal{U}))$, $\mathcal{C}_d = \{C \in \mathcal{C} : d \in C\} \in \mathcal{U}$. Now we can see that the collection $\{\mathcal{C}_d\}_{d \in C}$ covers $\beta\mathcal{C}$. Since $\beta\mathcal{C}$ is compact, there are finitely many members $\{d_1, d_2, ..., d_n\}$ from D, such that $\{\mathcal{C}_{d_i} : i \in \{1, 2, ..., n\}$ is a subcover for $\beta\mathcal{C}$. So we have a finite set $F \subset D$, namely $\{d_1, d_2, ..., d_n\}$, such that $F \cap C \neq \emptyset$ for all $C \in \mathcal{C}$, which shows that $\mathcal{C}$ is compactlike. $\square$

Now we can prove the next important result.

Theorem 2.15. The property of being Markov SS is finitely productive.

Proof. Let X and Y have σ-compactlike bases $\mathcal{B} = \bigcup_n \mathcal{B}_n$ and $\mathcal{C} = \bigcup_n \mathcal{C}_n$ respectively. We use Lemma 2.14 to show that the collection $\mathcal{A}_n = \{B \times C : B \in \mathcal{B}_n, C \in \mathcal{C}_n\}$ is compactlike. Let $\mathcal{W}$ be any ultrafilter on $\mathcal{A}_n$. We will show that,

$$S(\mathcal{A}_n, \mathcal{W}) = \{(x, y) \in X \times Y : \{(B \times C) \in \mathcal{A}_n : (x, y) \in B \times C\} \in \mathcal{W}\}$$

has interior. Let us define $\mathcal{W}_0$ and $\mathcal{W}_1$ by

$$\mathcal{W}_0 = \{W \subset \mathcal{B}_n : \pi_X^{-1}(W) = W \times \mathcal{C}_n \in \mathcal{W}\}$$

$$\text{and}$$

$$\mathcal{W}_1 = \{V \subset \mathcal{C}_n : \pi_Y^{-1}(V) = \mathcal{B}_n \times V \in \mathcal{W}\}.$$

Since $\mathcal{W}$ is an ultrafilter, $\mathcal{W}_0$ and $\mathcal{W}_1$ are both ultrafilters on $\mathcal{B}_n$ and $\mathcal{C}_n$ respectively. We claim that $S(\mathcal{B}_n, \mathcal{W}_0) \times S(\mathcal{C}_n, \mathcal{W}_1) \subset S(\mathcal{A}_n, \mathcal{W})$. Let us choose any $(x, y) \in S(\mathcal{B}_n, \mathcal{W}_0) \times S(\mathcal{C}_n, \mathcal{W}_1)$. Then $x \in S(\mathcal{B}_n, \mathcal{W}_0)$ and $y \in S(\mathcal{C}_n, \mathcal{W}_1)$, hence $(\mathcal{B}_n)_x \in$

$\mathcal{W}_0$ and $(\mathcal{C}_n)_y \in \mathcal{W}_1$. Since $\mathcal{W}$ is an ultrafilter $(\mathcal{B}_n)_x \times (\mathcal{C}_n)_y = {\pi_X}^{-1}((\mathcal{B}_n)_x) \cap {\pi_Y}^{-1}((\mathcal{C}_n)_y) \in \mathcal{W}$. Since $(\mathcal{A}_n)_{(x,y)} \supset (\mathcal{B}_n)_x \times (\mathcal{C}_n)_y$ we have shown that $(x,y) \in S(\mathcal{A}_n, \mathcal{W})$. Therefore $S(\mathcal{A}_n, \mathcal{W})$ contains $S(\mathcal{B}_n, \mathcal{W}_0) \times S(\mathcal{C}_n, \mathcal{W}_1)$. Since $\mathcal{B}_n$ and $\mathcal{C}_n$ are compactlike, both of $S(\mathcal{B}_n, \mathcal{W}_0)$ and $S(\mathcal{C}_n, \mathcal{W}_1)$ have interior which implies that $S(\mathcal{A}_n, \mathcal{W})$ also has interior. Therefore $\mathcal{A}_n$ is compactlike. $\qquad\square$

We also notice the following important observation about countable SS$^+$ spaces.

Proposition 2.16. The finite product of countable SS$^+$ spaces is again SS$^+$.

The extensive use of ultrafilters does seem somewhat unnatural in dealing with finite products, so we thought it helpful to provide a proof of Theorem 2.15 with more similarity to the standard proof of compactness for the product of two compact spaces. However, we still rely on ultrafilters by using Lemma 2.14. We begin with the following consequence of a collection being compactlike.

Proposition 2.17. Suppose that $\mathcal{E}$ is a family of finite subsets of a space S with the property that for all non empty open $U \subset S$, there exists $e \in \mathcal{E}$ such that $e \subset U$. Then for each compactlike collection $\mathcal{C}$ of open subsets of S there exists a finite collection $\mathcal{E}' \subset \mathcal{E}$ such that for all $C \in \mathcal{C}$, there exists $e \in \mathcal{E}'$ with $e \subset C$.

Proof. We apply Lemma 2.14 as follows. For each ultrafilter $\mathcal{U}$ on $\mathcal{C}$, we have that $S(\mathcal{C}, \mathcal{U})$ has interior. Therefore, there is an $e_\mathcal{U} \in \mathcal{E}$ which is contained in $S(\mathcal{C}, \mathcal{U})$. Similarly, there is a subcollection $\mathcal{C}_\mathcal{U} \in \mathcal{U}$ satisfying that $e_\mathcal{U} \subset C$ for all $C \in \mathcal{C}_\mathcal{U}$. As in Lemma 2.14, there is a finite set, $\{\mathcal{U}_i : i < n\}$, of ultrafilters on $\mathcal{C}$ such that $\mathcal{C}$ is covered by $\bigcup\{\mathcal{C}_{\mathcal{U}_i} : i < n\}$. It follows immediately, that $\mathcal{E}' = \{e_{\mathcal{U}_i} : i < n\}$ is the desired finite subset of $\mathcal{E}$. $\qquad\square$

Proposition 2.18. If $\mathcal{B}$ and $\mathcal{C}$ are compactlike families of open subsets of X, Y respectively, then $\mathcal{B} \times \mathcal{C}$ is compactlike in $X \times Y$.

Proof. Let π_X denote the projection map from $X \times Y$ onto X, and fix any dense subset D of $X \times Y$. Let U be any non-empty open set in $X \times Y$. Since $\mathcal{C}$ is compactlike in Y, it is trivial to check that the family $\mathcal{C}_U = \{U \times C : C \in \mathcal{C}\}$ is compactlike in

$X \times Y$. Therefore there is a finite $D_U \subset D \cap (U \times Y)$ which meets every member of $\mathcal{C}_U$. Observe that this means that D_U meets $\pi_X(D_U) \times C$ for every $C \in \mathcal{C}$.

Now the family $\mathcal{E} = \{\pi_X(D_U) : \emptyset \neq U \subset X$ is open$\}$ (where π_X is the projection onto X) satisfies the hypothesis of Proposition 2.17, and so we may select open sets $\{U_i : i < n\}$ of X so that each $B \in \mathcal{B}$ contains $\pi_X(D_{U_i})$ for some $i < n$. Since D_{U_i} meets $\pi_X(D_{U_i}) \times C$ for all $C \in \mathcal{C}$, this shows that D_{U_i} meets $B \times C$ for all $C \in \mathcal{C}$. Thus $\bigcup_{i<n} D_{U_i}$ is the desired finite set to show that $\mathcal{B} \times \mathcal{C}$ is compactlike. $\qquad \square$

CHAPTER 3: CONSISTENCY RESULTS AND FORCING WITH SELECTIVE SEPARABILITY

3.1   Introduction

The method of forcing was introduced by Paul Cohen in his proof of independence of the Continuum Hypothesis and of the Axiom of Choice. Forcing proved to be a remarkably general technique for producing a large number of models and consistency results.

The main idea of forcing is to extend a transitive model $\mathbf{V}$ of set theory (the ground model) by adjoining a new set G (a generic set) in order to obtain a larger transitive model of set theory $\mathbf{V}[G]$ called a generic extension. The generic set is approximated by forcing conditions in the ground model, and a judicious choice of forcing conditions determines what is true in the generic extension.

Definition 3.1. For a set $\mathbf{V}$ (usually a model of most of ZF) and a generic filter G of a poset P,

$$\mathbf{V}[G] = \{val_G(\dot{X}) : \dot{X} \in \mathbf{V} \text{ is a P-name}\}$$

Let us mention the forcing theorem, which will be used in this often.

Theorem 3.1. *The Forcing Theorem:* Let $(P, <)$ be a notion of forcing in the ground model $\mathbf{V}$. If σ is a sentence of the forcing language, then for every $G \subset P$ generic over $\mathbf{V}$,

$\mathbf{V}[G] \models \sigma$ if and only if $(\exists p \in G)p \Vdash \sigma$

We have seen that if $S \subset 2^\kappa$, and we force $\kappa < \mathfrak{d}$, then S becomes SS. Also if S is a countable dense in $C_p(X, 2)$, where $X = \mathbb{Q} \cup \{f_\alpha : \alpha \in \mathfrak{b}\}$, then S is SS. Hence any

forcing which preserves the value of $\mathfrak{b}$ (more precisely preserving that the unbounded families of functions remain unbounded) will preserve that S is SS.

Here we can ask a question: Can we force to destroy selective separability? The answer to this question is an immediate consequence of the following result of A. Miller. This result is derived using Sacks Poset, defined below,

Definition 3.2. The elements of *Sacks poset* P satisfies : $p \in P$ iff p is a subtree of $2^{<\omega}$ such that

$$(\forall s \in p)(\exists t \in p) : (s \subseteq t) \wedge (t^\frown 0 \in p) \wedge (t^\frown 1 \in p);$$

and the ordering is by inclusion: $p \leqslant q$ iff $p \subseteq q$.

Fix $p, q \in P$ and $n \in \omega$. A node $t \in p$ is an nth branching point of p if $t^\frown 0, t^\frown 1 \in p$ and $|\{s \in p : (s \subseteq t) \wedge (s^\frown 0, s^\frown 1 \in p)\}| = n$.

$p \leqslant_n q$ means that $p \leqslant q$ and that every nth branching point of q is a branching point of p.

If $\langle p_n : n \in \omega \rangle$ is a fusion sequence ($p_{k+1} \leqslant_k p_k$, for all $k < \omega$), then $p := \bigcap_n p_n$ (the fusion of the sequence) satisfies $p \in P$ and $p \leqslant_n p_n$ for every $n \in \omega$.

Now we prove the next theorem,

Theorem 3.2. [Miller] If $\mathbf{x}$ is Sacks generic over $\mathbf{V}$, then in $\mathbf{V}[\mathbf{x}]$ the set $\mathbf{V} \cap 2^\omega$ does not have the Menger Property.

Proof. Let us define $Q = \{T \subseteq 2^{<\omega} \text{ infinite} : \forall \sigma, \tau (\sigma \subset \tau \in T \to \tau \in T)\}$. Note that Q is a closed subspace of $P(2^{<\omega})$ (when identified as a subspace of $2^{2^{<\omega}}$) and is homeomorphic to 2^ω.

