ALGEBRAIC THINKING OF SIXTH GRADERS THROUGH THE LENS OF MULTIMODALITY

by

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ABSTRACT

UTE LENTZ. Algebraic thinking of sixth graders through the lens of multimodality. (Under the direction of DR. ANTHONY FERNANDES and DR. MICHELLE STEPHAN)

This study explores ways in which sixth graders without prior formal algebra instruction attempt to generalize algebraic growing patterns. In a teaching experiment setting, two pairs of students solved the growing pattern tasks while having access to a variety of manipulatives from which they could choose. Using multimodal analysis, two different levels of the students' generalization skills were highlighted: (a) recursive-local and (b) functional-global generalization. Multimodality is an interdisciplinary approach to discourse analysis that treats communication and forms of representation to be more than about language and gestures. This theory defines communication practices in relation to the linguistic, written, auditory, spatial, haptic, and visual resources—or modes—used to communicate ideas. The findings suggest that students who immediately used manipulatives to model the patterns have not developed the skills to move from the concrete recursive-local stage to the abstract functional-global stage. The students with spatial thinking skills and strong number sense arrived at the functional-global stage without the help of concrete materials. Implications of these findings point to the importance of training elementary students in number sense to enable them for a successful start into formal algebra.

Keywords: mathematics education, multimodality, mode affordances, algebraic generalization, early algebra, growing patterns, number sense, manipulatives

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER ONE: INTRODUCTION

Algebraic understanding has taken prominence in mathematics curricula worldwide. In the United States, there is a push for all students to learn algebra, and it has assumed the role of gatekeeper in higher education (Chazan, 2008; Kilpatrick & Izsák, 2008). Recent policy initiatives push for the fundamentals of algebra to be taught in grades as early as Kindergarten, in the hope that a longer exposure will provide the students with an easier transition into algebraic thinking (Carpenter, Franke, & Levi, 2003; Kaput, 1995, 1998; NCTM, 1989, 2000; Paul, 2005). Thus, algebraic thinking was introduced into the national elementary curriculum with the implementation of the Common Core State Standards in Mathematics (CCSSM) in 2010 (NGA & CCSSO, 2010). One of the five learning domains in the recent Common Core document for elementary school is titled *Operations & Algebraic Thinking*. With the introduction of formal pre-algebra in grades six through eight, the domain title changes to *Expressions & Equations* (NGA & CCSSO, 2010). This implies that during the elementary years the emphasis lies in teaching arithmetic with algebraic thinking interwoven rather than the more abstract formal algebra.

Algebraic Thinking

"Algebraic thinking is a very sophisticated cultural type of reflection and action, a way of thinking that was refined … over the centuries" (Radford, 2011, p. 319).

Defining Algebraic Thinking

Algebraic thinking commonly involves the process of generalizing arithmetic operations and, as it becomes more sophisticated, operating on unknown quantities. The five categories of algebraic thinking are (a) generalization and formalization of arithmetic processes, (b) the manipulation and transformation of certain equality problems through inverse operations and guiding syntax, (c) the analysis of mathematical structures, (d) the study of relations and functions including numbers and letters, and (e) algebra specific language and representation (Bednarz, Kieran, & Lee, 1996; Blanton & Kaput, 2005; Kaput, 1995; NCTM, 2000; Radford, 2000, 2011; Schliemann, Carraher, & Brizuela, 2007; Stephens, Blanton, Knuth, Isler, & Gardiner, 2015; Usiskin, 1999). Teachers must provide instruction that carefully guides students' algebraic thinking in terms of pattern recognition and mathematical generalization while acquiring arithmetic skills (Carraher, Schliemann, & Brizuela, 2000). For example, students discover the multiplicative identity property by exploring the concrete equation $5 \times 1 = 5$ with several different quantities. Then they recognize the pattern leading to the rule that any number multiplied by one keeps its value, its identity. Finally, they learn to generalize it in form of $a \times 1 = a$ using the same letter as the symbolic representation of the same number of any value.

Early Algebra

Given the abstract nature of algebra, there has been a regular debate among mathematicians, mathematics educators, and policymakers about the appropriate age for the introduction of algebraic thinking to younger students. Advocates of early algebra claim that arithmetic and algebra are not separable and that the young students have the capability of solving problems with unknown values and formulating their ways of thinking in their common language (Cai & Knuth, 2011; Carraher, Schliemann, Brizuela, & Earnest, 2006; Carraher & Schliemann, 2007; Kieran, 2007, 2011; Lent, Wall, & Fosnot, 2006; Martinez & Brizuela, 2006; Radford, 2012). Radford (2011) worked with second graders, followed them through fourth grade, and pointed out that even though young students have not yet learned algebra specific language and symbolism, they can use a combination of informal language, gestures, drawings, concrete objects, and different forms of representations to communicate their algebraic thinking.

Supporters of early algebra propose a sensitive and age-appropriate introduction of algebraic thinking (Blanton, Stephens, Knuth, Gardiner, Isler, & Kim, 2015; Cai & Moyer, 2008; Radford & Sabena, 2015). Teachers can provide instruction that carefully guides students' algebraic thinking in terms of pattern recognition and mathematical generalization while acquiring arithmetic skills (Brizuela & Schliemann, 2004; Carraher, Schliemann, & Brizuela, 2000; Schliemann, Carraher, & Brizuela, 2007). Paul (2005) argues that young children can learn the foundations of arithmetic and algebraic thinking combined during their elementary and middle grades rather than as separate mathematics courses. She suggests the revision of mathematics curricula and instructional strategies to engage younger children in algebraic thinking.

Algebraic thinking starts with the concrete experience of numbers and through activities moves towards generalization and abstract reflection (Mason, 2008; Radford & Sabena, 2015; Vygotsky, 1997). Even without access to the academic language and symbolism, younger children can be challenged to express their thinking related to general patterns like the one shown in Figure One (Carraher, Schliemann, & Schwartz, 2008; Mason, 2008).

Mason (2008) stated that when children begin to discover numbers, teachers could use this natural curiosity to channel the sense-making process toward algebraic thinking. Thus, algebraic thinking evolves from the arithmetic recognition of number patterns, which the child begins to generalize. Over time and with targeted instruction, young students' algebraic thinking becomes more sophisticated (Schliemann, Carraher, Brizuela, & Jones, 1998). Research suggests that multiple studies in early algebra, many of them long-term, examined the aspect of how well younger students can solve algebraic growing pattern tasks by analyzing their discourse, work samples, and assessment results.

Growing Pattern Generalization

Many studies have shown that pattern generalization lies at the core of algebraic thinking and lends itself to hands-on manipulation of materials. Radford (2011) stated "...generality is not specific to algebra. Generality is a typical general trait of human and animal cognition and can be of diverse nature—arithmetic, geometric or other" (p. 308). Thus, algebraic thinking comes naturally as a way of making sense of pattern structures and then generalizing them. The formal algebraic structures need to be made accessible so that students properly communicate mathematically about large number sequences that go past the required limits (Radford & Sabena, 2015). A typical example is the pyramid dot pattern regularly increasing in size (Kindt, 2004).

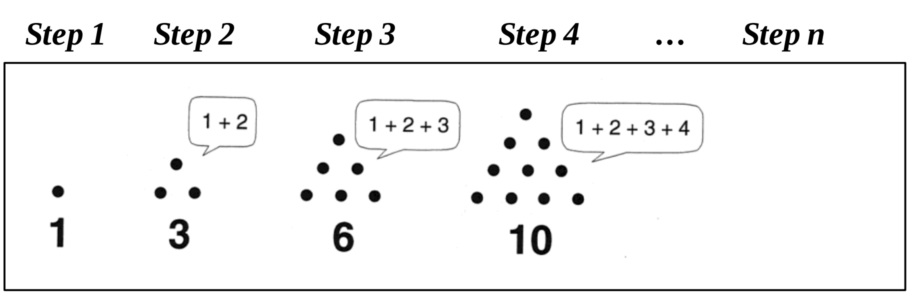


Figure 1. Pyramid pattern.

Multimodality

Modes of Communication

"The doing and communicating of mathematics is never a purely intellectual activity; it involves a wide range of bodily actions …" (Edwards & Robutti, 2014, p. 2). Developing and communication related to algebraic thinking entails much more than speech and writing, especially in young students who have not yet been introduced to the formal language and symbolism of algebra (Radford, 2009, 2010). Recent developments in the theory of multimodality show that communication goes beyond the speech and written modes to include other modes like gestures, drawings, and manipulation of concrete objects (Edwards & Robutti, 2014; Smith, 2014). The framework of multimodality has its roots in psychology, theater, and linguistics where modes of nonverbal communication often are equally if not more important than spoken words. Only recently, researchers in the field of mathematics education have begun applying multimodal frameworks, which provide insight into the spectrum of affordances of modes for mathematics education. Given that younger students are still learning algebra and the associated academic language, a multimodal approach has potential to highlight the mathematical thinking of these students extending beyond their speech and writing.

Mathematical Affordances of Modes

Learners make meaning of new situations based on prior experiences by comparing them to the observations and consequences from a similar event before. Learning mathematical concepts of any kind requires meaning-making of a number relationships. The learning process involves the entire body, not just the brain. Younger children initially use their fingers as support until they have developed the concept of counting and are able to move past the concrete gestures (Alibali, Church, Kita, & Hostetter, 2014; Bazzini & Sabena, 2015; Goldin-Meadow, Levine, & Jacobs, 2014;). The modes identified by Edwards and Robutti (2014) as the most common affordances for mathematics learning, specifically for the generalization activities in this study, are language in speech and text, formal notations of mathematical symbols, visual imagery such as diagrams or highlighting marks, and motor actions with or without artifacts. A more detailed description of the characteristics of the individual affordances will be given in the following chapters.

Purpose of the Study

This study builds upon a pilot study conducted with 10- and 11-year-old students who solved a growing pattern problem. The students demonstrated a move towards analytical thinking using symbols, though not in the traditional way. The students in the pilot had developed a quasi-equation with a variable despite a lack of formal instruction in algebra. It was evident that they simply converted what they had expressed verbally and through actions into an abbreviated format using the letter *t* for table preceded by the numeral 2 describing the second example from their chart (Figure 2). The respective signs symbolized the operations, addition and multiplication, and the value comparison. The fifth graders used recursive reasoning going along with the pattern they analyzed rather than developing a direct formula by viewing the variables as unknown values.

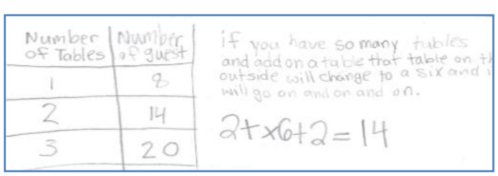


Figure 2. Quasi-formula expressing algebraic thinking.

This study involved the extension of a pilot study to examine the benefits of using manipulatives to enhance algebraic thinking. Such materials are often recommended for and used in elementary and middle grades mathematics to support students with tactile, spatial, and visual learning styles. Investigated were activities leading up to algebraic generalization of sixth graders (age 11) from regular education mathematics classes who all had performed at a level 2 on their most recent standardized test. The participants had not received formal algebra instruction before engaging in a variety of growing pattern tasks for this study. The tasks varied in structure to elicit the use of a variety of modes in students' explanations.

Research Question

The following research question guided this study.

In what ways do rising sixth graders draw on multiple modes to engage in algebraic generalization as they solve pattern tasks?

Conclusion

The following chapter involves a discussion of the prior research related to algebraic thinking and more specifically generalization. The multimodal framework guiding this study will also be outlined in Chapter 2. Chapter 3 contains the discussion of the design of the study, the tasks, data collection, and data analysis. The teaching experiment methodology, referenced in detail in Chapter 3, will guide the students' partner activities with growing pattern tasks. Pictorial growing pattern tasks are chosen for this study for two reasons. One, sixth graders can relate to the growing patterns, as they have seen and experienced such situations many times in their mathematics classrooms. It is a useful means to guide them to thinking about the abstract *n*th step of a pattern. Two, growing pattern tasks lend themselves to solicit the students' use of a variety of sensory and motor modes. Multimodality is utilized as the framework for analyzing the activities. This offers insight into the choices of bodily and material modes and allows inferences concerning the thought processes. Chapter 4 involves a discussion of the results from the study, and Chapter 5 offers a conclusion and possible implications for practitioners and future research.

CHAPTER TWO: LITERATURE REVIEW

Algebra is a gatekeeper to high school graduation that places the responsibility on mathematics educators to make it manageable for all students (National Mathematics Advisory Panel, 2008). In recent decades, there has been a push for the early implementation of algebraic concepts to ease students' access to formal algebra. Much research has been done on how students of all grade and age levels are able to develop algebraic thinking given age-appropriate instruction. The purpose of this study is to extend this research by investigating how younger students draw on multiple modes to communicate their algebraic thinking about growing patterns. The following research question will guide the study:

> In what ways do sixth graders draw on multiple modes to engage in algebraic generalization as they solve growing pattern tasks?

The first part of the literature review recaps research in algebraic thinking specifically geared to young students' learning. The second section is taking a closer look at a growing pattern generalization. The third part of the review outlines the multimodal framework and involves a discussion of the intersection of research in mathematics education and multimodality.

Algebraic Thinking in Elementary Grades

Patterns are typically categorized into number patterns, patterns in computational procedures, repetition patterns, linear and quadratic patterns, geometric/pictorial patterns, and others in the literature (Radford, 2011; Radford & Sabena, 2015; Schliemann et al., 2007; Stephens et al., 2015). It is a natural trait to human and animal cognition to recognize and generalize patterns in the environment. The ability to extend patterns and sequences is not specific to algebra but has been adapted to it. Algebraic thinking begins with children's physical experience of numbers in familiar situations as simple as needing two shoes to get dressed or discovering that they can count to 10 on their fingers. Many toys are designed to teach finding similarities of colors, shapes, and quantities. As they begin their journey as students, they learn through instruction to generalize arithmetic, then to think analytically about undetermined quantities, and finally to predict outcomes. The rule, or function, behind the calculation is recognized as a method of making use of the structure they know.

Scholars agree that generalization happens in phases. When students first begin to analyze a growth pattern, they connect their prior experiences to the new task. They take the first steps to discover the growth rate from one position to the next. The natural strategy used by most students is recursive addition. By working their way to higher position numbers, the students come to the realization that it becomes more and more burdensome to add or multiply step-by-step to reach the goal. Eventually a formula is developed that allows the substitution of the variables with any position number to calculate the total of elements in that position. Several different terms for the phases of pattern generalization can be found in the literature, for example tentative and refined (Polya, 1957), concrete and abstract, recursive and implicit (Lannin, 2005), or empirical and theoretical (Doerfler, 1991). For this study, I am using the approach by Amit and Neria (2008) who summarized the two strategies to generalization as recursive-operational-local and functional-conceptual-global.

Research has uncovered that pattern generalization begins with simple one-dimensional repeating patterns during early childhood. The recognition of a pattern happens with the discovery of the unit that repeats (Threlfall, 1999). For example, the pattern in XYZXYZXYZ can be identified as three repeats of the unit XYZ. As the brain develops and the pool of references grows, pattern recognition becomes more sophisticated in number of elements, size of the unit, or shape, as well as form, color, or orientation of the elements. The next step from identifying the unit of repeat is to recognize the rule with which greater elements of the pattern can be found (e.g., the 50th or the 1000th figure). From about third grade on, students are capable of generalizing linear patterns in such a way, but the strategies may not necessarily be algebraic in nature. In studies with 900 high school students, ages 13 to 16, and over 300 elementary and middle grades students, ages 7 to 11, it was determined that students used arithmetic problem-solving approaches based on their experiences in earlier years. Reasons for difficulties students had with algebraic thinking were the lack of knowledge of algebraic meaning of the unknown, the perception of the equal sign as command to calculate numbers, and the fact that often a trial and error approach with arithmetic calculations was preferred (Russell, Schifter, & Bastable, 2011; Schoenfeld, 1992; Stacey & MacGregor, 1999; Threlfall, 1999). Even a group of 36 pre-service elementary teachers demonstrated a preference for non-algebraic problem-solving strategies. The ability to discover the growth rate from one element to the next was not difficult to do, but it required some training in algebraic thinking that participants eventually found a way to the general formula for the *n*th element (Zazkis & Liljedahl, 2002).

Radford (2010) suggests there is an early stage of formula development during the generalization process where variables do not come into play yet. In this phase, one is just discovering the rule behind the pattern progression from one figure to the next. Formal notation is not yet part of the process, only the trial-and-error and mental organization of the growth pattern. Radford (2011) calls this specific part of generalization the early stage and describes it as in-action-formula. He argued that this is embodied generalization (Figure 3), which can occur in the form of gestures, utterances, and other actions such as pointing to elements of a figure in a growth pattern while thinking about larger figures that are not depicted. Although this kind of activity is seemingly of concrete nature, it requires higher order thinking that is communicated through a variety of coordinated modes such as words, gestures, tools, imagery, and interactive visuals. The student is "showing thereby the multi-modal nature of factual algebraic thinking" (Radford, 2010, p. 7).

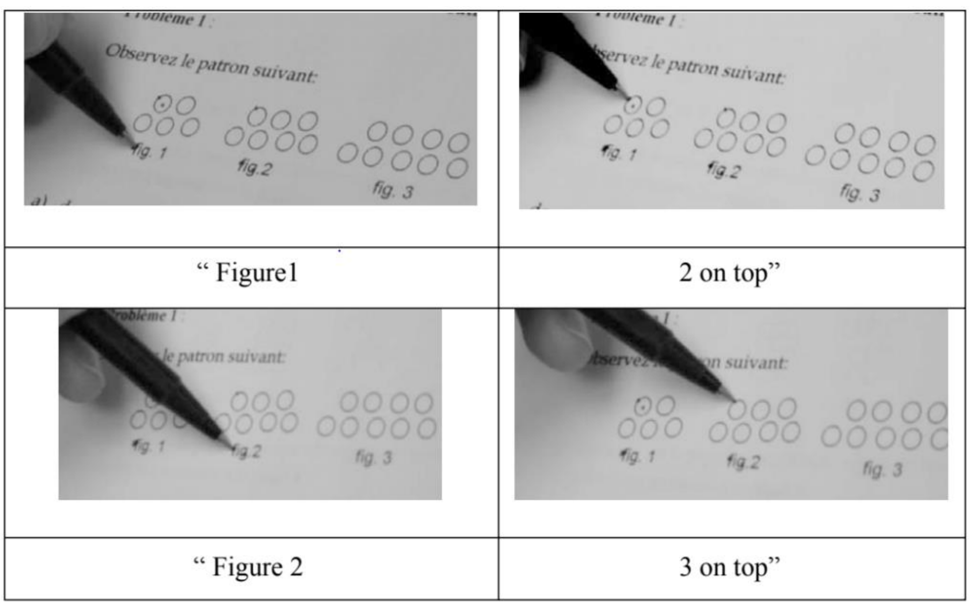


Figure 3. In-action embodied formula (Radford, 2011).