Given the Sacks real $\mathbf{x} \in 2^\omega$ and $n \in \omega$, we define in $\mathbf{V}[\mathbf{x}]$ an open cover of $Q \cap \mathbf{V}$ by

$$U(n,m) = \{T \in Q : \mathbf{x}{\restriction}m \notin T \text{ or } |\{\ell < m : \{(\mathbf{x}{\restriction}\ell)^\frown 0, (\mathbf{x}{\restriction}\ell)^\frown 1\} \subset T| \geqslant n+2\} .$$

A Sacks real has the property that it is not a member of any ground model closed set which does not contain a perfect set. This implies that for each $T \in Q \cap \mathbf{V}$ such that $\mathbf{x}{\restriction}m \in T$ for all m, then the set $\{\ell : \{(\mathbf{x}{\restriction}\ell)^\frown 0, (\mathbf{x}{\restriction}\ell)^\frown 1\} \subset T\}$ is infinite. Therefore, for each n, the family $\{U(n,m) : m \in \omega\}$ is an increasing open cover of $Q \cap \mathbf{V}$.

It is well-known that the family $\mathbf{V} \cap \omega^\omega$ is dominating in $\mathbf{V}[\mathbf{x}]$ (see [21]). Therefore to show that $Q \cap \mathbf{V}$ is not SS in $\mathbf{V}[\mathbf{x}]$, we consider a strictly increasing function $g \in \omega^\omega$ from V and show there is a $T \in Q$ such that $T \notin U(n, g(n))$ for all n. To prove this it is enough to know that if, working in $\mathbf{V}$, $\mathcal{C}$ is a collection of compact perfect subsets of 2^ω with the property that each perfect set contains one, then there is some $C \in \mathcal{C}$ such that $\mathbf{x} \in \overline{C}$. Set $\mathcal{C}$ to be the collection of all perfect subsets C of 2^ω with the property that if x_0, y_0, x_1, y_1 are distinct members of C, and $\ell < m$ are minimal such that $x_0(\ell) \neq x_1(\ell)$ and $y_0(m) \neq y_1(m)$, then there is an n such that $\ell \leqslant g(n) < g(n+1) < m$. Given such a perfect set C, $T_C = \{t \in 2^{<\omega} : (\exists y \in C) t \subset y\}$ will be a member of Q, and the Sacks real $\mathbf{x}$ will be in $\overline{C}$ precisely if for all m, $\mathbf{x} \restriction m \in T_C$. It is routine to see that each perfect set K contains a perfect set in $\mathcal{C}$, hence there is some such C such that $\mathbf{x} \restriction m \in T_C$ for all m. The definition of $\mathcal{C}$ ensures that for each n, $\{\ell \leqslant g(n) : \{(\mathbf{x} \restriction \ell)^\frown 0, (\mathbf{x} \restriction \ell)^\frown 1\} \subset T_C\}$ will have cardinality less than $n + 2$.

This of course completes the proof that $Q \cap \mathbf{V}$ fails to have Menger property in $\mathbf{V}[\mathbf{x}]$. $\qquad\square$

From this result, we observe the interesting fact that there is an SS^+ space, namely, $S = (C_p(2^\omega, 2), \tau^{\mathbf{V}})$, for which the SS property is also destroyed by adding a Sacks real.

3.2 On Product of Selectively Separable Spaces

Let us begin by the following result in the product space,

Theorem 3.3. If X and Y are both countable, selectively separable and $\pi w(Y) < \mathfrak{b}$, then $X \times Y$ is selectively separable.

Proof. Let $\{\mathcal{B}_\alpha : \alpha < \kappa\}$ where $\kappa < \mathfrak{b}$ be a π-base for Y. Let $\{D_k = \{d_{k,m} : m \in \omega\} : k \in \omega\}$ be the countable sequence of dense subsets of $X \times Y$. Let π_x and π_y be the natural projection onto the spaces X and Y respectively. Now the set $G_k^\alpha = \pi_x[D_k \cap (X \times B_\alpha)]$ is dense in X. Since X is selectively separable, there is

a selection $F_k^\alpha \subseteq D_k$ $(k \in \omega)$ so that $\pi_x[F_k^\alpha] \subseteq G_k^\alpha$ and $\overline{\bigcup \pi_x[F_k^\alpha]} = X$. Since F_k^α is

finite, $\exists f_\alpha(k) \in \omega$ so that $F_k^\alpha \subseteq \{d_{k,m} : m < f_\alpha(k)\}$. Therefore we have a sequence

$\{f_\alpha : \alpha < \kappa\}$ where $f_\alpha : \omega \to \omega$. Since $\kappa < \mathfrak{b}$, there exists a function $f \in \omega^\omega$ such

that $\forall \alpha < \kappa$, $f_\alpha <^* f$. Let us define $F_k = \{d_{k,m} : m < f(k)\} \subset D_k$. We claim that

$\overline{\bigcup_{k \in \omega} F_k} = X \times Y$. Let us choose a basic open set $U \times B_\alpha$ of $X \times Y$, then $\exists l \in \omega$

such that $\forall i > l, f(i) > f_\alpha(i)$. Since $U \cap \bigcup \Pi_x" F_k^\alpha \neq \emptyset$, there exists a $z \in F_k$ such

that $\pi_x(z) \in U \cap \bigcup \pi_x[F_k^\alpha]$, which implies that $z \in F_k \cap (U \times B_\alpha)$. Therefore $\bigcup F_k$ is

dense in $X \times Y$. $\qquad \square$

One of our main results is to confirm the conjecture in [9] that SS is not productive

in general.

Theorem 3.4. ($\mathsf{MA}_{\text{ctble}}$) There exists two countable SS spaces whose product is not

SS.

Proof. Let us consider the set $\mathbb{Q} = \{q_i : i \in \omega\}$ with the standard zero-dimensional

topology generated by a countable base $\mathcal{B}_0^0 = \mathcal{B}_0^1$ of clopen sets. Let τ_0^0 and τ_0^1

denote the topologies so generated. Obviously $(\mathbb{Q}, \tau_0^0)$ and $(\mathbb{Q}, \tau_0^1)$ are SS. We will

enlarge our topology in such a way that the product space $\mathbb{Q} \times \mathbb{Q}$ will not be SS. Let

$\{E_n : n \in \omega\}$ be a countable family of dense sets in $\mathbb{Q} \times \mathbb{Q}$ such that E_n hits every row

and column in a singleton set, in fact for any $q \in \mathbb{Q}$, $| E_n \cap [(\{q\} \times \mathbb{Q}) \cup (\mathbb{Q} \times \{q\})] | \leqslant 1$.

Moreover we ensure that for each $q \in \mathbb{Q}$, there is at most one integer n such that

$E_n \cap [(\{q\} \times \mathbb{Q}) \cup (\mathbb{Q} \times \{q\})]$ is non-empty. In order to ensure the product is not SS, we

let $\{\langle F_n^\alpha : n \in \omega \rangle : \alpha \in \mathfrak{c}\}$ be an enumeration of all selections $\{F_n \in [E_n]^{<\omega} : n \in \omega\}$.

Let $\{S_\alpha : \alpha \in \mathfrak{c}\}$ be a listing of all the countable subsets of $\mathfrak{c}$ so that for each α,

$S_\alpha \subset \alpha$. Of course the family $\{Y_\alpha = \{q_i : i \in S_\alpha \cap \omega\} : \alpha \in \mathfrak{c}$ and $S_\alpha \subset \omega\}$ is also a

listing of $\mathcal{P}(\mathbb{Q})$.

By induction on $\alpha \in \mathfrak{c}$, we define families $\langle \mathcal{B}_\beta^0 : \beta < \alpha \rangle$, $\langle \mathcal{B}_\beta^1 : \beta < \alpha \rangle$, $\langle D_\beta^0 : \beta < \alpha \rangle$,

and $\langle D_\beta^1 : \beta < \alpha \rangle$ so that, for each $i \in 2$ and $\beta < \gamma < \alpha$,

   1. $\mathcal{B}_\beta^i \subset \mathcal{B}_\gamma^i$,

2. $\mathcal{B}_\beta^i$ has cardinality at most $|\beta + \omega|$ and is a base of clopen sets for a topology, τ_β^i, on $\mathbb{Q}$,

3. $\{D_\xi^i : \xi < \beta\}$ is a family subsets of $\mathbb{Q}$ which are dense in the τ_β^i topology.

4. for each n, E_n is dense in the product topology $\tau_\beta^0 \times \tau_\beta^1$, and $\bigcup_n F_n^\beta$ is not dense in the product $\tau_\gamma^0 \times \tau_\gamma^1$

5. if $S_\beta \subset \omega$ and Y_β is dense in $(\mathbb{Q}, \tau_\beta^i)$, then $D_\beta^i = Y_\beta$,

6. if S_β is infinite and not contained in ω, then there is a sequence $\{E_\xi^i \in [D_\xi^i]^{<\omega} : \xi \in S_\beta\}$ such that $D_\beta^i = \bigcup_{\xi \in S_\alpha} E_\xi^i$.

To complete the $\alpha = 0$ stage of the induction, we may let $D_0^0 = D_0^1$ be any dense subset of $\mathbb{Q}$ (with the usual topology). Now we assume that $\alpha > 0$. If α is a limit and $i \in 2$, then $\mathcal{B}_\alpha^i = \bigcup_{\beta < \alpha} \mathcal{B}_\beta^i$. If α is a successor, we define $\mathcal{B}_\alpha^0$ and $\mathcal{B}_\alpha^1$ below. The choices of D_α^0 and D_α^1 do not depend on whether or not α is a limit. If S_α is finite, then $D_\alpha^i = D_0^i$ for each $i \in 2$. If S_α is a subset of ω, then, independently for $i \in 2$, we set $D_\alpha^i = Y_\alpha$ if Y_α is dense in τ_α^i, and otherwise, let $D_\alpha^i = D_0^i$. If S_α is infinite and is not a subset of ω, then, again independently for $i \in 2$, we let D_α^i be any τ_α^i-dense set satisfying the last condition. Such a set exists since τ_α^i is SS because of Lemma 1.3 and, by the hypothesis of the theorem, $\mathfrak{d} = \mathfrak{c}$.

Finally, in the case that $\alpha = \beta + 1$ we consider the construction of $\mathcal{B}_\alpha^0, \mathcal{B}_\alpha^1$ in order to satisfy condition 2. We will choose two sets A_0 and A_1 such that $(A_0 \times A_1) \cap (\cup_n F_n^\alpha) = \emptyset$. Then $\mathcal{B}_\alpha^i$ is the topology generated by $\mathcal{B}_\beta^i \cup \{A_i, \mathbb{Q} \setminus A_i\}$.

Let us consider the countable poset,

$$P = \{\langle a_j, b_j \rangle_{j < M} \in [\omega^2]^{<\omega} :$$

$$(\forall j < M - 1)\ (a_j < a_{j+1} \text{ and } b_j < b_{j+1}),$$

$$\text{and } (\{q_{a_j}\}_{j<M} \times \{q_{b_j}\}_{j<M}) \cap \bigcup_n F_n^\alpha = \phi\} . \quad (3.1)$$

We will define a family of fewer than $\mathfrak{c}$ many dense subsets of P and, applying $\mathsf{MA}_{\text{ctble}}$, select a generic filter G meeting that family of dense sets. Given such a G, we let

$$A_0 = \{q \in \mathbb{Q} : (\exists \langle a_j, b_j \rangle_{j<M} \in G) \; q \in \{q_{a_i} : i < M\}$$

and

$$A_1 = \{q \in \mathbb{Q} : (\exists \langle a_j, b_j \rangle_{j<M} \in G) \; q \in \{q_{b_j} : j < M\}.$$

We must define dense sets to ensure that each E_l remains dense which requires con-sidering all combinations from $\{A_0, \mathbb{Q} \setminus A_0\} \times \{A_1, \mathbb{Q} \setminus A_1\}$.