With reference to early algebra, this study is aimed at contributing findings about how elementary students can develop the means of communicating algebraic thinking. Mathematics and algebra have their specific language. While acquiring basic pattern recognition and counting concepts might come naturally, more complex mathematical thinking and communication in terms of generalized patterns, unknown quantities, placeholders, relationships, formulas, function systems, and others, must be learned much like a foreign language. Mathematical language also has a hierarchical structure in which concepts build logically on each other (Halliday, 1985). In addition to learning the vocabulary and symbols with their respective meanings, the young student also must develop the conceptual understanding in both language systems. Only when the basics are learned, the student can understand mathematical text whose grammatical structures give it the special scientific meaning (Blanton & Kaput, 2000; Carraher et al., 2006; Dougherty, 2003; Harper, 1978; Halliday, 1976; O'Halloran, 2015; Radford & Sabena, 2015; Usiskin, 1999).

Mathematical discourse research in the early development of algebraic thinking, due to its social interactive nature, relies on spoken and written language accompanied by gestures that students are using at specific ages to generalize mathematical ideas and to solve problems. Non-verbal behaviors and influences of the physical surroundings are generally explained as context of the process (Carraher et al., 2008; Radford, 2002, 2009, 2011; Mason, 2008; Radford & Sabena, 2015). Algebraic thinking needs to be introduced and made accessible to students. Such activities can happen mentally, but just as well by using motor skills while drawing or manipulating objects (Radford, 2011).

Students typically begin with a counting strategy known as recursive sense-making, since they are most familiar with it as they have demonstrated in the pilot study (Figure 3). They draw the objects. Then they count both the tables and the chairs of each situation and compare the change in the relationship. Once they have counted three or four steps in the growing pattern, they realize that they gain three seats but lose one each time a table is added. Their strategy changes to recursion, meaning they build on one or more previous terms to construct the following term. The realization may be that the end tables provide three seats, while those in between only allow two.

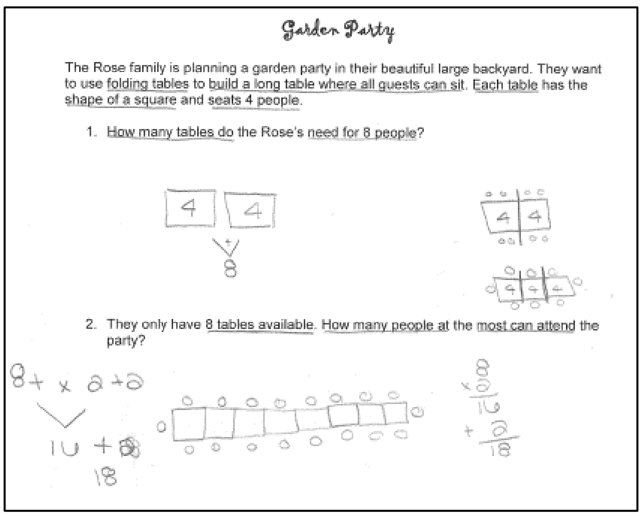


Figure 4. Pilot study: Counting and adding.

This is the whole-object strategy where students are using multiples of a unit to create a larger portion of the long table. Here it can happen that they over-count or under-count depending on the choice of unit. In a next step, students arrive at the contextual construction of a rule based on the relationship between table and chair numbers. Guessing and checking without consideration follow this strategy (Radford, 2011, 2013).

The commonly proposed path to teach generalization is from concrete to abstract, realistic mathematics education developed by the Freudenthal Institute in the Netherlands being one of many curricula with this approach (Gravemeijer & Doorman, 1999; Hough & Gough, 2007; Van den Heuvel-Panhuizen & Drijvers, 2014). Students gradually move from concrete, hands-on activities toward generalization by applying recursive reasoning to abstract algebraic thinking expressed in direct formulas containing variables. This requires knowledge of specific mathematical symbols and language. As concepts are logically building upon each other, the content-specific language provides the vertical, hierarchical structure needed to understand the mathematical text with its specific meanings (Carraher et al., 2008; Halliday, 1985; Mason, 2008; Sabena, 2008). In grade four, students learn to use letters as placeholders for unknown values, although the term variable is neither explicitly mentioned in the fourth nor in the fifth-grade learning goals; variables are officially introduced as representation for unknown numbers in grade six (CCSSI, 2010). The fifth graders in the pilot study demonstrated the first steps toward abstract thinking (Figure 4). However, the variable was naturally not acknowledged as a symbol for the unknown. Rather the students used the *t* as abbreviation for tables when rewriting their verbal description in symbols.

Exemplary tasks as suggested by Carraher and Schliemann (2007) can be utilized to gather experience with the teaching experiment methodology by sequencing missing addend problems such as "Ben had 8 baseballs and caught some more during a Minor League game he watched. He went home with 11 baseballs. How many did he catch that night?" By developing the number sentence $8 + \bullet = 11$, the students most likely use subtraction to solve it and are utilizing the additive inverse axiom. Now it is not a simple arithmetic addition problem anymore but turns into an algebraic relationship (Carraher & Schliemann, 2007). This technique can lead to students' development of a general form of their representations and thus of an understanding of the basic algebraic properties of the number system. The participants of the pilot study arrive at a final expression containing a variable without the intervention of the teacher-researcher. However, the notation reveals that they utilize the letter *t* as an initial for "table" instead of a placeholder for an undetermined number of tables (Figure 4). Some participants chose to draw and write out their thoughts while others organize the discovered pattern in a chart.

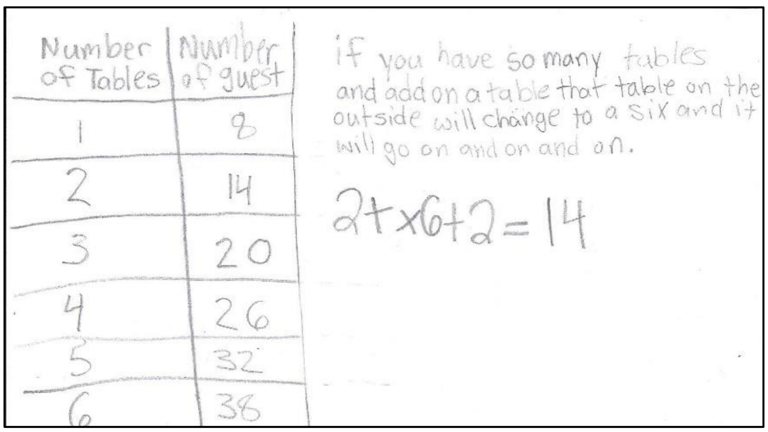


Figure 5. Attempt of more general description.

The participants in the pilot study quickly engaged in explaining their views and drawing and writing their thoughts on paper. The pilot was based on research at the time that was looking at early algebra. The students in the pilot study only had paper and pencil available. Growing pattern tasks lend themselves to be utilized for a multimodal framework because students of all ages have prior experiences with discovering and thinking about patterns. For this study, a broad variety of material media will be readily accessible to the students and they will make their choices of which to use independently.

Multimodality

The multimodal framework has its origins in linguistics, psychology, and other medicinal fields. The multimodal approach to researching communication during mathematical meaning-making, and interaction goes beyond language and gestures. It has gained importance for communicating and representing meaning in education in conjunction with the fast-growing communication technology over the past two decades. A multimodal framework not only spans embodied modes such as aural expression, gesture, posture, facial expression, actions, and gaze, it also includes any kind of physical artifacts and environments, as well as the selection of modes a person will make in a certain situation to communicate meaning. Multimodality takes also into account the social norms of meaning making set by the respective community in which the person is acting (Bezemer, 2012).

The multimodal methodology (Edwards & Robutti, 2014) provided a tool to differentiate among modes the students were choosing to communicate their algebraic thinking. Modes are not set entities and can be defined in variations, and the human mind does not operate linear and can lead others to intended perceptions (Norris, 2004). Discourse analysis is a staple in mathematics education. Multimodality as a framework in the fields of linguistics, art, and others has gained importance in conjunction with technological progress we have experienced since the 1990s. Children articulate topics they want to talk about on their level of languag**e** development, but verbal expression is only one component of communication. I used a lens of multimodality to understand how they draw on various modes including speech, gesture, posture, facial expression, actions, and gaze (Edwards & Robutti, 2014) to explain notions of generalization in specially designed tasks involving algebraic growing patterns.

The idea behind observing multimodality in mathematics activities is that it is not solely an abstract subject that requires mental activity. Doing mathematics and reasoning about thoughts and actions is just as important. The multimodal methodology (Edwards and Robutti, 2014) provides a tool to differentiate among modes the students are choosing to communicate their algebraic thinking. Modes are not set entities and can be defined in variations, and the human mind does not operate linear and can lead others to intended perceptions (Norris, 2004). Discourse analysis is a staple in mathematics education. Multimodality as a framework in the fields of linguistics, art, and others has gained importance in conjunction with technological progress we have experienced since the 1990s. Children articulate topics they want to talk about on their level of languag**e** development, but verbal expression is only one component of communication. I am using a lens of multimodality to understand how they draw on various modes including speech, gesture, posture, facial expression, actions, and gaze (Edwards and Robutti, 2014) to explain notions of generalization in specially designed tasks involving algebraic growing patterns.

Communication in verbal or non-verbal form is first on the list as a concept that will never leave the one who has the understanding unless the person shares it. Sharing thoughts serves several purposes. The person who communicates the concept to someone else goes through a reflection process while speaking or reading her or his work. Then a connection is established to the other person who might or might not find the same importance in the concept. Symbols are the tools for communication, but there needs to be a consensus about the meaning of them. Language and culture barriers can hinder the proper exchange of information. When new concepts are being communicated, it matters if it is a primary concept that can simply be taught by pointing at visuals and making connections to spoken and written words. Since mathematical concepts are exclusively secondary concepts, a learner's mind must be trained for the use of word and symbol combinations through a multitude of examples. We can use meanings of symbols to our advantage by choosing a meaning appropriate for the circumstance in which it is utilized. The author calls this classification. The function of explanation is rather important for someone who wants to teach a concept.

Three situations can lead to failure here. The explaining person either uses an inappropriate schema, the new idea might be too far removed from the schema it is supposed to fit in, or the schema does not have the capacity to assimilate the new concept. The way in which a new concept can be transferred into a reflective activity depends on the developmental stage of the learner, especially when he or she has not left child age yet. A suggested strategy is to think aloud in addition to more individualized visual activities. Symbols are means to communicate structure, which is helpful to transfer often used exercises into a stage of routine and to remember learned symbols or formulas later. Visual organizers support this kind of structured memorization (Pimm, 1987, 1995; Skemp, 1987; Wertsch, 1990).

Communication is the only possibility to let the world know about concepts because it is an individual, internal effort to develop conceptual structure. The symbols change their appearance from a mental object into a physical object brought onto the paper or transformed into sound waves. The internal symbols are the basis of what

Skemp (1987) calls the deep structure of mathematical thinking (p. 177). External symbol systems represent the surface structure, the external systems for which we have much less variety available than for the deep structure within our minds. Thus, mathematical manipulatives are symbols that enable the learner to do mathematics by forming complements to concepts using hands and fingers (Pimm, 1995). The deep structures are the conceptual, the schema development that occurs internally. Both surface and deep structures are connected in constant exchange of information.

Skemp (1987) distinguishes 10 functions of symbols that provide the ability to learn concepts and develop schemata. He presents the examples of the numerals 1 2 3 … As natural numerals they represent an ascending order from left to right each where of them counts one less than the following. As natural numbers, it is possible that each represents a place value ten times more than the neighboring number to the right. This analysis shows the enormity of the deep structure compared to the limited possibilities to communicate mathematical ideas through the surface structure. Symbols are the tools for communication, but there needs to be a consensus about the meaning of them. Since mathematical concepts are exclusively secondary concepts, a learner's mind must be trained for the use of word and symbol combinations through a multitude of examples (Skemp, 1987; Pimm, 1987, 1995). The function of the symbols is connected to the concepts that provide them with distinct meanings (*Figure 5*).

Multimodality and Affordances for Mathematics Education

> The starting point for multimodality is to extend the social interpretation of language and its meanings to the whole range of representational and communicational modes or semiotic resources for making meaning that are employed in a culture – such as image, writing, gesture, gaze, speech, posture. (Jewitt, 2011, p. 1)

Multimodality as a framework has gained importance in conjunction with the technological progress we have experienced since the 1990s. Children articulate topics they want to talk about on their level of languag**e** development. Considering that algebra for 700 to 800 years has been communicated purely rhetorically before its language entered the syncopated stage, verbal descriptions of tasks and the possible solving steps thereof should be the natural way for young students to verbalize their algebraic thinking (Katz & Barton, 2007). Words, gestures, facial expressions, utilizing various forms of representation, and more are modalities that allow inferences concerning the student's mathematical maturity (Radford, 2011; Radford & Sabena, 2015; Schliemann et al., 2007; Vygotsky, 1997). Interaction, communication, and representation within and between societies go far beyond language. The framework of multimodality considers the multitude of modes that a person uses in the context of an activity leading to socio-academic meaning-making (Norris, 2004; O'Halloran, 2015). Besides the linguistic interaction, participants in discussion rounds display non-verbal behaviors such as gestures, facial expressions, or postures that contribute to what they communicate. Another aspect affecting communication of any kind is the material environment. Room temperature, noise level, or the placements of furniture play an often-underestimated role during a conversation (Bazzini & Sabena, 2015; Gana, Stathopoulou, & Chaviaris, 2015; Gerofski, 2015; Norris, 2009).

The idea behind observing multimodality in mathematics activities is that it is not solely an abstract subject that requires mental activity. Doing mathematics and reasoning about thoughts and actions is just as important. Mathematics on every level is a combination of cognitive, sensory, and bodily activities and thus controlled by the mind. For this study, I will use Edwards & Robutti's (2014) four-category framework.

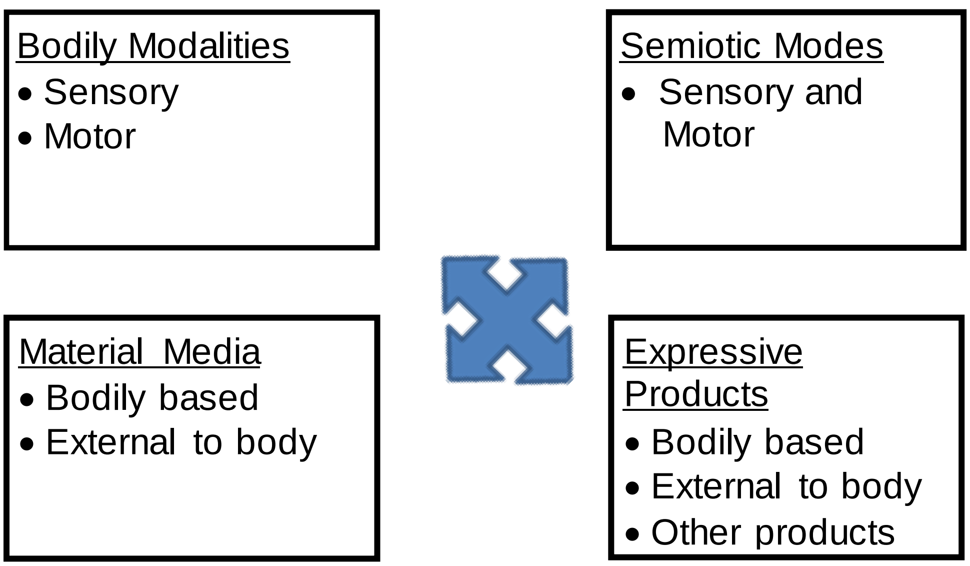


Figure 6. A framework for multimodality (Edwards & Robutti, 2014).

The diagram (Figure 7) shows the complexity of multimodality as all four areas must be considered parallel over the course of a growing pattern activity. The bodily modalities include the senses and the motoric capabilities of the acting person. The material media are the means the person chooses and controls through physical actions. These can be parts of the body as well as external materials such as paper and pencil or math manipulatives. Relating to the selected material media, the person also chooses one or more semiotic modes to convey her ideas. The semiotic modes are combinations of both motor and sensory since they require bodily action and stimulate one or more senses at the same time. The outcomes are the expressive products. They can be bodily based or external to the body and result from the combined choices of bodily modalities, semiotic modes, and material media. Gestures as expressive products, bodily based, for example are motoric movement (bodily modality, motor) by a hand or hands (material media, bodily based) and can occur to reinforce the expression of a sound (semiotic mode). Edwards and Robutti (2014) are referring to several studies on doing geometry that observed differences of students' learning experiences regarding the material media that were available to them and conclude that conceptual learning may be quite different.

For the purpose of observing and interpreting students' modes while solving algebraic growing pattern tasks, the affordances of modalities visible in the respective expressive products for mathematics (Edwards & Robutti, 2014) were adapted as follows (Figure 7). One of the challenges of multimodal analysis is to grasp and interpret the context in which the interaction is happening. When gestures are observed in isolation, they may not necessarily support the discourse and actions that occur simultaneously. Speech, the choice of words and tone, a pencil stroke, moving an object for a demonstration are all layers of meaning-making initiated by the task on hand. All these blended actions determine the following reactions in the collaborating group (Smith, 2014). During social interaction events, the mode of communication often is not speech alone and, in many situations, it is not used at all. A variety of other modes is coming into play such as but not limited to, gesture, posture, facial expression, actions, and gaze (Edwards & Robutti, 2014; Ferrara, Robutti, & Edwards, 2014; Gather, Alibali,

& Goldin-Meadow, 1998; Jewitt, 2008; Kress, 2005; Nemirovsky, 2003; Radford, 2009, 2011). Even the physical environment is considered as providing modes that are an important aspect to the interaction. Furniture placement, light conditions, acoustics, or temperature in a classroom are just a few examples (Gana et al., 2015; Kress, 2009; Norris, 2004; Smith, 2014;).

Modalities	**Expressive Products**	**Affordances for Mathematics**
• Language	• Speech • Written text	• Momentary; linear, analytic subunits; accompanied by volume, rhythm, prosody for emphasis • Permanent; linear, analytic
• Formal notations (FN)	• Written mathematical symbols	• Permanent; linear and synthetic; compression of complex, abstract ideas
• Visual Imagery (VI)	• Static diagrams, graphs, geometric, conventional • Marks drawn for emphasis	• Permanent; global/holistic, analytical; meaningful parts by convention; • Often spontaneous; can be permanent or momentary; global/holistic, synthetic
• Motor Actions (MA)	• Gestures empty-handed • Gestures holding artefacts • Gestures involving objects in environment • Other bodily actions such as posture, gaze	• Momentary; global, synthetic • More limited boundaries and location while gesturing • Objects can be incorporated into the meaning of the gestures • Momentary; global, synthetic; have specific affordances

Figure 7. Affordances and expressive products of modalities adapted from Edwards and Robutti (2014).

Kress (2009) and Norris (2004) state that speech does not communicate everything a person experiences, feels or thinks. Instead, in social interaction we use a combination of multiple modes such as speech, gestures, gaze, tactility or various forms of representations (Edwards & Robutti, 2014; Fernandes, Kahn, & Civil, 2017; O'Halloran, 2015; Norris, 2004; Radford, 2010). Researchers have demonstrated that in addition to their informal language, younger students can draw on various modes like gestures, body movement, and concrete objects to explain their mathematical thinking (Nemirovsky, 2003; Radford, 2005, 2010, 2011; Radford & Sabena, 2015).