For each $B, B' \in \mathcal{B}_\beta^0 \times \mathcal{B}_\beta^1$ let

$$D(\ell, B, B') = \{\langle a_j, b_j \rangle_{j<M} \in P : (\exists j < M - 4)$$

$$(q_{a_j}, q_{b_j}) \in E_\ell \cap (B \times B')$$

$$(\exists i \in (a_j, a_{j+1})) \; (q_i, q_{b_{j+1}}) \in E_\ell \cap (B \times B')$$

$$(\exists i \in (b_{j+1}, b_{j+2})) \; (q_{a_{j+2}}, q_i) \in E_\ell \cap (B \times B')$$

$$(\exists i \in (a_{j+2}, a_{j+3}), i' \in (b_{j+2}, b_{j+3})) \; (q_i, q_{i'}) \in E_\ell \cap (B \times B')\} \,. \quad (3.2)$$

The special properties of the family $\{E_k : k \in \omega\}$ ensure that each $D(\ell, n, B, B')$ is a dense subset of P. To see this, fix any $p = \langle a_j, b_j \rangle_{j<M} \in P$. For each $j < M$, there are at most four points in E_ℓ which have q_{a_j} or q_{b_j} in one of their coordinates. Let E'_ℓ be E_ℓ minus these at most $4M$ many points. Since E_ℓ is $\tau_\alpha^0 \times \tau_\alpha^1$-dense, there is a $(q_{a_M}, q_{b_M}) \in (E'_\ell \setminus F_\ell^\alpha) \cap (B \times B')$. Furthermore, since $(q_{a_M}, q_{b_M}) \in E_\ell$, it follows that $(\{q_{a_M}, q_{b_M}\} \times \mathbb{Q}) \cup (\mathbb{Q} \times \{q_{a_M}, q_{b_M}\})$ is disjoint from E_k for all $k \neq \ell$. Therefore it follows that $\langle a_i, b_i \rangle_{i \leqslant M}$ is an extension of p in P. Similarly, repeat this process and choose pairs $(a_{M+j}, b_{M+j}) \in E_l \cap (B \times B')$ (for $j < 6$) with exactly the same requirements (so as to ensure no intersection with $\bigcup_n F_n^\alpha$). The desired extension q of p which is in the set $D(\ell, n, B, B')$ is $\langle a'_j, b'_j \rangle_{j<M+4}$ where

1. $a'_j = a_j$ and $b'_j = b_j$ for $j \leqslant M$,
2. $a'_{M+1} = a_{M+2}$ and $b'_{M+1} = b_{M+1}$,
3. $a'_{M+2} = a_{M+2}$ and $b'_{M+2} = b_{M+3}$,

4. $a'_{M+3} = a_{M+5}$ and $b'_{M+3} = b_{M+5}$.

By suitably skipping members of $\langle a_j, b_j \rangle_{j < M+6}$ we have ensured that each of the conditions in $D(\ell, n, B, B')$ are met by one of the pairs (a_i, b_i) $(M \leqslant i < M + 6)$. Next, to show that each of D^0_γ and D^1_γ for $\gamma \leqslant \alpha$ remain dense, we define

$$D(\gamma, B, B') = \{\langle a_j, b_j \rangle_{j < M} \in P : (\exists j < M-1)(\exists i, i')$$

$$\text{such that } \{q_i, q_{a_j}\} \subset D^0_\gamma \cap B, \{q_{i'}, q_{b_{j+1}}\} \subset D^1_\gamma \cap B',$$

$$i \in (a_j, a_{j+1}) \text{ and } i' \in (b_j, b_{j+1})\} . \quad (3.3)$$

By a similar but easier argument as above, one can show that $D(\gamma, B, B')$ is a dense subset of P.

This completes the inductive construction of the topologies $\tau^0 = \tau^0_{\mathfrak{c}}$ and $\tau^1 = \tau^1_{\mathfrak{c}}$. The family $\{E_n : n \in \omega\}$ is a family of dense subsets of the product space, and by condition 2, it is a witness to the fact that the product is not SS. Condition 5 ensures that, for each $\ell \in 2$, $\{D^\ell_\gamma : \gamma \in \mathfrak{c}\}$ lists all τ^ℓ-dense sets. Finally, condition 6 ensures that τ^ℓ is SS. $\qquad\square$

Let us remark that we have learned that the above result has been established independently by Bella and Gruenhage. In addition, L. Babinkostova has a stronger result from CH, namely that there are spaces X, Y such that $C_p(X)$ and $C_p(Y)$ are SS but the product is not.

In light of the fact that separable Fréchet spaces are SS, it is natural to wonder if the SS property is productive if the factors are Fréchet. We will show, this time from the continuum hypothesis that it is not. Although it is a stronger topological statement than 3.4, we include both proofs since the set-theoretic assumption is stronger and the ZFC questions remain open. Before dealing with the Fréchet product question, we turn our attention to maximal spaces and will use one of the methods from these results for the Fréchet result.

Again motivated by the results in [9], we turn our attention to maximal spaces. A

space is said to be *maximal* if it is crowded and it has no strictly finer crowded topology. We restrict our interest to maximal spaces which are also regular. Let us recall van Douwen's well-known result that there are regular maximal spaces. One can deduce more from his proof.

Proposition 3.5. [7] For any countable crowded regular space X, there is a stronger regular topology on X which contains a dense subspace D which is a maximal space.

The following result was proven from the hypothesis that $\mathfrak{d} = \omega_1$ in [9].

Theorem 3.6. There is a countable maximal space which is not selectively separable.

Proof. Let us start with the countable non-SS subspace $S \subset \square(\omega + 1)^{\omega}$ as discussed in Example 1.1. Apply Proposition 3.5 to expand the topology (on a dense subspace) D to a maximal regular topology. We check that D can not be SS. Of course D maps continously into a dense subset of S. Although a non-SS space can have a preimage which is SS, the reason that does not happen in this example is that the dense subsets, $\{D_n\}_{n \in \omega}$, of S from 1.1 which witness that S is not SS are dense open sets. It follows then that the sequence $\{D \cap D_n\}_{n \in \omega}$ are also dense in the maximal topology. The fact that there is no appropriate dense selection of finite sets for D follows easily from the fact that no such selection exists for the coarser topology on S. □

The next two results establish that the existence of a maximal SS space is independent of ZFC.

Theorem 3.7. It is consistent with ZFC that there is no maximal SS space.

Proof. Assume that X is a maximal crowded SS space and assume that ω is a dense subset. Let $\mathcal{F}$ be the filter of dense open subsets of ω. Since X is a maximal space, every dense subset of X is open (see [7]), hence $\mathcal{F}$ is also the (free) filter of dense subsets of ω. Since X is SS, it follows easily then that $\mathcal{F}$ is a P-filter in the (usual) sense that if $\{F_n : n \in \omega\} \subset \mathcal{F}$, then there is an $F \in \mathcal{F}$, such that $F \setminus F_n$ is finite for all n. Such an F can be chosen simply by applying the SS property applied to the descending sequence of dense sets $\{F_0 \cap \cdots \cap F_n : n \in \omega\}$. In addition, since X

is maximal (and every dense set is open), if $I \in \mathcal{F}^+$ (i.e. $I \cap F \neq \emptyset$ for all $F \in \mathcal{F}$) then its complement is not dense, hence I must have interior in X. In $\beta\mathbb{N}$ terminology, we have shown that $\mathcal{F}$ gives rise to a ccc P-set in ω^*. That is, the subset $K = \bigcap\{F^* : F \in \mathcal{F}\}$ is a P-set in ω^* which has the ccc (in fact, it is separable). To finish the proof, we note that it was shown in [16] that it is consistent that there are no such P-sets. $\square$

Theorem 3.8. ($\mathsf{MA}_{\mathrm{ctble}}$) There exists a maximal SS space.

Proof. Let us start with ω endowed with a crowded metric topology, let τ_0 be the countable base of clopen sets. Let $\{D_\alpha : \alpha < \mathfrak{c}\}$ be the listing of all dense τ_0-dense sets. Suppose that at stage α we have a zero-dimensional topology τ_α such that for each $\alpha < \mathfrak{c}$ the following conditions are satisfied,

1. If $\beta < \alpha$ then $\tau_\beta \subset \tau_\alpha$

2. If $\beta < \alpha$ and $\mathcal{D}_\beta$ is dense in τ_β, they remain dense in τ_α

3. For $\beta < \alpha$, B_β is either open or it has an isolated point in $\tau_{\beta+1}$.

At stage α, along with τ_α we also have the listing $\{D_\beta : \beta \in \alpha\}$ of dense subsets. If we are given a countable $S_\alpha \in [\alpha]^\omega$, hence a list $\{D_n : n \in \omega\} = \{D_\beta : \beta \in S_\alpha\}$, we use $\mathsf{MA}_{\mathrm{ctble}}$ to pick a new countable dense set D_α such that D_α can be expressed as a countable union of finite sets selected from each D_α. This ensures selective separability. Now to ensure maximality, if we are given any $B_\alpha \subset \omega$, we first assume that B_α is not currently open,then there is some $b_\alpha \in B_\alpha$ which is also in the closure of $\omega \setminus B_\alpha$ in τ_α. Let $D_\alpha =\mathrm{int}(B_\alpha) \cup (\omega \setminus B_\alpha)$ - which is of course dense. Now we use $\mathsf{MA}_{\mathrm{ctble}}$ to partition $D_\alpha = \bigcup_n D(\alpha, n)$ into dense sets. Also let $\{b(\alpha, n) : n \in \omega\}$ be the listing of complement of D_α. By assumption, $b(\alpha, n) \in \overline{D(\alpha, n) \setminus B\alpha}$ for all n. Now let us define a countable familily of disjoint sets, for each n and 0,1 $U(\alpha, n, 0) = D(\alpha, n) \cap \mathrm{int}(B_\alpha)$ and $U(\alpha, n, 1) = \{b(\alpha, n)\} \cup D(\alpha, n) \setminus B_\alpha$. Now we add them to our topology τ_α to get to $\tau_{\alpha+1}$ and see that $U(\alpha, n, 1) \cap B_\alpha = \{(b_\alpha)\}$. So b_α becomes an isolated point of B_α. $\square$

It will be useful to extract the following lemma from the previous proof. However we need a strengthening of it for use with Fréchet spaces. This also necessitates a strengthing of the set-theoretic assumption beyond $\mathsf{MA}_{\mathrm{ctble}}$.

Lemma 3.9. If X is a countable crowded space of weight less than $\mathfrak{p}$, $\mathcal{D} \subset \mathcal{P}(X)$ is a family almost disjoint converging sequences of X, $|\mathfrak{D}| < \mathfrak{p}$, and $S \subset X$ has dense complement and is almost disjoint from each member of $\mathcal{D}$, then there is an expansion of the topology obtained by adding countably many (crowded) clopen sets, in which S is a closed nowhere dense set, and each member of $\mathcal{D}$ is again a converging sequence.

Proof. Fix any countable subcollection $\mathcal{B}$ of clopen subsets of X which separates points (and assume that $\mathcal{B}$ is closed under the operations of complements and finite unions and intersections). We have the set S which is almost disjoint from each $D \in \mathcal{D}$ and what we want to do is to introduce new clopen sets which will preserve that each $D \in \mathcal{D}$ is converging, and which will ensure that S is closed and discrete. If S is finite there is nothing to do, so let $S = \{s_i : i \in \omega\}$ (a faithful enumeration). The plan, like in Theorem 3.8, is to produce countably many disjoint dense subsets of X. The difficulty is to ensure that the members of $\mathcal{D}$ are not split.

Define a poset P by $p \in P$ if there is an $n_p \in \omega$ and a finite sequence $\{A_i^p : i < n_p\}$ such that these sets are pairwise disjoint, and for each $i < n_p$, $s_i \in A_i^p$ is a compact subset of $\{s_i\} \cup X \setminus S$ which satisfies that for some finite set $F \subset X$, there is a finite set $\mathcal{D}' \subset \mathcal{D}$ such that $A_i^p \setminus F = \bigcup \mathcal{D}' \setminus F$

We define $p < q$ if $n_q \leqslant n_p$ and for each $i < n_q$, $A_i^q \subset A_i^p$. We show below that P is σ-centered, from which we deduce that we can find "generic" filters that meet any collection of fewer than $\mathfrak{p}$ dense sets. In particular, we see easily that for each $D \in \mathcal{D}$ and $x \in X$, $\{p \in P : (\exists i, j < n_p)x \in A_j^p \text{ and } |D \setminus A_i^p| < \omega\}$ is dense. Furthermore, for each non-empty open $U \subset X$ and each $i \in \omega$, the set $\{p \in P : A_i^p \cap U \neq \emptyset\}$ is dense. Given a filter $G \subset P$ meeting each of these dense sets, we define $A_i = \bigcup\{A_i^p : p \in G\}$ and observe that A_i will be dense and meet S at the point s_i. Furthermore, the family

$\{A_i : i \in \omega\}$ will be a partition of X. It follows easily that the topology we obtain by adding each $\{A_i, X \setminus A_i\}$ to the base will be as desired. It remains only to show that P is σ-centered.

Given any $p \in P$, we may choose a finite sequence $\{B_i^p : i < n_p\}$ of pairwise disjoint members of $\mathcal{B}$ so that $A_i^p \subset B_i^p$ for each $i < n_p$. If $p, q \in P$ are such that $n_p = n_q$ and $B_i^p = B_i^q$ for each $i < n_p$, then it is easy to see that $r = \{A_i^p \cup A_i^q : i < n_p\}$ is a common extension which is again separated by the same sequence $\{B_i^p : i < n_p\}$. Clearly then the poset P is σ-centered. $\qquad \square$

Theorem 3.10. (CH) There exist two countable Fréchet spaces whose product is not SS.

Proof. Let us start with ω as our base set and a standard countable base $\tau_0 = \sigma_0$ of clopen sets for a zero-dimensional crowded topology on ω. Choose the sequence $\{E_n : n \in \omega\} \subset \omega^2$ just as we did in Theorem 3.4. Let π_0 and π_1 denote the two coordinate projections on $\omega \times \omega$. For a set $Y \subset \omega$, define

$$E(Y, 0) = \pi_0[(\omega \times Y) \cap \bigcup_n E_n]$$

$$\text{and}$$

$$E(Y, 1) = \pi_1[(Y \times \omega) \cap \bigcup_n E_n].$$

Fix an enumeration $\{(x_\alpha, S_\alpha) : \alpha \in \omega_1\}$ for $\omega \times [\omega]^\omega$. We inductively choose countable bases τ_β, σ_β for crowded 0-dimensional topologies on ω. We also inductively choose families $\{Y_\beta : \beta < \alpha\}$ and $\{Z_\beta : \beta < \alpha\}$ of converging sequences with respect to the τ_α and σ_α topologies, respectively. For convenience we assume that $\lim(Y_\beta) \in Y_\beta$ and $\lim(Z_\beta) \in Z_\beta$ for each $\beta < \alpha$ (the limits are uniquely determined by the $\tau_0 = \sigma_0$ topology). Let $\{\langle F_n^\alpha : n \in \omega \rangle : \alpha \in \omega_1\}$ be an enumeration of all selections $\{F_n \in [E_n]^{<\omega} : n \in \omega\}$.

Suppose that at stage $\alpha < \omega_1$ of our induction the following conditions are satisfied for $\gamma < \beta < \alpha$:

  1. $\tau_\gamma \subset \tau_\beta$ and $\sigma_\gamma \subset \sigma_\beta$ are countable bases on ω,

2. for each n, E_n is dense in the product topology $\tau_\beta \times \sigma_\beta$, and $\bigcup_n F_n^\gamma$ is not dense in the product $\tau_\beta \times \sigma_\beta$.

3. Y_γ is a τ_β-converging sequence, $E(Y_\gamma, 1)$ is σ_β closed discrete, and if x_γ is a τ_β-limit of S_γ, then for some $\xi \leqslant \gamma$, $Y_\xi \cap S_\gamma$ is infinite and $\lim(Y_\gamma) = x_\gamma$,

4. Z_γ is a σ_β-converging sequence, $E(Z_\gamma, 0)$ is τ_β closed discrete, and if x_γ is a σ_β-limit of S_γ, then for some $\xi \leqslant \gamma$, $Z_\gamma \cap S_\gamma$ is infinite and $\lim(Z_\gamma) = x_\gamma$.

5. each of the families $\{Y_\xi : \xi < \beta\}$ and $\{Z_\xi : \xi < \beta\}$ are almost disjoint.

If α is a limit, then $\tau_\alpha = \bigcup_{\beta < \alpha} \tau_\beta$, $\sigma_\alpha = \bigcup_{\beta < \alpha} \sigma_\beta$, and all the inductive conditions are preserved. For the successor stage, i.e. $\alpha = \beta + 1$, we define τ_α and σ_α as follows. We have the sequence $\{F_n = F_n^\beta : n \in \omega\} \in [E_n]^{<\omega}$. Our plan is to first choose new clopen sets A to be added to τ_α and B to be added to σ_α with the property that $A \times B$ is disjoint from each F_n.

We will define A and B by a countable induction. Let $\{\xi_k : k \in \omega\}$ be an enumeration of α. Let $\{U_j : j \in \omega\}$ enumerate a clopen base for $\tau_\beta \times \sigma_\beta$. Finally, let $\{(i_k, j_k) : k \in \omega\}$ enumerate $\omega \times \omega$. For each $n \in \omega$, we define τ_0-closed sets A_n, A_n^-, B_n, B_n^-, so that

1. for $k < n$, $A_k \subset A_n$, $A_k^- \subset A_n^-$, $B_k \subset B_n$, and $B_k^- \subset B_n^-$,

2. $n \subset A_n \cup A_n^-$ and $n \subset B_n \cup B_n^-$,

3. $A_n \cap A_n^- = \emptyset$, $B_n \cap B_n^- = \emptyset$,

4. each of A_n and A_n^- is, mod finite, equal to a finite union of members of $\{Y_{\xi_k} : k \in \omega\}$ and, for each $k < n$, Y_{ξ_k} is, mod finite, contained in one of A_n, A_n^-,

5. each of B_n and B_n^- is, mod finite, equal to a finite union of members of $\{Z_{\xi_k} : k \in \omega\}$ and, for each $k < n$, Z_{ξ_k} is, mod finite, contained in one of B_n, B_n^-,

6. $A_n \times B_n$ is disjoint from $\bigcup_\ell F_\ell$,

7. each product from $\{A_n, A_n^-\} \times \{B_n, B_n^-\}$ meets $E_{i_k} \cap U_{j_k}$ for each $k < n$.

To start the induction, we can let each of A_0, A_0^-, B_0, and B_0^- be empty. Assume that $n \in \omega$ and we have chosen the sets A_n, A_n^-, B_n, and B_n^- satisfying the inductive conditions. Each of the conditions are preserved if we add the singleton n to A_n^- providing $n \notin A_n$, and similarly add n to B_n^- if $n \notin B_n$. With this possible change then, we may assume that $n+1$ is a subset of each of $A_n \cup A_n^-$ and $B_n \cup B_n^-$. We begin by considering the last inductive condition. Since $E(\{\ell\} \cup Y_\xi, 1)$ and $E(\{\ell\} \cup Z_\xi, 0)$ are nowhere dense in σ_β and τ_β respectively (for all $\ell \in \omega$ and $\xi \in \alpha$), it follows that the set

$$\left((A_n \cup A_n^- \cup E(B_n, 0)) \times \omega \right) \cup \left(\omega \times (B_n \cup B_n^- \cup E(A_n, 1)) \right)$$

is a nowhere dense set in the topology $\tau_\beta \times \sigma_\beta$. Since $E_{i_n} \setminus F_{i_n}$ is dense, we can choose a point $(a_n^0, b_n^0) \in U_{j_n} \cap E_{i_n} \setminus F_{i_n}$ which is not in that product. Consider any point (a_n^0, b) for $b \in B_n$. Since $a_n^0 \notin E(B_n, 0)$, it follows that $(a_n^0, b) \notin E_\ell$ for all $\ell \in \omega$. Similarly, for all $a \in A_n$, $(a, b_n^0) \notin E_\ell$ for all $\ell \in \omega$. In addition, the family $\{E_\ell : \ell \in \omega\}$ are pairwise disjoint, hence $(A_n \cup \{a_n^0\}) \times (B_n \cup \{b_n^0\})$ is disjoint from F_ℓ for all ℓ. It is routine to recursively repeat this process to similarly choose points $\{(a_n^i, b_n^i) : i < 4\} \subset E_{i_n} \cap U_{j_n}$ (so that each of $\{a_n^i : i < 4\}$ and $\{b_n^i : i < 4\}$ have four elements). It will then follow that $(A_n \cup \{a_n^0, a_n^1\}) \times (B_n \cup \{b_n^0, b_n^2\})$ will be disjoint from $\bigcup_\ell F_\ell$ (and of course that each of $(A_n \cup \{a_n^0, a_n^1\}) \cap (A_n^- \cup \{a_n^2, a_n^3\})$ and $(B_n \cup \{b_n^0, b_n^2\}) \cap (B_n^- \cup \{b_n^1, b_n^3\})$ are empty).