I used a multimodal framework (Edwards & Robutti, 2014; Jewitt, 2009; Kress, 2009; Jewitt, Kress, Ogborn, & Tsatsarelis, 2000; Norris, 2004; Radford & Sabena, 2015) to develop a better understanding of how the 11-year-old children embodied and expressed their algebraic thinking as they engaged in generalization problems through the modes they are choosing. Due to the nature of the activities, the students likely draw on more than one mode at a time. This is referred to as mode overlays, also called co-occurrences or density, of modes, which will be expected indicators of the students' level of engagement (Norris, 2004; Radford & Sabena, 2015; Smith, 2014). The combined analysis of the modes the students are choosing during the key moments of their problem-solving activity and their oral and written work may indicate the gain of mathematical understanding.

Growing Pattern Tasks through the Lens of the Multimodal Framework

Multimodality is a framework that has been used in the field of social sciences for some time. In education, it is rooted in the discourse study of the subject areas English language and literacy. More recently, multimodality has informed research in areas such as science and mathematics education. In mathematics education, several studies have been conducted researching gesturing of students during problem-solving processes (Alibali, Kita, & Young, 2000; Cobb, Yackel, & McClain, 2000; Goldin-Meadow, 2003; McNeill, 1992; Rasmussen, Stephan, & Allen, 2004; Sabena, Radford, & Bardini, 2005; Singer, Radinsky, & Goldman, 2008). Regardless, usually the teacher or researcher was making the choice of which tangible material media, such as paper and pencil, shapes, computers, etc., they want to provide for the students and prepare one or two media. The students then were expected to use them, but it brings up the question whether the teacher truly can conclude that the students gain a better conceptual understanding by utilizing the specifically provided manipulatives for example. If the students had a variety of options, what would they choose and in what modes would the students "communicate" during the process of solving the task? It will be interesting to see what differences and commonalities will arise between the pairs of sixth graders when they work on their growing pattern tasks.

Conclusion

Multiple studies have been conducted on algebraic thinking and generalization in all age groups from elementary school to secondary level, including discourse analysis and gesturing. Only recently has the multimodal framework been utilized in mathematics education research to develop an understanding of the complexity of action and interaction for a young student during a sense-making process. The following methodology chapter explains the research design. It gives the rational for the choice and sequence of algebraic growing pattern tasks leading to a better understanding of the multiple simultaneous modes of embodied mathematics. Results from this study are expected to provide information and clarification of students' ways of mathematical sense-making and doing mathematics through the lens of multimodality to inform best practices for the teacher-researcher to facilitate the group activities with the most useful modifications to the coaching.

CHAPTER THREE: METHODOLOGY

Overview of This Study

This section lays out the research design, setting and participants, the data collection process, and the interpretive framework for the data analysis. Two pairs of 11-year-old sixth grade students participated in an after-school teaching experiment setting at the middle school where the teacher-researcher taught. The growing pattern activities took place before they had received the formal introduction to pre-algebra, which was scheduled toward the end of the school year. During the activities, the participants had a variety of material media available and could choose freely which ones to use for the respective growing pattern tasks. The 16 sessions were digitally videotaped from two different angles to warrant two perspectives of bodily modalities for better analysis. The purpose of the data analysis was to interpret the algebraic activities of and conversations between the group partners considering both the choices of affordances of modalities and the level of algebraic thinking the students reached.

Research Design

The constructivist nature of the research presented itself as an observational study. The teaching experiment methodology appeared to be the best fit for a teacher-researcher to investigate the embodied means the children were using to express their algebraic thinking processes. The teaching experiment has been designed as a method that provides teacher-researchers with the flexibility of combining the interviewer role with the possibility of intervention. The method has been developed for the field of mathematics education based on Piaget's clinical interview methodology (Creswell, 2005; (Kelly, Lesh, & Baek, 2014; Komorek & Duit, 2004; Piaget, 1969; Steffe, 1983).

The characteristics of the teaching experiment are that it can stretch over multiple sessions and that the interview setting is transformed into a learning session. It can include two or more participants who interact with each other and allows the teacher-researcher to intervene, but she does not necessarily have to lead the session. This was a well-suited methodology for this study.

The students were working on the growing pattern task and likely needed a little guidance toward generalizing their findings. The intention was to give them room to develop their ideas about the rules behind the growing patterns and to provide ample opportunity to choose the sensory and motor modes they deemed fitting for the task at hand (Radford, Bardini, Sabina, Diallo, & Simbagoye, 2005; Radinsky, Goldman, & Singer, 2008). Intervention in the form of guiding questions became necessary here and there for several reasons. The conversation between students sometimes died down due to the lack of ideas how to approach the task, but the opposite was the case at times as well, when one or both participants talked excitedly and did not take the time to listen to each other. On few occasions, the teacher-researcher monitored students' concept limitations pointing them in the right direction (Steffe & Thompson, 2000). The teaching experiment methodology was suitable to investigate answers to the research question: In what ways do sixth graders draw on multiple modes to engage in algebraic generalization as they solve growing pattern tasks?

Setting and Participants

Setting

The study was conducted at the middle school of a small North Carolina city school district. As the teacher-researcher, I chose this site for several reasons. Since this was my work place, it did not take much time to organize and conduct the group sessions, and I did not need to spend time building a rapport with the participating students. The principal was supportive of teacher-research and collaboration with UNC Charlotte. I was familiar with the facilities and the technology, and I could ask for help from specialists in case of an emergency, as they had experience with the equipment and with videotaping research sessions. There was no financial aspect involved.

The research took place in the district's only middle school with a low socio-economic and culturally diverse population. About 75% of the students were eligible for free or reduced lunch. The make-up of the student body was roughly 30% African-American, 30% Latino, 35% White, and 5% other ethnicities. The eight sessions per group took place in my classroom in the format of an after-school club. Extracurricular school activities were common at the school and students were motivated to participate in a math club.

Rationale for Participant Selection

Investigated was the algebraic thinking of sixth grade students as they engage in growing pattern generalization. A lens of multimodality was chosen to understand how 11-year-olds draw on various modes including speech, gesture, posture, facial expression, actions, and gaze (Edwards & Robutti, 2014; Ferrara et al., 2014; Gather et al., 1998; Jewitt, 2008; Kress, 2005; Nemirovsky, 2003; Radford, 2009, 2011) to explain notions of generalizations in growing pattern tasks. In a 3-year longitudinal study, Radford (2000) followed a small group of three students from Grades 7 to 9 observing the problem-solving processes during similar tasks. In another 3-year study, he investigated the algebraic thinking processes of a group of three students from Grades 2 to 4 (Radford, 2010). That leaves a gap of Grades 5 and 6, which partially has been satisfied here. Considering that the introduction of pre-algebra is scheduled for Grade 6 (CCSSM, 2010), the four sixth graders participating in this study were finding themselves at the doorstep from elementary to middle and secondary algebra instruction.

Participants

Potential participants were selected from 16 sixth grade classes, which were considered regular education classes. Students with learning disabilities or giftedness received mathematics instruction in separate classes and were not intended to be subjects of this study. Honors and gifted math students had pre-algebra instruction beginning in fifth grade and students with learning and other disabilities would have needed much more guidance, which was beyond the scope of this study.

All interested students from the 16 regular education classes received a recruiting letter including a parental permission form explaining the timeline and purpose of the activities. Students and their parents that agreed to the participation received an assent as well as a parent consent form. The final selection of eight participants was made in the order in which the completed forms were returned. The selected participants worked in four pairs, each on a different day of the week. Only having one group each afternoon allowed me as the teacher-researcher to direct my full attention on each pair of students while conducting and video recording the sessions.

Due to attendance issues, two groups of the four did not completed all eight sessions in partner work as planned. Thus, the sessions were excluded from this study. The four remaining participants were Group A consisting of an African-American girl and a Latino boy, and Group B of a Latino boy and an African-American boy. All four were 11 years old at the time and had performed on Achievement Level 2 on their most recent standardized mathematics assessment at the end of fifth grade.

Data Collection

In this study, I looked at the algebraic generalization of sixth graders with Achievement Levels 2 on their most current standardized test who were placed in regular mathematics classes. The participants were working on four multi-step, thought-eliciting tasks with growing patterns they could connect to their prior experiences (Lesh, Hoover, Hole, Kelly, & Post, 2000). The first task presented dot patterns in V- and W-shapes related to flying geese and airplane formations. The second growing pattern depicted ascending stairsteps. The third task was about table and chair arrangements at a garden party. Finally, they investigated a growing triangular toothpick pattern. The students were expected to discover the growth rate of the patterns from the first few figures quickly by applying a recursive-operational-local strategy. As they were working through the first task with the growing V-pattern, they were guided toward formulating the rule and writing a general formula. Thus, they would move to the functional-conceptual-global stage, if they could successfully complete this step.

Data Condensation

The process of qualitative data analysis is tri-fold. It consists of the condensation of the data, the display, and the conclusion (Miles, Huberman, & Saldaña, 2014). I first deselected the sessions when participants could not be there. This eliminated the dot pattern I task for Group B and dot pattern II for Group A. The dot pattern tasks were closely related, II being the extension of I, and a group comparison based on correct and complete solutions were not of interest in this study. Thus, the recordings were kept in the selection. The remaining recordings were narrowed down to eight, four each for groups A and B. I took the data condensation even further, because I was specifically interested in certain moments, or episodes, during the students' algebraic thinking and generalization process in combination with their uses of communicative modes. I then selected specific episodes of 7 to 10 minutes to go into depth concerning the overlapping actions and conversations.

The coding process requires two cycles of viewing and listening to the recordings and studying the transcripts. Saldaña (2013) offers a multitude of coding methods for the first cycle but points out that the method chosen by the researcher must be the best fit for the purpose and research question of the study. The codes are utilized to either find or align with themes. For my study, the descriptive coding method appears to be most fitting. Since the context is mathematical generalization, the themes in this case are predetermined. The students are either applying recursive generalization by thinking of larger figures in the patterns in additive manner, or they can jump to explicit generalization and set up a global formula. I want to pinpoint certain key moments when the students recognize the growth rate of a pattern. A second possible event that might occur is the step from concrete to a general approach of describing the pattern growth, which would mark the move toward abstract thinking. The coding of video/audio transcripts and image frames analyzed in the context of the participants' interactions is an established data analysis method in the field of discourse analysis (Jewitt, 2011; Norris, 2004).

Sequencing of the Growing Pattern Tasks

Amit and Neria (2007) worked with students of the same age group as in this study. The 50 participants ages 11-13 also had not received formal algebra instruction yet. To determine eligibility for a math club, the students had to demonstrate their ability to generalize growing patterns. The authors distinguished between two generalization strategies the students would apply solving the tasks. Complete solutions in the sense of formal algebra were not expected since the students lacked the knowledge.

The basic operational adding strategy students first applied to find the rate of growth of a pattern was labeled "recursive-local." The repetitive action of systematically adding to one figure to create the following figure is commonly called "recursive." The term "local" referred to the first few figures of a pattern, possibly printed on the paper in front of the students, manageable for anyone (Amit & Neria, 2007; Becker & Rivera, 2004; Rivera, 2007; Stacey, 1989). In this study, the strategy is referred to as recursive-operational-local with the intention to capture the local activities of adding in numerical form, drawings, or creating tables.

The students advanced in the generalization process, which means the thinking and reasoning turned to greater figures not visible on their papers anymore and became more abstract and holistic. The advanced strategies evolved to functional thinking finding the unique rule that governed a growing pattern. As the figures were not right there anymore, the thinking had to turn global; and the operational background was now multiplicative thinking (Amit & Neria, 2007; Becker & Rivera, 2004; Rivera, Knott, & Evitts, 2007; Stacey, 1989). Thus the strategy describing the advancement in generalization is labeled "functional-conceptual-global" in the data analysis.

The decision on the sequencing of the growing pattern tasks for this study was made based on the criteria of the levels of linearity and imagery included in the problems (Amit & Neria, 2007; Rivera, Knott, & Evitts, 2007). The multimodal aspect was added grounded on whether the use of mathematics material in for of manipulatives was required or not. The first three tasks did not direct the students to use manipulatives, here described as non-material. Thus, the fourth task was chosen, because it gives explicit instructions to build the figures from toothpicks.

1. Dot patterns (I and II): linear, pictorial, non-material
2. Stair-like structure problem: non-linear, pictorial, non-material
3. Garden party task: linear, non-pictorial, non-material
4. Toothpick pattern task: linear, pictorial, material

I expected the growing pattern tasks to elicit the choice of a variety of material media. Depending on the modes, expressive products such as students' drawings and the manipulations of concrete materials should emerge and allow multimodal analyses of the students' actions and interactions. The multimodal analysis, as it was used in this study, entailed the dissection of certain generalization episodes into the overlaying modes and the simultaneous algebraic thinking. The purpose was to discover possible correlations between the choice of material media, modes, and algebraic thinking leading to generalization.

The growing pattern tasks were intended to be student-lead to allow time for thinking and interacting. As the teacher-researcher, I observed the activities and only sporadically provided pointers or answered questions as needed. The groups had materials available to address various modes of communicating their ideas as listed in

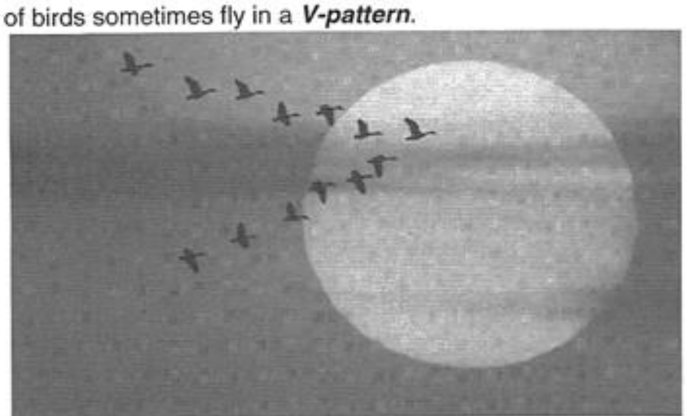
Figure 2. The tasks were designed for middle and high school students (Gravemeijer et al., 2010; Kindt, 2004). The participants were used to manipulatives and math talks as this had been an integral part of their mathematics learning during the elementary years in the school district. From the experiences in the pilot study, I was aware that younger students would be able to engage in the tasks and likely describe a generalized solution, albeit in an untrained notation. Thus, I chose the growing pattern tasks and sequence for the following reasons.

Dot patterns I and II: V- and W-pattern. The two dot patterns served as introduction to the generalization sessions. The dot pattern activities were taken from the workbook *Positive Algebra* by Kindt (2004) published by the Freudenthal Institute at the Utrecht University, Belgium. The Dot Pattern I mimicked a V-shaped flight pattern of geese, and in the extension in Dot Pattern II a W-shaped flight pattern of an airplane squadron. The worksheets provided images of the geese and airplanes flying in distinct formations. The images were familiar to the students and the growing rates of the patterns could be simply expressed by using addition and multiplication.

The worksheets spelled out in detail which steps the students had to take to arrive at the generalized formulae. Following the picture, the first three or four positions of the growing pattern were depicted and explained. The double-faced round chips with a yellow and a red side were the expected choice of manipulative, if the students would choose to use something from the material media menu. The first bullet point requiring action was a larger number of dots printed in the V-shape without the position number. Dot Pattern II was a table to be completed with position numbers and the responding number of dots. In four more steps, the participants were directed to find the number of dots in larger positions, 85 and 25, and finally to create direct formulae with the provided variables.

Dot patterns (I)

Groups of birds sometimes fly in a ***V-pattern***.



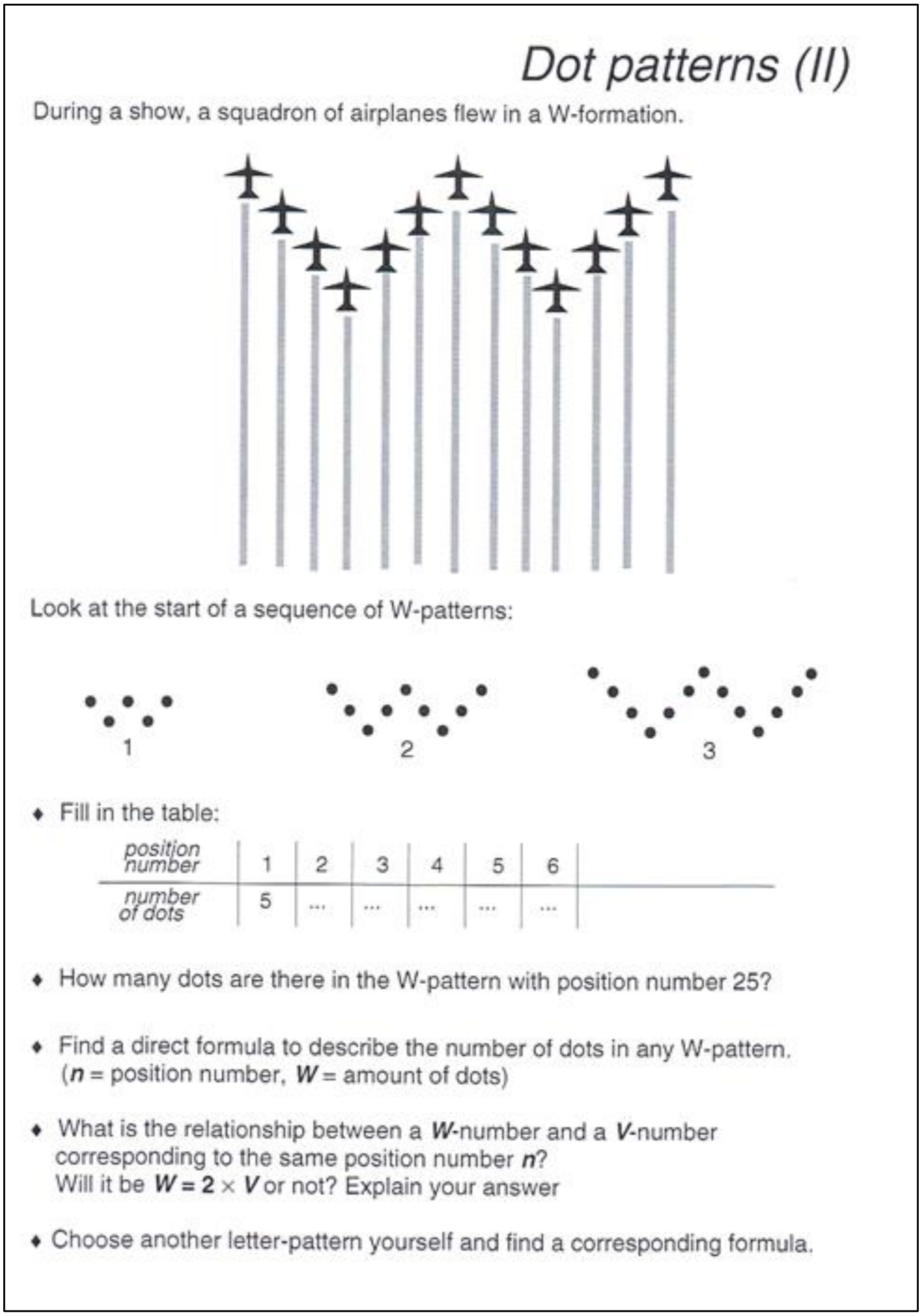
V-pattern with dots:

The figure shows the first four V-patterns. Each pattern has a position number (starting at 1). Below you see a V-pattern with 17 dots.