We next consider the converging sequence Y_{ξ_n} with limit y_n. Since $E((n \cup B_n \cup B_n^- \cup \{b_n^i : i < 4\}), 0)$ is closed discrete, there is an integer m_n so that $Y_{\xi_n} \setminus m_n$ is disjoint from $E((n \cup B_n \cup B_n^- \cup \{b_n^i : i < 4\}), 0)$. If $y_n \in A_n \cup \{a_n^0, a_n^1\}$, then we define $A_{n+1} = A_n \cup \{a_n^0, a_n^1\} \cup (Y_{\xi_n} \setminus m_n)$ and $A_{n+1}^- = A_n^- \cup \{a_n^2, a_n^3\}$. Otherwise, $y_n \notin A_n \cup \{a_n^0, a_n^1\}$, and we set

$$A_{n+1} = A_n \cup \{a_n^0, a_n^1\}$$

and

$$A_{n+1}^- = A_n^- \cup \{a_n^2, a_n^3\} \cup \{y_n\} \cup (Y_{\xi_n} \setminus m_n).$$

We have maintained the requirements that $A_{n+1} \times (B_n \cup \{b_n^0, b_n^2\})$ is disjoint from F_ℓ for all ℓ. We proceed similarly with Z_{ξ_n} and $z_n = \lim(Z_{\xi_n})$. There is an integer m_n' so that $Z_{\xi_n} \setminus m_n'$ is disjoint from $E((n \cup A_{n+1}), 1)$. If $z_n \in B_n \cup \{b_n^0, b_n^2\}$, then we define

$$B_{n+1} = B_n \cup \{b_n^0, b_n^2\} \cup (Z_{\xi_n} \setminus m_n')$$

and

$$B_{n+1}^- = B_n^- \cup \{b_n^1, b_n^3\}.$$

Otherwise, $z_n \notin B_n \cup \{b_n^0, b_n^2\}$, and we set $B_{n+1} = B_n \cup \{b_n^0, b_n^1\}$ and $B_{n+1}^- = B_n^- \cup \{b_n^1, b_n^3\} \cup \{z_n\} \cup (Z_{\xi_n} \setminus m_n')$. We have maintained the requirements that $A_{n+1} \times B_{n+1}$ is disjoint from F_ℓ for all ℓ. It should be clear that A_{n+1}, B_{n+1}, A_{n+1}^-, and B_{n+1}^- meet all the inductive requirements. Let $A = \bigcup_n A_n$ and $B = \bigcup_n B_n$ (hence $\omega \setminus A = \bigcup_n A_n^-$ and $\omega \setminus B = \bigcup_n B_n^-$). We generate new topologies from $\tau_\beta \cup \{A, \omega \setminus A\}$ and $\sigma_\beta \cup \{B, \omega \setminus B\}$ which we will temporarily denote by τ_α' and σ_α'. Of course we have ensured that $A \times B$ is disjoint from $\bigcup_\ell F_\ell$ and we have preserved that each E_ℓ is dense in $\tau_\alpha' \times \sigma_\alpha'$.

Now we define Y_β and take care to ensure that $E(Y_\beta, 1)$ is closed discrete in σ_α. Before starting, we select countably many σ_α' converging sequences to temporarily add to the collection $\{Z_\xi : \xi < \beta\}$ so that for each $\ell \in \omega$ and each $(n, m) \in \omega \times \omega$, there is a sequence, $T(\ell, n, m)$, in this collection, and a function from $T(\ell, n, m)$ into E_ℓ so that the range converges to (n, m). Now choose Y_β so as to be almost disjoint from each member of $\{Y_\xi : \xi \in \beta\}$, and to be a sequence which τ_α'-converges to x_β and, if possible, is contained in S_β. By a simple inductive thinning out process of Y_β, we can additionally enusure that $T(\ell, n, m) \setminus E(Y_\beta, 1)$ is infinite for each $\ell, n, m \in \omega$ (which uses the fact that $E(\{y\}, 1)$ is finite (even a singleton) for each $y \in \omega$). Now we apply Lemma 3.9 to expand the countable base σ_α' to a countable base σ_α so as to ensure $E(Y_\beta, 1)$ is closed discrete and while preserving that each member of the collection $\{Z_\xi : \xi \in \beta\} \cup \{T(\ell, n, m) \setminus E(Y_\beta, 1) : \ell, n, m \in \omega\}$ remains converging. The existence of the converging sequences $T(\ell, n, m)$ and the fact that τ_α' is not changing, ensures that each E_ℓ is dense in $\tau_\alpha' \times \sigma_\alpha$. Next, working with the topologies τ_α' and

σ_α, we repeat the process to suitably choose a σ_α converging Z_α (satisfying condition 4) so that by expanding τ'_α to a countable base τ_α, $E(Z_\alpha, 0)$ is closed discrete. This completes the inductive construction. $\qquad\square$

We had shown that, $\mathsf{MA}_{\mathrm{ctble}}$ implies that there are countable SS spaces whose product is not SS but we required CH to construct two countable Fréchet spaces whose product was not SS. Of course it is well-known that the Fréchet property itself is not finitely productive. In this section we begin by establishing that $\mathsf{MA}_{\mathrm{ctble}}$ is not sufficient by studying Fréchet spaces in the well-known Cohen model. This first result is certainly of independent interest.

Let us first introduce elementary submodels, before going into the CH proof.

Definition 3.3. For a set or class M, N, M is an elementary submodel of N, denoted by $M \prec N$, if $M \subset N$ and for all $n \in \omega$ and formulas ϕ with at most n free variables and all $\{a_1, ..., a_n\} \subset M$ the formula $\phi(a_1, ..., a_n)$ is absolute for M, N. That is $M \models \phi(a_1, ..., a_n)$ if and only if $N \models \phi(a_1, ..., a_n)$.

Definition 3.4. For a cardinal κ, the set $H(\kappa)$ is the set of all sets whose transitive closures has size less than κ.

Let us also recall the Cohen Forcing,

Definition 3.5. Let I and J be sets. We define

$$\mathrm{Fn}(I, J) := \{p \in [I \times J]^{<\omega} : p \text{ is a function}\}.$$

The Cohen forcing is $\mathrm{Fn}(I, 2)$ ordered by $p \leqslant q$ iff $q \subseteq p$.

If I, J are in the ground model $\mathbb{V}$ such that I is infinite, $J \neq \emptyset$ and G is $\mathrm{Fn}(I, J)$-generic over $\mathbb{V}$, then $f = \bigcup G$ is a new function from I onto J, called by Cohen Real.

We use the term "Adding a Cohen Real" to mean that forcing with the Cohen poset and getting a generic function as described above.

Now we can state the result and show the proof using an elementary submodel.

Theorem 3.11. In any model obtained by adding Cohen reals over a model of CH all countable Fréchet spaces have π-weight at most ω_1.

Proof. We assume our ground model satisfies CH and we consider forcing with $P = \mathrm{Fn}(\kappa, 2)$ where κ is some cardinal greater than ω_1. Let $\dot{\tau}$ be a P-name of a topology on ω so that $X = (\omega, \dot{\tau})$ is forced to be a Fréchet space. Let $\dot{A}_n$ denote the P-name which is forced to be the collection of all sequences converging to n. Let $\theta = 2^{\mathfrak{c}^+}$ and $M \prec H_\theta$ be an elementary submodel such that $M^\omega \subset M$ and $|M| = \omega_1$. Suppose also that $X, \dot{\tau}, \{\dot{A}_n : n \in \omega\}$ are in M. We will prove that $\dot{\tau} \cap M$ is forced to be a π-base for $\dot{\tau}$. This will rely heavily on the fact that the elementary submodel M is closed under ω-sequences. In particular, we have that if G is a P-generic filter, then $V[G \cap M]$ is a submodel of $V[G]$ which will satisfy that the interpretation of $\dot{\tau} \cap M$ will be a Fréchet topology on ω in which, for each n, the interpretation of $\dot{A}_n \cap M$ will be the collection of sequences converging to n (see [17, 4.5] for more explanation).

We now proceed by working within the model $V[G \cap M]$ (which we refer to as the ground model) and using that $V[G]$ is obtained by forcing over this model with $\mathrm{Fn}(\kappa \setminus M, 2)$. Through a standard abuse of notation, we may let $\dot{\tau}$ continue to denote the name for the final topology in $V[G]$. Now suppose that $\dot{U}$ is a name of a set forced to be non-empty and a member of $\dot{\tau}$. For each condition p, let $\dot{U}_p^-$ denote the set $\{x \in \omega : p \Vdash x \in \dot{U}\}$. Notice that $\dot{U}_p^-$ is a set in the ground model and is forced by p to be contained in $\dot{U}$. Also, by the elementarity assumptions on M, it also follows that p would force that the ground model closure of $\dot{U}_p^-$ would be contained in the closure of $\dot{U}$.

For a contradiction, let us assume that it is forced that the closure of $\dot{U}$ contains no ground model open set. In particular, by the assumptions on M, we then have that there is a condition p_0 and an integer n such that $p_0 \Vdash n \in \dot{U}$ and for all conditions $p \leqslant p_0$, $\dot{U}_p^-$ is nowhere dense.

Since $\dot{U}$ is a name of a subset of ω, we may choose a countable set $L \subset \kappa \setminus M$ so that

$dom(p_0) \subset L$ and for each $k \in \omega$ and each condition $p \in \text{Fn}(\kappa, 2)$, $p \Vdash k \in \dot{U}$ implies $p \restriction L \Vdash k \in \dot{U}$. In effect, $\dot{U}$ is a $\text{Fn}(L, 2)$-name, and let $\{p_\ell : \ell \in \omega\}$ enumerate those members of $\text{Fn}(L, 2)$ which extend p_0. Since, for each n, $\dot{U}_{p_0}^- \cup \dot{U}_{p_1}^- \cup ..... \cup \dot{U}_{p_n}^-$ is nowhere dense, it follows that, the complement of the closure of this union, D_n, is dense. As mention, [8, 2.9], each countable Fréchet space is SS, so there is a selection $F_n \in [D_n]^{<\omega}$ such that $\bigcup_n F_n$ is dense.

Since the space is Fréchet and $x \in \overline{\cup_n F_n}$, there is a sequence $S_x \subset \cup_n F_n$ converging to x. By the definition of the D_n's, we have that S_x is almost disjoint from $\dot{U}_p^-$ for each $p \in \text{Fn}(L, 2)$ which extends p_0. On the other hand, since S_x converges to x, we have, by elementarity, S_x converges to x in the final model, and so there must be a condition p which forces that S_x is almost contained in $\dot{U}$. This is the desired contradiction. $\square$

Corollary 3.12. In the Cohen model, finite products of countable Fréchet spaces are SS.

Proof. It was shown in [3], that if a space is separable and has π-weight less that $\mathfrak{d}$ then it is SS. Our last theorem shows that in the specified Cohen model, all countable Fréchet spaces have π-weight at most ω_1. So the product will also have π-weight at most ω_1, which is less than $\mathfrak{d}$. Therefore the product is SS. $\square$

Now for the next result, let us say something about proper posets.

Definition 3.6. Let P be a poset and let M be a countable elementary submodel of $H(\theta)$ for some cardinal number θ so that $P \in M$. We say that a condition $q \in P$ is (M, P)-generic if for each dense subset $D \in M$ of P and for all $r < q$ there exists a condition $p \in D \cap M$ so that r is compatible with p.

P is proper if for each regular $\theta > 2^{|P|}$ and each countable $M \prec H_\theta$ with $P \in M$ there is an (M, P)-generic condition below each $p \in M \cap P$.

The proper forcing axiom, PFA is very similar to Martin's Axiom, which is already introduced.

Definition 3.7. *PFA is the statement :* For any proper poset P and predense sets D_α,

$\alpha < \omega_1$, there is a filter $G \subset P$ so that for each $\alpha < \omega_1$, $G \cap A_\alpha \neq \emptyset$.

The open coloring axiom (OCA) is defined below,

Definition 3.8. *OCA is the statemnt:* Let X be a subset of $\mathbb{R}$. For any partition $[X]^2 = K_0 \cup K_1$ with K_0 open, either there is an uncountable $Y \subset X$ such that $[Y]^2 \subset K_0$, or there exist sets $\{H_n : n \in \omega\}$, such that $X = \bigcup_{n=0}^{\infty} H_n$ and $[H_n]^2 \subset K_1$ for all n.

The **Open Coloring Axiom** is a consequence of PFA. If $[X]^2 = K_0 \cup K_1$ with K_0 open, let us call $Z \subset X$ 0-homogeneous if $[Z]^2 \subset K_0$ and 1-homogeneous if $[Z]^2 \subset K_1$. The next theorem shows us the same conclusion as before assuming PFA.

Theorem 3.13. The proper forcing axiom, PFA, implies that products of finitely many countable Fréchet spaces are SS.

Proof. Let X and Y be countable Fréchet spaces and we assume that their product is not SS. There is no loss of generality to assume that neither X nor Y has isolated points. Let $\{E_n : n \in \omega\}$ be a sequence of dense subsets of $X \times Y$. It is known ([8, 2.7]) and easy to see that it is sufficient to show that each point $(x_0, y_0) \in X \times Y$ is in the closure of the union of some sequence of finite selections. So we fix a point (x_0, y_0). Without loss of generality, we may also arrange that the E_n's are a descending sequence. Let $\mathcal{A}_{x_0} \subset [X]^\omega$ and $\mathcal{B}_{y_0} \subset [Y]^\omega$ be the collection of all sequences converging to x_0 and y_0 respectively. Let $\{x_i : i \in \omega\}$ and $\{y_i : i \in \omega\}$ be enumerations of X and Y respectively. Since there is no harm to shrink the E_n's, we will assume that each E_n is disjoint from the closed nowhere dense sets $\{x_i : i < n\} \times Y$ and $X \times \{y_i : i < n\}$. For each $(A, B) \in \mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$, we may assume there is an m such that $E_m \cap (A \times B)$ is empty, because otherwise there is a suitable selection of finite choices accumulating to (x_0, y_0). To see this, first notice that there must be an n such that (x_0, y_0) is not in the closure of $E_n \cap (A \times B)$. It follows that such an E_n will satisfy that, for some $m > n$,

$$E_n \cap (A \times B) \subset ((\{x_i\}_{i<m} \times Y) \cup (X \times \{y_i\}_{i<m})).$$

Then we choose our $m > n$ by our additional assumption that E_m is disjoint from $\{x_i\}_{i<m} \times Y$ and $X \times \{y_i\}_{i<m}$.

Now we consider the poset P defined by the following:

$$P = \bigcup_n \Pi_{k<n} E_k$$

where P is ordered by set inclusion. Of course the members of P are just finite partial selections from the sequence $\langle E_k : k \in \omega \rangle$ and forcing with P gives rise to a name of a generic selection $\dot{F} = \{p(k) : k \in \omega\}$. Notice also that no x and no y will appear as a coordinate in infinitely many of the pairs $\{p(k) : k \in \omega\}$.

We will prove, using an auxiliary proper poset extending P, that there is a family of ω_1-dense sets which are sufficient to ensure that (x_0, y_0) is forced to be in the closure of $\dot{F}$. Establishing this completes the proof of the theorem since PFA implies there is a filter meeting each of those dense sets. The methodology is borrowed from the theory behind the development of the Open Coloring Axiom.

In the generic extension by P, notice that for any $A \in \mathcal{A}_{x_0}$ and $B \in \mathcal{B}_{y_0}$, we have that $F \cap ((A \times Y) \cap (X \times B)) = F \cap (A \times B)$ is finite (since some E_m misses $A \times B$). For $A \in \mathcal{A}_{x_0}$, let $\tilde{A} = F \cap (A \times Y)$ and, for $B \in \mathcal{B}_{y_0}$, let $\tilde{B} = F \cap (X \times B)$. Let $\mathfrak{X} = \{(\tilde{A}, \tilde{B}) : \tilde{A} \cap \tilde{B} = \emptyset\}$. Now we define $K_0 \subset [\mathfrak{X}]^2$ as follows:

$$\langle (\tilde{A}_0, \tilde{B}_0), (\tilde{A}_1, \tilde{B}_1) \rangle \in K_0$$

if

$$(\tilde{A}_0 \cap \tilde{B}_1) \cup (\tilde{B}_0 \cap \tilde{A}_1) \neq \emptyset.$$

The separable metric topology on $\mathfrak{X}$ is defined by the following: for finite subsets u_0, u_1, v_0, v_1 of $X \times Y$, the basic open sets are of the form

$$[(u_0, u_1), (v_0, v_1)] = \{(\tilde{A}, \tilde{B}) \in \mathfrak{X} : u_0 \subset \tilde{A}, u_1 \cap \tilde{A} = \emptyset, v_0 \subset \tilde{B}, v_1 \cap \tilde{B} = \emptyset\}.$$

Notice that K_0 is an open set in this topology.

Let $K_1 = [\mathfrak{X}]^2 \setminus K_0$. Since K_0 is open in $[\mathfrak{X}]^2$, then by [18], we can say that either $\mathfrak{X}$ is a countable union of 1-homogeneous sets or there is a proper poset, Q which introduces an uncountable 0-homogeneous set.

First we show that if indeed $\mathfrak{X}$ can not be covered by a countable union of 1-homogeneous sets then we obtain our desired selection F from the E_n's accumulating to (x_0, y_0). In this case then, there exists a P-name $\dot{Q}$ for a proper poset such that $\dot{Q}$ introduces an uncountable 0-homogeneous set. That is, there is a $P * \dot{Q}$-name of a sequence, $\langle (\dot{A}_\alpha, \dot{B}_\alpha) : \alpha \in \omega_1 \rangle$ of pairs from $\mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$ so that (it is forced that) $\{ (\widetilde{\dot{A}_\alpha, \dot{B}_\alpha}) : \alpha \in \omega_1 \}$ is a K_0-homogeneous subset of $\mathfrak{X}$. It is somewhat routine to verify that there is a family of ω_1-many dense subsets of $P * \dot{Q}$ so that an application of PFA ensures that we obtain an infinite selector F from $\langle E_n \rangle_n$ and a sequence $\{ (A_\alpha, B_\alpha) : \alpha \in \omega_1 \} \subset \mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$ satisfying that for each $\alpha \neq \beta \in \omega_1$,

$$(F \cap \tilde{A}_\alpha) \cap (F \cap \tilde{B}_\alpha) = \emptyset$$

and

$$F \cap [(\tilde{A}_\alpha \cap \tilde{B}_\beta) \cup (\tilde{A}_\beta \cap \tilde{B}_\alpha)] \neq \emptyset \text{ and is finite.}$$

The above properties are the requirements that the families $\{ F \cap (A_\alpha \times Y) : \alpha \in \omega_1 \}$ and $\{ F \cap (X \times B_\alpha) : \alpha \in \omega_1 \}$ form a Luzin gap and so, [20], cannot be mod finite separated in $\mathcal{P}(X \times Y)$. Now we show that if $U \times W$ is a neighborhood of (x_0, y_0), then $U \times W$ meets F – as required. Notice that $U \times Y$ will contain, mod finite, $F \cap (A_\alpha \times Y)$ for all $\alpha \in \omega_1$. Therefore there must be some $\alpha \in \omega_1$ such that $U \times Y$ meets $F \cap (X \times B_\alpha)$ in an infinite set. Since $X \times W$ will contain a cofinite subset of $F \cap (X \times B_\alpha)$, we then have that $U \times W$ meets $F \cap (X \times B_\alpha)$ (and hence F) in an infinite set.

So finally we complete the proof by showing that (in the extension by P) the family $\mathfrak{X}$ is not a countable union of 1-homogeneous sets. To see this, first we fix a P-name $\dot{\mathfrak{X}}$, for $\mathfrak{X}$. Suppose we have a P-name of such a sequence $\langle \dot{\mathfrak{X}}_n \rangle_n$ and a condition $p_0 \in P$ such that,

$$p_0 \Vdash \bigcup_n \dot{\mathfrak{X}}_n = \dot{\mathfrak{X}},$$

and for each n,

$$p_0 \Vdash [\dot{\mathfrak{X}}_n]^2 \subset K_1.$$

For better readability, let $A \setminus m$ abbreviate $A \setminus \{x_i : i < m\}$ for $A \subset X$ and $m \in \omega$, and similarly let $B \setminus m$ abbreviate $B \setminus \{y_j : j < m\}$ for $B \subset Y$. Recall that we showed that, for each $(A, B) \in \mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$, there exists m such that $((A \times Y) \cap (X \times B)) \cap E_m = \emptyset$. Therefore it follows that there is a sufficiently large m such that p_0 forces that $(\widetilde{A \setminus m}, \widetilde{B \setminus m})$ is a member of $\dot{\mathfrak{X}}$. Furthermore, there is an n and a $p < p_0$ in P, such that $p \Vdash (\widetilde{A \setminus m}, \widetilde{B \setminus m}) \in \dot{\mathfrak{X}}_n$. Let us define

$$\mathfrak{X}_{p,n,m} = \{(A, B) \in \mathcal{A}_{x_0} \times \mathcal{B}_{y_0} : p \Vdash (\widetilde{A \setminus m}, \widetilde{B \setminus m}) \in \dot{\mathfrak{X}}_n\}.$$

It is obvious that $\bigcup\{\mathfrak{X}_{p,n,m} : p \in P, n, m \in \omega\}$ should equal $\mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$. We will prove our claim by proving that this is not the case. First let us enumerate $P \times \omega \times \omega$ in order type ω as $\{(p_k, n_k, m_k) : k \in \omega\}$ and we will construct, by induction on k, a descending sequence $\{X_k \times Y_k : k \in \omega\}$ of subspaces of $X \times Y$ (with $X_0 = X$ and $Y_0 = Y$). To guide this induction we fix an ultrafilter $\mathcal{W}$ on $\omega \times \omega$ which is not a P-filter. We also choose a sequence $\{a_j : j \in \omega\}$ converging to x_0 and $\{b_l : l \in \omega\}$ a sequence converging to y_0. At any stage k in the induction we will let (p, n, m) denote the triple (p_k, n_k, m_k) and we deal with $\mathfrak{X}_{p,n,m}$. For each k, let $\mathbb{A}_k = \{A \setminus m : \exists B \ (A, B) \in \mathfrak{X}_{p,n,m}\}$ and $\mathbb{B}_k = \{B \setminus m : \exists A \ (A, B) \in \mathfrak{X}_{p,n,m}\}$ for $n \in \omega$. As an induction hypothesis we will assume that, for all m, $\{(j, l) : (a_j, b_l) \in [E_m \cap (X_k \times Y_k)]'\} \in \mathcal{W}$. This is true for X_0 and Y_0 as E_m is a dense set of $X \times Y$ for all $m \in \omega$. The construction of X_{k+1} and Y_{k+1} will also ensure that, for each pair $(A, B) \in \mathfrak{X}_{p_k,n_k,m_k}$, one of $A \cap X_{k+1}$ and $B \cap Y_{k+1}$ will be finite.