- What is the position number for this V-pattern?

- How many dots has the V-pattern with position number 85?
- Does a V-pattern with 35778 dots exist? Why or why not?
- Find a rule for describing the number of dots of a V-pattern, knowing the position number.
- Represent this rule by a direct formula; use the letters ***n*** and ***V*** (***n*** = position number, ***V*** = number of dots for that position number)

Figure 8. Dot pattern I, V-pattern (Kindt, 2004).



Dot patterns (II)

During a show, a squadron of airplanes flew in a W-formation.

Look at the start of a sequence of W-patterns:

1 2 3

- Fill in the table:

position number	1	2	3	4	5	6
number of dots	5	...	...	...	...	...

- How many dots are there in the W-pattern with position number 25?
- Find a direct formula to describe the number of dots in any W-pattern. (***n*** = position number, ***W*** = amount of dots)
- What is the relationship between a ***W***-number and a ***V***-number corresponding to the same position number ***n***? Will it be $W = 2 \times V$ or not? Explain your answer
- Choose another letter-pattern yourself and find a corresponding formula.

Figure 9. Dot pattern II, W-pattern (Kindt, 2004)

The groups did both dot patterns. Due to attendance issues, only the episodes with Group A doing the V-pattern and Group B doing the W-pattern were used for the data analysis. The dot pattern tasks were similar in their design and solicited comparable modes. The purpose of the study was not to compare the groups' solutions to problems but rather the processes to reach the solutions.

Stair-like structure problem. The second task was a growing stairstep pattern published on the *Youcubed* website (Milvidskaia & Tebelman, 2015). Like the previous tasks, the situation was familiar to the participants and more elements were added to each figure. The worksheet came with three guiding questions that were less specific as to what the students were supposed to do with the intention to encourage independent thinking. With the prior experience though, the students did not miss a beat. They quickly built the first several figures to find the growth rate.

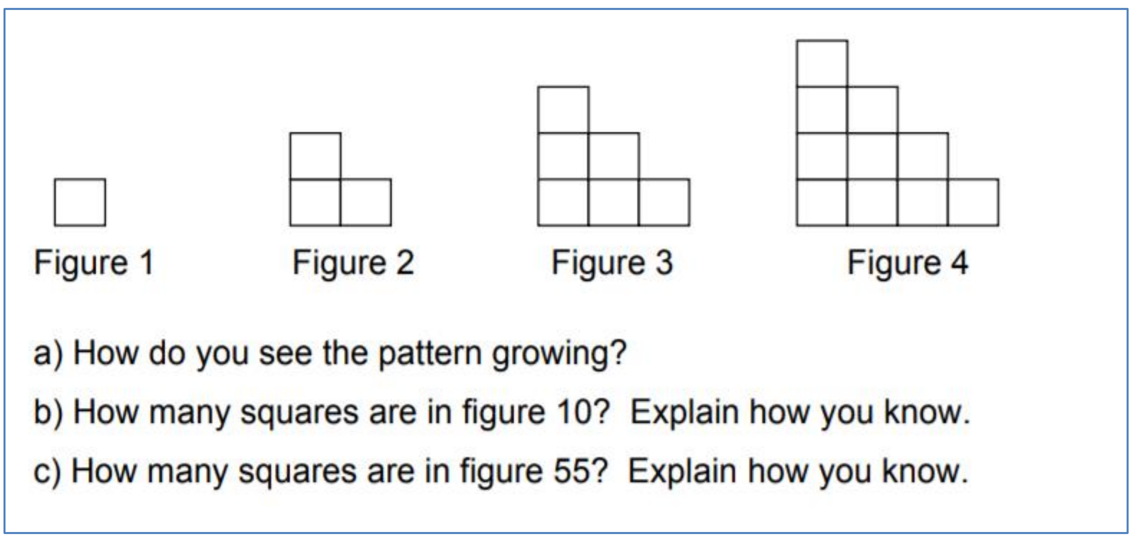


Figure 10. Stair-like structure problem (Milvidskaia & Tebelman, 2015).

Plastic 1-inch squares in primary colors that can be found in many elementary classrooms were available to the students. I expected that it would be the students' first action to take some square tiles and start building.

The garden party task: Tables and chairs. The garden party story had been created for the pilot study. Again, the situation was familiar to the participants. I chose it as the third task, because here the students had to add not only one but two kinds of items, tables and chairs, plus two additional chairs at the end tables. The students could think about the growing pattern in different ways. Either the number of tables could have been multiplied by the total of seats, and the seats that got lost by pushing the tables together would then have been subtracted, with exception of the two end seats, or the two lost seats at each table would have been subtracted, the difference multiplied by the number of tables, and the two end seats added.

Garden Party

The Rose family is planning a cook-out for July 4th in their backyard. They want to push together square tables to make one long table where all their guests can sit. Each table by itself seats four people.

a. How many tables are needed for eight people?

b. How many people can attend the party, if the Roses have six tables available?

c. Mr. and Mrs. Rose decide to start their own party business. How can they come up with a chart that helps them to quickly figure out how many tables they need for different amounts of party guests of their customers' parties?

d. What would change, if the Roses would use tables that seat six people? Or eight people?

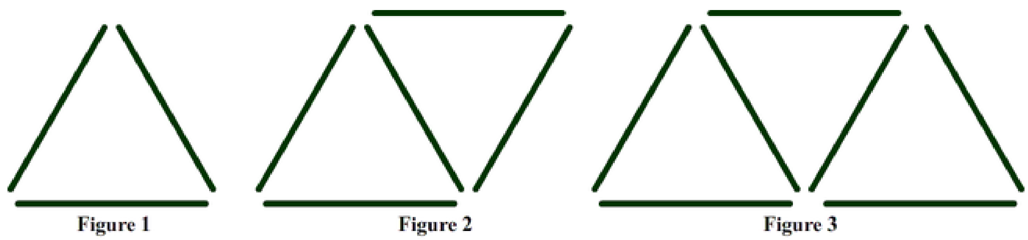
Figure 11. Garden party: Tables and chairs.

Fitting manipulatives were square tiles, rectangular *Cuisenaire* tiles, round chips for seats, or some drawings would be created for this task. The problem was presented in form of a story to reinforce the mental imagery based on a familiar environment. Both groups actually asked at some point, how the tables would be set up, and whether they would be pushed together. I used the example of the school's cafeteria where rectangular tables were positioned in long rows across the room.

Toothpick pattern. The toothpick pattern task retrieved online from the Open Education Resources Commons (Wells, 2016), concluded the sessions. It explicitly instructed the students to recreate the first three figures with toothpicks. In this episode, written text as a mode is only available in form of the printed task; the participants are not creating their own text yet. The modes of formal notation and visual imagery is not utilized either, this short into the task solving process. The episode captures the initial thinking about the problem and the concrete recreation of the pattern with the available manipulatives, the toothpicks.

The Task - Toothpick Patterns

Create the following toothpick pattern with the stack of toothpicks you have been given. Record how many toothpicks it takes to create each figure and record the information in your table.



2. Follow the pattern and create the next 3 figures using your toothpicks and fill in your table with the number of toothpicks needed for each figure.

3. Draw the six figures on your paper so you have a drawing of the pattern.

4. Now see if you can jump to the 10th figure without drawing or creating the figures in between. Create and draw the 10th figure in the pattern without creating figures 7,8,9, and 10.

5. Try to develop a rule that would allow you to calculate the number of toothpicks needed for any figure any the pattern. For example, how many toothpicks would you need for the 1000th figure in the pattern.

Perimeter

Now let's look at the perimeter of the toothpick figures. For this pattern we will not count the toothpicks in the interior of each figure.

Record the perimeter of each figure in your table.
Develop a rule to determine the number of toothpicks in the perimeter of any figure in that pattern. For example, how many toothpicks are there in the perimeter of the 1000th figure?
Compare the two patterns. What is the same and what is different?

Figure 12. Toothpick pattern (Wells, 2016).

Environment and Materials

Location. The 16 weekly sessions with Groups A and B lasted 30 to 40 minutes and the teams took two sessions working on each of the four growing pattern tasks. When the team partners arrived, they took their seats side-by-side at a rectangular table. The table was a computer table closed off on three sides. Before each session, the table was covered with colored bulletin board paper, and the students were informed that they could write or draw on the table cover. Each team had a different color and the covers were labeled with the session date and code and kept as artefacts.

Group A and Group B met on different days to avoid distractions. This allowed me to concentrate on videotaping and zooming in or out based on the current discussions and interactions within the group to identify multiple modes. Two digital video cameras were set up facing the participants from different angles. The 40-minute sessions for each group were digitally recorded from two angles. One camera was set to capture the actions of all participants during their collaborations. I operated a second camera as needed to zoom in on actions where students use different modes.

Material media and expected use. During the duration of the sessions, the students had access to a wide variety of manipulatives, such as double-sided chips representing dots or colored squares useful for stair models of table rows. They could choose to use any or all material media. Only in the toothpick task were explicit directions given to build the figures depicted on the worksheet from toothpicks. Along the side of the table opposite from the participants' seats, an array of material media were placed in reach for both team members. Figure 14 lists the materials and the modalities the materials were intended to address.

Material Media	Modalities	Expressive Products
Colored bulletin board paper covering table	• Language • Formal notations (FN) • Visual imagery (VI)	• Written text • Written mathematical symbols • Static diagrams, graphs, geometric, conventional
Pencils, colored pencils, markers, highlighters, dry-erase markers	• Language • Formal notations (FN) • Visual imagery (VI) • Motor Actions (MA)	• Written text • Written mathematical symbols • Static diagrams, graphs, geometric, conventional • Marks drawn • Gestures holding artifacts
Paper: Plain, colored plain, ruled, gridded Other: Dry-erase board (handheld)	• Language • Formal notations (FN) • Visual imagery (VI) • Motor actions (MA)	• Written text • Written mathematical symbols • Static diagrams, graphs, geometric, conventional • Gestures holding artifacts
Manipulatives: Double-sided (red-yellow) circular plastic chips, plastic 1-inch squares (four colors), toothpicks, rulers, calculators, connecting centimeter cubes	• Language • Formal notations (FN) • Visual imagery (VI) • Motor actions (MA)	• Written text • Written mathematical symbols • Static diagrams, graphs, geometric, conventional • Gestures holding artifacts

*Figure 13***.** Material media offered to participants (Edwards & Robutti, 2015).

Data Analysis

The data were analyzed continuously after each session. Field notes, coding of the student actions, and conversations were included serving as parts of the retrospective conceptual analysis (Cobb et al., 2000; Lesh & Kelly, 2000; Steffe & Thompson, 2000). The analysis after each session informed adjustments for the following session. Literature specifically points out that student interaction not only provides them with an additional function in the learning process of their peers, but their communication also simplifies the researcher's work as the cycles become step-by-step better observable and can be more efficiently guided by the teacher-researcher (Dekker & Elshout-Mohr, 1998; Ellis, 2011; Good, Mulryan, & McCaslin, 1992).

Multimodal Framework

The multimodal methodology (Edwards & Robutti, 2014) provided a tool to differentiate among modes the students are choosing to communicate their algebraic thinking. Modes are not set entities and can be defined in variations, and the human mind does not operate linearly and can lead others to intended perceptions (Norris, 2004). Discourse analysis is a staple in mathematics education. Multimodality as a framework in the fields of linguistics, art, and others has gained importance in conjunction with technological progress we have experienced since the 1990s. Children articulate topics they want to talk about on their level of languag**e** development, but verbal expression is only one component of communication. I am using a lens of multimodality to understand how they draw on various modes including speech, gesture, posture, facial expression, actions, and gaze (Edwards & Robutti, 2014) to explain notions of generalization in specially designed tasks involving algebraic growth patterns.

The lens of the multimodal framework posts some specific challenges concerning data collection and analysis. Qualitative data collection relies on more than one method. The richness of data comes to light by triangulating the data collected in different ways, such as video filming, audio recordings, transcripts, and field notes during the same time frame. Digital recordings grant reproducibility and availability for later thorough analysis (Hall, 2000). During the pilot study, I had the opportunity to acquire some experience in videotaping group activities for research purposes. Selecting excerpts from the digital recordings with rich data was central. This is especially important for the multimodal lens I applied to this study. It was crucial to review the recordings immediately and repeatedly to recognize the simultaneously used modes during the problem-solving activities. Special attention was paid to the mathematical explanations that the students provided. I tracked how the students drew on various modes to build mathematical meaning within the context of the growing pattern tasks. Episodes best revealing the overlay of multiple modes (Norris, 2004; Smith, 2014) of specific modes were isolated and transcribed frame by frame to support the analysis.

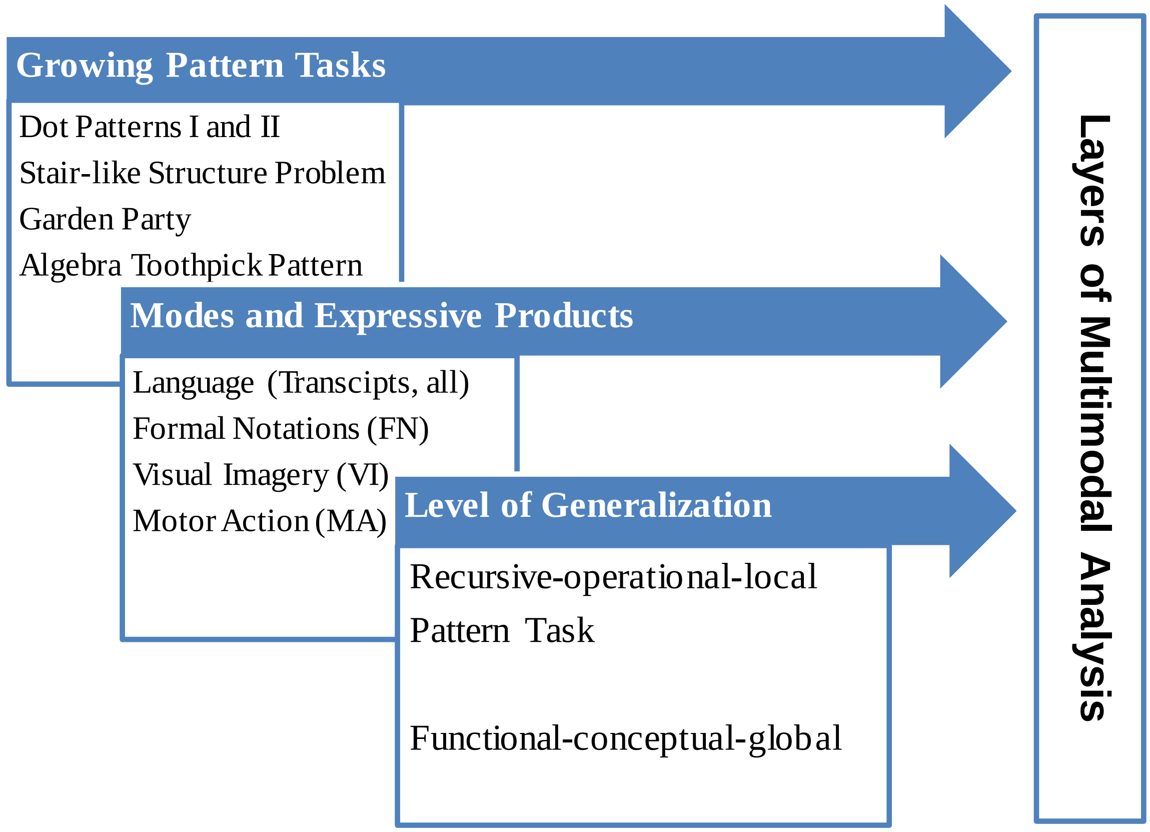


Figure 14. Layers of tasks, modes, and generalization.

The methodological framework of multimodality allowed for the differentiation among modes the students were choosing to communicate their algebraic thinking. The coding of audio transcripts analyzed in the context of the participants' interactions had the purpose of identifying the simultaneous layers of the three processes: (a) solving the tasks, (b) modes and expressive products, and (c) steps of generalization reached by the students (Figure 12).

Multimodality adds another layer of complexity to the data analysis. Multimodal transcriptions consider visual evidence of actions in context, even if there is no audible contribution at a certain instant during a problem-solving session (Flewitt, Hampel, Hauck, & Lancaster, 2009). This is an established data analysis method in the field of discourse analysis (Jewitt, 2009; Norris, 2004). The multimodal transcripts consist of individual frames of the digital video recordings. This opens the possibility to pinpoint which sensory and motor modes each participant was choosing at a certain time of the interaction. It also allowed the detection of the modes occurring simultaneously during key phases of the problem-solving process (Norris, 2009). Analyzing digital recordings frame by frame let me observe shifts in attention or for that matter awareness of the active students, and the effect on other group members. At the same time, I had to be mindful of some constraints. Modes are not set entities and can be defined in variations, and the human mind does not operate linearly and can lead others to intended perceptions (Norris, 2004).

Assumptions, Limitations, and Scope

Ethical considerations. The problem-solving meetings were organized as extra activities, which to them were a fun-filled extracurricular club activity. The school offered a variety of after-school clubs and students in general are very eager to participate in as many as possible. This contributed to not making them feel pressured to perform. They could approach the activities more playful than they might have in a regular classroom setting. Since there was no risk of harm from any treatment given to the participants, ethical considerations are limited to the responsibility of the researcher to make the participants unidentifiable in the research papers and records and to follow the protocol of safe storage of the recorded data in a locked cabinet separate from the letters and coding with the participants' real names. Eight students were selected to participate. However, only four regularly attended.

Internal and external validation. As strategies for quality, the presence of a second and possibly third researcher is of great importance as they will be able to triangulate their observations under inclusion of their field notes and the video recordings, called the retrospective conceptual analysis. The full attention of the teacher-researcher is on the children doing mathematics and to respond spontaneously and appropriately to each of their steps without imposing his or her own mathematical thinking on them. During these intensive episodes, it is possible that self-observations are missing details or interpret them differently (Cobb et al., 2000; Lesh & Kelly, 2000; Steffe & Thompson, 2000).

Summary

The purpose of this qualitative study was to present a multimodal view of a variety of sixth graders' strategies leading to growing pattern generalization. The research design of choice was the teaching experiment. It allowed the teacher researcher to interject questions for clarification or to guide students to the next steps should they get trapped in actions or thought processes. The data analysis presented chapter four has been organized by tasks in order of the sequencing by linearity, pictures provided, and instructions to use manipulatives. Key episodes of each group session have been extracted and were analyzed by applied modes and their affordances for the growing pattern generalization process. The layers of multimodality were observed simultaneously and interpreted in task-specific discussions. Focal point was the parallel modes and generalization strategies the students used categorized by recursive-operational-local and functional-conceptual-global approaches.

CHAPTER FOUR: DATA ANALYSIS AND DISCUSSION

The purpose of the study was to address the simultaneous events of the participants applying generalization strategies and various modes of communication (Edwards & Robutti, 2014). This chapter is organized in sections around how each group engaged with the growing pattern tasks. The interpretations of students' thinking intend to capture the layers of the multimodal analysis of the observations. The mode of language expressed through speech is presumed to be the most common mode of communication. Spoken statements students made over the course of the selected episodes of the growing pattern activities are mentioned in the findings and interpretations as a way to illustrate and provide evidence for the interpretations. The additional modes of interest for the problem-solving process and the generalization strategies are listed in task-specific tables accompanied by frames extracted from the videos recorded during the sessions.

Table 1

Abbreviations for Modalities, Expressive Products and Generalization Strategies

MA	**Motor Action**
- GH	Gesture with empty hand/s
- GA	Gesture with Artifact in hand/s
- GE	Gesture with non-standard objects in the environment
- BA	Other bodily actions/postures/gaze
VI	**Visual Imagery**
- SG	Static graphs
- SD	Static diagrams
- SM	Static marks highlighting or emphasizing
FN	**Formal Mathematical Notations**
FCG	**Functional Conceptual General reasoning**
ROL	**Recursive Operational Local reasoning**

The Dot Pattern Tasks

The dot pattern activities (Figure 16) from the workbook *Positive Algebra* by Kindt (2004) presented linear growing patterns accompanied by pictorial representations of possible origins of the patterns. Looking through the multimodal lens, the dot pattern tasks were categorized as non-material. They contained no instructions to use manipulatives or even to physically build models of the figures. If the students used manipulatives, it was their personal choice. For quick reference, I use the following abbreviations to communicate the types of actions and modes students' used throughout their activity.

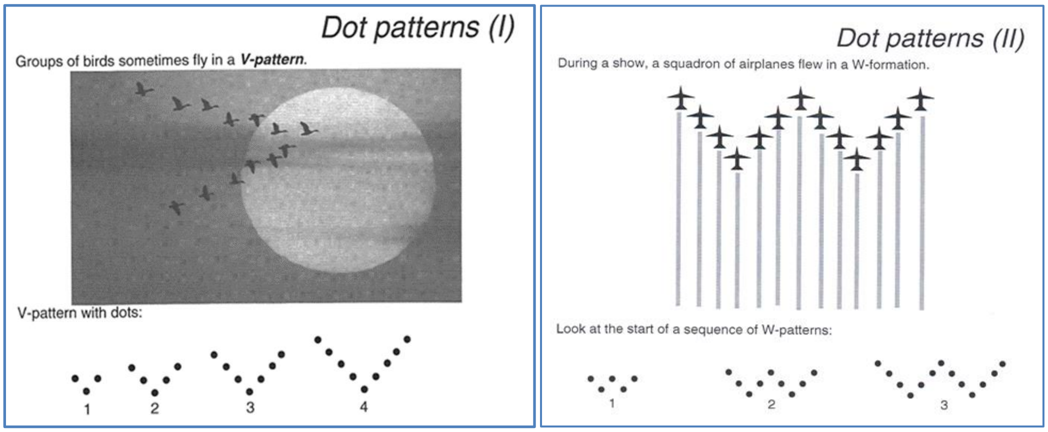


Figure 15. Dot patterns I and II (Kindt, 2004).

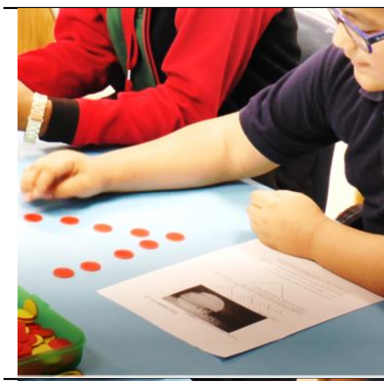
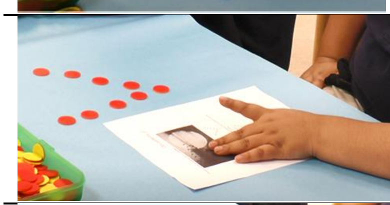
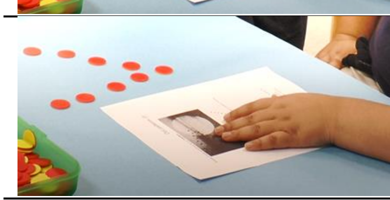
Group A: Findings and Interpretations of V-pattern Task

The students received the printed task worksheet with the picture of the pattern's origin and the graphic representation of the first four growing pattern positions. Before he even read the directions, Noah (N) tapped on Iris' (I) arm saying "Look" while taking a random amount of double-sided chips out of the tub. He began building Position 4 skipping the first three positions. He used both hands to build the model straightening both sides of the V but added the fourth dots on either side with his right hand. Then he read the directions aloud. He went back to study the figures one by one tapping his left index finger on the paper counting the positions first saying "1, 2, 3, 4" and then the number of dots "3, 5, 7, 9".

As expected, Noah learned the pattern and began working by applying recursive-operations-global strategies. He chose to build Position 4 right away and later went back to the lower positions to confirm his thought that the pattern's growth rate is two dots (Table 1). While he checked for correctness of his thinking, he kept tapping onto the dot images on his paper. He was doing so with his left index finger although he was right-handed. He held additional chips he had left over in his right hand. In a later conversation, he stated that he liked to have something in his hands because he had difficulties sitting still and this was his displacement action that would not get him in trouble.

Table 2

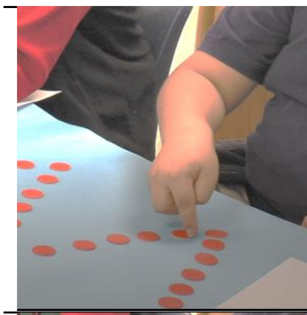
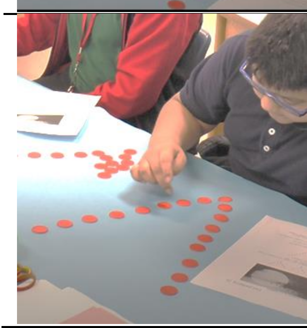
Creating the Model for Position 4

Actions	**Mode Description**	**Generalization Strategy**
	Motor Action (MA): - Gestures holding artifact (GA): building the figure; holding chip surplus - Gestures empty-handed (GH): tapping on chips and paper - Gestures involving environment (GE): table top	ROL
 	MA: - GE: tapping left index finger up and down while re-counting dots - GA: holding chip surplus in right hand	ROL

Next, Noah proceeded to building the figure with 17 dots. The task was to find the position number. While he added one chip to either side of the V, he still counted only one side including the leading dot and recounted the leader going up the other side. He looked back at the smaller figures for reference and now counted the sides of the V skipping the leader dot. Then he announced "Ah, I found it out. It's the number eight. The order of the number is number eight. Order of the 17 dots." By "order of the number" he meant the position of the number. Iris asked him how he found that, and he began explaining that he had added chips to figure four until he reached eight.

Table 3

Beginning to Discover the Function of the Growing Pattern

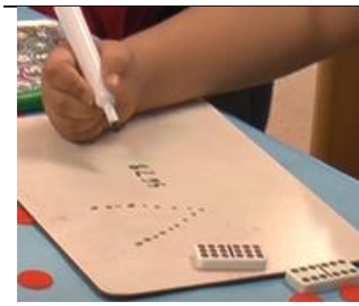
Actions	**Mode Description**	**Generalization Strategy**
	Motor Action (MA): - GA: building the figure; holding chip surplus while tapping and counting	ROL
	MA: - GA: building the figure; holding chip surplus	ROL
	MA: - GA: building the figure; holding chip surplus	ROL yet beginning functional thinking; still local position

While Noah was explaining, he again pointed to both sides of the V but only counting only one-sided. He still added two every time, which was the growth rate. Finally, he recounted the 17 just to make sure. Noah was still operating with addition but was beginning to generalize. He recognized that each side of the V contained the number of chips equivalent to the position number. However, his thinking was becoming more complex. As he added two chips at a time, one to each side (Table 3), he named the position number, which he calls "the order of the number." He also began not to count the leading chip any longer. Thus, he had comprehended that the pattern grew by two and had one additional dot in the leading position.

The next step of the task was to find the total number of dots in Position 85. Noah realized that he could not create a model as before, picked up a dry-erase board and marker, and began drawing. After an unsuccessful attempt to calculate the total involving the prime factors of 85, he went back to the additive strategy. When asked how many more he had added to Figure 4 to get to Figure 8, he draws Figure 4 on the board and while adding two dots he writes the current position number beside the graph. During the entire episode, Noah was talking about the number of dots he was adding on one side, but he always demonstrated that he was thinking both sides simultaneously by pointing to each side of the V (Table 4).

Table 4

Explaining the Operational Strategy Applied to Both Sides of the V

Actions	**Mode Description**	**Generalization Strategy**
	MA: - GA: holding marker FN: tracking position numbers VI: - SM: marks dots in V-shape	ROL Early FCG: marks 2 dots while counting by 1 to track position number
	MA: - GA: holding marker in right hand - GA: points with left hand that holds the cap VI: - SG: analyses printed V-pattern	ROL Early FCG: points to the two sides of position 4 on the paper
 *Teacher*: Where did you add the 4 dots?	MA: - GA: holding marker in right hand - GA: points with left hand that holds the cap VI: - SG: analyses printed V-pattern	ROL Early FCG: points to the two sides of position 4 on the paper
 *Noah:* Up here, where the line is.	MA: - GA: holding marker in right hand - GA: holds the cap in left hand GA: holds his hands in V position	ROL Early FCG: talks about one line but gestures both

Based on Figure 8, Noah then figured out that he could add 77 to the eight dots to reach 85. When prompted, he explained his strategy saying that he had to add 77 to each side. This showed that he knew the relation between the growth rate and the position number, but he was not ready to formulate a general statement. Meanwhile Iris had started building a model of Figure 8 from connecting centimeter cubes and draws Noah's attention. She was thinking of the growth rate as the total number of dots she was adding. Noah stuck to his approach of talking about adding one but keeping both sides in mind. He clarified his viewpoint by pointing to her cube model while explaining.

Table 5

Pointing to Partner's Cube Model

Actions	**Mode Description**	**Generalization Strategy**
 *Teacher:* How many did you add? *Iris*: Two. *Noah*: One.	MA Iris: - GA: stacking centimeter cubes in V-shape MA Noah GE: points to left side of her V-model	ROL Early FCG: marks 2 dots while counting by 1 to track position number
 *Noah*: One on this one and one on here	MA Iris: - GA: stacking centimeter cubes in V-shape MA Noah GE: points to right side of her V-model	ROL Early FCG: marks 2 dots while counting by 1 to track position number

T: How many do you have on each side?
Iris (counts): Nine. On both sides.
T: So which position number is this.
Iris (without hesitation): Nine.

Noah paused for a moment crossing his arms in front of his torso thinking about her response. Then he reacted with a smile, and while he was talking about the two sides of the V, he was moving his head to his left and his right as if he is looking at either side of an imaginary V (Table 6).

Table 6

Noah's Body Language Referring to Both Sides of the V

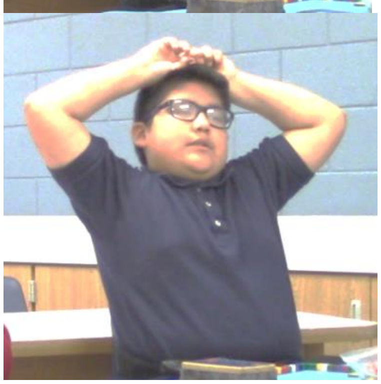
Actions	**Mode Description**	**Generalization Strategy**
 Noah: Oh!	MA: - GE: hides both hands under his crossed arms	ROL Early FCG: gestures 2 dots while counting by 1 to track position number
 They're both...	MA: - GE: points to imaginary sides of the V	ROL Early FCG: gestures 2 dots while counting by 1 to track position number
  Noah: .... the same number.	MA: - GE: hides both hands under his crossed arms BA: moves his head left, then right as he talks about the sides of the V	ROL Early FCG: marks 2 dots while counting by 1 to track position number

The teacher now prompted Noah to re-visit his approach of adding 77 more dots to the sides of Position 8. He moved his arms up and rested his connected hands on his head while thinking. When she asked "So, where exactly do you add on the 77 [dots]?" he lifted his hands and moves them up and down repeatedly signaling that he was adding. Then he formed his fists while talking about the dots on each side of the V (Table 7). He explained that the 77 dots must be added to either side of the V, to the dotted lines. However, he never managed to think further and calculating the total dots of Figure 85.

Table 7

Gesturing Adding to Either Side of the V

Actions	Mode Description	Generalization Strategy
 *Noah*: Do you mean 'add more at the top?	MA: - GH: points with both hands to imaginary V BA: moves both hands up the sides of the V	ROL Early FCG: marks 2 dots while counting by 1 to track position number
	MA: GH: connects hands and moves them up and down touching his head	ROL Early FCG: marks 2 dots while counting by 1 to track position number

 **Noah:** On the dotted lines.	MA: GH: makes fists with both hands representing dots of the 2 dotted lines	ROL Early FCG: marks 2 dots while counting by 1 to track position number
  *Noah:* On each side	MA: - GH: connects hands and rests them on his head BA: moving his head to the left, then to the right representing the 2 sides of the V	ROL Early FCG: marks 2 dots while counting by 1 to track position number

Discussion: Group A, V-pattern

The analysis of the linear, pictorial, and non-material dot pattern revealed three phenomena. During the 40-minute session, Noah used the recursive-operational-local strategy of adding two dots to each position forming the next. He demonstrated the ability to double-count as well as to count one side of the V he modeled and simultaneously keeping track of the total number of dots in the growing pattern. His words and his actions, however, did not agree. When he explained his steps, he counted by ones but gestured or moved his body to both sides of the V signaling pairs of dots (Tables 5-7).

Part 3 of the V-pattern task was designed to encourage the move from the recursive-operational-local to functional-conceptual-global strategies. The students were instructed to find the total number of dots in Position 85. Noah realized immediately that he could not build a model of that size and began drawing on a dry-erase board. He attempted a multiplicative solution by finding factors of 85, but his trials remained unsuccessful. Returning to the additive strategy, he used Position 8 as his starting point and calculated that he needed 77 more dots on either side to reach Position 85. He did not compute the total number of dots in Position 85.

Beside the language mode of speech, he communicated his thinking mostly through motor actions. For most of the time he gestured holding and placing a variety of artifacts. He was quick picking up a handful of double-sided chips. It was a random amount and he did not connect the number of chips he picked up to the mathematics he was about to explore. There were only a few minutes here and there when his hands were empty, either pointing or resting close to his body. In a later conversation, Noah shared that he had difficulties sitting still in class. He found an outlet of that energy by fidgeting with objects. Manipulatives provided for mathematics activities were great, he said, because he would not get in trouble for playing with them in class.

Group B: Findings and Interpretations of W-pattern Task

The W-pattern task (Kindt, 2004) was an extension of the V-pattern task. It was a linear pattern. Pictorial representation was provided, but students were not instructed to use material. Isaiah had successfully completed the V-pattern task by himself during a previous session for which Omar was absent. So that Isaiah did not have to repeat the V pattern, he and Omar were presented with the W pattern problems. Isaiah explained to Omar how he had identified the rate of growth and the direct formula for the V-pattern. Isaiah quickly explained that there was a leader and each figure would grow by two compared to the previous figure. Omar had no difficulties following Isaiah's explanations and only needed a few minutes to understand the process from recursive-operational-local to functional-conceptual-global strategies to the point of establishing a direct formula. Now they were presented with the W-pattern, an extension of the V-pattern problem. They had looked at the pictorial representation of the figures only briefly and exclaimed that this was the same as in the V-pattern.

The two students in Group B went to work on the task by studying the pattern figures and filling in the function table printed on the worksheet (Figure 17).

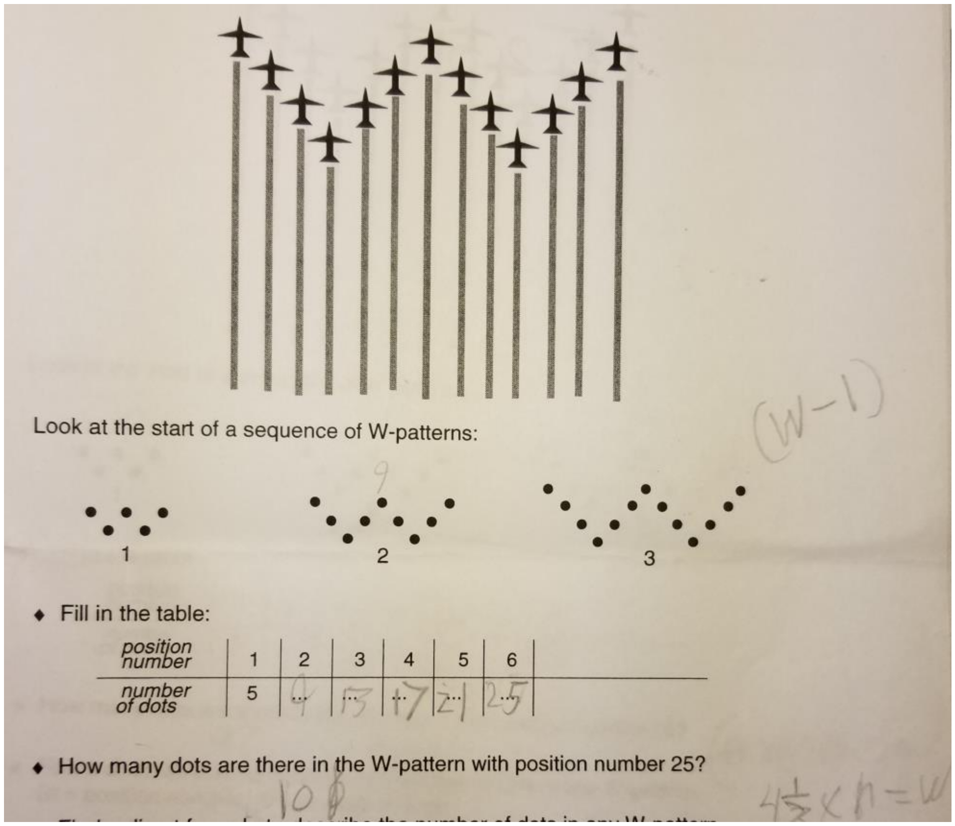


Figure 16. W-pattern, Parts 1 and 2, Group B.

Isaiah went straight to the formula activity in Part 3 and began dividing the total number of dots in Position 4 by two, which he remembered from the V-pattern. The result was 4½ times the position number. He argued that for odd-numbered positions the total had to be rounded up because there obviously could not be a half a plane in the formation. After the teacher prompted him about the leader dot that we had observed in the V-pattern task Isaiah re-visited the pictorial representations of positions 1 through 4. He covered one dot of the W in figure 1 and noted $W – 1$ beside Position 3. He covered the dot to the top right, not the one in the center (Table 7). After thinking about the picture and the function table for a while, he recognized that the number of the remaining dots was four times the position number.