Now we show the inductive step. Let $S_k = \bigcup \mathbb{A}_k$ and $T_k = \bigcup \mathbb{B}_k$. Now a key step in the proof is that since $p_0 \Vdash [\dot{\mathfrak{X}}_n]^2 \subset K_1$, there must exist $\overline{m}$ such that $(S_k \times T_k) \cap E_{\overline{m}} = \emptyset$. In fact choose $\overline{m}$ larger than each of m and $dom(p)$ and assume that $(x, y) \in (S_k \times T_k) \cap E_{\overline{m}}$ is not empty. Extend p to some $\overline{p}$ so that $\overline{p}(\overline{m}) = (x, y)$ and observe that $\overline{p} \Vdash (x, y) \in \dot{F}$. Since $(x, y) \in S_k \times T_k$ there are (A_0, B_0) and (A_1, B_1) in $\mathfrak{X}_{p,n,m}$ such that $x \in A_0 \setminus m \in \mathbb{A}_k$ and $y \in B_1 \setminus m \in \mathbb{B}_k$ so that $\overline{p} \Vdash (x, y) \in \dot{F} \cap ((A_0 \setminus m) \times (B_1 \setminus m))$. However notice also that $(x, y) \in (\widetilde{A_0 \setminus m} \cap \widetilde{B_1 \setminus m})$ and

so $\bar{p} \Vdash \langle (\widetilde{A_0 \setminus m}, \widetilde{B_0 \setminus m}), (\widetilde{A_1 \setminus m}, \widetilde{B_1 \setminus m}) \rangle \in K_0$. Of course this contradicts that p forces that this pair is in K_1.

Now we are ready to define $X_{k+1} \subset X_k$ and $Y_{k+1} \subset Y_k$. If for all $\bar{m} > m$,

$$\{(j,l) : (a_j, b_l) \in [E_{\bar{m}} \cap ((X_k \setminus S_k) \times Y_k)]'\} \in \mathcal{W}$$

then put $X_{k+1} = X_k \setminus S_k$ and $Y_{k+1} = Y_k$. Otherwise we set $X_{k+1} = X_k$ and $Y_{k+1} = Y_k \setminus T_k$. To show that this works we must show that for all $\bar{m} > m$,

$$\{(j,l) : (a_j, b_l) \in [E_{\bar{m}} \cap (X_k \times (Y_k \setminus T_k))]'\} \in \mathcal{W} \ .$$

If this fails, then there is an $\bar{m} > m$ such that

$$\{(j,l) : (a_j, b_l) \notin [E_{\bar{m}} \cap ((X_k \setminus S_k) \times Y_k)]' \cup [E_{\bar{m}} \cap (X_k \times (Y_k \setminus T_k))]'\} \in \mathcal{W} \ .$$

However this implies that

$$\{(j,l) : (a_j, b_l) \in \overline{E_{\bar{m}} \cap (S_k \times T_k)}\} \in \mathcal{W},$$

which is impossible since it contradicts the fact that $S_k \times T_k$ is disjoint from E_m.

So we select all the X_k's and Y_k's satisfying our induction hypothesis. According to our construction, for each k, there is $j_k > k$ such that the sequence a_{j_k} is in X_k'. Now is the place where we use the hypothesis that X is Fréchet. For each k, choose a sequence $J_k \subset X_k$ converging to a_{j_k}. Since the sequence $\{a_{j_k}\}_k$ converges to x_0, we have that x_0 is in the closure of $\bigcup_k J_k$. Therefore there is a sequence $A \subset \bigcup_k J_k$ converging to x_0. By construction we have that $A \setminus X_k$ is finite for all k. By the similar argument as above we get a sequence B converging to y_0 with the property that $B \setminus Y_k$ is finite for all k. Therefore $(A, B) \in \mathcal{A}_{x_0} \times \mathcal{B}_{y_0}$ but clearly $(A, B) \notin \bigcup \mathfrak{X}_{p,n,m}$. □

CHAPTER 4: COMPLETELY SEPARATED IN RANDOM MODEL

4.1 Introduction

Ohta and Yamazaki asked [27] if every C^*-embedded subset of a first countable space is C-embedded. It is known [25] that a counterexample can be derived from the assumption $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$ and that if the Product Measure Extension Axiom (PMEA) holds then the answer is affirmative in some special cases.

We show that in the model obtained by adding supercompact many reals the question of Ohta and Yamazaki has a positive answer with no extra assumptions needed. It is well known that this model satisfies PMEA and therefore this result improves the one from [25].

One of the key devices in this section is that adding random reals does not introduce a continuous function that separates two ground model sets that were not completely separated.

4.2 Preliminaries

The purpose of this section is to establish the basic terminology. Our primary sources are [23] for topology; [2] and [4] for forcing and set theory (large cardinals and elementary embeddings).

Let X be a topological space. A subset $A \subseteq X$ is *C-embedded* if every continuous real-valued function with domain A can be extended continuously to X. If every continuous function from A into $[0,1]$ has a continuous extension to X then A is *C^*-embedded* in X.

A *zero-set* in X is a set of the form $f^{-1}(0)$ for some continuous $f : X \to [0,1]$. Two sets $A, B \subseteq X$ are *completely separated* if there is a continuous $f : X \to [0,1]$ such

that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$; equivalently, A and B are contained in disjoint zero-sets.

If j is a function whose domain is transitive we will denote by $j(a)$ the value that j assigns to the element $a \in dom\, j$ and $j''a$ will be used to represent $\{j(x) : x \in a\}$.

Let κ be a cardinal. We say that X has *character less than* κ (in symbols, $\chi(X) < \kappa$) if any point in X has a local base of cardinality $< \kappa$. 2^{κ} denotes the set of all functions from κ into $2 = \{0, 1\}$. For each $\alpha < \kappa$ the set $a_{\alpha} := \{f \in 2^{\kappa} : f(\alpha) = 0\}$ is a clopen subset of the topological product 2^{κ}.

Let $\mathcal{B}$ be the σ-algebra generated by $\{a_{\alpha} : \alpha < \kappa\}$. For each $\alpha < \kappa$ define $\mu(a_{\alpha}) = \mu(2^{\kappa} \setminus a_{\alpha}) = 1/2$. One can extend μ to obtain a probability measure on $\mathcal{B}$. This μ is called the Haar measure on 2^{κ}.

$2^{<\omega}$ is the set of all functions whose domain is an integer. Observe that when $\kappa = \omega$, $\mathcal{B}$ is generated by $\{[t] : t \in 2^{<\omega}\}$, where $[t] := \{f \in 2^{\omega} : t \subseteq f\}$, i.e. all the functions that extend t. Each $[t]$ will be called a basic clopen set for 2^{ω}.

Random real forcing is the poset $\mathbb{M}_{\kappa}$ obtained by identifying two members of $\mathcal{B} \setminus \{\emptyset\}$ if the measure of their symmetric difference is zero. $\mathbb{M}_{\kappa}$ is ccc and complete, i.e. if $S \subseteq \mathbb{M}_{\kappa}$ is not empty then S has a supremum in $\mathbb{M}_{\kappa}$, denoted by $\bigvee S$. In particular, if Φ is a formula and $\sigma_1, \ldots, \sigma_n$ are names so that $a \Vdash \Phi(\sigma_1, \ldots, \sigma_n)$, for some $a \in \mathbb{M}_{\kappa}$, then we define

$$\llbracket \Phi(\sigma_1, \ldots, \sigma_n) \rrbracket := \bigvee \{b \in \mathbb{M}_{\kappa} : b \Vdash \Phi(\sigma_1, \ldots, \sigma_n)\}$$

If S is a non-empty subset of $\mathbb{M}_{\kappa}$ and has a lower bound in $\mathbb{M}_{\kappa}$ then S has an infimum which will be denoted by $\bigwedge S$.

If τ is a topology for X and $\mathbb{P}$ is any forcing notion, then it could be the case that, in the generic extension, τ is no longer a topology for X due to the presence of new subsets of τ but τ will always be a base for some topology for X. Hence, whenever we refer to the topological space (X, τ) (or simply X) we will be referring to the topology on X that has τ as a base.

We recall that A submodel N of a model M is *elementary* if all formulas are absolute between N and M with respect to every set of parameters in N.

An embedding of V into a model M is an *elementary embedding* if its image is an elementary submodel of M.

If $j : V \to M$ is a non-trivial elementary embedding with M transitive, then M is inner, and induction on rank shows that there is a least ordinal κ moved by j, that is, $j(\alpha) = \alpha$ for all $\alpha < \kappa$, and $j(\kappa) > \kappa$. Such a κ is called the *critical point* of j, and is necessarily a measurable cardinal.

For a set X and a cardinal κ, let $\mathcal{P}_\kappa(X)$ be the set of subsets of X of cardinality less that κ. A cardinal κ is called λ-*supercompact*, where λ is an ordinal, if the set $\mathcal{P}_\kappa(\lambda)$ admits a normal fine measure. A cardinal κ is *supercompact* if it is λ-supercompact for every ordinal λ.

Instead of recalling the definition of a normal measure, we recall that a cardinal κ is λ-supercompact if and only if there is an elementary embedding $j : V \to M$ such that $j(\alpha) = \alpha$ for all $\alpha < \kappa$ and $j(\kappa) > \kappa$, where M is an inner model such that $M^\lambda \subset M$. For more information on supercompact cardinals, see [4] and [29].

4.3   Consistency Modulo a Supercompact Cardinal

We start this section with an auxiliary result which is itself of significant interest.

Theorem 4.1. Let κ be a cardinal. If X is a topological space and $A, B \subseteq X$, then the following are equivalent.

1. A and B are completely separated in the ground model.

2. $\mathbb{M}_\kappa \Vdash$ "A and B are completely separated"

Proof.   To show that (1) implies (2) note that any continuous function from the ground model remains continuous in the generic extension.

Now assume (2) and let $\dot{f}$ be a name for a real-valued continuous function on X so that

$$\mathbb{M}_\kappa \Vdash \text{``} \dot{f}[A] \subseteq \{0\} \wedge \dot{f}[B] \subseteq \{1\} \wedge \dot{f}[X] \subseteq [0, 1]\text{''}.$$

For each $0 < r < 1$ define $U_r := \{x \in X : \mu([\![\dot{f}(x) < r]\!]) > 1 - r\}$. We show below that $\{U_r : r \in (0,1)\}$ is a family of open sets satisfying $\overline{U_r} \subseteq U_s$ whenever $s < t$, and $A \subseteq U_r \subseteq X \setminus B$ for every r. And therefore the map $h : S \to [0,1]$ given by $h(x) := \inf(\{1\} \cup \{r \in (0,1) : x \in U_r\})$ is continuous, $h[A] \subseteq \{0\}$ and $h[B] \subseteq \{1\}$.

Let r be arbitrary. If $x \in U_r$ and $b := [\![\dot{f}(x) < r]\!]$, then there exists a name for an open set $\dot{W}$ so that $b \Vdash$ "$x \in \dot{W} \wedge \dot{f}[\dot{W}] \subseteq [0, r)$". Fix an antichain $\{b_n : n < \omega\}$ and a family $\{W_n : n < \omega\}$ of open sets from the ground model so that $b = \bigvee\{b_n : n < \omega\}$ and $b_n \Vdash x \in W_n \subseteq \dot{W}$. Since $\sum_{n<\omega} \mu(b_n) = \mu(b) > 1 - r$ there is an integer m for which $\sum_{n<m} \mu(b_n) > 1 - r$. Define $a := \bigwedge\{b_n : n < m\}$ and $O := \bigcap\{W_n : n < m\}$. Hence $a \Vdash \dot{f}[O] \subseteq [0, r)$ and therefore $1 - r < \mu(a) \leqslant \mu([\![\dot{f}(y) < r]\!])$ for each $y \in O$. Clearly $x \in O \subseteq U_r$ so U_r is open.