Table 8

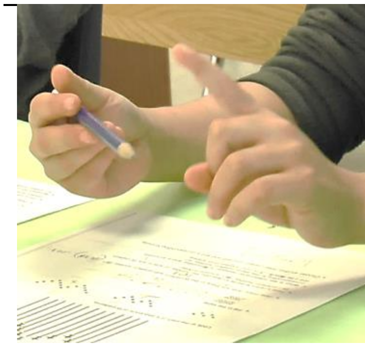
Using the Pictorial Representation

Actions	**Mode Description**	**Generalization Strategy**
 *Isaiah*: So you need to take one away. Then that will go into it four times. Minus one divided by four.	MA: - GA: Covering top right dot holding pencil - BA: Nodding his head toward Position 1 - BA: Gazing at direct formula on the bottom of the sheet	FCG
 *Isaiah: (whispering)*.. ohh, that's even better.	MA: - GA: Covering top right dot of position 2 holding pencil - BA: Gazing at direct formula on the bottom of the sheet	FCG

Both boys had filled out the function table in Part 2 individually without conversing about the activity. Isaiah advanced to the functional-conceptual-global (FCG) thinking by writing the direct formula with the variables given in Part 3. Both wrote the formula as directed by using position number *n* as the independent variable and the number of dots *W* as the dependent variable. Isaiah recognized that the formula has two unknowns, position number *n* and total number of dots *W*. He explained that there were two unknown entities, the position number and the number of dots and gestured accordingly by tapping first one and then two fingers against his pencil (Table 9).

Table 9

Gesturing Two Variables

Actions	**Mode Description**	**Generalization Strategy**
 Isaiah: We have to find out, one, the position number...  ... and the number of dots."	MA: GA: emphasizing his explanations about variables 1 and 2	FCG ROL

Discussion: Group B, W-Pattern

The boys in Group B only used the worksheet and pencils. Neither one felt the need to pick up manipulatives to support their thinking. The pictorial representations were sufficient. Starting with a review of the V-pattern, they rushed ahead assuming that the W-pattern had to be solved the same way. After the teacher's re-direction, Isaiah quickly moved on to the functional-conceptual-global strategy of applying the formula

$(4n) + 1 = W$ for finding the total number of dots in Position 25. Omar followed along learning from his partner. He remained critical though and asked for clarification when something did not seem right from his perspective, for example when he was not sure about Isaiah's initial calculations with factor 4 ½. Isaiah justified his thinking but later admitted that there was a better way. During the entire conversation, he kept his hands underneath the table holding his pencil. At times it could be heard moving against the table top from below.

Stair-like Pattern Task

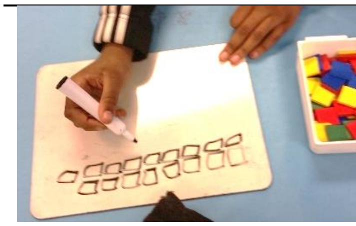
The stair-like pattern task (Milvidskaia & Tebelman, n.d.) was the selected example of a non-linear, pictorial, and non-material problem. The sixth graders were not expected to be able to solve the task since they had never encountered non-linear relationships in any formal way. Points of interest for the teacher-researcher were to what extent the students could generalize the growth rate of a quadratic function without prior knowledge, and whether they would use available manipulatives to help make sense of a more difficult task.

Group A: Findings and Interpretations of stair Steps Pattern

As expected, Iris and Noah went straight to the tub with the square inch tiles. Both began separately to build the models depicted on the worksheets. Iris discovered after a while that a row of squares equivalent to the position number was added underneath the previous figure. She kept working with the recursive-operational-local strategy of adding the next higher bottom row to her model. When she reached Position 7, she was short of space on the table. Her solution to the space issue was to continue the pattern on a dry-erase board (Table 10). She accomplished position 10 and counted the total number of tiles. The following part of the task was to find the total number of squares for Position 25. Iris could not find the answer, because she did not move away from the additive strategy and became overwhelmed with the number of tiles she would have needed.

Table 10

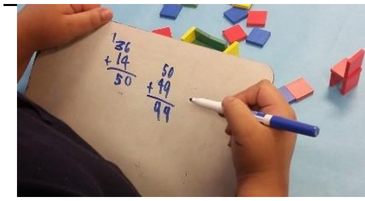
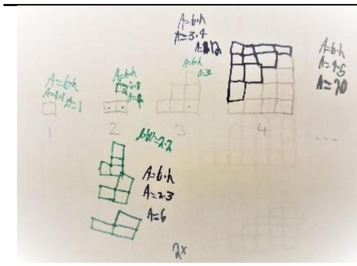
Continuing Model by Drawing Squares

Actions	Mode Description	Generalization Strategy
	VI: SM: Drawing marks representing stair steps	ROL

Noah initially used the tiles as well. Feeling overwhelmed, he began building three-dimensional objects and appeared to be giving up on the task. Then he took a dry-erase board and a marker and attempted the additive strategy as well with no success. After tinkering with the tiles and the board and marker for some time, he made a connection to arrays. He stated that the tiles reminded him of the time when they learned multiplication by building arrays with them. He then completed the corresponding rectangle to Position 4. He remembered the formula for the area of a rectangle and calculated a total of 20 squares for Figure 4. From there he went back to the lower positions (Table 11). However, despite prompts he could not take the next step of dividing the area of the rectangle by two to get back to the stair steps.

Table 11

Additive and Multiplicative Strategies

Actions	Mode Description	Generalization Strategy
	MA: - GA: Adding on dry-erase board with dry-erase marker FN: writing addition problems	ROL
	VI: - SD: drawing models of arrays according to position numbers FN: writing area formula	ROL FCG

Discussion: Group A, Stair Step Pattern

As expected, the students in Group A immediately resorted to the square-inch tiles to model the figures that resembled the stair steps the most. Iris did not advance past the recursive-operational-local strategy of adding another row to the bottom of her model and adding to the previous total number of tiles. She had to re-count multiple times for self-assurance. She gave up before she even reached Position 25 in Part 2 of the task.

Noah accidentally took a step toward the functional-conceptual-global strategy by extending Position 4 to a rectangle. He had learned and practiced multiplication facts in third grade where he had built arrays with the kind of tiles he was using here. The connection to his prior experience with the manipulative inspired him to double the total amount of tiles and apply the formula for area $A = b \times h$. The formula was fresh on his memory from the sixth-grade geometry unit. With his way of thinking about the problem, he accomplished the most difficult part of the task of thinking in two dimensions. Noah seemed satisfied with his findings and stopped without returning to the original problem, the stair steps. He did not realize that he had to divide the area by two. Based on making the connection to his prior knowledge of arrays, he more or less accidently moved into functional-conceptual-global strategies without understanding what his actions had to do with the growing pattern generalization.

Group B: Findings and Interpretations of Stair Steps Pattern

Omar and Isaiah began the task by looking at the pictorial representations and only picked up lined paper and pencils. They talked about the growth rate while Isaiah counted the total numbers of the first four positions. He counted the additional squares in each position diagonally in ascending direction of the stair steps. The connection he made was obviously based on experiences he had with climbing stairs (Table 12).

Both boys worked with the recursive-operational-local strategy of adding but tried to move on to functional-conceptual-global. Omar diligently kept adding the position number to the total number of the previous position using grid paper first and later a dry-erase board. He correctly worked up to Position 20 before he abandoned the additive strategy. He went back to Position 10 with 55 squares and doubled both numbers which did not give him the correct number of 210 squares. His next thought was to 55 and 55 again, but the sum was 220. Isaiah attempted functional-conceptual-global generalization by finding a direct formula with coefficient and constant similar to the task before. Neither student was successful.

Table 12

Emphasizing Increasing Diagonal

Actions	Mode Description	Generalization Strategy
  	MA: - GA: Isaiah using pencil as pointer VI: - SM: marking the increasing stair steps on the outside diagonal	ROL

Discussion: Group B, Stair Step Pattern

The unexpected finding from the stair steps pattern task was Noah's approach to forming an array. Connection to prior knowledge helped him taking the step to functional-conceptual-global thinking about a quadratic growing pattern. The fact that he did not follow through with the stair step pattern in the end may have been part of his difficulty to focus for a longer amount of time. The other group asked for the formula after the session. The boys wanted to know, if anyone else "got the answer."

Competitiveness was their motivator. Four students worked with four different approaches on which teachers could capitalize.

Garden Party Task

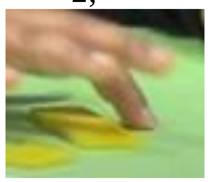
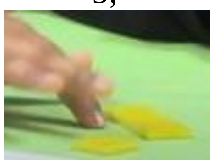
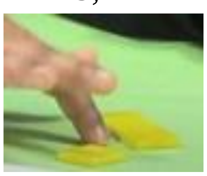
The garden party task had been created as a text problem only. The growing pattern was linear, but pictorial representations and instructions to use materials were not included. The purpose of the task design was to describe a situation familiar to most students to elicit their generalized thinking. The task had been used in the pilot to this study, but without the use of manipulatives. It had lent itself to drawing the various situations that could occur by pushing tables together and at the same time losing seats.

Group A: Findings and Interpretations of Table and Chairs Pattern

Both students began the session by picking up square tiles. After glancing at the title of the growing pattern task, Iris arranged alternating blue and yellow tiles in a superior square and a rectangle. Isaiah just held a hand full of yellow tiles. They read Part A of the task that asks about the number of tables needed for eight people. Isaiah lined up three yellow tiles representing tables and counted the imaginary seats beginning at the short end closest to him. Demonstrating the recursive-operational-local strategy of counting, he tapped his right middle finger on each seat alternating the sides of his rectangle (Table 13). He counted audibly in a soft voice ending with number eight at the far end from his position. His left hand was resting on the table surface.

Table 12

Counting Seats in Position 3

Actions	Mode Description	Generalization Strategy
  *Noah*: 1, 2,   3, 4,   5, 6,   7, 8.	MA: - MH: Noah counting seats by tapping with right middle finger; his hand moves in zick-zack motion	ROL

Part B of the task asked about the number of seats at six tables. Noah only had four tiles in front of him. He glanced over to the tub at the other end of the table considering taking more out but abandoned the thought. Instead he lined the four tiles up and began counting as before. When he reached the end of his table model though, he did not count the short end but kept tapping on the alternating sides. In doing so, he kept track of both the seat number, which he counted out in his soft voice, and the number of tables he had added in his mind. While he was concentrating on his activity (Table 14), his eyes did not look at the model nor his finger but gazed into the distance. His thinking had become independent from the model, but he still used motor action with his finger to support his thought process.

Table 13

Gazing into Distance

Actions	**Mode Description**	**Generalization Strategy**
	MA: - MH: Noah counting seats by tapping with right middle finger; his hand moves in zick-zack motion BA: gazing into distance	ROL

Discussion: Group A, Garden Party Task

Group A used square tiles to model the tables but did not choose additional manipulatives to represent chairs. The reason may have been that neither the word "chairs" nor "seats" appeared in the task description. The only variables mentioned were tables and people/guests. Neither student considered the latter as part of the static environment in the story context and did not see the need for their representation. The students demonstrated that they took the content of the story literal. "Square table" was the only expression in the text mentioning a specific shape. Both students immediately began using the square tiles and even pushed them together as superior squares before noticing that arranging them in a row was the way to go for greater numbers of people.

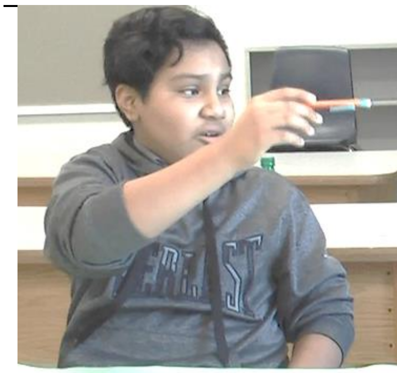
The students did not draw their own pictures of the table constellations nor did they write down any symbolic expressions. Apparently, the concrete models and verbal descriptions were what they needed to imagine the situation. Both Iris and Noah recognized that they were losing seats whenever two tables were pushed together. Without instructions though, neither student in Group A moved toward functional-conceptual-global strategies for this task.

Group B: Findings and Interpretations of Table and Chairs Pattern

After reading part a. of the task, Omar immediately answered that three tables were needed for eight people and six tables for 14 people. Isaiah disagreed saying that there were three seats to a table when pushed together. Jumping quickly to the multiplicative strategy of functional-conceptual-global generalization, he did not consider that this was only the case for the end tables, not the tables in between. Omar explained the situation to his partner and pointed to the sides of the table he "saw" in his mind. He turned his head and looked into the direction of each end of his imaginary table while pointing there with his pencil. He emphasized this thought by pointing to both ends at the same time saying "So, that gives you two."

Table 15

Developing Embodied Formula

Actions	Mode Description	Generalization Strategy
 *Omar:* The long way is one ...  *Omar*: ... and one.  *Omar*: So, that gives you two.	MA: - GA: using pencil upside down as pointer demonstrating the location of the seats; grouping them by short sides of the table BA: turning his head looking into the direction he is speaking about	FCG: Embodied formula

Then Omar proceeded to demonstrate where the two rows of six seats were located. In the process, he swung his right hand with the pencil along those two imaginary table sides, first the side away from him, then the side facing him (Table 16). Not only did he already generalize the growing pattern by moving toward functional-conceptual-global thinking when doubling the number of tables. He also revealed that he had strong spatial thinking skills. He literally moved along the table he had set up in his mind. Through his motor actions, he developed the embodied formula (Radford, 2011) of doubling the number of tables and adding the two seats at the end.

Table 16

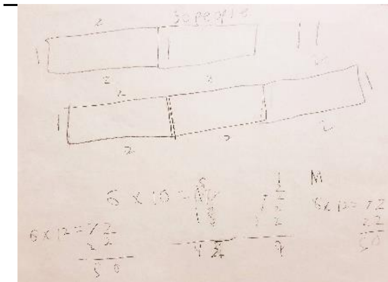
Demonstrating Long Sides of the Table

Actions	**Mode Description**	**Generalization Strategy**
  *Omar*: Then you add two sixes, on each side.	MA: - GA: using pencil upside down as pointer demonstrating the location of the seats; grouping them by long sides of the table motioning pencil forth and back BA: turning his head looking into the direction he is speaking about	FCG: Embodied formula

Part D of the task proposed the use of longer tables with six seats. The boys were not ready developing a direct formula. Both felt more comfortable using a concrete example, so the teacher suggested a table seating 50 people. Omar was the only one out of all four students who drew pictures of the table combinations and numbered the seats. Isaiah watched Omar free-handedly drawing a large rectangle which he partitioned to receive six adjacent rectangles (Table 17). After a lengthy discussion, they calculated with 10 tables and subtracted 2 x 8 for the seats lost in the middle and only one for each end table. That amounted to 42 seats. The boys finally agreed that they needed 12 tables based on the guess-and-check method. They calculated 6 x 12 = 72 and subtracted 22 seats lost. For the entire time of the concentrated discussion, Omar kept drawing and writing the calculations on his paper. In between he emphasized his thoughts and justifications by heavily gesturing based on his mental imagery. His goal was to convince Isaiah who sat with his hands in his lap for the most time listening and talking.

Table 17

Conversation Based on One Partner's Drawings

Actions	Mode Description	Generalization Strategy
	VI: - SD: Omar draws table and numbers the 6 seats on one side. FN: Calculates total seats for 10 and 12 tables	ROL FCG
	MA: - BA: Isaiah gazing at Omar's drawings during conversation - GA: Isaiah keeping hands underneath table holding his pencil; occasionally tapping pencil under table top - GA/GE: Omar gesturing according to his eaxplanations	ROL FCG

When Isaiah talked, he gave the impression that he was more talking himself through the thought process rather than wanting to prove Omar right or wrong. Isaiah was the one student to generalize the growing pattern by creating a direct formula as a functional-conceptual-global strategy. While he did not have the knowledge base for writing a formal equation with conventional variable use, he produced a general expression from his verbal descriptions of the rule for the growing pattern without being prompted. He explained that he was using P for people at each table, E for the estimate of the number of tables, *En* for the lost seats at the end of each table, and *A* would stand for the answer representing the total number of seats. The resulting formula was (*P* x *E*) – *En* = *A*. He assigned abbreviations for each possible value that could change but did not consider that the two end tables only lost one seat each. Neither Isaiah nor Omar attempted to use the formula for their calculations. Instead they continued with the recursive-operational-local calculations reported above.

Discussion: Group B, Garden Party Task

The use of modalities in this task could not have been more different. Iris and Noah worked with the familiar concrete manipulatives and showed a lot of motor action that did not result in a change of generalization strategies from recursive-operational-local to functional-conceptual-global. Omar's motor actions involved his pencil and hands. Whenever he talked, he used gestures that clearly expressed what his spatial imagery of the situation in the story of the task looked like. He created the most detailed pictorial representations for the garden party task, but only for three of the multiple examples the group discussed. Isaiah used his pencil sparsely for writing down fragmented answers to the questions on the worksheet, for sketching three possible table sizes, and for writing the generalized description of his solution on the back of the worksheet.

The garden party task was designed as a story problem without pictorial representations and instruction for the use of any materials. As the participants had demonstrated in the previous two growing pattern tasks, Group A turned to manipulatives to support their thinking about the problem but did not move past the additive strategy of recursive-operational-local generalization. Groups B demonstrated steps toward generalized thinking about the relationship between the numbers of tables and seats in this growing pattern.

Toothpick Pattern Task

The final task presented to the participants was the toothpick pattern task, Well's (2016) adaptation of a common linear algebraic growing pattern. The task opened with clear directions to use toothpicks to recreate concrete models of the first three figures of the growing pattern. The pictorial representation was provided on the worksheet. This was the only task that required the students to use specific manipulatives. Even the students who would usually not rely on or be interested in using concrete materials to picture a mathematical problem would have to involve the manipulatives.

Group A: Findings and Interpretations of Toothpick Pattern

For the first 20 minutes, Iris and Noah worked individually without saying a word. Iris took her time to read and follow the directions on the task sheet step by step. Meanwhile, Noah was busy building three-dimensional tables and chairs with the square tiles, stacking centimeter cubes together forming a right angle, and picking up transparent Cuisenaire tiles before he ever opened the toothpick box. He used blue and green markers to draw the figures on the butcher paper and the worksheet as directed in Part 3. of the task. He even "asked" Iris for the calculator by tapping on her arm, pointing, and forming the words without sound. Later, he used a red marker to draw the figures on the paper covering the table. He then copied the drawing over to his worksheet with a light green marker. In between he used the dry-erase board and marker. The dry-erase marker doubled as stylus for operating the calculator.

At this point, the teacher-researcher reminded the students that they could work together. Only then they started talking about their separate toothpick activities. Twenty-eight minutes into the session, Noah and Iris talked about their findings by stating that two adjacent triangles share one toothpick. Iris insisted that she had to count by threes from one position to the next, because she needed three toothpicks to form Figure 1. Noah stated that after Figure 1 he only had to add two more toothpicks each time. Both students were able to see the relation between the number of triangles and the number of toothpicks needed to form the triangles. Despite prompting questions by the teacher, the two could apply their rule to calculate a total number of toothpicks needed for Figure 1,000.

Discussion: Group A, Toothpick Pattern

It seemed as if neither student in Group A could make a connection to a concrete situation they knew where they had used toothpick in such way. Thus, recognizing the growth rate of the pattern without someone teaching or modeling the process seemed out of reach. The students did not move past the recursive-operational-local strategies of counting and adding on.

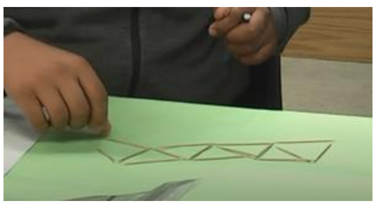
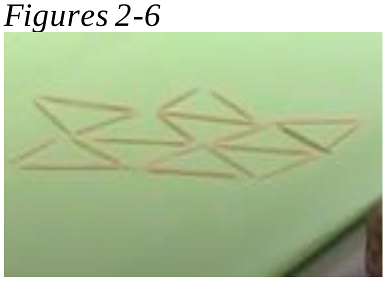
An interesting observation was that Noah appeared to recap the three previous growing patterns by handling the manipulatives he had used before. He recreated the V from the first task and build a table with two chairs on opposite sides, this time three-dimensional. While he did not lay out the stairs again, he used the square tiles and mentioned the stair step growing pattern when talking about finding the growth rate for the toothpick pattern. In a post-observation conversation, he explained that he just liked having things in his hands to help him focus. The presence of all the manipulatives appeared to be a distraction for him rather than an aide to guide him to generalizing the toothpick pattern.

Group B: Findings and Interpretations of Toothpick Pattern

For the toothpick pattern task, both Omar and Isaiah used materials with their motor actions. It was expected that the students would follow the directions and build the figures with the toothpicks. When instructed to draw additional figures on the paper, Isaiah took a ruler to do so. Omar drew free-handedly as before but used the calculator for the operations with greater numbers. The teacher-researcher left it up to the students to decide whether they wanted to build the toothpick models together. They asked for clarification whether they could add to the previous figure or should build each figure separately. The boys decided to work as a team. Both determined quickly that the first figure consisted of three toothpicks and every following figure required two additional toothpicks. Isaiah explained that they used the recursive-operational-local strategy of counting the toothpicks in Figure 6. Since it was four more figures up to Figure 10 and they needed 2 more for each figure, they added eight to the total number of Figure 6 to receive the total number of toothpicks for Figure 10. At this point, the students had demonstrated some functional-conceptual-global generalization strategy by incorporating the relationship between the triangles formed and the toothpicks needed.

Table 18

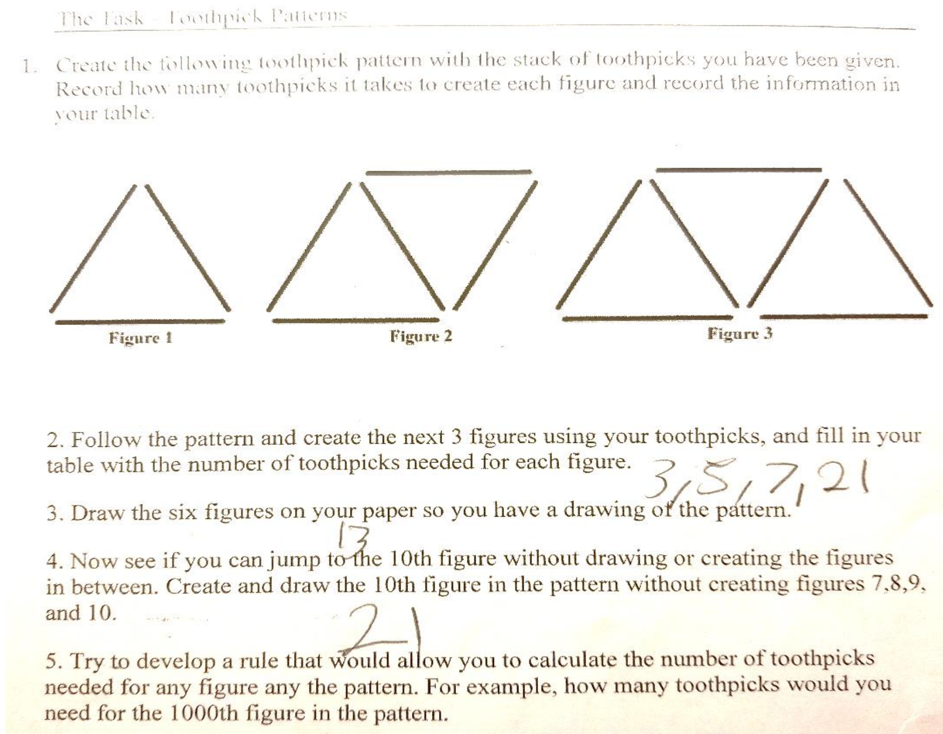
Non-linear Arrangement of Linear Toothpick Pattern

Actions	**Mode Description**	**Generalization Strategy**
 	MA: - GE: building toothpick model together	ROL
 *Figures 2, 3, and 4*  *Figures 2-6*  Figures 7-10 progressed out of one shape.	MA: - GE: building toothpick model together	ROL

The total the group calculated was incorrect because the boys did not extend the pattern in a linear manner past Figure 3. Figure 6 appeared as a hexagon, which only took one toothpick added to Figure 5 to complete. Thus, the growth rate was not the same and the pattern was interrupted. Since the students based the further calculations on Figure 6, the results for Figures 7 to 10 were one toothpick short. Part 5 of the task instructed the students to find the total number of toothpicks for figure 1,000. Isaiah summarized that they had calculated 22 x 100 = 2200, not realizing that a growing pattern builds form Figure 1 up. Although he said 2200, both boys wrote 22,000 on their worksheets as answer to Question 5.

Discussion: Group B, Toothpick Pattern

As expected, all four participants followed directions and built the toothpick figures with the toothpicks provided. However, there appeared to be several things that were not clear to the students. The directions read, "record the information" but did not specify how to record. In the following part, a table is mentioned. None of the students created a table although they had done so in previous sessions. Iris was the only one to write down any totals, skipping from Figure 3 to Figure 10 (Figure 18).


The Task – Toothpick Patterns

1. Create the following toothpick pattern with the stack of toothpicks you have been given. Record how many toothpicks it takes to create each figure and record the information in your table.

Figure 1 Figure 2 Figure 3

2. Follow the pattern and create the next 3 figures using your toothpicks, and fill in your table with the number of toothpicks needed for each figure. 3,5,7,21

3. Draw the six figures on your paper so you have a drawing of the pattern.

13

4. Now see if you can jump to the 10th figure without drawing or creating the figures in between. Create and draw the 10th figure in the pattern without creating figures 7,8,9, and 10.

21

5. Try to develop a rule that would allow you to calculate the number of toothpicks needed for any figure any the pattern. For example, how many toothpicks would you need for the 1000th figure in the pattern.

Figure 17. Iris' toothpick work.

Another point that was unclear to the participants was the instruction to develop a rule for any figure. The students did so, but by using the mode of speech and not in writing a direct formula. The terminology in the toothpick task was different from the previous tasks. Since the teacher intentionally did not prompt the students to write a formula, they skipped the step and rather work on the given example of finding the total number of toothpicks for Figure 1,000.

Summary of Findings and Discussions

It appears that the sixth-grade students in this study had developed their personal preferences of modalities. Informal conversations about the sessions in retrospect confirmed that students do or do not use manipulatives for various reasons. Iris was hiding her insecurity about her mathematical abilities behind a way of looking busy. It was instilled in her, as well as the others, that not doing anything in class was unacceptable and would draw the teacher's negative attention. Noah was compensating for his restlessness by fidgeting with the permissible manipulatives. Omar and Isaiah only picked up the toothpicks, because it was required for the task. Both boys worked the problems out mentally. Omar demonstrated strong special thinking through his gesturing and body language. Isaiah's body language was much subtler. Instead, he talked himself through the processes in a fast manner with a dim voice. His talking did not keep him from listening and both he and his partner learned a lot from each other.

CHAPTER FIVE: CONCLUSION

The findings of the study are founded on the observations of four sixth-grade students at the brink of experiencing their first formal algebra unit. Over the course of 16 sessions, the students collaborated in pairs on carrying out four growing pattern tasks. The first task was a series of linear growing patterns with pictorial representations. The students were not directed to use specific materials in the form of manipulatives to help with the generalization process. The second task was a non-linear growing pattern that was presented graphically but did not require the use of certain materials. The third task, a linear growing pattern, did not provide pictures or graphs in the task introduction and did not mention the use of manipulatives. The first three tasks were based on everyday situations the students likely had experienced before. The fourth and final task was a lesser known linear toothpick pattern with pictorial representations and specific instructions to use toothpicks to aid in their reasoning.

The participants volunteered for the study, which was set as an after-school math club in a more casual environment. A variety of materials were readily available to the students, specifically manipulatives that were commonly used in mathematics classrooms. The analysis of modes the students were using during the key moments of the problem-solving activities and their oral and written work were analyzed to understand their level of mathematical understanding. The purpose of the study was to offer some answers and implications for the following research question.

> In what ways do rising sixth graders draw on multiple modes to engage in algebraic generalization as they solve pattern tasks?

This study used a multimodal lens with the intention to contribute to a better understanding of how younger children embody and express their algebraic thinking as they engaged in generalization problems through the modes they chose (Edwards & Robutti, 2014; Jewitt, 2011; Kress, 2009; Norris, 2004; Radford & Sabena, 2015). Due to the nature of the activities, the students drew on more than one mode at a time. This is referred to as mode overlays, also called co-occurrences or density of modes that were expected to be indicators of the students' level of engagement (Norris, 2004; Radford & Sabena, 2015; Smith, 2014).

This chapter is organized in the following sections. First, I discuss the findings relating to the students' use of (a) recursive-operational-local strategies, (b) functional-conceptual-global strategies, and (c) manipulatives that emerged from the analysis, situated within the research on growing pattern generalization and multimodal reasoning. Second, I draw some conclusions about the role that multimodality plays in growing pattern generalization. Finally, I derive some implications for teaching algebra to elementary students from this work for both future research as well as teaching practices.

Findings

The findings from this study are three-fold. The layers observed during the sixth-graders' growing pattern generalization activities are (a) the selection of the growing pattern tasks, (b) the modes and expressive products the students exhibited, and (c) the generalization strategies they used. The tasks selection exposed the participants to various combinations of linear and non-linear tasks that came with or without pictorial representations. Only the toothpick pattern provided explicit instructions to use manipulatives. The analysis of the observations uncovers a possible correlation between the degree of recursive-operational-local reasoning and the nature of motor actions the students displayed. Those who used the recursive-operational-local strategies almost exclusively produced gestures involving objects from the environment, in this case a wide variety of manipulatives placed on the side of their workspace. The two students who were able to use functional-conceptual-global strategies toward generalization did not use Manipulatives.

Discussion of Recursive-Operational-Local Strategy Use

Without access to the academic language and symbolism, younger children can be challenged to express their thinking related to generalization (Carraher et al., 2008; Mason, 2008). Mathematics on every level is a combination of cognitive, sensory, and bodily activities, and thus controlled by the mind. Growing pattern generalization occurs in two steps. The first step of recognizing the growth rate of the pattern by analyzing the given, concrete figures that mark the beginning of the pattern. In this study, the growing pattern recognition was referred to as the recursive-operational-local level. The analysis of all four participants indicated that they could draw on their recursive-operational-local level of reasoning to attempt to solve pattern problems.

The students in Group A almost exclusively used the additive recursive-operational-local strategy, with one exception. At the beginning of a growing pattern activity, the rate of growth must be determined. It was natural to begin with addition and counting on. The students did not have problems to find the number of additional items needed to build the next higher position model. This was the case as long as the size of the model was manageable. Stepping up from recursive-operational-local reasoning to functional-conceptual-global strategies occurs when the students see a relationship between the ascending position numbers and the respective total numbers of items. At that point, the students begin thinking about a general rule, called direct formula in the dot pattern tasks (Kindt, 2004). Both students kept building greater positions and resorted to drawing them as the space was too small.

Neither student in Group A was capable of truly accomplishing this step with their prior knowledge levels. Noah demonstrated little more advanced thinking. He talked about the position number, but his gestures referred to the total number of items (e.g. Table 5 on page 58). Noah also made a surprising connection to his learning of multiplication facts by using arrays during the non-linear stair steps task. He doubled the number of squares, arranged them in a rectangle, and found the area by using the formula $A = b \times h$. However, he never referred back to the original stair steps by taking half of the area. Thus, true functional-conceptual-global generalization was not evident in this case.

Discussion of Functional-Conceptual-Global Strategy Use

It was a challenge for all four participants to take a step toward the functional-conceptual-global strategies to complete the generalization process. The dot pattern worksheets spelled out step-by-step instructions how to develop what was called a direct formula. The variables were given, and the W-pattern task provided a function table to be filled in. The only student who accomplished the goal of writing and understanding the formula was Isaiah. His partner Omar understood what needed to be done after Isaiah had explained it to him.

In the following tasks, Isaiah's motivation was to become faster in recognizing the rates of growth and the rules. He immediately saw the rate and began thinking of the relationship between the position number and the total number of items. He did not like that he did not have much success with the non-linear stair step task. After the session he asked for the solution, but although he was successful with the W-pattern, he was not able to create direct formulae for the other tasks. He used abbreviations of the words he used for the verbal expressions he gave. The result was an equation written with letters and operational symbols.

Omar caught on to Isaiah's thoughts since Isaiah talked through the process, seemingly more so to himself than to his partner. Omar appeared to develop a mental image of each situation evident through the gestures he made, both empty-handed and with his pencil. His spatial thinking guided the gestures. It almost looked as if he drew his images into the air and he physically showed Isaiah where the items he was talking about were located in the scheme. He sketched the tables and chairs free-handedly on his paper to re-enforce his explanations to his partner who did not draw at all. He was able to generalize (e.g. in the table and chair task, by relating the number of tables to the total number of seats). He did so verbally by multiplying seats per table by number of tables; then he subtracted the seats lost by pushing tables together. This is the type of solution Radford (2010) calls in-action embodied formula.

Discussion of Students' Use of Manipulatives

When higher concepts need to be formed than a person has acquired up to that point, a definition will not have meaning to this person. Only experiences with the new concept and ample examples will help this individual to recognize commonalities to prior knowledge and then to conceptualize the new situation. Manipulatives serve the purpose of aiding with concrete meaning making, mainly for young students with less personal experiences. A crucial step in mathematics teaching and learning is to move past the concrete stage to abstract thinking.

One of the first striking observations was that the two members of Group A picked up one or more of the available manipulatives immediately while or even before reading the task. Every time they chose material that was close to or matched in shape with the items shown or described in the task. For the stair steps and the tables in the following two problems, both chose the square inch tiles from a medium sized tub. Noah even went as far as to pick the colors he wanted to use to make the figures he laid out look more uniform. During the stair step task session, he tinkered with the square tiles and started building three-dimensional things. These things turned into table and chair arrangements in the garden party and the toothpick session.

The first thing both Iris and Noah did was to build the given initial figures of the growing patterns, no matter if it was asked of them or not. They both built onto the first figure and extended it to the following instead of creating individual figures as shown on the worksheets. Noah kept going in consecutive order with the following figures until he completed the one he was looking for, for example Figure 8 in the V-pattern. Iris on the other hand formed her Figures going from 4 to 8, which was pictured on the worksheet without acknowledging the position number. She only understood that 17 dots formed Position 8 after Noah had found the solution and explained it to her. Neither one succeeded in answering the questions about the total of dots in Position 85 or in reasoning whether there could be a position with 35778 dots. While both students in Group A found the rule by which the V-pattern grew, they could not take the next step to greater positions without concrete models.

Iris and Noah obviously had not reached the point where they could leave the concrete level behind and move on to abstract thinking. The use of manipulatives was familiar, even comforting, to them as they did an activity that had the potential to lead them to a solution. It was doing math the way it was supposed to be done. For Noah using manipulatives had become an outlet for his energy. He shared during an informal conversation about the recordings that it always had been difficult for him to sit still. So, he compensated by constantly fidgeting with something. Manipulatives were a welcome outlet for him since he was expected to use them in class where he could get away with playing when the teacher did not pay attention to him. He did exactly that during the sessions as well. Every time he took several square inch tiles and built the same three-dimensional chair and table set. It became apparent that these two students up to that point never had been instructed how to take the next step toward abstract thinking without using manipulatives. Manipulatives had turned into crutches for doing mathematics.

The two boys in Group B did not touch the manipulatives during the sessions until the last day when they were explicitly instructed to do so for the toothpick task. Both picked up pencils and wrote and made some marks on the paper while reading about the tasks. They simply thought about the growth rate by pointing to the pictorial representations on the printed papers or using their mental images. In retrospect, they individually stated that they could just think about a problem like this to get it. They did not need to manipulate anything for it. Isaiah more so than Omar felt the need to talk through his thinking. He was talking fast in a soft voice and quite often said "no wait" when he felt the need to correct himself. His thoughts seemed to be faster than he could articulate them at times.

Other Observations

The boys in Group B listened and reacted to each other's comments appropriately and showed evidence that they learned from each other, both correct and incorrect solutions. For the toothpick pattern, they formed Figure 6 into a hexagon rather than keeping the row of triangles linear. That inevitably led to a falsified rate of growth since there now was one less toothpick as there should have been. The students kept adding two toothpicks and arranged them in a large star pattern up to Figure 10. Then the confusion was too big, and they stopped working. This was evidence that students take things literal and instructions need to be specific and clear. It seemed that a toothpick pattern was not something for which the students necessarily had a reference, as it was the case with the previous tasks.

The participants had all scored in the mid to high range of Level 2 on their fifth-grade standardized exam. During the sessions, it manifested itself that not all of the participants had a well-developed number sense. In Group B, Isaiah quickly calculated 2 x 85 mentally, and Omar calculated mentally as well but questioned his result at first and needed validation from Isaiah to move on. Noah and Iris in Group A needed some time to complete the calculations on paper or a calculator.

Summary

Throughout the sessions, Isaiah was the one participant who ventured furthest into abstract thinking successfully. Omar followed along and seemed intrigued but never showed the confidence his partner reached. Rather, he demonstrated strong spatial thinking based on his gesturing when he explained something, which will be addressed in a following section. Group A seemed quickly overwhelmed by the tasks of finding the number of elements of larger figures they could not lay out with manipulatives anymore. More than once did they attempt to form for example Figure 20, or even Figure 50, not realizing that a multiple of 20 or 50 tiles was needed to build the figure. The intention was to count the elements, but both students had to realize that the task could not be accomplished the concrete way. Rather than giving up though, they kept trying to find the solution and their perseverance was remarkable.