To prove that $A \subseteq U_r \subseteq X \setminus B$ observe that $\mu([\![\dot{f}(x) = 0]\!]) = 1$ and $\mu([\![\dot{f}(y) = 1]\!]) = 1$ for all $x \in A$ and $y \in B$.

To finish the proof assume that $r < s$ and let $x \in \overline{U_r}$ be arbitrary. Let $\mathcal{W}$ be the collection of all open sets from the ground model that contain x. For each $W \in \mathcal{W}$ the condition $b_W := \bigvee\{[\![\dot{f}(y) < r]\!] : y \in W \cap U_r\}$ satisfies $b_W \Vdash$ "$\dot{f}[W] \cap [0, r) \neq \emptyset$" and $\mu(b_W) > 1 - r$. Set $b := \bigwedge\{b_W : W \in \mathcal{W}\}$. Since $\{b_W : W \in \mathcal{W}\}$ is closed under finite intersections, we obtain $\mu(b) \geqslant 1 - r > 1 - s$. We also have that $b \Vdash \dot{f}(x) \leqslant r$ which implies $1 - s < \mu(b) \leqslant \mu([\![\dot{f}(x) < s]\!])$. Thus $x \in U_s$. $\square$

Assume that $\nu : \mathbb{M}_\kappa \to [0,1]$ is a probability measure. Note that the argument given above shows that if $\dot{f}$ is an $\mathbb{M}_\kappa$-name for a continuous real-valued function with domain X, then $h : X \to [0,1]$ given by

$$h(x) := \inf(\{1\} \cup \{r \in (0,1) : \nu([\![\dot{f}(x) < r]\!]) > 1 - r\})$$

is continuous.

Before proving the main theorem let us discuss a simplification that will be used: Any real-valued continuous function f can be expressed as $f = (f^+ + 1) - (f^- + 1)$, where both, f^+ and f^-, are continuous and non-negative. This simple remark shows

that a set A is C-embedded in X iff any continuous function from A into $[1, \infty)$ has a continuous extension to X.

Theorem 4.2. Let κ be a supercompact cardinal. In the model obtained by adding κ many random reals, every C^*-embedded subset of space whose character $< \kappa$ is C-embedded.

Proof. Let $\dot{X}$, $\dot{\tau}$, $\dot{A}$ and $\dot{f}$ be $\mathbb{M}_\kappa$-names so that $\mathbb{M}_\kappa \Vdash$ "$\chi(\dot{X}, \dot{\tau}) < \kappa$, $\dot{A}$ is C^*-embedded in $\dot{X}$ and $\dot{f} : \dot{A} \to [1, \infty)$ is continuous." As remarked above, it is enough to show that $\dot{f}$ has a continuous extension to $\dot{X}$. In order to do this we may assume that $\dot{X}$ and $\dot{A}$ have been decided, i.e. there are two sets (in fact ordinals) X and A from the ground model satisfying

$$1 \Vdash \text{``} \dot{X} = \check{X} \wedge \dot{A} = \check{A} \text{''}$$

Let G be an $\mathbb{M}_\kappa$-generic filter. Working in $V[G]$ we observe that $1/f$ is a function from A to $[0, 1]$ and so can be extended to a continuous map $h : X \to [0, 1]$. Note that we only have to prove that A and $Z(h)$ are completely separated. Indeed, if $s : X \to [0, 1]$ is a continuous function so that $s[A] \subseteq \{0\}$ and $s[Z(h)] \subseteq \{1\}$, then $1/(s + h)$ extends f.

In $V[G]$ let ρ be a name for the canonical random real added by $\mathbb{M}_\omega$. In other words, $\mathbb{M}_\omega \Vdash \rho : \omega \to 2$ and $\mu([\![\rho(n) = i]\!]) = 1/2$ for all $n \in \omega$ and $i < 2$, where μ is the Haar measure on 2^ω described before. Also let $\dot{g}$ be an $\mathbb{M}_\omega$-name for the piecewise linear extension of ρ on $[1, \infty)$, i.e. $\dot{g} \restriction [n, n+1]$ is the line segment that connects the points $(n, \rho(n))$ and $(n + 1, \rho(n + 1))$ for each positive integer n.

Fix a cardinal $\lambda > \max\{|X|, \mathfrak{c}\}$. Since κ is supercompact, there exists an elementary embedding $j_0 : V \to M$, where M is a transitive class closed under λ-sequences, so that $j_0(\alpha) = \alpha$ for each $\alpha < \kappa$ and $j_0(\kappa) > \lambda$. Therefore (see [24]) G can be extended to G^*, an $\mathbb{M}_{j(\kappa)}$-generic filter over M, and j_0 can be exentended to an elementary embedding $j : V[G] \to M[G^*]$ in such a way that $V[G]$ and $M[G]$ have exactly the same sets of rank $< \lambda$. As a consequence of this we obtain that $j(A)$ is a C^*-

embedded subspace of $(j(X), j(\tau))$ and $j(f)$ is a continuous function from $j(A)$ into $[1, \infty)$. Since $\dot{g}$ can be interpreted as an $\mathbb{M}_{\kappa+\omega}$-name and $\kappa + \omega < j(\kappa)$ we get that $g := \mathrm{val}(\dot{g}, G^*)$ is a continuous function from $[1, \infty)$ into $[0, 1]$. Hence $g \circ j(f)$ has a continuous extension $\psi : j(X) \to [0, 1]$.

Elementarily, the fact that (X, τ) has character $< \kappa$, and our choice of λ imply that $j \restriction X : X \to j''X$ is a homeomorphism (proof of Lemma 2.4 of [24]) where $j''X$ is considered as a subspace of $j(X)$. Thus the function $\varphi_0 : X \to [0, 1]$ given by $\varphi_0(x) = \psi(j(x))$ is continuous. To show that it extends $g \circ f$ we only have to observe that if $x \in A$ then $j(f)(j(x)) = j(f(x))$ by elementarity and that $j(f(x)) = f(x)$ because $f(x)$ is a real number.

The argument given above proves that there is an $\mathbb{M}_{j(\kappa)}$-name, $\dot{\varphi}_0$, for a continuous extension of $\dot{g} \circ \dot{f}$. Using the fact that $\mathbb{M}_{j(\kappa)}$ is ccc and assuming that $\dot{\varphi}_0$ is a nice name we can find an ordinal α for which $\dot{\varphi}_0$ is an $\mathbb{M}_{\kappa+\omega} * \mathbb{M}_\alpha$-name and $\kappa + \alpha + \omega < j(\kappa)$. Since $\mathbb{M}_{\kappa+\omega} * \mathbb{M}_\alpha$ and $\mathbb{M}_{\kappa+\alpha} * \mathbb{M}_\omega$ are forcing equivalent we can arrange things in such a way that $\dot{\varphi}_0$ is an $\mathbb{M}_{\kappa+\alpha} * \mathbb{M}_\omega$-name, G is extended to $\overline{G}$, an $\mathbb{M}_{\kappa+\alpha}$-generic filter over V, and, in $V[\overline{G}]$, ρ is an $\mathbb{M}_\omega$-name for the canonical random real added by $\mathbb{M}_\omega$. The rest of the argument takes place in $V[\overline{G}]$.

Let $\dot{\varphi}_1$ and $\dot{\varphi}_2$ be names for the maps $1 - \dot{\varphi}_0$ and $|\dot{\varphi}_0 - 1/2|$, respectively. If $b \in \mathbb{M}_\omega$ then $\mu_b : \mathbb{M}_\omega \to [0, 1]$ defined by

$$\mu_b(a) = \frac{\mu(a \wedge b)}{\mu(b)},$$

where $a \wedge b$ is the infimum of $\{a, b\}$, is a probability measure and therefore (see the remark following Theorem 4.1) the function $\psi_{b,i} : X \to [0, 1]$ given by

$$\psi_{b,i}(x) = \inf(\{1\} \cup \{r \in (0, 1) : \mu_b([\![\dot{\varphi}_i(x) < r]\!]) > 1 - r\})$$

is continuous for all $i < 3$.

We claim that if b is a basic clopen set then there is an integer n_b so that $f^{-1}[n_b, \infty) \subseteq \psi_{b,i}^{-1}[1/3, 1]$ for all $i < 3$. To see that this is true let $t \in 2^{<\omega}$ be such that $b = [t]$ and let $n_b \in \omega \setminus \mathrm{dom}\, t$ be arbitrary. The arguments needed for each individual i are similar

so we will present here only the case $i = 0$. Start with arbitrary $x \in f^{-1}[n_b, \infty)$ and $r \in (0, 1/3)$.

Define $c := [\![\dot\varphi_0(x) < r]\!]$ and fix integers $m \geqslant n$ and $k < 3$ such that

$$m + k/3 \leqslant f(x) \leqslant m + (k + 1)/3.$$

If $k = 0$ or $k = 2$ we obtain $c = [\![\rho(m) = 0]\!]$ or $c = [\![\rho(m + 1) = 0]\!]$, respectively, and therefore $\mu_b(c) = 1/2$. When $k = 1$, $c = [\![\rho(m) = \rho(m + 1) = 0]\!]$ and hence $\mu_b(c) = 1/4$. In any case, $\mu_b(c) < 2/3 < 1 - r$ which implies that $\psi_{b,i}(x) \geqslant 1/3$.

For each basic clopen set b and each integer $i < 3$ define

$$Z(b, i) := h^{-1}[1/n_b, 1] \cup \psi_{b,i}^{-1}[1/3, 1]$$

to obtain a zero-set in X that contains A. We will show that $Z(h)$ and $\bigcap \{Z([t], i) : t \in 2^{<\omega} \wedge i < 3\}$ are disjoint and thus A and $Z(h)$ are completely separated (recall that two sets are completely separated iff they can be separated by disjoint zero-sets). Given $z \in Z(h)$ let $a \in \mathbb{M}_\omega$ and $0 \leqslant k \leqslant 3$ be so that

$$a \Vdash k/4 \leqslant \dot\varphi_0(z) \leqslant (k + 1)/4.$$

There is an integer $i < 3$ depending entirely on k so that $a \Vdash \dot\varphi_i(z) \leqslant 1/4$. Fix a real number $1/4 < r < 1/3$. Since μ is the Haar measure on 2^ω we can apply the analogue to Lebesgue Density Lemma for μ (see Section 17.B of [26]) and claim the existence of a basic clopen set b for which $1 - r < \mu_b(a)$. On the other hand, $a \leqslant [\![\dot\varphi_i(z) < r]\!]$ and therefore $1 - r < \mu_b([\![\dot\varphi_i(z) < r]\!])$. Clearly $\psi_{b,i}(z) < 1/3$ and hence $z \notin Z(b, i)$. We just showed that, in $V[G]$, A and $Z(h)$ are completely separated after adding (an additional) $\alpha + \omega$ many random reals. According to Theorem 4.1 this implies that A and $Z(h)$ are completely separated in $V[G]$ and this finishes the argument. $\qquad\square$

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