It is evident that there is a correlation between the modalities, specifically the expressive products shown by the students and their ability of thinking in general ways about growing pattern tasks. Gestures involving objects almost exclusively occurred in Group A. These participants demonstrated less developed number sense and lower capability to generalized thinking.

Group B ignored the manipulatives unless the directions called for using them. The pair had the better developed number sense. Their gesturing happened mostly empty-handed or with pencils. Edwards and Robutti (2014) call pencils (pen, pointers, etc.) artifacts, because the students did not use them as primary objects to reconstruct a situation. The pencils just happened to stay in the hands for convenience in case the students had to go back to writing momentarily.

General Conclusions about Research Question

The idea behind observing multimodality in mathematics activities is that it is not solely an abstract subject that requires mental activity. Doing mathematics and reasoning about thoughts and actions is just as important. The multimodal methodology (Edwards

& Robutti, 2014) provides a tool to differentiate among modes the students are choosing to communicate their algebraic thinking. Modes are not set entities and can be defined in variations, and the human mind does not operate linear and can lead others to intended perceptions (Norris, 2004). Discourse analysis is a staple in mathematics education. Multimodality as a framework in the fields of linguistics, art, and others has gained importance in conjunction with technological progress we have experienced since the 1990s. Children articulate topics they want to talk about on their level of languag**e** development, but verbal expression is only one component of communication. I am using a lens of multimodality to understand how they draw on various modes including speech, gesture, posture, facial expression, actions, and gaze (Edwards and Robutti, 2014) to explain notions of generalization in specially designed tasks involving algebraic growth patterns.

Kress (2009) and Norris (2004) and other researchers in multimodality state that speech does not communicate everything a person experiences, feels or thinks. In social interaction, we use a combination of multiple modes such as speech, gestures, gaze, tactility or various forms of representations (Norris, 2004; O'Halloran, 2015). Researchers have demonstrated that in addition to their informal language, younger students can draw on various modes like gestures, body movement, and concrete objects to explain their mathematical thinking (Nemirovsky, 2003; Radford & Sabena, 2015).

Algebraic thinking starts with the concrete experience of numbers and through activities moves towards generalization and abstract reflection (Mason, 2008; Radford & Sabena, 2015; Vygotsky, 1997). The multimodal approach provided a lens for this study for looking at the mathematics learning process from a holistic standpoint. Iris and Noah obviously had not reached the point where they could level the concrete level behind and move on to abstract thinking. The use of manipulatives was familiar, even comforting, to them as they did an activity that had the potential to lead them to a solution. It was doing math the way it was supposed to be done. For Noah, using manipulatives had become an outlet for his energy. He shared during an informal conversation about the recordings that it always had been difficult for him to sit still. So, he compensated by constantly fidgeting with something. Manipulatives were a welcome outlet for him since he was expected to use them in class where he could get away with playing when the teacher did not pay attention to him. He did exactly that during the sessions as well. Every time he took several square inch tiles and built the same three-dimensional chair and table set. It became apparent that these two students up to that point never had been instructed how to take the next step toward abstract thinking without using manipulatives. Manipulatives had turned into crutches for doing mathematics.

Implications for Research and Teaching

Implications for my study are to raise the awareness of the multitude of modes that add to the mathematical thought processes, with or without the use of manipulatives. Teachers put a lot of time and thought into lessons that are intended to guide students from the concrete and hands-on doing of mathematics to conceptual and abstract thinking. While one pair of participants chose for all but one growing pattern tasks not to use any manipulatives, only paper and pencil, both students of the other pair used everything in reach but struggled to think about Figures 20 or 50. The question remains how we as educators can make sure that activities and manipulatives they consider best for a task truly help students in their development of abstract thinking.

Communication in verbal or non-verbal form is first on the list, as a concept will never leave of the one who has the understanding unless he shares it. Sharing thoughts serves several purposes. The person who communicates the concept to someone else goes through a reflection process while speaking or reading his work. Then a connection is established to the other person who might or might not find the same importance in the concept. Symbols are the tools for communication, but there needs to be a consensus about the meaning of them. Culture and language can be barriers that hinder the proper exchange of information. When new concepts are being communicated, it matters if it is a primary concept that can simply be taught by pointing at visuals and making connections to spoken and written words.

Since mathematical concepts are exclusively secondary concepts, a learner's mind must be trained for the use of word and symbol combinations through a multitude of examples. We can use meanings of symbols to our advantage by choosing a meaning appropriate for the circumstance in which it is utilized. The function of explanation is rather important for someone who wants to teach a concept. Three situations can lead to failure. The explaining person either uses (a) an inappropriate schema, (b) the new idea might be too far removed from the schema it is supposed to fit in, or (c) the schema does not have the capacity to assimilate the new concept. The way in which a new concept can be transferred into a reflective activity depends on the developmental stage of the learner, especially when he or she has not left child age yet. A suggested strategy is to think aloud in addition to more individualized visual activities. Symbols are means to communicate structure, which is helpful to transfer often-used exercises into a stage of routine and to remember learned symbols or formulas later. Visual organizers support this kind of structured memorization (Pimm, 1987, 1995; Skemp, 1987; Wertsch, 1990).

For mathematics teachers, it is important to study interrelated structures and the nature of these relations because this affects the psychology of learning mathematics directly. While the integrative function for previously learned things is rather obvious, the tool function for advanced learning and understanding can occur on various levels (Pimm, 1987, 1995; Skemp, 1987). The basic, or primary, concepts are based on experiences we make through our senses and actions. We then make connections to examples for concepts that are at a similar hierarchal level. The more abstracted a concept, the higher is its order. While this is usually not difficult to realize for concrete every day experiences, the learning of mathematical concepts eventually must move almost entirely onto the abstract level.

Social interaction and cultural heritage are extremely influential in this process since we naturally include experiences of others into our own learning. It empowers us to add to common knowledge rather than having to start at point zero every time. At this intersection, teachers of mathematics are playing a crucial role, as they must provide suitable examples and resources that help their students to classify their experiences and abstract the new concepts. Implications for this study are to raise the awareness of the multitude of modes that add to the mathematical thought processes, with or without the use of manipulatives. Teachers put a lot of time and thought into lessons that are intended to guide students from the concrete and hands-on doing of mathematics to conceptual and abstract thinking. Group B chose for all other pattern tasks not to use manipulatives for growing pattern tasks, only paper and pencil, while Group A used everything in reach but struggled to think about larger Figures such as 20 or 50. The question remains how we as educators can make sure that it really helps students in their development of abstract thinking.

Through constant training and doing algebra, students ideally reach a stage where they do not have to think through every step anymore before and while acting. A thinking process causes reflection on different levels of the process. Learners are re-thinking the concept structure of an activity and either teach it to someone, correct themselves, if necessary, or add a new concept or schema. This reflective intelligence consists of activities that lead into mathematical generalization, which occurs when a concrete example is transformed and noted as an algebraic formula for future use in similar examples (Pimm, 1987, 1995; Skemp, 1987). A teacher must be aware not only of his or her own intuitive and reflective intelligence, but also of the stage of development of the individual student. The goal is to address the variety of learning styles in a classroom and to cater to visual and tactile learners.

Mathematics teachers have the difficult task on hand as their teaching and learning success is based on agreement and reasoning. Skemp (1987) elaborates on the benefits of verbal communication and group work for the learning process where the teacher takes the role of a discussion leader. Teachers may prompt students to reflect upon and communicate their thinking. Knowledge must be transformed into action, and it is a process that follows through three stages. The “knowledge that” requires the existence of a schema useful for the application. The “knowledge how” is the ability to extract a plan of action from the schema. “Being able” is the stage of putting this plan into action. In this context, knowledge means to have an organized structure thereof rather than a random collection of factual knowledge. The acquisition of knowledge can be compared to a first look at a road map, a system of understanding that never will happen the same way again (Meadows, 2008). As soon as this established in the brain's directory system, the transition to deeper understanding begins and the knowledge is eventually conceptualized. It is the first piece of a new experience to which any similar experiences will later be connected. Finally, the process of abstraction forms a concept that will become part of a generalized schema from where the knowledge is retrievable. This is called the resonance model (Meadows, 2008; Pimm, 1995; Skemp, 1987).

True understanding only can happen through connections. Many students are turned off at a young age when it comes to mathematics learning. Too many mathematicians and mathematics educators are not willing to take the route of teaching relational understanding of mathematical concepts. At first, this approach requires more time, but it enables the students to expand this knowledge of relations to other areas later and makes them more independent, motivated and self-directed learners. A mathematics learning system can be prior knowledge being connected to a newly presented task, which then initiates new learning through multiple modes of communication such as spoken and written language, interactive visual inquiry, motor actions, formal notation, imagery, and others (Edwards & Robutti, 2014; Meadows, 2008; Radford, Bardini, Sabena, Diallo, & Simbagoye, 2005; Skemp, 1987). Skemp, (1987), based on the realization that "to understand something means to assimilate it into an appropriate schema" (p. 29), defines three kinds of understanding. In instrumental understanding (a), we find rote memorization of rules and their mechanical application as the main goal. It is not of importance here to know how this rule works; only delta one is engaged to reach the primary goal of getting the correct answer and possibly to please the teacher. Relational understanding (b) allows the learner to draw from general understanding of mathematical relationships. This is mostly achieved through delta two activities that result in self-motivation and personal satisfaction of reaching goals although the process takes longer than instrumental understanding, and logical, or formal, understanding (c) is directly related to the use of symbols especially occurring in mathematics. It describes the ability to connect steps of reasoning that are connected through symbols to complete a logical sequence. The connections between the types of understanding are categorized into two modes of mental activity, the intuitive dimension occurring in delta one and the reflective dimension of delta two activities. This system is based on feedback by the teacher functioning as facilitator, other students, objects that can be manipulated for demonstrations or calculations, or other environmental modes having an influence on the momentary learning process (Edwards & Robutti, 2014; Meadows, 2008; Pimm, 1995).

Communication is the only possibility to let the world know about concepts because it is an individual, internal effort to develop conceptual structure. The symbols change their appearance from a mental object into a physical object brought onto the paper or transformed into sound waves. The internal symbols are the basis of what Skemp (1987) calls the deep structure of mathematical thinking (p. 177). External symbol systems represent the surface structure, the external systems for which we have much less variety available than for the deep structure within our minds. Thus, mathematical manipulatives are symbols that enable the learner to do mathematics by forming complements to concepts using hands and fingers (Pimm, 1995). The deep structures are the conceptual, the schema development that occurs internally. Both surface and deep structures are connected in constant exchange of information.

The observations in this study indicate that students with less ability and confidence in their generalization skills communicate this through a higher frequency of motor actions involving objects. Future research can contribute to the understanding of how individual elementary students perceive the process of learning algebraic generalization by focusing in on differences within motor actions. One possible application could be incorporating multimodal analysis based on affordances for mathematics learning in diagnostics for young students' skill levels.

Conclusion

Vygotky (1997) is rather clear about the fact that the potential of each child is different, depending on physical and mental development, and on personality and subject preferences. Even when the individual task is performed on the same level, the potential for growth between two candidates can vary tremendously. Vygotsky's (1997) realization that academic instruction cannot be given under the premise that one method fits all has not lost any actuality. It is still the driving force in the mathematics education research debate as to why "what works" studies are not useful for the development of generalized curricula for students living in different contexts. The multimodal methodology does by far not provide the convenience of reaching a general rule for children of certain age groups, but it can contribute to teachers' awareness of their students' individualities in processing and solving algebraic generalization tasks.

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APPENDIX A: IRB APPROVAL RENEWAL

IRB **uncc-irb@uncc.edu via adminliveunc.onmicrosoft.com** A pr 1 7

to me, uncc-irbis, afernan2, mstepha1

To: Ute Lentz
Curriculum and Instruction

From: IRB

Approval Date: 4/17/2018
Expiration Date of Approval: 4/16/2019
RE: Notice of IRB Approval by Expedited Review (under 45 CFR 46.110)
Submission Type: Renewal
Expedited Category: 7.Surveys/interviews/focus groups
Study #: 15-0414

Study Title: Algebraic Thinking in Sixth Grade

This submission has been approved by the IRB for the period indicated.

Study Description:

The purpose of the study is to investigate ways in which sixth-grade students contribute to the algebraic thinking of individual group members when finding particular and general solutions to problems. The small group setting requires the students to solve problems in cooperation with each other. Another focus of the study is the common language the nine to eleven-year-old children will be using while solving pre-algebraic tasks. Will the students apply the appropriate vocabulary implemented through the Common Core Curriculum Standards?

Investigator's Responsibilities:

Federal regulations require that all research be reviewed at least annually. It is the Principal Investigator's responsibility to submit for renewal and obtain approval before the expiration date. You may not continue any research activity beyond the expiration date without IRB approval. Failure to receive approval for continuation before the expiration date will result in automatic termination of the approval for this study on the expiration date.

If applicable, your approved consent forms and other documents are available online at http://uncc.myresearchonline.org/irb/index.cfm?event=home.dashboard.irbStudyManagement&irb_id=15-0414.

You are required to obtain IRB approval for any changes to any aspect of this study before they can be implemented.

Data security procedures must follow procedures as approved in the protocol and in accordance with ITS Guidelines for Data Handling.

Any unanticipated problem involving risks to subjects or others (including adverse events) should be reported to the IRB by contacting the Compliance Office. uncc-irb@uncc.edu

This study was reviewed in accordance with federal regulations governing human subjects research, including those found at 45 CFR 46 (Common Rule), 45 CFR 164 (HIPAA), 21 CFR 50 & 56 (FDA), and 40CFR 26 (EPA), where applicable.

APPENDIX B: PARENTAL CONSENT LETTER

UNCCHARLOTTE

College of Education
9201 University City Blvd.
Charlotte, NC 28223

Informed Consent for Dissertation Research on
Algebraic thinking in sixth grade

Project Title and Purpose:

Your child is invited to participate in a dissertation research study titled *Algebraic thinking in sixth grade*. This is a study to investigate how sixth grade students learn and talk about algebraic problem solving individually and by cooperating in a small group setting.

Investigator(s):

This study will be conducted by Mrs. Ute Lentz, teacher at Kannapolis Middle School and doctoral student at the College of Education in Curriculum and Instruction. Dr. Michelle Stephan, professor in the Department of Middle and Secondary Education, is the responsible UNCC faculty members and co-investigator.

Description of Participation:

If you give permission for his or her participation, your child will be asked to join a focus group of two students attending the sixth grade of Kannapolis Middle School. The group will meet for three to four weekly meetings in the afternoons and work on an activity for approximately 60 minutes after school (until 3:15).

The sessions will be videotaped, the work samples collected, and your child may be invited for an individual audio-taped exit interview about his or her experiences during the activities. The video and audio interview materials will be used for observation and transcript purposes only. It is important for you to know that the data collected will be kept confidential and used in a way that will not reveal the identity of your child.

The data collected by the investigator will be de-identified and kept confidential. The following steps will be taken to ensure this confidentiality. Your child's name will not appear in any documents. All collected data will be stored on password-protected electronic memory devices in a locked cabinet only accessible by the investigator and the immediate research staff listed above. Data will be destroyed after three years. All paper data will be shredded, and electronic data will be dismantled and, or rendered useless.

Length of Participation

Your child's participation in this project will begin on February 8, 2018 (please see attached schedule), and last four weeks. If you agree to the participation of your child, he or she will be one of up to 18 participants in this study. The participation has been determined by the order in which the permission forms have been received.

Risks and Benefits of Participation:

There are no known risks of physical or emotional harm during the participation in this study. However, there may be risks which are currently unforeseeable. Through data gathered during this study, we hope to identify the students' perceptions of learning and doing algebraic problem solving.

Volunteer Statement:

Your child is a volunteer. The decision for his or her participation in this study is completely up to both of you. If you and/or your child decide to be in the study, you may stop at any time. He or she will not be treated any differently if one or both of you decide not to participate or to stop once the study has started.

Confidentiality versus Anonymity:

The data collected by the Investigator will not contain any identifying information or any link back to your child or his or her participation in this study. The following steps will be taken to ensure this confidentiality:

- His or her name will never be mentioned in the reported results.
- You and/or your child may end the participation at any time.
- He or she may choose not to respond to any question.
- He or she may choose not to participate in group discussions.
- Only the immediate research staff will have access to the raw data.
- All gathered data will be password-protected and stored in a locked cabinet.

Fair Treatment and Respect:

UNC Charlotte wants to make sure that you and your child are treated in a fair and respectful manner. Contact the University's Research Compliance Office (704.687.1871, uncc-irb@uncc.edu) if you have any questions about how you are treated as a study participant. If you have any questions about the project, please contact Mrs. Ute Lentz (704.932.6102 ext. 31402, ute.lentz@kcs.k12.nc.us) or Dr. Michelle Stephan (704.687.8875, michelle.stephan@uncc.edu).

Approval Date

This form was approved By the UNC Charlotte Internal Review Board for Research with Human Subjects on May 17, 2017, for use of one year. Protocol # 165015.

Participant and Parent Consent

I have read the information in this consent form. I have had the chance to ask questions about this study, and those questions have been answered to my satisfaction. I am at least 18 years of age, and I agree to the participation of my child in this research project. I understand that I will receive a copy of this form after it has been signed by me and the principal investigator.

_____________________________ _____________________________ ____________
Participant Name (PLEASE PRINT) Participant Signature DATE

_____________________________ _____________________________ ____________
Parent Name (PLEASE PRINT) Parent Signature DATE

____ ____________
Investigator Signature DATE

APPENDIX C: MINOR ASSENT LETTER

UNC CHARLOTTE

College of Education
9201 University City Blvd.
Charlotte, NC 28223

Dear ________________________,

My name is Mrs. Ute Lentz, and I am teacher at Kannapolis Middle School. I am also a student at the University of North Carolina at Charlotte where I am studying mathematics education to become a philosophical doctor of curriculum and instruction. My final research project is to find more out about how sixth grade students learn and do mathematics when they solve algebra problems together in a small group and how they talk about math.

If you want to be part of the math group, I will ask you to participate in an afternoon group with up to four other students from your grade level. Your group will meet four to six times, once a week after school for 60 minutes. During each session you will work together solving a math problem and discussing ways to get to a solution. This is a club and there will be no test or grades of your work. In the end, I may interview you individually to ask some questions about your personal experiences during the group sessions.

Your group will be videotaped by a camera and microphone to help me remember everything that you talked about the group activities when I will write my research paper. There may be a second teacher, Dr. Michelle Stephan from UNCC, in the room to help with the observations. The videos and your work will be locked up in a safe place. The study is confidential, which means that your name will not be mentioned in my research paper and nobody will be able to identify you.

You can ask questions at any time. Your participation is voluntary. You can stop participating at any time without consequences. I hope that this study will show how you and other students learn and talk about mathematics when you are working together. You will be one of eighteen total participants who will meet after school for four to six afternoon sessions. There will be no risk for you to get hurt during this study.

If you want to participate in my study, please sign your name and return this letter to me.
Thank you,
Mrs. Lentz

_______________________________________ _____________________

Signature of Participant DATE

_______________________________________ _____________________

Signature of Investigator DATE